ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર–ક્રમાંક
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MATHEMATICS

Standard 11

(Semester I)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board has prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is a pleasure for the Gujarat State Board of School Textbooks, to place before the sudents this textbook of **Mathematics** for **Standard 11 (Semester I)** prepared according to the new syllabus.

Before publishing the textbook, its manuscript has been fully reviewed by experts and teachers teaching at this level. Following suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to ensure that this textbook is interesting, useful and free from errors. However, we welcome any suggestions, from people interested in education, to improve the quality of the textbook.

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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India

(A) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(B) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(C) to uphold and protect the sovereignty, unity and integrity of India;

(D) to defend the country and render national service when called upon to do so;

(E) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(F) to value and preserve the rich heritage of our composite culture;

(G) to protect and improve the natural environment including forests, lakes, rivers and wild life, and to have compassion for living creatures;

(H) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(I) to safeguard public property and to abjure violence;

(J) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement;

(K) to provide opportunities for education by the parent or the guardian, to his child or a ward between the age of 6-14 years as the case may be.

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About This Textbook...

The Gujarat Secondary and Higher Secondary Education Board has prepared syllabus for 10 + 2 level in accordance with NCERT syllabus. The syllabus for science stream at higher secondary level has been prepared by school teachers, college teachers and university teachers keeping in mind the national level entrance examination and the core syllabus prepared by COBSE.

The State Government has decided to implement semester system at 10 + 2 level and accordingly this is the textbook based on syllabus for semester I. The Gujarat Secondary and Higher Secondary Education Board has decided to hold OMR system examination for semester I and III and detailed examination for semester II and IV. Hence at the end of each chapter objective questions have been given.

The first draft of the textbook was prepared in English so that the students can have advantage of current affairs at national level originally in English language and reviewed by a panel of experts from schools, colleges and universities. This workshop lasted for four days and suggestions from experts had been obtained and necessary amendments were made in both the English draft and Gujarati translation of the textbook draft.

Then there was a workshop for four days to review the translation by the teachers from schools, colleges and universities of Gujarat. Again suggestions from experts were invited and discussed and necessary changes were made. Then the final draft was prepared and placed before the Gujarat Secondary and Higher Secondary Education Board. A panel of experts and members of syllabus committee reviewed the draft and had discussions with the authors and finally the textbook was ready for printing. The book was also reviewed by a language expert for the English language.

The chapter I deals with mathematical reasoning. It is the base of logical thinking to be applied furthur in development. The chapters on set theory and functions apply these ideas and construct a base for foundation of mathematics. The fourth and the fifth chapters introduce basic trigonometric functions. The chapter six deals with coordination of geometry and consolidates the base studied in std. 10. The chapter seven deals with permutations and combinations and they are useful for study of probability and the binomials to be studied in semester II. Combinatorial approach is also explained. The chapter eight deals with graphs of linear inequalities useful in statistics and optimisation problems. The four colour diagrams are used to explain concepts in this chapter. The last two chapters are based on statistics and probability.

Necessary diagram, graphs and figures are used to explain concepts. Attractive two colour printing and four colour title page and four colour graphs for linear inequalities add to the features of this book. The concepts are explained in lucid language. Plenty of illustrations and graded exercises are given. The concepts are explained so easily and elaborately that any student residing at the extreme end of the state can understand by own efforts.

Some teachers of CBSE schools also reviewed the draft and praised it whole heartedly. The information given in the textbook is at par with the CBSC textbook and the whole text is completely written according to NCERT syllabus. Plenty of examples serve the purpose of escalating the level of understanding.

When the state is celebrating golden jubilee of the birth of the Gujarat State, this is a revolutionary step by the state to put a student of Gujarat State on national map. This is an humble try to enable students to appear in various competitive examinations. We have concentrated on giving concepts in details and not sketchy information.

The textbook has been written in a very short time. Inspite of all the precautions, some errors may have entered through. We welcome suggestions to enhance the quality of the textbook.

- Authors

Chapter 1

MATHEMATICAL REASONING

1.1 Introduction

In this chapter we will study an important tool to study mathematics. The ability to reason gives us proper guidance to study mathematics.

In mathematics, mainly two kinds of reasoning occur. One is inductive reasoning. Here we observe some pattern and prove some results inductively. We will develop this method in the chapter on **mathematical induction** later on. Another is deductive reasoning which we intend to study in this chapter. Here is an example.

If $\alpha\beta = \alpha\gamma$ and $\alpha \neq 0$, prove $\beta = \gamma$ for $\alpha, \beta, \gamma \in R$.

Proof : Since $\alpha \neq 0$, α^{-1} exists,

$\therefore \quad \alpha^{-1}(\alpha\beta) = \alpha^{-1}(\alpha\gamma)$ **(multiply both sides by α^{-1})**

$\therefore \quad (\alpha^{-1}\alpha)\beta = (\alpha^{-1}\alpha)\gamma$ **(associative law for multiplication)**

$\therefore \quad 1 \cdot \beta = 1 \cdot \gamma$

$\therefore \quad \beta = \gamma$

Here, we argue step-wise using known mathematical results and deduce $\beta = \gamma$.

Let us take another example. Every non-zero real number is either non-negative or negative. Suppose x is not non-negative in a problem under some conditions. Hence x is negative. This is as well an example of deductive reasoning.

1.2 Statements

Consider following sentences :

(1) In 2010, India had a woman president.

(2) India won world cup in T-20 cricket in 2010.

First is true and second is false. Such sentences are called statements.

Statement : A sentence is called a statement, if it is either true or false but not both and its truth or falseness can be definitely decided upon without ambiguity. It is also called a mathematically acceptable statement.

Following are some of the examples of statements. Whether the statement is true or false is given in the bracket.

(1) The product of two positive numbers is positive. (True)

(2) 1 is a prime (False)

(3) $2 + 2 = 5$ (False)

(4) The square of every real number is non-negative. (True)

(5) 1 is the only real number whose square is itself. (False)

Consider the following sentence.

For real numbers x and y, $xy > 0$.

This depends upon x and y. If $x = 3$, $y = 2$, it is true.

If $x = 2$, $y = -1$, it is false. This is an ambiguous sentence. So it is not a statement, because its truth or falseness depends upon the value taken by the variable.

Now consider the following sentences.

(1) Please pray for peace for Martyrs in 26/11 attack on Bombay.

(2) What a beautiful sunset it is !

(3) Get lost.

(4) Where is Gandhinagar ?

Here (1) is a request, (2) is an exclamation, (3) is an order, (4) is a question. None of them can be decided as true or false. So they are not statements.

Consider, 'Today is Monday'. This sentence is true on Mondays and false otherwise. So sentences involving variable time like 'today', 'tomorrow', 'yesterday' etc. are not statements.

Similarly, sentences involving variable places like 'here', 'there', etc. pronouns like 'he', 'she' etc. are also not statements.

For example (1) 'Gandhinagar is near'. But from where ?

(2) 'He is brilliant.' But who ?

Now consider the following sentence :

There are $25 \times 60 \times 60$ seconds in a day. This also involves variable time, 'a day'. But it is certainly false, as there are only 24 hours in a day. So this sentence is a statement.

Thus ultimate criteria to decide whether a sentence is a statement or not is to determine its truth or falseness without doubt. Generally statements are denoted by small letters p, q, r,... etc. For example,

p : There are 35 days in February.

p is a false statement.

Example 1 : Determine which of the following are statements and give reasons for your answer.

(1) Tomorrow is a holiday.

(2) For real number x, $[x]$ is an integer.

(3) He died young.

(4) Galois was a mathematician. He died young.

(5) For real x, $x \cdot 0 = 0$

(6) How far is Himalayas ?

(7) Stand in a queue !

(8) For non-zero numbers x and y, $x^2 + y^2 \neq 0$.

(9) $3^2 = 9$

(10) Pythagoras was a mathematician.

(11) Let us unite !

(12) Save energy !

Solution :

(1) It is not a statement. It involves variable time.

(2) Though x is a variable, it is a true statement for every x. So it is a statement.

(3) Pronoun is used. It is not a statement. Whom does 'he' refer to ?

(4) Here 'he' refers to Galois. This is a statement.

(5) Though x is variable, the result is true for all real x, so it is a statement.

(6) It is a question. Hence it is not a statement.

(7) It is an order. Hence it is not a statement.

(8) This is true for all non zero numbers x and y. Hence it is a statement.

(9), (10) are statements.

(11), (12) Both are requests. They are not statements.

Exercise 1.1

Determine which of the following are statements and give reasons ?

1. $3^2 + 4^2 = 5$
2. If x is a natural number, 2^x is even.
3. Please stand up !
4. What a horrible movie this is !
5. Socrates was a mathematician.
6. He is a scientist.
7. Venice is in Rome.
8. 2011 world cup for one-day cricket was jointly held by India, Sri-lanka and Bangladesh.

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1.3 Simple Statement and its Negation

A statement which cannot be broken into two statements is called a simple statement.

For example, '2 + 2 = 5' is a simple statement.

Negation : The negation of a statement is denial of the truth of the statement i.e. a statement which is true or false according as the given statement is false or true respectively.

p : Ahmedabad is the business capital of Gujarat.

Its negation would be, 'Ahmedabad is not the business capital of Gujarat' or 'It is false that Ahmedabad is the business capital of Gujarat,' or 'It is not true that Ahmedabad is the business capital of Gujarat'.

If a statement p is true, its negation would be false and vice versa. Negation of p is denoted by $\sim p$.

If p is true, we say it has truthvalue T and if p is false, we say it has truthvalue F.

Thus $\sim p$ has truthvalue T or F, according as p has truthvalue F or T respectively.

Example 2 : Give negations of following statements.

(1) $3 \times 2 = 6$

(2) Christmas is celebrated on 25th of December.

(3) Diwali marks the end of current Hindu year.

(4) To get admission to engineering course, mathematics is a compulsory subject at $10 + 2$ level.

(5) Mathematics is the queen of sciences.

Solution : (1) $3 \times 2 \neq 6$ or it is false that $3 \times 2 = 6$

(2) Christmas is not celebrated on 25th of December.

(3) Diwali does not mark the end of current Hindu year.

(4) To get admission to engineering course, mathematics is not a compulsory subject at $10 + 2$ level.

(5) Mathematics is not the queen of sciences.

Exercise 1.2

Give negations of the following statements.

1. $2 + 2 = 5$
2. Area of a square is given by the formula $A = \pi r^2$.
3. A cube is a plane figure.
4. Georg Cantor developed the set theory.
5. Amitabh Bachchan is the brand ambassador of Gujarat tourism.
6. $2 + 2 = 2^2$
7. For natural number $x \geq 3$, $x + x = x^2$
8. Ice is hot.

*

1.4 Compound Statements using Connectives

Sometimes simple statements are combined to yield a new statement called a compound statement. **'Or' and 'And' are such connectives called logical connectives.** They are used to form a compound statement.

Conjunction : When two or more simple statements are joined using the logical connective 'and', the compound statement obtained is called the conjunction of two given simple statements and the constituent simple statements are called component statements.

Let $p : 3 + 2 = 5$, $q : 5 \cdot 2 = 10$.

Their conjunction is '$3 + 2 = 5$ and $5 \cdot 2 = 10$'.

Conjuction of p and q is denoted by symbol $p \wedge q$. (Read as p and q)

Thus $p \wedge q : 3 + 2 = 5$ and $5 \cdot 2 = 10$.

The compound statement $p \wedge q$ has the truthvalue T when both of its component statements are true, otherwise it has truthvalue F.

When all of the component statements are true then and only then, the compound statement obtained by conjuction of several simple statements applying logical connective 'and' is true. If at least one of the component statement is false, then the compound statement obtained by applying 'and' is false.

Example 3 : Identify component statements in the following compound statement and find truthvalue of the compound statement.

Ahmedabad is in Gujarat and $3 + 2 = 6$.

Solution : Let p : Ahmedabad is in Gujarat

$q : 3 + 2 = 6$

The given statement is $p \wedge q$. Obviously $q : 3 + 2 = 6$ is false.

Hence the conjunction $p \wedge q$ is false.

Example 4 : Identify component statements in the following compound statements and determine the truthvalue of the compound statement.

(1) Kanpur is in India and $7 \times 5 = 35$.

(2) Delhi is the capital of Gujarat and $7 \times 5 = 75$.

(3) Ahmedabad and Vadodara are cities of Gujarat.

(4) For real x, $x^2 \geq 0$ and $1^2 = 1$.

(5) Every angle is acute and has measure less than 90.

Solution : (1) Let p : Kanpur is in India.

$q : 7 \times 5 = 35$

Given statement is $p \wedge q$. p is true and q is true.

Hence $p \wedge q$ is true.

(2) Let p : Delhi is the capital of Gujarat.

$q : 7 \times 5 = 75$

p and q are both false and hence given statement $p \wedge q$ is also false.

(3) Let p : Ahmedabad is a city of Gujarat.

q : Vadodara is a city of Gujarat.

p and q both are true and hence $p \wedge q$ is also true.

(4) Let p : For real x, $x^2 \geq 0$

$q : 1^2 = 1$

Both p and q are true and hence $p \wedge q$ is true.

(5) Let p : Every angle is acute.

q : Every angle has measure less than 90.

p and q are both false and hence $p \wedge q$ is false.

Disjunction : When two or more simple statements are joined by using logical connective 'or', a compound statement is formed. It is called the disjunction of given statements. If p and q are given statements, their disjunction is denoted by $p \vee q$. (Read as p or q). The constituent statements are called component statements.

If a compound statement is formed using connective 'or', it is true if any one or more of its constituents are true i.e. if at least one of its constituent statements has truthvalue T.

Example 5 : Identify constituent statements in the following compound statements and determine truthvalue of the compound statements.

(1) $3 + 4 = 7$ or $2 + 2 = 4$

(2) Every prime is odd or Every odd number is a prime.

(3) Gujarat is a state of India or Ahmedabad is in Maharashtra.

(4) There are 5 days in a week or there are 24 hours in a day.

(5) Socrates was a mathematician or Socrates was a philosopher.

Solution : (1) Let $p : 3 + 4 = 7$

$q : 2 + 2 = 4$

Obviously p and q are true. Since at least one of p or q (here both) is true, $p \vee q$ is true.

(2) Let p : Every prime is odd.

q : Every odd number is a prime.

p is false as 2 is an (in fact only) even prime.

q is false as 9 is odd but not a prime.

$\therefore$ $p \vee q$ is false.

(3) Let p : Gujarat is a state of India.

q : Ahmedabad is in Maharashtra.

Since p is true, $p \lor q$ is true.

(4) Let p : There are 5 days in a week.

q : There are 24 hours in a day.

Since q is true and hence $p \lor q$ is true.

(5) Let p : Socrates was a mathematician.

q : Socrates was a philosopher.

Since q is true and hence $p \lor q$ is true.

Negation of a compound statement :

We know how to write negation of a simple statement and we remember, it is denoted by $\sim p$. Naturally $\sim(\sim p) = p$.

$\sim p$ has truthvalue F or T respectively according as p has truthvalue T or F.

$\therefore$ $\sim(\sim p)$ has truthvalue T or F according as $\sim p$ has truthvalue F or T i.e. p has truthvalue T or F.

$\therefore$ $\sim(\sim p) = p$

Now we will find out negation of disjunction or conjunction of simple statements.

Rule (1) : $\sim(p \land q) = (\sim p) \lor (\sim q)$ (2) : $\sim(p \lor q) = (\sim p) \land (\sim q)$

Negation of conjuction of two simple statements is the disjunction of negations of component statements.

Negation of disjunction of two simple statements is the conjunction of negations of component statements.

Example 6 : Find negations of the following statements.

(1) 5 is an integer and $5^2 = 25$

(2) $3^2 = 9$ and $(-3)^2 = 9$

(3) $1^2 = 1$ or $1^3 = 1$

(4) $\frac{1}{2} \in N$ or $\frac{1}{2} \in Q$

(5) $\sqrt{2} \in R$ or $\sqrt{2} \notin R$

(6) Gandhiji was born in Porbandar and Porbandar is in Gujarat.

Solution : (1) Let p : 5 is an integer.

q : $5^2 = 25$

$\sim p$: 5 is not an integer.

$\sim q$: $5^2 \neq 25$

Given statement is $p \wedge q$. Its negation is $(\sim p) \vee (\sim q)$.

Thus the negation is : 5 is not an integer or $5^2 \neq 25$.

(2) Let $p : 3^2 = 9$

$q : (-3)^2 = 9$

$\sim p : 3^2 \neq 9$

$\sim q : (-3)^2 \neq 9$

Given statement is $p \wedge q$. Its negation is $(\sim p) \vee (\sim q)$.

The negation of the compound statement is : $3^2 \neq 9$ or $(-3)^2 \neq 9$.

(3) Let $p : 1^2 = 1$

$q : 1^3 = 1$

$\sim p : 1^2 \neq 1$

$\sim q : 1^3 \neq 1$

$\sim(p \vee q) = (\sim p) \wedge (\sim q)$

$\therefore$ The negation of the compound statement is : $1^2 \neq 1$ and $1^3 \neq 1$.

(4) Let $p : \frac{1}{2} \in N$

$q : \frac{1}{2} \in Q$

$\sim p : \frac{1}{2} \notin N$

$\sim q : \frac{1}{2} \notin Q$

$\sim(p \vee q) = (\sim p) \wedge (\sim q)$

$\therefore$ The negation of the compound statement is : $\frac{1}{2} \notin N$ and $\frac{1}{2} \notin Q$.

(5) Let $p : \sqrt{2} \in R$

$q : \sqrt{2} \notin R$

$\sim p : \sqrt{2} \notin R$

$\sim q : \sqrt{2} \in R$

$\sim(p \vee q) = (\sim p) \wedge (\sim q)$

$\therefore$ The negation of the compound statement is : $\sqrt{2} \notin R$ and $\sqrt{2} \in R$.

(6) Let p : Gandhiji was born in Porbandar.

q : Porbandar is in Gujarat.

$\sim p$: Gandhiji was not born in Porbandar.

$\sim q$: Porbandar is not in Gujarat.

$\sim(p \wedge q) = (\sim p) \vee (\sim q)$

$\therefore$ $(\sim p) \vee (\sim q)$: Gandhiji was not born in Porbandar or Porbandar is not in Gujarat.

A note on use of 'and' and 'or' :

Sometimes the word 'and' is not used as a logical connective but it joins two words.

(1) 'A mixture of oil and water cannot be used for maintenance of instruments.' Here 'oil' and 'water' are joined by 'and'. This is not a conjunction of two statements.

(2) 'Team of Charlie and Chaplin entertained children and adults alike'.

Here two words 'Charlie' and 'Chaplin' are connected by 'and'. Similar is the case with 'children and adults'. Here 'and' connects two words and does not act as a logical connective.

(1) In higher secondary science stream a student can opt for mathematics or biology.

Here 'or' is used in **'inclusive'** sense. A student can opt for both the subjects.

(2) Bharat will go abroad for further studies or will study advanced mathematics in India immediately after passing 12th standard.

Here the word 'or' is used in **'exclusive'** sense. Both the actions cannot take place at the same time.

Example 7 : Determine where 'or' used in following example is in inclusive sense or exclusive sense.

'Two distinct coplanar lines intersect in a point or are parallel.'

Solution : Let p : Two distinct coplanar lines intersect in a point.

q : Two distinct coplanar lines are parallel.

The given disjunction is $p \vee q$ and it is in **exclusive** sense as lines intersecting in a point and to be parallel are exclusive events.

Example 8 : Find the constituent statements in the given compound statement and determine whether the connective 'or' is used in inclusive or exclusive sense. Also determine truthvalue of the compound statement.

'Real number π is irrational or rational.'

Solution : Let p : Real number π is irrational

q : Real number π is rational.

Obviously being a rational number and an irrational number at a time is impossible. Hence 'or' is used in exclusive sense.

The given statement is $p \vee q$. p is true and q is false. Hence $p \vee q$ is true.

Example 9 : Find the constituent statements of the following compound statement and determine whether 'or' is inclusive sense or exclusive sense. Find truthvalue, negation and the truthvalue of the negation.

'A proof of identity is PAN card or a bank passbook.'

Let p : A proof of identity is PAN card.

q : A proof of identity is a bank pass-book.

Here the statement is $p \vee q$. 'or' is used in the inclusive sense. A person can have both the identity proofs.

p and q both are true. Hence $p \vee q$ is true.

The negation is $(\sim p) \wedge (\sim q)$: A proof of identity is neither a PAN card nor a bank pass-book. Obviously $\sim p$ and $\sim q$ are false. Hence $(\sim p) \wedge (\sim q)$ is false.

Exercise 1.3

1. Write the constituent component statements and find the truthvalue of the compound statement.
   (1) $3 + 7 = 5$ and $5^2 = 25$
   (2) $3 + 7 = 10$ and $10^2 = 100$
   (3) A triangle has three sides and three angles.
   (4) A quadrilateral has four sides and four angles.
   (5) Sum of the measures of all the angles of a triangle is 180 or 360.
   (6) $2 + 2 = 5$ and $5 + 2 = 25$
   (7) 1 and 2 are roots of $x^2 - 3x + 2 = 0$
   (8) $1^3 = 1$ or $3^2 = 9$
   (9) $x^2 = x$ is satisfied by 1 and 0.
   (10) 0 is the identity for addition and 1 is the identity for multiplication.
2. Write negations of the statements in example 1.
3. Determine whether 'or' is used in inclusive sense or exclusive sense in the following statements.
   (1) School is closed on Sundays or on holidays.
   (2) Having passed the entrance examination, you can get admission to medical course or engineering course.
   (3) Roses are yellow or pink.
   (4) India got gold in CWG shooting events and India got silver in CWG hockey.
   (5) Pizza is served with cold drink or cold coffee.
4. Give two examples of statements where 'and' is used but not used as a logical connective.
5. '30 is divisible by 2, 3 and 5.' What are the constituent simple statements of this statement ? Express them in symbols and determine the truthvalue. Give its negation also.
6. '1 is prime or composite.' What are the constituent statements of this statement ? Determine the truthvalue and give the negation.

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1.5 Quantifiers and their Negations

In mathematics some statements as given below occur.

(1) There exists smallest natural number in every non-empty subset of natural numbers.

(2) For every real number x, $x^2 \geq 0$.

The use of phrases like '**there exists**' and '**for all**' or '**for every**' is abundant in mathematics. **These phrases are called quantifiers.**

A symbol for 'there exists' is $\exists$ and a symbol for 'for all' is $\forall$.

'$\exists$' is called an existential quantifier and '$\forall$' is called a universal quantifier.

For example 'for every $x \in$ R, $x^2 \geq 0$' can be written as '$\forall x \in$ R, $x^2 \geq 0$'.

'There exists x, $x \in$ R such that $x^2 = -1$'. This can be written as '$\exists x$, $x \in$ R, $x^2 = -1$'.

The negation of such sentences is to be done carefully. For example the negations of above two statements (1) and (2) are,

(1) There exists a non-empty subset of natural numbers which does not have the smallest number.

(2) There exists a real number x such that $x^2 < 0$.

Negations of Universal quantifier and Existential quantifier

A rule for the negation of a statement with a quantifier is as follows,

let p be any statement.

~(there exists p) = for all ~p. i.e. $\sim(\exists p) = \forall(\sim p)$

~(for all p) = there exist ~p, i.e. $\sim(\forall p) = \exists(\sim p)$.

Example 10 : Give negations of following statements.

(1) For every set A, $\emptyset \subset$ A.

(2) For every $x \in$ R, $x + y = y + x$ where $y \in$ R

(3) For every $x \in$ R, $x + 0 = 0 + x$

(4) For every $x \in$ R, $x \cdot 1 = 1 \cdot x$

Solution : Every statement is of type 'for every' p. So the negations would be of type 'there exists ~p.'

(1) There exists a set A such that $\emptyset \not\subset$ A.

(2) There exists $x \in$ R such that $x + y \neq y + x$ where $y \in$ R

(3) There exists $x \in$ R such that $x + 0 \neq 0 + x$

(4) There exists $x \in$ R such that $x \cdot 1 \neq 1 \cdot x$

Example 11 : Give negations of the following.

(1) There exists $x \in$ N such that $x^2 = x$.

(2) There exists $x \in$ R such that $x^2 = -1$.

(3) There exists a man who is not mortal.

Solution : ~(there exists p) = for all ~p.

(1) For all $x \in$ N, $x^2 \neq x$.

(2) For all $x \in$ R, $x^2 \neq -1$.

(3) For all men, man is mortal.

1.6 Implication and Biconditional

In mathematics, we come across many statements like 'If all the three sides of a triangle are congruent, all of its angles are congruent.'

If p and q are any two statements the statement, of the form 'if p, then q' is called an implication and is denoted by $p \Rightarrow q$. It is called a conditional statement.

Implication is expressed by any one of the following :

(1) If p then q.

(2) q, if p.

(3) p only if q.

(4) p is a sufficient condition for q.

(5) q is a necessary condition for p.

p is called the hypothesis or the antecedent and q is called the conclusion or the consequent.

$p \Rightarrow q$ does not say anything about q when p is false. Also $p \Rightarrow q$ does not mean p happens. Implication means if p is true, then q must be true.

When p is true and q is false then $p \Rightarrow q$ is false and true otherwise.

Example 12 : Express the following in the implication form.

(1) The square of an even number is even.

(2) The sum of digits of an integer is divisible by 9, if it is divisible by 9.

(3) The square of a prime number is not a prime.

(4) The last digit being zero is a necessary condition for an integer to be divisible by 10.

(5) The last digit being 5 is a sufficient condition for an integer to be divisible by 5.

(6) The roads will be wet only if it rains.

(7) Nirav failed only if he did not take the examination.

Solution : (1) Let p : x is an even number.

q : x^2 is even.

Implication $p \Rightarrow q$ is : if x is an even number, x^2 is even.

(2) Let p : An integer is divisible by 9.

q : The sum of its digits is divisible by 9.

The implication $p \Rightarrow q$ is : if an integer is divisible by 9, the sum of its digits is divisible by 9.

(3) Let p : A number x is prime.

q : Square of x is not prime.

The implication $p \Rightarrow q$ is : if a number is prime, its square is not a prime.

(4) Let p : An integer is divisible by 10.

q : Last digit of the integer is zero.

Thus the implication $p \Rightarrow q$ is : if an integer is divisible by 10, then its last digit is zero.

(5) Let p : The last digit of an integer is 5.

q : The integer is divisible by 5.

The implication $p \Rightarrow q$ is : if the last digit of an integer is 5, it is divisible by 5.

(6) Let p : The roads will be wet.

q : It rains.

$p \Rightarrow q$ is : if the roads are wet, it has rained.

(7) Let p : Nirav failed.

q : Nirav did not take the examination.

$p \Rightarrow q$ is : if Nirav failed, he did not take the examination.

Biconditional statement : In geometry, we come across statements like, 'A triangle is equilateral if and only if it is equiangular.' Here we have two statements connected by connective 'if and only if.'

Let p : A triangle is equilateral.

q : A triangle is equiangular.

Then given statement is **p if and only if q.**

It is called a biconditional statement or double implication and denoted by $p \Leftrightarrow q$.

Here the statements are equivalent. p is true if q is true and vice versa. Thus **$p \Leftrightarrow q$ is true if both p and q are true or both are false.** It is a conjunction of, 'If p then q' and 'if q then p'.

Example 13 : Combine the following statements to form a biconditional.

p : If a rectangle is a square, all its four sides are congruent.

q : If all the four sides of a rectangle are congruent, it is a square.

Solution : $p \Leftrightarrow q$ is 'a rectangle is a square if and only if all its four sides are congruent'.

Example 14 : For each of the following statements, identify component statements and state whether the compound statements are true or not.

(1) If a triangle is equiangular, all its sides are congruent.

(2) If $a \in$ N, $b \in$ N then $a + b \in$ N and $ab \in$ N.

(3) If a number is a real number, it is a natural number.

(4) If m and n are odd, then $m + n$ is even.

(5) If m and n are odd, then mn is odd.

Solution : (1) Let p : A triangle is equiangular.

q : All the sides of the triangle are congruent.

We have $p \Rightarrow q$. Now it cannot happen that p is true and q is false. If p is true, then q is true.

Therefore, $p \Rightarrow q$ is true.

(2) Let p : $a \in$ N, $b \in$ N

q : $a + b \in$ N and $ab \in$ N.

$p \Rightarrow q$ is true, since if p is true, q is true.

(3) Let p : A number is a real number.

q : The number is a natural number.

Here $\sqrt{2} \in R$ and $\sqrt{2} \notin N$.

Thus p is true and q is false for the number $\sqrt{2}$.

Thus there exists a real number for which $p \Rightarrow q$ is false.

$\therefore$ $p \Rightarrow q$ is false.

(4) Let p : m and n are odd.

q : $m + n$ is even.

$p \Rightarrow q$ is true because if p is true, then q is true.

(5) Let p : m and n be odd.

q : mn is odd.

$p \Rightarrow q$ is true because if p is true, then q is true.

Exercise 1.4

1. Identify the quantifier in following statements and give negations of the statements :

(1) For every pair of natural numbers a and b, $a + b$ is an even integer.

(2) Every income-tax payer must have a PAN card.

(3) There exists a positive integer x such that $\sqrt{x} \in R$.

(4) There exists some element x such that $x \in \emptyset$.

(5) For every $\theta \in R$, $sin^2\theta + cos^2\theta = 1$.

(6) Every angle can be constructed using a straight edge and compass only.

(7) Every person of age exceeding 18 years is a voter.

(8) There exists a smallest integer in every subset of N.

(9) Every number ending in zero is divisible by 10.

(10) There exists a multiple of 5 not ending in 5.

2. Identify constituent component statements and verify whether following implications are true or not.

(1) If n is odd, n^2 is odd.

(2) If 2 divides n, 4 divides n.

(3) If 9 divides n, 3 divides n.

(4) If all the angles of a quadrilateral have measure 90, it is a rectangle.

(5) If all the angles of a triangle have same measure, it is equilateral.

(6) If a triangle is isosceles, it is equilateral.

(7) If a triangle is a right angled triangle, the largest side occurs opposite to the right angle.

(8) If a triangle has sides $2uv$, $u^2 - v^2$, $u^2 + v^2$ for integers u, v ($u > v$), it is a right angled triangle.

(9) If a triangle has sides $2mn$, $m^2 - n^2$, $m^2 + n^2$ for $m, n \in N$, $m > n$ it is a right angled triangle.

(10) If a number is divisible by 1001, it is divisible by 7, 11 and 13.

3. Form biconditionals from following statements and verify their truth.

(1) p : Quadrilateral ABCD is a rectangle.
q : Quadrilateral ABCD is a square.

(2) p : ΔABC is isosceles.
q : ΔABC is equilateral.

(3) p : Quadrilateral ABCD has all angles and all sides congruent.
q : Quadrilateral ABCD is a square.

(4) p : Integer n is positive.
q : Integer n is even.

(5) p : Real number x is positive.
q : Real number x is a square of another real number.

*

Contrapositive and Converse :

Sometimes equivalent implication of given implication is used by taking negations of antecedents and consequents and it is useful in logical proof.

$p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$

$\sim q \Rightarrow \sim p$ is called contrapositive statement of $p \Rightarrow q$.

Thus suppose we want to prove $A \subset B$. i.e. if $x \in A$, then $x \in B$.

Contrapositive would be if $x \notin B$ then $x \notin A$. i.e. $B' \subset A'$.

Converse of $p \Rightarrow q$ is $q \Rightarrow p$.

Thus $p \Leftrightarrow q$ is the conjunction of implication and its converse.

Example 15 : Give contrapositive and converse of following.

(1) If it rains, the roads are wet.
(2) If $ab = 0$, then $a = 0$ or $b = 0$.
(3) If $ab = ac$ and $a \neq 0$, then $b = c$.
(4) If x is prime, then x is odd.
(5) If $\square$ABCD is a square, its diagonals are congruent.

Solution : (1) Let p : It rains.
q : The roads are wet.

Contrapositive is, 'if the roads are not wet, it has not rained'.
Converse is, 'if the roads are wet, it has rained'.

(2) $p : ab = 0$
$q : a = 0$ or $b = 0$
$\sim q \Rightarrow \sim p$ is 'if $a \neq 0$ and $b \neq 0$, then $ab \neq 0$' **(see 'or' changes to 'and'. Why?)**
Converse is 'if $a = 0$ or $b = 0$, then $ab = 0$'.

(3) $p : ab = ac$ and $a \neq 0$
$q : b = c$.
Contrapositive is 'if $b \neq c$, then $ab \neq ac$ or $a = 0$' **('and' changes to 'or'. Why ?)**
Converse is 'if $b = c$, then $ab = ac$ and $a \neq 0$'.

(4) p : x is prime.

q : x is odd.

$\sim q \Rightarrow \sim p$ is 'if x is not odd, then x is not a prime'.

Converse is 'if x is odd, then x is a prime'.

(5) p : □ ABCD is a square

q : Diagonals of □ ABCD are congruent.

$\sim q \Rightarrow \sim p$ is 'if diagonals of □ ABCD are not congruent, it is not a square'.

Converse is 'if diagonals of □ ABCD are congruent, then it is a square'.

Exercise 1.5

1. Write converse and contrapositive of the following.

(1) If 30 divides n, then 2 divides n.

(2) If 8 divides n, then 16 divides n.

(3) If Sanjay does not take the examination, he will fail.

(4) If n is the square of an integer, its square-root is an integer.

(5) If n is the cube of an integer, it has three real cube roots.

(6) If two lines in a plane intersect, they are not parallel.

(7) In a triangle, if two angles are not congruent, the sides opposite to them are also not congruent.

(8) In a plane if $l \parallel m$ and $m \parallel n$ then $l \parallel n$ or $l = n$.

(9) If $a^2 = b^2$, then $a = \pm b$ $(a, b \in R)$

(10) If $a^3 = b^3$, then $a = b$ $(a, b \in R)$

*

1.7 Validating Statements

In this section we will discuss validity of statements. We know a statement is either true or false and not both. We wish to decide when a statement is true or false depends upon logical connectives used 'or' and 'and'; implications 'if...then' and 'if and only if' and quantifiers 'for every' and 'there exists'.

We recollect some rules.

(1) If logical connective is 'and' then if p and q both are true, the conjunction is true and otherwise false.

(2) If logical connective used is 'or' then if p and q both are false, the disjunction is false and otherwise true.

(3) To prove a statement with 'if p then q'. (i) Assuming p is true, prove q is true. This is called direct method of proof. Here we use the fact that implication is false only when p is true and q is false. (ii) Assuming $\sim q$ is true, prove $\sim p$ is true. This is called the method of contrapositive statements since $\sim q \Rightarrow \sim p$ is equivalent to $p \Rightarrow q$.

(4) To prove $p \Leftrightarrow q$ i.e. p if and only if q, we have to prove (i) q is true assuming p is true, and (ii) p is true assuming q is true.

(5) **Method of contradiction** (also called method of reductio ad absurdum). Here we assume that q is not true and come to a contradiction against hypothesis. Thus $\sim q$ is true is not possible and we get q is true.

(6) We note $\sim(p \Rightarrow q) = p \wedge (\sim q)$

Example 16 : Prove if $x, y \in$ N and x, y are odd, then xy is odd.

(1) by using direct method, $p \Rightarrow q$.

(2) by using method of contrapositive, $\sim q \Rightarrow \sim p$.

Solution : (1) Let $x = 2m - 1$, $y = 2n - 1$, $m, n \in$ N

(odd numbers are of the form $2n \pm 1$...)

$\therefore$ $xy = (2m - 1)(2n - 1)$

$= 4mn - 2m - 2n + 1$

$= 2(2mn - m - n) + 1$

$= 2k + 1$ where $k = 2mn - m - n$

$\therefore$ xy is odd.

(2) Let $\sim q$ be true.

$\therefore$ xy is not odd.

$\therefore$ xy is even.

Let $xy = 2m$

$\therefore$ 2 divides xy

$\therefore$ 2 divides x or 2 divides y

$\therefore$ x is even or y is even

$\therefore$ $\sim p$ is true **(remember p : x and y are odd)**

$\therefore$ $\sim q \Rightarrow \sim p$ is true.

$\therefore$ $p \Rightarrow q$ is true.

$\therefore$ xy is odd, if x, y are odd.

Example 17 : Prove by method of contrapositive that if xy is odd then x and y are odd.

Solution : Here p : xy is odd.

q : x and y are odd.

$\therefore$ $\sim q$: x is even or y is even. **(both can be even)**

Let $x = 2m$, $m \in$ N **(or similarly if $y = 2m$)**

$\therefore$ $xy = 2my$

$\therefore$ xy is even.

$\therefore$ $\sim p$ is true.

$\therefore$ $\sim q \Rightarrow \sim p$ is true.

$\therefore$ $p \Rightarrow q$ is true.

$\therefore$ If xy is odd, then x and y are odd.

Example 18 : Using the method of contradiction, prove that sum of an irrational number and a rational number is irrational.

Solution : Let x be an irrational number and y be a rational number.

Let $x + y = z$ be a rational number, if possible.

Since z and y are both rational, $z - y$ is also rational.

$\therefore \quad x = z - y$ is also rational.

But x is irrational.

$\therefore$ We come to a contradiction.

$\therefore \quad z = x + y$ is irrational.

Example 19 : Prove that if $x > 3$, then $x^2 > 9$ using the method of contradiction. $(x \in R)$

Solution : Let it be false that '$x > 3 \Rightarrow x^2 > 9$'

We recall that $\sim(p \Rightarrow q) = p \wedge (\sim q)$

$\therefore \quad x > 3$ and $x^2 \leq 9$

$\therefore \quad x > 3$ and $x^2 - 9 \leq 0$

$\therefore \quad x > 3$ and $-3 \leq x \leq 3$

$\therefore \quad x > 3$ and $x \leq 3$

This is false.

Since $\sim(p \Rightarrow q)$ is false, $p \Rightarrow q$ is true.

Example 20 : Prove that if $x = y$, then $x^2 = y^2$ for real numbers x and y by direct method.

Solution : Let $x = y$

$\therefore \quad xx = xy$

Also since, $x = y$

$xy = yy$

$\therefore \quad xx = xy = yy$

$\therefore \quad x^2 = y^2$

Example 21 : Prove if n is even, n^2 is even.

Solution : Let $p : n$ is even.

$q : n^2$ is even.

$\sim(p \Rightarrow q) = p \wedge (\sim q)$

$\therefore \quad p \wedge (\sim q) : n$ is even and n^2 is not even.

Let $n = 2m$

$\therefore \quad n^2 = 4m^2$ is even.

$\therefore \quad p \wedge (\sim q)$ i.e. 'n is even and n^2 is not even' is false.

$\therefore \quad \sim(p \Rightarrow q)$ is false

$\therefore \quad p \Rightarrow q$ i.e. n is even $\Rightarrow n^2$ is even is true.

Example 22 : Show by counter example that following are not true.

(1) If n is prime, n is odd.

(2) If $m^2 = n^2$, then $m = n$.

Solution : (1) $n = 2$ is prime, but not odd. Hence given statement is false.

(2) $3^2 = (-3)^2 = 9$, but $3 \neq -3$.

By counter example given statement is not true.

Example 23 : Show by direct method that if $x^3 + x = 0$ for real number x, then $x = 0$.

Solution : Let $p : x^3 + x = 0$ and $q : x = 0$.

$\therefore$ p is $x(x^2 + 1) = 0$

$\therefore$ $x = 0$ or $x^2 + 1 = 0$

But for real x, $x^2 \geq 0$

$\therefore$ $x^2 + 1 > 0$

$\therefore$ $x^2 + 1 \neq 0$

$\therefore$ $x = 0$

$\therefore$ q is true.

$\therefore$ $p \Rightarrow q$ is true.

Exercise 1

1. Determine whether the following sentences are statements or not :

(1) If n is an integer, n is odd or even.

(2) An angle is a right angle if and only if its measure is 90.

(3) Follow the traffic rules.

(4) What a fine show it is !

(5) India is a developed country.

(6) Gujarat is a vibrant state.

(7) Where does Tajmahal stand ?

(8) Who is the tourism brand ambassador of Gujarat ?

(9) If n is an even integer, $n + 1$ is odd.

(10) If n is an even integer, $n + 1$ is not zero.

2. Give negations of the following statements :

(1) Science and mathematics are useful for development.

(2) One can opt for engineering or medicine course.

(3) If n is a perfect square, last digit of n can not be 3.

(4) All primes are odd.

(5) All odd numbers are primes.

(6) Every integer is a rational number.

(7) There exists an even integer which is a prime.

(8) There exists a real number x such that $x^2 = -1$.

(9) For all $a \in$ R, $a + 0 = a$

(10) There exists $a \in$ R, $a \cdot 1 \neq a$

(11) For every real number x, $x^2 \neq x$

(12) There exists $x \in$ R such that $x^3 < x$

3. Write the following sentence using 'if ... then' in five different ways :

(Hint : If p then q, q if p, p only if q, necessary, sufficient)

'If a number n is odd, $n^2 + 1$ is even.'

4. Give converse and contrapositive of the following :

(1) If it is raining outside, you must have an umbrella.

(2) If a positive integer is composite, it has at least three factors.

(3) If n is not a prime or not composite, $n = 1$.

(4) If a quadrilateral is a parallelogram, its opposite sides are congruent.

(5) If a quadrilateral has diagonals bisecting each other, it is a parallelogram.

(6) If it is Friday, I will go to watch a new movie.

(7) If x is negative, x^2 is positive.

(8) If x and y are negative, xy is positive.

(9) If a quadrilateral is equiangular, it is a square.

(10) If $x - a$ is a factor of polynomial $p(x)$, then $p(a) = 0$.

5. Show that following statement is true by

(1) direct method (2) method of contrapositive (3) method of contradiction

if $x^5 + 16x = 0$, then $x = 0$

6. Prove by contradiction method that $\sqrt{2}$ is irrational.

7. Prove by counter example that following statement is false.

p : if a triangle is a right angled triangle, it can not be isosceles.

8. Prove by counter example that following statement is false.

$\sqrt{x}$ is always irrational, whenever x is a natural number.

9. Prove by direct method that for any integer n, $n^3 - n$ is even.

10. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Which of the following is a statement ? ☐

(a) x is positive. (b) -1 is negative.

(c) Stand up. (d) Where are you ?

(2) Which of the following is not a statement ? ☐

(a) $2 \times 3 = 6$

(b) $2 \times 4 \neq 8$

(c) Let the truth win.

(d) The square of an odd number is odd.

(3) The negation of the statement, '3 is odd or 3 is prime' is... ☐

(a) 3 is not odd and 3 is not a prime. (b) 3 is not odd or 3 is not a prime.

(c) 3 is odd and 3 is not a prime. (d) 3 is not odd and 3 is a prime.

(4) Converse of the statement, 'if $x^2 = y^2$, then $x = y$' is... ☐

(a) if $x^2 = y^2$, then $x \neq y$ (b) if $x = y$, then $x^2 = y^2$

(c) if $x \neq y$, then $x^2 = y^2$ (d) if $x^2 \neq y^2$, then $x = y$

(5) The contrapositive of, 'if $x > y$, then $3x > 3y$' is... ☐

(a) if $x > y$, then $3x \leq 3y$ (b) if $3x > 3y$, then $x > y$

(c) if $3x \leq 3y$, then $x \leq y$ (d) if $x < y$, then $3x < 3y$

(6) The contrapositive of $p \Rightarrow q$ is... ☐

(a) $q \Rightarrow p$ (b) $\sim q \Rightarrow \sim p$ (c) $p \Rightarrow \sim q$ (d) $\sim p \Rightarrow q$

(7) The negation of $p \Rightarrow q$ is... ☐

(a) p and $\sim q$ (b) p or q (c) $\sim p$ or q (d) $q \Rightarrow p$

(8) The converse of $p \Rightarrow q$ is... ☐

(a) $p \Rightarrow \sim q$ (b) $\sim q \Rightarrow p$ (c) $q \Rightarrow p$ (d) $\sim p \Rightarrow q$

(9) The negation of 'for all x, p' is... ☐

(a) there exists x, $\sim p$ (b) for all x, $\sim p$

(c) $\sim p$ (d) p

(10) The negation of '12 is a multiple of 3 and 12 is a multiple of 4' is... ☐

(a) 12 is a multiple of 3 or 4

(b) 12 is not a multiple of 3 or 12 is not a multiple of 4

(c) 12 is not a multiple of 3 and not a multiple of 4

(d) 12 is a multiple of 3 and a multiple of 4.

(11) Biconditional $p \Leftrightarrow q$ is...

(a) conjunction of $p \Rightarrow q$ and $q \Rightarrow p$ (b) disjunction of $p \Rightarrow q$ and $q \Rightarrow p$

(c) converse of $p \Rightarrow q$ (d) contrapositive of $q \Rightarrow p$

(12) $p \wedge q$ is valid (true) when,

(a) p and q are true (b) p and q are false

(c) p is true and q is false (d) p is false and q is true

(13) $p \vee q$ is false, when,

(a) p and q are true (b) p and q are false

(c) p is true and q is false (d) p is false and q is true

(14) $p \Rightarrow q$ is false, when,

(a) p and q are true (b) p and q are false

(c) p is true and q is false (d) p is false and q is true

(15) $\sim p \Rightarrow \sim q$ is false, when,

(a) p and q are true (b) p and q are false

(c) p is true and q is false (d) p is false and q is true

(16) $\sim(p$ or $q)$ is...

(a) p and q (b) $(\sim p)$ and $\sim q$ (c) $(\sim p)$ or $\sim q$ (d) p or $\sim q$

(17) $\sim(p$ and $q)$ is...

(a) p or q (b) $(\sim p)$ or $(\sim q)$ (c) $(\sim p)$ and q (d) $(\sim p)$ and $\sim q$

Summary

We studied following points in this chapter :

1. Statements, their negations, conjuctions and disjunctions.
2. Negation of a compound statement
3. Universal and Existential quantifiers and their negations.
4. Implications and double implications and their negations and contrapositive and converse.
5. Various methods of proof like direct method and indirect method.

   Method of counter-example and method of reductio ad absurdum in indirect method.

Chapter 2

SET THEORY

2.1 Introduction

Elementary introduction to set theory was done in standard 8 and 9. In this chapter logical approach to the set theory is discussed. In this section we will review what has been discussed earlier.

The term set falls in the category of undefined terms in mathematics. Also to be an element of a set is also an undefined term. However a set means a well-defined collection of objects. A set is expressed by enclosing its members within a pair of braces; for example when we write A = $\{a, b, c\}$, the set is labelled as A and a, b, c are members of this set. The sets are usually denoted by A, B, C, etc., while members of the sets are denoted by a, b, c, x, y, z etc. The objects in a set are called elements of the set or members of the set. If x is a member (or element) of a set A, then we write $x \in$ A, which is read as 'x belongs to A', and $y \notin$ A (read as 'y does not belong to A') means y is not a member of the set A.

Some known sets are

N = the set of all natural numbers = {1, 2, 3, 4, 5,...}

Z = the set of all integers = {...–2, –1, 0, 1, 2, 3,...}

Q = the set of all rational numbers.

R = the set of all real numbers.

There are two methods of expressing a set.

(1) Listing Method (Roster Form) : In this method, elements of the set are explicitly written (listed) separated by commas. For example A = {1, 11, 111, 1111} has elements 1, 11, 111, 1111. The sets N and Z are described as N = {1, 2, 3,...} and Z = {....–2, –1, 0, 1, 2,...}.

In the listing method or roster form no element is repeated. For example, the set of letters of the word BEGINNING is α = {B, E, G, I, N}. Also the order in which elements are listed is not important.

In roster form of writing a set, when the list of elements is very large, it can be abbreviated by listing some initial elements followed by three dots and then some elements in the end. For example the set of even natural numbers less than 1000 is represented by {2, 4, 6, 8,..., 996, 998}. Further, if the list is to be continued indefinitely, then three dots in the end are sufficient.

(2) Property Method (Set Builder Form) : In this method, **a set is expressed by some common characteristic property P(*x*) of elements *x* of the set. We have the notation $\{x \mid P(x)\} = \{x \mid \text{The property of } x\}$ which is read as the set of all *x* possessing given property P(*x*).** For example the set of rational numbers Q is described as

$$Q = \left\{x \mid x = \frac{p}{q},\ p \in Z,\ q \in N\right\}$$

If we write, $M = \{x \mid x \text{ is an integer}, -2 < x < 3\}$, then $M = \{-1, 0, 1, 2\}$

A set consisting of only one element is called a singleton. For example $\{-2\}$ is a singleton. Let $D = \{x \mid x \text{ is an even prime number}\}$, then since all even numbers except 2 are composite numbers, we have $D = \{2\}$. Hence D is a singleton.

A set which does not contain any element is called an empty set (or null set). An empty set is denoted by { } or ∅. In the set builder form, it is likely that there is no object satisfying the property. In this case the set will be an empty set. For example the set $\{x \mid x^2 = -4; x \in R\}$ is an empty set as square of any real number is non-negative. Hence no number satisfies the property mentioned to describe the set.

A set which is not empty is called a non-empty set.

2.2 Universal Set

Generally when we consider many sets of similar nature, the elements in the sets are selected from a definite set. This set is called the universal set and it is denoted by U. The universal set depends upon the context. While working with a set containing some integers, universal set is taken as Z. In the case of working with real *n*th roots of a number, universal set is R, the set of real numbers.

2.3 Subset

Definition : A set A is said to be subset of a set B if every element of A is also an element set B. The fact that A is a subset of B is denoted by $A \subset B$.

In logical notations we write (for every x, $x \in A \Rightarrow x \in B$) $\Rightarrow A \subset B$.

We can also write $(\forall x,\ x \in A \Rightarrow x \in B) \Rightarrow A \subset B$.

All natural numbers are integers and hence we write $N \subset Z$. In a similar manner, it is easy to see that $Z \subset Q$ and $Q \subset R$.

To understand the concept of a subset let us understand the following examples.

Obviously $\{1, 2, 4\} \subset \{1, 2, 3, 4, 5, 6\}$

$\{1, 3, 9\} \not\subset \{1, 2, 4, 8, 9\}$ because $3 \in \{1, 3, 9\}$ but $3 \notin \{1, 2, 4, 8, 9\}$.

Here we note that to confirm a set A to be subset of the other set B, it is necessary to verify that all the elements of set A are the elements of set B. But to prove that $A \not\subset B$, all that is needed is to find an element of A which is not in B.

The equivalent implication of if ($\forall x$, $x \in A \Rightarrow x \in B$), then $A \subset B$ is if $A \not\subset B$, then there exists x, such that $x \in A$ and $x \notin B$ since $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are logically equivalent implications. (contrapositives of each other) and $\sim(p \Rightarrow q) = p \wedge (\sim q)$.

Theorem 2.1 : $A \subset A$

Proof : Clearly $\forall x$, $x \in A \Rightarrow x \in A$ and hence $A \subset A$.

Theorem 2.2 : For any set A, $\emptyset \subset A$

Proof : Suppose $\emptyset \not\subset A$.

$\therefore$ There exists x, $x \in \emptyset$ and $x \notin A$.

but $x \in \emptyset$ is not possible for any x, hence our supposition that $\emptyset \not\subset A$ is wrong. Thus $\emptyset \subset A$.

Above two theorems state that **any non-empty set has at least two subsets namely $\emptyset$ and the set itself. These subsets are called improper subsets. Other subsets (if any) of a set are called proper subsets.**

If a set A is a subset of a set B, then set B is called a superset of A. In this terminology the universal set is superset of all sets and all sets are subsets of the universal set.

In the following table an interesting relation between number of elements of a set and number of subsets of the set is demonstrated.

Set	Subsets	No. of elements (n)	No. of subsets	No. of proper subsets
$\emptyset$	$\emptyset$	0	$1 = 2^0$	0
$\{a\}$	$\emptyset$, $\{a\}$	1	$2 = 2^1$	$2 - 2 = 0$
$\{a, b\}$	$\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$	2	$4 = 2^2$	$4 - 2 = 2$
$\{a, b, c\}$	$\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ $\{a, b, c\}$	3	$8 = 2^3$	$8 - 2 = 6$

In general, **if a set has n elements, the number of its subsets is 2^n and if $n \geq 1$, the number of proper subsets is $2^n - 2$.**

Power set : For any set A, the set consisting of all the subsets of A is called the power set of A and it is denoted by P(A). As discussed above, if A has n elements, then the number of elements in P(A) is 2^n. The power set of a set A can be described as, $P(A) = \{B \mid B \subset A\}$. Power set P(A) is also denoted by 2^A.

We note that for any set A, $\emptyset \subset A$, hence $\emptyset \in P(A)$, thus the **power set of any set is never an empty set.**

If $A = \{d, e, f\}$, then $P(A) = \{\emptyset, \{d\}, \{e\}, \{f\}, \{d, e\}, \{d, f\}, \{e, f\}, \{d, e, f\}\}$.

Subsets of set of real numbers :

The set R of real numbers has some important subsets. The illustrations of some subsets are given below.

The set of natural numbers, N = {1, 2, 3,...}.

The set of integers, Z = {..., –3, –2, –1, 0, 1, 2,...}.

The set of rational numbers, $Q = \left\{x \mid x = \frac{p}{q}, p \in Z, q \in N\right\}$.

All numbers $\frac{p}{q}$ where $p \in Z$ and $q \in N$ are called rational numbers.

All natural numbers and integers are members of Q (as each of them can be expressed with denominator 1, for example $3 = \frac{3}{1}$, $-5 = \frac{-5}{1}$, etc.). In general if $n \in Z$ then $n = \frac{n}{1}$. Thus it obviously follows that

$N \subset Z \subset Q \subset R$

Further, there are some real numbers like $\sqrt{2}$, $\sqrt{5}$, π etc which cannot be expressed as a quotient of integers, i.e. $\frac{p}{q}$ form, and hence they are not members of Q. They are called irrational numbers. The set of all irrational numbers is denoted by I. Thus,

$I = \{x \mid x \in R, x \notin Q\}$

i.e. I is the set of all real numbers that are not rationals.

Another important type of a subset of R is an **interval.**

Definition : Let $a, b \in R$ and $a < b$, then the set $\{x \mid x \in R, a < x < b\}$ is called an open interval and is denoted by (a, b).

In this case all real numbers between a and b belong to (a, b) but a and b themselves do not belong to this interval.

Definition : Let $a, b \in R$ and $a < b$, then the set $\{x \mid x \in R, a \le x \le b\}$ is called a closed interval and is denoted by $[a, b]$.

Here $[a, b]$ consists of all real numbers between a and b including a and b. The numbers a and b are called end-points of the interval.

Intervals which contains one of the end-points are

$[a, b) = \{x \mid x \in R, a \le x < b\}$ and $(a, b] = \{x \mid x \in R, a < x \le b\}$

Other intervals are given below.

$[0, \infty)$ = Set of non-negative real numbers.

$(-\infty, 0)$ = Set of negative real numbers.

$(-\infty, \infty) = R$

$(a, \infty) = \{x \mid x \in R, x > a\}$

$[a, \infty) = \{x \mid x \in R, x \ge a\}$

$(-\infty, a) = \{x \mid x \in R, x < a\}$

$(-\infty, a] = \{x \mid x \in R, x \le a\}$

On the real number line, various types of intervals are depicted below.

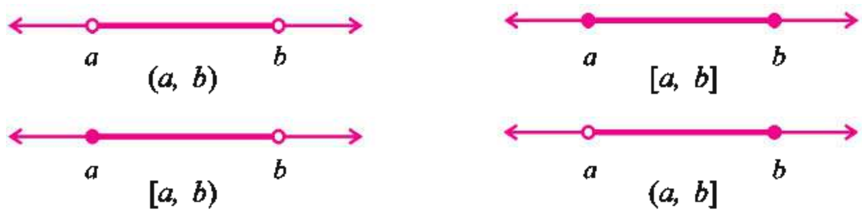


Figure 2.1

Here, we note that an interval is an infinite set.

The number $(b - a)$ is called the length of any of the intervals (a, b), $[a, b]$, $[a, b)$ or $(a, b]$.

Equal sets : Two sets A and B are said to be equal sets, if they have the same elements. Thus if for all x, if $x \in A$, then $x \in B$ and if for all x, if $x \in B$, then $x \in A$, then $A = B$. In other words if $A \subset B$ and $B \subset A$, then $A = B$.

From this definition, it is clear that the order in which elements in a set are listed is not important. For example if $A = \{a, b, c\}$, $B = \{b, c, a\}$, then it is easy to verify that $A \subset B$ and $B \subset A$. Hence $A = B$.

The condition for equality of two sets can be stated as,

if $\forall x, x \in A \Rightarrow x \in B$ and $\forall x, x \in B \Rightarrow x \in A$ then $A = B$.

Example 1 : Prove that $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

Solution : Since $A \subset B$, $\forall x, x \in A \Rightarrow x \in B$

Further as $B \subset C$, $\forall x, x \in B \Rightarrow x \in C$

$\therefore \quad \forall x, x \in A \Rightarrow x \in C$

$\therefore \quad A \subset C$

Thus, **the relation of being a subset has the transitive property.**

Example 2 : $\alpha = \{x \mid x$ is a letter in the word FELLOW$\}$

$\beta = \{x \mid x$ is a letter in the word FLOW$\}$, then which one of the following is correct ?

(a) $\alpha \subset \beta$ (b) $\alpha = \beta$ (c) $\beta \subset \alpha$ (d) None of the these

Solution : Here the sets can be written as $\alpha = \{F, E, L, O, W\}$, $\beta = \{F, L, O, W\}$ and it is clear that $\beta \subset \alpha$, but $\alpha \not\subset \beta$ and hence $\alpha \neq \beta$. Hence alternative (c) $\beta \subset \alpha$ is true.

Example 3 : $A = \{x \mid x \in N, x < 5\}$ and $B = \{x \mid x \in N, x^2 < 25\}$ show that $A = B$.

Solution : Method-1 : Here $A = \{1, 2, 3, 4\}$. Also $1^2 < 25$, $2^2 < 25$, $3^2 < 25$, $4^2 < 25$, $5^2 \nless 25$ etc. Hence $B = \{1, 2, 3, 4\}$.

So $A = B$.

Method-2 : This method is rather abstract.

Let $x \in A$

$\therefore \quad x < 5$ and $x \in N$

$\therefore \quad x^2 < 25$ **(squaring both sides)**

$\therefore \quad x \in B$

thus $\forall x, x \in A \Rightarrow x \in B$

$\therefore \quad A \subset B$ **(i)**

Conversely, let $x \in B$

$\therefore \quad x^2 < 25$

$\therefore \quad |x| < 5$ **(taking square root)**

$\therefore \quad x < 5$ **($x \in N$)**

$\therefore \quad x \in A$

$\therefore \quad \forall x, x \in B \Rightarrow x \in A$

$\therefore \quad B \subset A$ **(ii)**

$\therefore \quad A \subset B$ and $B \subset A$ **((i) and (ii))**

Hence $A = B$.

Example 4 : $A = \{x \mid 4 < x^2 < 40, x \in N\}$ and $B = \{x \mid 4 < x^3 < 40, x \in N\}$. Show that $A \not\subset B$ and $B \not\subset A$.

Solution : For $x \in A$, $4 < x^2 < 40 < 49$

$\therefore \quad 2 < x < 7$ **($x \in N$)**

Thus possible values of x are 3, 4, 5, 6.

Also $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$ and 9, 16, 25, 36 all lie between 4 and 40.

$\therefore \quad A = \{3, 4, 5, 6\}$

For $x \in B$, $1 < 4 < x^3 < 40 < 64$

$\therefore \quad 1 < x < 4$ **($x \in N$)**

Hence $x = 2$ or 3 is possible.

Also $2^3 = 8$, $3^3 = 27$ and 8 and 27 lie between 4 and 40.

$\therefore \quad B = \{2, 3\}$

It can be seen clearly that $4 \in A$ and $4 \notin B$

$\therefore \quad A \not\subset B$

Also, $2 \in B$ and $2 \notin A$

$\therefore \quad B \not\subset A$.

Exercise 2.1

1. Write the following sets in roster form :

(1) $\{x \mid x$ is a natural number less than $10\}$

(2) $\{x \mid x^2 - 5x - 6 = 0, x \in N\}$ (3) $\{x \mid x^2 - 5x - 6 = 0, x \in R\}$

(4) $\{x \mid x^3 - x = 0, x \in Z\}$ (5) $\{x \mid -3 \le x \le 3, x \in Z\}$

2. List all subsets of A = {1, a, b}
3. A = {a, b, c, d}, B = {a, b, c}, C = {b, d}. Find all possible sets X which satisfy the given conditions.
   (1) $X \subset B, X \not\subset C$ (2) $X \subset B, X \not\subset C, X \neq B$
   (3) $X \subset A, X \not\subset B, X \not\subset C$
4. The relation $\le$ on numbers, has the following properties :
   (1) $a \le a, \forall a \in R$ **(Reflexivity)**
   (2) If $a \le b$ and $b \le a$, then $a = b$, $\forall a, b \in R$ **(Antisymmetry)**
   (3) If $a \le b$ and $b \le c$, then $a \le c$, $\forall a, b, c \in R$ **(Transitivity)**
   Which of the above properties the relation $\subset$ on P(A) has ?
5. Let $A = \{x \mid x = 2y - 1, y \in Z\}$, $B = \{x \mid x = 2y + 1, y \in Z\}$. Show that A = B.

*

2.4 Operations on Sets

Let U be the universal set and P(U) its power set. Now, we define some operations on P(U) and study their properties.

(1) Union : Let A, B ∈ P(U). The set consisting of all elements of U which are in A or in B, is called the union set of sets A and B and it is denoted by A ∪ B. The operation of taking the union of two sets is called the union operation.

Thus, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Here the word 'or' is used in the inclusive sense. That is A ∪ B is the set consisting all those elements x for which x is either in A or in B or in A and B both. Hence x must be an element of at least one of A or B. For example if A = {1, 2, 5}, B = {2, 3, 4, 5}, then A ∪ B = {1, 2, 3, 4, 5}.

We recall that in figure 2.2 coloured portion is Venn diagram of A ∪ B.

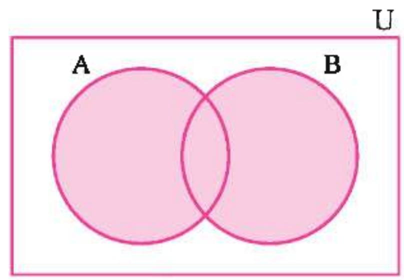


Figure 2.2

We now discuss some interesting results regarding union operation. We will assume A, B, C, D ∈ P(U).

(1) The union operation is a binary operation on P(U); that is if A, B ∈ P(U), then A ∪ B ∈ P(U).

$x \in A \cup B \Rightarrow x \in A$ or $x \in B$

Since A, B ∈ P(U), $A \subset U$ and $B \subset U$

thus, $x \in (A \cup B) \Rightarrow x \in U$

$\therefore \quad (A \cup B) \subset U$

$\therefore \quad (A \cup B) \in P(U)$

This is also called **closure property** of the operation.

(2) $A \subset (A \cup B)$ and $B \subset (A \cup B)$

$x \in A \Rightarrow x \in (A \cup B)$ and $x \in B \Rightarrow x \in (A \cup B)$

$\therefore$ $A \subset (A \cup B)$, $B \subset (A \cup B)$

(3) Idempotent law : $A \cup A = A$

To prove this we write

$A \cup A = \{x \mid x \in A \text{ or } x \in A\}$

$= \{x \mid x \in A\} = A$

(4) If $A \subset B$ and $C \subset D$, then $(A \cup C) \subset (B \cup D)$

Let $x \in A \cup C$

$\therefore$ $x \in A$ or $x \in C$

$\therefore$ $x \in B$ or $x \in D$ **($A \subset B$ and $C \subset D$)**

$\therefore$ $x \in B \cup D$

Thus for $\forall x$, $x \in (A \cup C) \Rightarrow x \in (B \cup D)$

$\therefore$ $(A \cup C) \subset (B \cup D)$

(5) Commutative law : $A \cup B = B \cup A$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$= \{x \mid x \in B \text{ or } x \in A\}$

$= B \cup A$

(6) Associative law : $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cup B) \cup C = \{x \mid x \in A \cup B \text{ or } x \in C\}$

$= \{x \mid (x \in A \text{ or } x \in B) \text{ or } x \in C\}$

$= \{x \mid x \in A \text{ or } (x \in B \text{ or } x \in C)\}$

$= \{x \mid x \in A \text{ or } x \in B \cup C\}$

$= A \cup (B \cup C)$

Therefore we write $A \cup (B \cup C)$ or $(A \cup B) \cup C$ as $A \cup B \cup C$.

(7) $A \cup \emptyset = A$ (Thus, $\emptyset$ is an identity element for the union operation)

In property (2) it was shown that $A \subset (A \cup B)$ for any set B. Thus in particular for $B = \emptyset$ we write

$A \subset (A \cup \emptyset)$ **(i)**

Also, $A \subset A$, $\emptyset \subset A \Rightarrow (A \cup \emptyset) \subset (A \cup A)$ **(property (4))**

$\Rightarrow (A \cup \emptyset) \subset A$ **(property (3))** **(ii)**

From (i) and (ii) we get,

$A \cup \emptyset = A$

Another Method :

Already $A \subset (A \cup \emptyset)$ **(i)**

Now let $x \in A \cup \emptyset$.

$\therefore$ $x \in A$ or $x \in \emptyset$

But for no x, $x \in \emptyset$ is true.

$\therefore$ $x \in A$

$\therefore$ $(A \cup \emptyset) \subset A$ **(ii)**

$\therefore$ $A \cup \emptyset = A$ **(i) and (ii)**

(8) $A \cup U = U$

$A \subset U$ and $U \subset U$

$\therefore$ $(A \cup U) \subset (U \cup U)$ **(property (4))**

$\therefore$ $(A \cup U) \subset U$ as $U \cup U = U$ **(i)**

we have $U \subset (A \cup U)$ **(ii)**

From (i) and (ii) we get

$A \cup U = U$

(2) Intersection : Let A, B $\in$ P(U). Then the set consisting of all elements of U which are in both A and B is called the intersection set of sets A and B and is denoted by $A \cap B$. The operation of finding the intersection of two sets is called the intersection operation.

Thus, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

We note here that $A \cap B$ is the set of all elements common to both A and B.

Venn diagram for $A \cap B$ is shown in figure 2.3 by coloured region.

Some properties of intersection operation are discussed below.

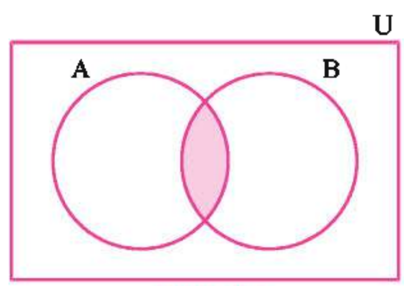


Figure 2.3

Let A, B, C, D $\in$ P(U)

(1) Intersection operation is a binary operation :

Let $x \in A \cap B$

$\therefore$ $x \in A$ and $x \in B$

$\therefore$ $x \in U$ **(A, B $\subset$ U)**

$\therefore$ $(A \cap B) \subset U$

$\therefore$ $(A \cap B) \in P(U)$

(2) $(A \cap B) \subset A$, $(A \cap B) \subset B$

This result is obvious.

(3) Idempotent law : $A \cap A = A$

$$A \cap A = \{x \mid x \in A \text{ and } x \in A\}$$
$$= \{x \mid x \in A\}$$
$$= A$$

(4) If A ⊂ B, C ⊂ D, then (A ∩ C) ⊂ (B ∩ D)

Let $x \in A \cap C$

∴ $x \in A$ and $x \in C$

∴ $x \in B$ and $x \in D$ **(A ⊂ B, C ⊂ D)**

∴ $x \in B \cap D$

∴ $(A \cap C) \subset (B \cap D)$

(5) Commutative law : A ∩ B = B ∩ A

(6) Associative law : A ∩ (B ∩ C) = (A ∩ B) ∩ C

Properties (5) and (6) can be proved exactly in a similar manner as in the case of union. $A \cap (B \cap C)$ or $(A \cap B) \cap C$ is written as $A \cap B \cap C$.

(7) A ∩ ∅ = ∅

As seen in property 2, $(A \cap \emptyset) \subset \emptyset$ **(i)**

Further $\emptyset$ is a subset of all the sets.

In particular $\emptyset \subset (A \cap \emptyset)$ **(ii)**

From (i) and (ii) we get $A \cap \emptyset = \emptyset$

(8) A ∩ U = A (U is the identity element for the intersection operation).

We know $(A \cap U) \subset A$ **(i)**

Now, $A \subset A$ and $A \subset U$

∴ $(A \cap A) \subset (A \cap U)$

∴ $A \subset (A \cap U)$ **(ii)**

From (i) and (ii) we get $A \cap U = A$

Distributive laws :

(1) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

Proof : Let $x \in A \cap (B \cup C)$

∴ $x \in A$ and $x \in B \cup C$

∴ $x \in A$ and ($x \in B$ or $x \in C$)

∴ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

∴ $x \in A \cap B$ or $x \in A \cap C$

∴ $x \in (A \cap B) \cup (A \cap C)$

∴ $(A \cap (B \cup C)) \subset (A \cap B) \cup (A \cap C)$ **(i)**

In similar manner, it can be proved that

$((A \cap B) \cup (A \cap C)) \subset (A \cap (B \cup C))$ **(ii)**

From (i) and (ii), we get

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

This law is called distributive law of intersection operation over union operation.

(2) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof of this result is similar to that of (1) and it is left as an exercise. This law is called distributive law of union over intersection.

Definition : Non-empty sets A and B are said to be disjoint if their intersection is the empty set.

An Important Result :

The following statements are equivalent :

(1) $A \subset B$

(2) $A \cup B = B$

(3) $A \cap B = A$

[**Note :** $p \Leftrightarrow q$, $q \Leftrightarrow r$, $r \Leftrightarrow p$ can be proved in following sequence, $p \Rightarrow q$, $q \Rightarrow r$, $r \Rightarrow p$. Due to transitive property. $(p \Rightarrow q$ and $q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ etc.]

Proof : (1) $\Rightarrow$ (2)

It is known that $B \subset (A \cup B)$ **(i)**

Now let $x \in A \cup B$

$\therefore$ $x \in A$ or $x \in B$

If $x \in A$ then $x \in B$ because we are given that $A \subset B$

$\therefore$ $x \in B$ or $x \in B$

$\therefore$ $x \in B$

$\therefore$ $(A \cup B) \subset B$ **(ii)**

From (i) and (ii), $A \cup B = B$

Another Proof : We know $B \subset (A \cup B)$ **(i)**

Also $A \subset B$ and $B \subset B$

$\therefore$ $(A \cup B) \subset (B \cup B)$

$\therefore$ $(A \cup B) \subset B$ **(ii)**

$\therefore$ From (i) and (ii), $(A \cup B) = B$

(2) $\Rightarrow$ (3)

It is clear that, $(A \cap B) \subset A$ **(i)**

Let $x \in A$

$\therefore$ $x \in A \cup B$

$\therefore$ $x \in B$

$\therefore$ $x \in A$ and $x \in B$

$\therefore$ $x \in A \cap B$

Thus $x \in A \Rightarrow x \in A \cap B$

$\therefore$ $A \subset (A \cap B)$ **(ii)**

$\therefore$ From (i) and (ii) $A \cap B = A$

Another Proof : We know $(A \cap B) \subset A$ **(i)**

Also $A \subset A$ and $A \subset B$

$\therefore$ $(A \cap A) \subset (A \cap B)$

$\therefore$ $A \subset (A \cap B)$ **(ii)**

From (i) and (ii), $(A \cap B) = A$

(3) $\Rightarrow$ (1)

It is known that, $(A \cap B) \subset B$

$\therefore$ $A \subset B$ **$(A \cap B = A)$**

Thus we have proved logical equivalence of three statements (1), (2), (3).

(3) Complementation : For $A \in P(U)$, the set consisting of all those elements of U which are not in A, is called complement of A and is denoted by A'. The operation of finding the complement of a set is called complementation operation. Here $A' = \{x \mid x \in U \text{ and } x \notin A\}$

The operation of complementation assigns a unique set A' to every set A.

This operation is a unary operation on P(U).

Coloured region in figure 2.4 represents A'.

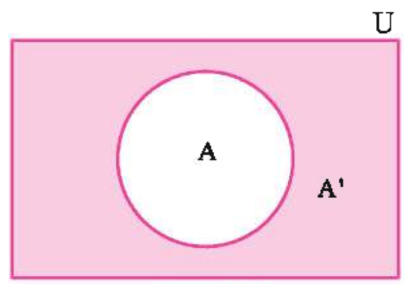


Figure 2.4

Some properties of complementation are given below.

(1) $A' \in P(U)$.

This follows from the definition.

(2) $A \cap A' = \emptyset$, $A \cup A' = U$

From the definition of complement of a set A, it follows that

$x \in A \Rightarrow x \notin A'$ and $x \in A' \Rightarrow x \notin A$

$\therefore$ $A \cap A' = \emptyset$

To prove second result we note that since $A \subset U$, $A' \subset U$

$(A \cup A') \subset U$ **(i)**

Further, if $x \in U$ then $x \in A$ or $x \in A'$

$x \in A \cup A'$

$U \subset (A \cup A')$ **(ii)**

From (i) and (ii) it follows that $A \cup A' = U$

(3) $\emptyset' = U$, $U' = \emptyset$.

These results obviously follow from the definition of complementation operation.

(4) $(A')' = A$

Proof of this result is very easy. Try yourself.

De Morgan's Laws :

(1) $(A \cup B)' = A' \cap B'$ (2) $(A \cap B)' = A' \cup B'$

These results are proved below :

(1) $(A \cup B)' = \{x \mid x \in U, x \notin A \cup B\}$

$= \{x \mid x \in U \text{ and } (x \notin A \text{ and } x \notin B)\}$ $(\sim(p \vee q) = (\sim p) \wedge (\sim q))$

$= \{x \mid x \in U \text{ and } (x \in A' \text{ and } x \in B')\}$

$= A' \cap B'$

(2) $(A \cap B)' = \{x \mid x \in U \text{ and } x \notin A \cap B\}$

$= \{x \mid x \in U \text{ and } (x \notin A \text{ or } x \notin B)\}$ $(\sim(p \wedge q) = (\sim p) \vee (\sim q))$

$= \{x \mid x \in U \text{ and } (x \in A' \text{ or } x \in B')\}$

$= A' \cup B'$

Using mathematical reasoning, De Morgan's laws can easily be proved. We will prove (1) as under :

$x \in (A \cup B)' \Leftrightarrow x \notin A \cup B$

$\Leftrightarrow \sim(x \in A \cup B)$

$\Leftrightarrow \sim(x \in A \text{ or } x \in B)$

$\Leftrightarrow \sim(x \in A) \text{ and } \sim(x \in B)$ $(\sim(p \vee q) = (\sim p) \wedge (\sim q))$

$\Leftrightarrow x \notin A \text{ and } x \notin B$

$\Leftrightarrow x \in A' \text{ and } x \in B'$

$\Leftrightarrow x \in A' \cap B'$

$\therefore$ $(A \cup B)' = A' \cap B'$

(4) Difference set : For the sets A, B $\in$ P(U), the set consisting of all elements of A which are not in B, is called the difference set of A and B. This set is denoted by A–B. The operation of taking the difference of two sets is called the difference operation.

Hence, $A - B = \{x \mid x \in U, x \in A \text{ and } x \notin B\}$

$\therefore$ $A - B = \{x \mid x \in A \text{ and } x \in B'\}$

$\therefore$ **$A - B = A \cap B'$**

From this it is clear that $(A - B) \subset A$. In figure 2.5, Venn diagram of $A - B$ and $B - A$ are represented by coloured regions.

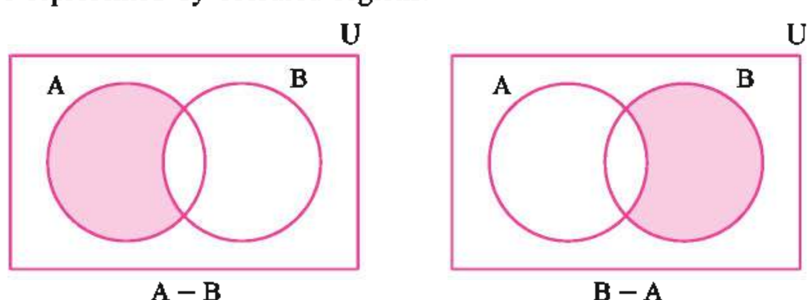


Figure 2.5

From these diagrams it is clear that $A - B \neq B - A$, if $A \neq B$.

If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$ then $A - B = \{1, 3, 5\}$, $B - A = \{8, 10\}$

Some properties of difference operation are as under :

(1) $U - A = A'$ **(quite obvious !)**

(2) $A \subset B \Rightarrow A - B = \emptyset$

$$\begin{aligned} A - B &= \{x \mid x \in A \text{ and } x \notin B\} \\ &= \{x \mid x \in A \text{ and } x \in B'\} \\ &= \{x \mid x \in B \text{ and } x \in B'\} \quad (A \subset B) \\ &= \emptyset \end{aligned}$$

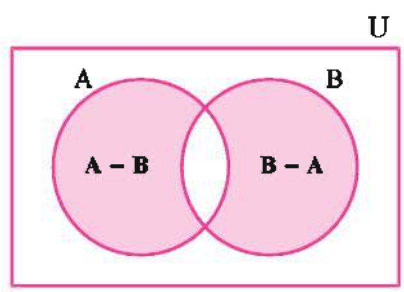
(5) Symmetric Difference set : For sets A, B ∈ P(U), the set consisting of all elements which are in the set A or in the set B, but not in both is called the symmetric difference of the sets A and B. Symmetric difference of two sets is denoted by A Δ B.

Thus $A \Delta B = (A \cup B) - (A \cap B)$

We will prove that, $A \Delta B = (A - B) \cup (B - A)$

$$\begin{aligned} (A \cup B) - (A \cap B) &= (A \cup B) \cap (A \cap B)' \quad (A - B = A \cap B') \\ &= (A \cup B) \cap (A' \cup B') \quad \text{(De Morgan's law)} \\ &= ((A \cup B) \cap A') \cup ((A \cup B) \cap B') \quad \text{(Distributive law)} \\ &= [(A \cap A') \cup (B \cap A')] \cup [(A \cap B') \cup (B \cap B')] \\ &= [\emptyset \cup (B \cap A')] \cup [(A \cap B') \cup \emptyset] \\ &= (B \cap A') \cup (A \cap B') \\ &= (B - A) \cup (A - B) \\ &= (A - B) \cup (B - A) \end{aligned}$$

This set is depicted below by the coloured region in the Venn diagram 2.6.



A Δ B

Figure 2.6

Example 5 : Find the power set of $A = \{x \mid x \in Z, x^3 - 4x = 0\}$.

Solution : Here $x^3 - 4x = 0$

$\therefore \quad x(x^2 - 4) = 0$

$\therefore \quad x(x - 2)(x + 2) = 0$

$\therefore \quad x = 0, x = 2, x = -2$

$\therefore \quad A = \{0, 2, -2\}$

Hence $P(A) = \{\emptyset, \{0\}, \{2\}, \{-2\}, \{0, 2\}, \{0, -2\}, \{2, -2\}, A\}$

Example 6 : $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 3, 5, 7, 9\}$, $B = \{1, 5, 6, 8\}$, $C = \{1, 4, 6, 7\}$

Verify the following,

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(2) $(A \cup B)' = A' \cap B'$

(3) $A - B = A \cap B'$

(4) $A \Delta B = B \Delta A$ taking $A \Delta B = (A \cup B) - (A \cap B)$ and $B \Delta A = (B - A) \cup (A - B)$

(5) $A - C = A - (A \cap C)$

Solution : (1) Here $B \cap C = \{1, 6\}$

$A \cup (B \cap C) = \{1, 3, 5, 6, 7, 9\}$

Now, $A \cup B = \{1, 3, 5, 6, 7, 8, 9\}$

$A \cup C = \{1, 3, 4, 5, 6, 7, 9\}$

$(A \cup B) \cap (A \cup C) = \{1, 3, 5, 6, 7, 9\}$

Thus, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(2) $A \cup B = \{1, 3, 5, 6, 7, 8, 9\}$

$\therefore \quad (A \cup B)' = \{2, 4, 10\}$

Now $A' = \{2, 4, 6, 8, 10\}$

$B' = \{2, 3, 4, 7, 9, 10\}$

$\therefore \quad A' \cap B' = \{2, 4, 10\}$

Thus, $(A \cup B)' = A' \cap B'$

(3) $A - B = \{3, 7, 9\}$

$B' = \{2, 3, 4, 7, 9, 10\}$

$A \cap B' = \{3, 7, 9\}$

Hence $A - B = A \cap B'$

(4) $A \Delta B = (A \cup B) - (A \cap B)$

Now $A \cup B = \{1, 3, 5, 6, 7, 8, 9\}$

$A \cap B = \{1, 5\}$

∴ $A \Delta B = \{3, 6, 7, 8, 9\}$

$B \Delta A = (B - A) \cup (A - B)$

$B - A = \{6, 8\}$

$A - B = \{3, 7, 9\}$

∴ $(B - A) \cup (A - B) = \{3, 6, 7, 8, 9\}$

Hence, $A \Delta B = B \Delta A$

☛ Note Incidentally this also verifies that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.

(5) $A - C = \{3, 5, 9\}$

$A \cap C = \{1, 7\}$

$A - (A \cap C) = \{3, 5, 9\}$

Thus, $A - C = A - (A \cap C)$

Example 7 : Prove that $A - B = A - (A \cap B)$

Solution : We have, by definition, $A - B = A \cap B'$

$A - (A \cap B) = A \cap (A \cap B)'$

$= A \cap (A' \cup B')$ **(De' Morgan's law)**

$= (A \cap A') \cup (A \cap B')$ **(Distributive law)**

$= \emptyset \cup (A \cap B')$

$= A \cap B'$

$= A - B$

Example 8 : Prove that $(A \cap B) \cup (A - B) = A$

Solution : This can be easily proved using distributive law.

We have $A - B = A \cap B'$

∴ $(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B')$

$= A \cap (B \cup B')$

$= A \cap U = A$

Example 9 : Prove that if $A \subset B$, then $B' \subset A'$ hence deduce that $A = B \Leftrightarrow A' = B'$.

Solution : We are given that $A \subset B$

$\forall x, x \in B' \Rightarrow x \in U$ and $x \notin B$

$\Rightarrow x \in U$ and $x \notin A$ **$(A \subset B)$**

$\Rightarrow x \in A'$

∴ $B' \subset A'$

Also, $A = B \Leftrightarrow A \subset B$ and $B \subset A$

$\Leftrightarrow B' \subset A'$ and $A' \subset B'$

$\Leftrightarrow A' = B'$

Example 10 : If $A = \{x \mid x \in R, x^2 - 3x - 4 = 0\}$ and $B = \{x \mid x \in Z, x^2 = x\}$.

Find the following :

(1) $A \cup B$ (2) $A \cap B$ (3) $A \Delta B$

Solution : For $x \in A$, we have

$x^2 - 3x - 4 = 0$

$\therefore$ $(x - 4)(x + 1) = 0$

$\therefore$ $x = 4$ or $x = -1$

$\therefore$ $A = \{-1, 4\}$

For $x \in B$, it is given that, $x^2 = x$

$\therefore$ $x^2 - x = 0$

$\therefore$ $x(x - 1) = 0$

$\therefore$ $x = 0$ or $x = 1$

$\therefore$ $B = \{0, 1\}$

Now, (1) $A \cup B = \{-1, 0, 1, 4\}$

(2) $A \cap B = \emptyset$

(3) $A \Delta B = (A \cup B) - (A \cap B) = \{-1, 0, 1, 4\}$

Example 11 : If $A = \{4k + 1 \mid k \in Z\}$, $B = \{6k - 1 \mid k \in Z\}$, find $A \cap B$.

Solution : Taking $k = 0, \pm 1, \pm 2, ...$

$A = \{ ..., 1, 5, 9, 13, 17, 21, ... \}$ and $B = \{ ..., -1, 5, 11, 17, 23, ... \}$

$\therefore$ It seems that $A \cap B = \{..., 5, 17, ...\} = \{ 12k + 5 \mid k \in Z \}$.

We prove this.

Let $x \in A \cap B$.

Then $x \in B$. Hence $x = 6k - 1, k \in Z$.

If k is even, let $k = 2m, m \in Z$. Then $x = 6(2m) - 1 = 12m - 1$

(Taking $k = 2m, m \in Z$)

$\therefore$ $x - 1 = 12m - 2 = 2(6m - 1)$, which is not a multiple of 4.

$\therefore$ $x - 1 \neq 4k'$ for any $k' \in Z$.

$\therefore$ $x \neq 4k' + 1$ for any $k' \in Z$.

$\therefore$ $x \notin A$

$\therefore$ $x \notin A \cap B$, which contradicts our assumption.

$\therefore$ k can not be even.

$\therefore$ k has to be odd.

Let $k = 2m + 1$, where $m \in Z$.

then $x = 6(2m + 1) - 1 = 12m + 5$

$= 12m + 4 + 1$

$= 4(3m + 1) + 1 \in A$ **($3m + 1 \in Z$)**

$\therefore$ If $x \in A \cap B$, then it is necessary that x is of the form $x = 12m + 5$ ($m \in Z$).

Also $12m + 5 = 4(3m) + 4 + 1 = 4(3m + 1) + 1 \in A$.

And $12m + 5 = 12m + 6 - 1 = 6(2m + 1) - 1 \in B$.

$\therefore \quad 12m + 5 \in A \cap B$

$\therefore \quad A \cap B = \{12k + 5 \mid k \in Z\}$

Exercise 2.2

1. If $A = \{x \mid x$ is a natural number less than $5\}$

$B = \{x \mid x$ is a prime number less than $15\}$, find $A \cup B$ and $A \cap B$.

2. $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 6\}$, $C = \{1, 2, 3\}$, $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Verify the following :

(1) $(A - B) \cup B = A \cup B$ (2) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(3) $A - (B - C) = (A - B) \cup (A \cap C)$

(4) $A \Delta A = \emptyset$ and $A \Delta \emptyset = A$

(5) $A - (B \cap C) = (A - B) \cup (A - C)$

3. If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 5, 7\}$ and $B = \{2, 5, 7, 8\}$, verify De Morgan's laws.

4. If $A = \{a, b, c, d, e\}$ and $B = \{c, d, e, f\}$, find (1) $A \cup B$ (2) $A \cap B$ (3) $A - B$ (4) $B - A$ (5) $A \Delta B$.

*

2.5 Cartesian Product of Sets

In day-to-day life, ordered pairs occur naturally. In an auditorium a typical seat number is say A5 where A denotes the row label and 5 is the fifth chair in that row. This can be represented as an order pair (A, 5) or A5. Here we note that in this pair alphabet (indicating row) occurs first and the number (indicating chair number) occurs second and this order is important.

In a result sheet of an examination, (35, 100) denotes that student with seat number 35 gets 100 marks. But (100, 35) indicates 35 marks are obtained by the student with seat number 100. As stated earlier $\{p, q\} = \{q, p\}$ but $(p, q) \neq (q, p)$.

$\{p, q\}$ is an unordered pair, a set with elements p and q.

Definition : Let A and B be two non-empty sets. Then the set of all ordered pairs (x, y), where $x \in A$, $y \in B$ is called cartesian product of A and B; and cartesian product of A and B is denoted by $A \times B$ (read : 'A cross B').

Thus, $A \times B = \{(x, y) \mid x \in A, y \in B\}$.

If A or B or both are empty sets then we take $A \times B = \emptyset$. Also $A \times A$ is written as A^2.

Just as we have ordered pairs (x, y), we can have ordered triplet or ordered n-tuple $(x_1, x_2, x_3, \ldots, x_n)$. If A, B and C are non-empty sets, their cartesian product is defined as

$A \times B \times C = \{(x, y, z) \mid x \in A, y \in B, z \in C\}$

In analogy with $A^2 = A \times A$, we will write $A^3 = A \times A \times A$.

Example 12 : Find $A \times B$, if $A = \{a, b, c\}$, $B = \{a, b\}$

Solution : $A \times B = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b)\}$

Example 13 : If $A = \{1, 2, 3\}$, $B = \{2, 6, 7\}$, $C = \{2, 7\}$,

verify $A \times (B - C) = (A \times B) - (A \times C)$

Solution : Here $B - C = \{6\}$

$\therefore \quad A \times (B - C) = \{(1, 6), (2, 6), (3, 6)\}$

Now $A \times B = \{(1, 2), (1, 6), (1, 7), (2, 2), (2, 6), (2, 7), (3, 2), (3, 6), (3, 7)\}$

$A \times C = \{(1, 2), (1, 7), (2, 2), (2, 7), (3, 2), (3, 7)\}$

$\therefore \quad (A \times B) - (A \times C) = \{(1, 6), (2, 6), (3, 6)\}$

Thus, $A \times (B - C) = (A \times B) - (A \times C)$

Example 14 : If $A \neq \emptyset$ and $A \times B = A \times C$, show that $B = C$.

Solution : If $B = C = \emptyset$, then $A \times B = A \times C = \emptyset$ and $B = C$.

Obviously, only one of B or C being empty is not possible since $A \neq \emptyset$.

So suppose $B \neq \emptyset$, $C \neq \emptyset$.

Since $A \neq \emptyset$, there exists an $x \in A$.

$\therefore \quad$ For every $y \in B, (x, y) \in A \times B$

$\therefore \quad (x, y) \in A \times C \qquad$ **$(A \times B = A \times C)$**

$\therefore \quad x \in A, y \in C$

Hence, $\forall y, y \in B \Rightarrow y \in C$

$\therefore \quad B \subset C$

Similarly, it can be proved that $C \subset B$

$\therefore \quad B = C$

Example 15 : $A = \{ 1, 2, 3, 4 \}$, $B = \{ (a, b) \mid b$ is divisible by a; $a, b \in A \}$.

List elements of B.

Solution : Here, 1, 2, 3, 4 are divisible by 1; 2, 4 are divisible by 2; 3 is divisible by 3; and 4 is divisible by 4.

$\therefore \quad B = \{ (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) \}$

Example 16 : If $A \times A = B \times B$, then prove that $A = B$.

Solution : If $A = \emptyset$, then $\emptyset = B \times B \Rightarrow B = \emptyset$. This shows that $A = B$.

Suppose $A \neq \emptyset$. Let $x \in A$

$\therefore \quad (x, x) \in A \times A$

$\therefore$ $(x, x) \in B \times B$ **(A × A = B × B)**

$\therefore$ $x \in B$

Hence, $\forall x,\ x \in A \Rightarrow x \in B$.

$\therefore$ $A \subset B$.

Similarly, $B \subset A$.

$\therefore$ $A = B$

Exercise 2.3

1. If $A = \{1, 2, 3\}$, $B = \{4, 7\}$, find $A \times B$ and $B \times A$.

2. If $A = \{1, 2, 3\}$, $B = \{3, 5\}$, $C = \{2, 6\}$, verify $A \times (B - C) = (A \times B) - (A \times C)$

3. If $A = \{x \mid x$ is natural number less than $5\}$, $B = \{x \mid x = 3a - 1,\ a \in A\}$, find $A \times B$.

*

2.6 Number of Elements of a Finite Set

We recall that $n(A)$ denotes the number of elements in a finite set A. Clearly if A and B are disjoint sets, we have $n(A \cup B) = n(A) + n(B)$. Similarly if $A \cap B = B \cap C = A \cap C = \emptyset$, then $n(A \cup B \cup C) = n(A) + n(B) + n(C)$.

For example, if $A = \{a, b, c\}$, $B = \{d, e, f\}$

then $A \cup B = \{a, b, c, d, e, f\}$

Here $n(A) = 3$, $n(B) = 3$ and $A \cap B = \emptyset$

and $n(A \cup B) = 6 = 3 + 3 = n(A) + n(B)$.

As shown in the Venn diagram 2.7,

$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$

and $A - B$, $A \cap B$, $B - A$ are mutually disjoint sets.

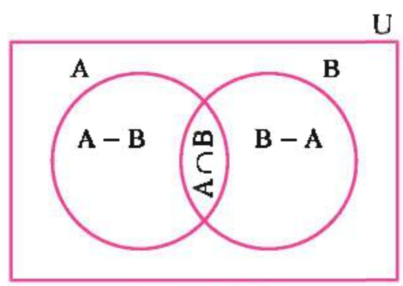


Figure 2.7

$\therefore$ $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$ **(i)**

Now $A = (A - B) \cup (A \cap B)$ with $(A - B) \cap (A \cap B) = \emptyset$

$\therefore$ $n(A) = n(A - B) + n(A \cap B)$

$\therefore$ $n(A - B) = n(A) - n(A \cap B)$

Similarly $n(B - A) = n(B) - n(A \cap B)$

Using these in (i) we get

$n(A \cup B) = [n(A) - n(A \cap B)] + n(A \cap B) + [n(B) - n(A \cap B)]$

$\qquad = n(A) + n(B) - n(A \cap B)$

☛ Note It can be proved easily, without the use of Venn diagram that $A - B$, $B - A$ and $A \cap B$ are mutually disjoint sets and their union is $A \cup B$.

Similarly, $n(A \cup B \cup C) = n(A) + n(B \cup C) - n(A \cap (B \cup C))$

$= n(A) + \{n(B) + n(C) - n(B \cap C)\} - n[(A \cap B) \cup (A \cap C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) -$

$[n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$

$= n(A) + n(B) + n(C) - n(A \cap B) -$

$n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Example 17 : A and B are sets such that $n(A \cup B) = 75$, $n(A) = 50$, $n(B) = 50$. Find $n(A \cap B)$.

Solution : It is known that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore \quad 75 = 50 + 50 - n(A \cap B)$

$\therefore \quad n(A \cap B) = 100 - 75 = 25$

Alternatively, as shown in the Venn diagram 2.8 :

$n(A - B) = a$, $n(A \cap B) = b$, $n(B - A) = c$

$a + b + c = 75$

$a + b = 50$

$b + c = 50$

$\therefore \quad a + b + b + c = 100$

$\therefore \quad b + 75 = 100$

$\therefore \quad n(A \cap B) = b = 25$

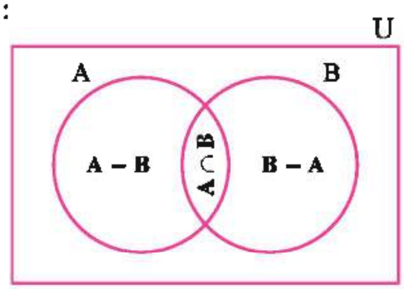


Figure 2.8

Example 18 : Prove that

(1) If non-empty, $A - B$ and $A \cap B$ are disjoint sets

(2) $A = (A - B) \cup (A \cap B)$

(3) $n(A - B) = n(A) - n(A \cap B)$

(4) If $B \subset A$, then $n(A - B) = n(A) - n(B)$

(5) $n(A') = n(U) - n(A)$

Solution : (1) $(A - B) \cap (A \cap B) = (A \cap B') \cap (A \cap B)$

$= A \cap (B' \cap B)$

$= A \cap \emptyset$ $\quad (B \cap B' = \emptyset)$

$= \emptyset$

$\therefore$ If non-empty, $A - B$ and $A \cap B$ are disjoint sets.

(2) R.H.S. $= (A - B) \cup (A \cap B) = (A \cap B') \cup (A \cap B)$

$= A \cap (B' \cup B)$

$= A \cap U$

$= A =$ L.H.S.

(3) From (1) and (2), $n(A) = n(A - B) + n(A \cap B)$

$\therefore \quad n(A - B) = n(A) - n(A \cap B)$

(4) Given that $B \subset A$

$\therefore \quad A \cap B = B$

Now from the result (3) proved above we have

$n(A - B) = n(A) - n(A \cap B)$

$= n(A) - n(B)$ **($A \cap B = B$)**

(5) $A \cap A' = \emptyset$ and $A \cup A' = U$

$\therefore \quad n(U) = n(A) + n(A')$

$\therefore \quad n(A') = n(U) - n(A)$

Note that if **$A \subset B$ then $n(A) \leq n(B)$.**

For finite sets A and B, **$n(A \times B) = n(A)\, n(B)$.**

Example 19 : If $A = \{1, 2, 3, 4\}$, $B = \{2, 4\}$, verify $n(A \times B) = n(A)\, n(B)$.

Solution : Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4\}$

Here $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4), (4, 2), (4, 4)\}$

$\therefore \quad n(A \times B) = 8$

Also $n(A) = 4$, $n(B) = 2$, $n(A \times B) = 8$

$\therefore \quad n(A \times B) = n(A)\, n(B)$.

Example 20 : A and B are not singleton and $n(A \times B) = 21$. Also $A \subset B$. Find $n(A)$ and $n(B)$.

Solution : $n(A \times B) = 21 = 3 \times 7 = 1 \times 21$

But, $n(A) \neq 1$, $n(B) \neq 1$

$\therefore \quad n(A) = 3$ and $n(B) = 7$ or $n(A) = 7$ and $n(B) = 3$.

But, $n(A) \leq n(B)$ **($A \subset B$)**

$\therefore \quad n(A) = 3$, $n(B) = 7$

Example 21 : In a troop of 20 dancers performing Bharatnatyam or Kuchipudi, 12 dancers perform Bharatnatyam and 4 perform both Bharatnatyam and Kuchipudi. Find the number of dancers performing Kuchipudi.

(**Note :** Each dancer performs Bharatnatyam or Kuchipudi)

Solution : Let A = Set of dancers performing Bharatnatyam

B = Set of dancers performing Kuchipudi

Then given that, $n(A) = 12$, $n(A \cap B) = 4$, $n(A \cup B) = 20$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore \quad 20 = 12 + n(B) - 4$

$\quad 20 = n(B) + 8$

$\therefore \quad n(B) = 12$

Thus number of dancers performing Kuchipudi is 12.

$\therefore$ Number of Dancers performing only Kuchipudi $= n(B) - n(A \cap B)$

$= 12 - 4 = 8$

Example 22 : In a group of people, 28 like Gujarati movies, 30 like Hindi movies, 42 like English movies; 5 like both Gujarati and Hindi movies, 8 like Hindi and English movies, 8 like Gujarati and English movies and 3 like Gujarati, Hindi and English movies. What is the least number of people in the group ?

Solution :

Here let G = the set of people who like Gujarati movies

H = the set of people who like Hindi movies

E = the set of people who like English movies

then given that

$n(G) = 28$, $n(H) = 30$, $n(E) = 42$

$n(G \cap H) = 5$, $n(E \cap H) = 8$, $n(G \cap E) = 8$, $n(G \cap E \cap H) = 3$

Now, $n(G \cup E \cup H) = n(G) + n(H) + n(E) - n(G \cap H) - n(E \cap H) - n(G \cap E) + n(G \cap E \cap H)$

$= 28 + 30 + 42 - 5 - 8 - 8 + 3$

$= 103 - 21 = 82$

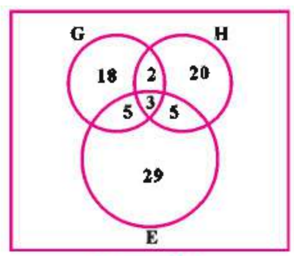


Figure 2.9

Some persons may not like to watch a movie. Hence there are at least 82 people in the group.

Miscellaneous Problems

Example 23 : If $A \cap B = A \cap C$, $A \cup B = A \cup C$, then prove that $B = C$ $(B \neq \emptyset, C \neq \emptyset)$

Method 1 : Let $x \in B$

$\therefore \quad x \in A \cup B$

$\therefore \quad x \in A \cup C$ **$(A \cup B = A \cup C)$**

there are two possibilities

(1) $x \in A$ or (2) $x \in C$

(1) $x \in A$

Thus, $x \in$ A and $x \in$ B

$\therefore$ $x \in A \cap B$

$\therefore$ $x \in A \cap C$ **(A ∩ B = A ∩ C)**

$\therefore$ $x \in C$

(2) Here $x \in C$ is what we want.

$\therefore$ In both alternatives $x \in C$

$\therefore$ $\forall x, x \in B \Rightarrow x \in C$

$\therefore$ $B \subset C$

Similarly it can be shown that $C \subset B$.

Thus, B = C.

Method 2 : It is known that $X \subset Y \Rightarrow X \cup Y = Y$

$(A \cap B) \subset B$

Now, $B = (A \cap B) \cup B$

$= (A \cap C) \cup B$ **(A ∩ B = A ∩ C)**

$= (A \cup B) \cap (B \cup C)$

$= (A \cup C) \cap (B \cup C)$ **(A ∪ B = A ∪ C)**

$= (A \cap B) \cup C$

$= (A \cap C) \cup C = C$ **((A ∩ C) ⊂ C)**

Method 3 : This result can be proved using $X \subset Y \Rightarrow X = X \cap Y$ The proof is left as an exercise.

Example 24 : $A - B = A - C$ and $B - A = C - A$. Prove that $B = C$. $(B \neq \emptyset, C \neq \emptyset)$

Solution : If $B \not\subset C$, then there exists a member $p \in B$ such that $p \notin C$.

Since $p \in U$, $p \in A$ or $p \notin A$.

(1) : Let $p \in A$. Since $p \notin C$, $p \in A - C$

$\therefore$ $p \in A - B$ **(A − B = A − C)**

$\therefore$ $p \notin B$.

Contradiction as $p \in B$

(2) : If $p \notin A$, then $p \in B - A$

$\therefore$ $p \in C - A$ **(B − A = C − A)**

$\therefore$ $p \in C$ which contradicts $p \notin C$

Thus none of the alternatives is possible.

$\therefore$ B $\not\subset$ C is not true.

$\therefore$ B $\subset$ C.

Similarly C $\subset$ B.

Hence B = C.

Example 25 : Prove that P(A) = P(B) $\Rightarrow$ A = B

Solution : A $\subset$ A $\Rightarrow$ A $\in$ P(A)

$\Rightarrow$ A $\in$ P(B) **(P(A) = P(B))**

$\Rightarrow$ A $\subset$ B

Similarly B $\subset$ A.

$\therefore$ A = B

Example 26 : n(A $\times$ A) = 9. (a, b) $\in$ A $\times$ A. Also $c \in$ A. Write the set A.

Solution : Let n(A) = k

Now, n(A $\times$ A) = k^2 = 9

$\therefore$ k = 3

(a, b) $\in$ A $\times$ A

$\therefore$ $a \in$ A, $b \in$ A

Further we are given that $c \in$ A.

Thus A has 3 elements namely a, b, c.

$\therefore$ A = {a, b, c}

Example 27 : A $\cap$ B = $\emptyset$ and A $\cup$ B = U, prove that A' = B.

Solution : Let $x \in$ B

$\therefore$ $x \notin$ A as A $\cap$ B = $\emptyset$

$\therefore$ $x \in$ A'

$\therefore$ B $\subset$ A' **(i)**

Let $x \in$ A'

$\therefore$ $x \notin$ A

but $x \in$ U

$\therefore$ $x \in$ A $\cup$ B **(A $\cup$ B = U)**

$\therefore$ $x \in$ A or $x \in$ B

$\therefore$ $x \in$ B **(x $\notin$ A)**

$\therefore$ A' $\subset$ B **(ii)**

From (i) and (ii) A' = B.

Exercise 2.4

1. In a group of students, 100 know Hindi and 50 know English and 25 know both the languages. Every student knows at least one of these languages. How many students are there in the group ?
2. Out of 600 residents in a society, 500 read Gujarati newspapers, 300 read English newspapers and 50 read both newspapers. Is this data correct ? Explain.
3. In a survey, it was found that 21 people liked product A, 26 liked product B and 29 liked product C. 14 people liked both A and B, 12 liked C and A, 14 liked B and C and 8 liked all three products. How many liked the product C only ? How many people did not like any of the products ?
4. In my school there are three sports teams. There are 21 players in the basketball team, 26 in the hockey team and 29 in the football team. 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all three games. How many players are there in all ?
5. $A \subset U$ and $B \subset U$, such that $n(A) = 20$, $n(B) = 30$, $n(U) = 100$, $n(A \cap B) = 10$. Find $n(A' \cap B')$.

*

Exercise 2

1. Write the following sets in the roster form :
   (1) $A = \{x \mid x \text{ is a prime number less than } 20\}$.
   (2) $\beta = \{x \mid x \text{ is a vowel in English alphabet}\}$.
   (3) $X = \{x \mid x \in N, 5 < x < 11\}$.
   (4) $X = \{x \mid x \in R, x^2 - 1 = 0\}$.
   (5) $X = \{x \mid x \in N, x^2 + 3x + 2 = 0\}$.
2. Write the following sets in set builder form :
   (1) $A = \{5, 10, 15, 20\}$
   (2) $P = \{1, 3, 5, ...\}$
3. If $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 7, 11\}$, find (1) $A - B$, (2) $B - A$, (3) $A \cup B$.
4. Prove the following
   (1) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
   (2) $A - (B \cup C) = (A - B) \cap (A - C)$
   (3) $A \cap (B - C) = (A \cap B) - (A \cap C)$

5. $U = \{x \mid x \in N, 1 \le x \le 10\}$

$A = \{1, 3, 5, 7, 9\}$, $B = \{2, 5, 8\}$, $C = \{2, 7, 8, 10\}$.

Verify the following :

(1) $(A \cup B)' = A' \cap B'$ (2) $(A \cap B)' = A' \cup B'$

(3) $A - (B - C) = (A - B) \cup (A \cap C)$

6. List all the subsets of the set $A = \{1, 5, 9\}$.

7. If $A \cup B = A \cap B$, prove that $A = B$.

8. Draw a Venn diagram for the sets A, B, C such that $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$, $A \cap C \neq \emptyset$ but $A \cap B \cap C = \emptyset$.

9. A and B are subsets of a universal set U. Let $n(A) = 20$, $n(B) = 30$, $n(U) = 80$, $n(A \cap B) = 10$. Find $n(A' \cap B')$.

10. In a class of 60 students 35 play Kabaddi and 40 play Kho-kho and 20 play both. Find the number of students who play neither of these.

11. Prove the following :

(1) $A - \emptyset = A$, $\emptyset - A = \emptyset$

(2) $A \cup \emptyset = A$ (see null set acts like zero !)

(3) $A \subset B \Leftrightarrow A - B = \emptyset$

(4) $A - B = B - A \Leftrightarrow A = B = \emptyset$

12. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) In column-A sets are described in roster form and in column-B sets are in the set-builder form.

A	B
(1) {L, A, T}	(A) $\{x \mid x$ is a natural number less than $4\}$
(2) $\{-2, -1, 0, 1, 2\}$	(B) $\{x \mid x$ is a letter of the word LATA$\}$
(3) {1, 2, 3}	(C) $\{x \mid x \in Z, x^2 < 5\}$

Which one of the following matches is correct ? ☐

(a) (1) - (A), (2) - (B), (3) - (C) (b) (1) - (B), (2) - (A), (3) - (C)

(c) (1) - (B), (2) - (C), (3) - (A) (d) None of these

(2) Let A = The collection of even numbers less than 100.

B = The collection of best athletes of 20th century

C = The collection of poems written by Umashankar Joshi.

Which one of the following is correct ? ☐

(a) A and B are sets (b) B is not a set

(c) A and C are not sets (d) A, B and C are sets

(3) If $A = \{x \mid x \in Z, x^4 - 16 = 0\}$, then

(a) $A = \{-2, 2\}$ (b) $A = \{2\}$

(c) $A = \{-4\}$ (d) $A = \{-4, 4, -2, 2\}$

(4) Let $A = \{y \mid y \in N, y^3 - 27 = 0\}$. Select the correct statement.

(a) $9 \in A$ (b) $-3 \in A$ (c) $3 \in A$ (d) $-9 \in A$

(5) Let $B = \{x \mid x \in Z, x^2 - 16 = 0\}$. Select the correct statement.

(a) $4 \in B$ (b) $-4 \notin B$ (c) $-2 \in B$ (d) $2 \in B$

(6) Let $B = \{ \emptyset \}$, then

(a) B is the null set (b) B is a finite set

(c) B is an infinite set (d) B is not a set

(7) $A = \{x \mid x \in N, x^2 + 4 = 0\}$. Then

(a) $A = \{-2, 2\}$ (b) $A = \{2\}$ (c) $A = \emptyset$ (d) $A = \{ \emptyset \}$

(8) $\alpha = \{x \mid x$ is a letter of word ALPHA$\}$,

$\beta = \{x \mid x$ is a letter of word ALPA$\}$ and $\gamma = \{L, P, A, H\}$

Select the incorrect statement.

(a) $\alpha = \gamma$ (b) $\beta = \{A, L, P\}$ (c) $\alpha = \beta$ (d) $\beta \cap \gamma \neq \emptyset$

(9) In column-A some sets are given and in column-B subsets are given

Column-A	Column-B
(1) $\{1, 3, 5, 7, ...\}$	(A) $\{1, 19, 21\}$
(2) $\{2, 4, 6, 8, ...\}$	(B) $\{2, 5, 6, 8, 19\}$
(3) $\{1, 2, 3, 4, ...\}$	(C) $\{8, 28, 38\}$

if sets in A are matched with their corresponding subsets in B, then which of the following match is correct ?

(a) (1) - (C), (2) - (B), (3) - (A) (b) (1) - (A), (2) - (C), (3) - (B)

(c) (1) - (C), (2) - (A), (3) - (B) (d) (1) - (A), (2) - (B), (3) - (C)

(10) Number of elements in the power set of $A = \{x \mid x \in N, x^2 < 9\}$ is

(a) 9 (b) 4 (c) 1 (d) 8

(11) For the set R of real numbers, which one of the following is incorrect ?

(a) $N \subset R$ (b) $(a, b) \subset R; a < b$

(c) $\pi \notin R$ (d) $\emptyset \subset R$

(12) Of which set A = {1, 5, 7}, B = {1, 10}, C = {11, 12,..., 20} are subsets ? ☐

(a) {1, 2, 3,..., 20} (b) {1, 3, 5,..., 21}

(c) ∅ (d) {1, 11, 111, 1111}

(13) Of which set, A = {1, 2, 3, 4}, B = {−1, 1, 0, −2, 2}, C = {1, 3, 4}; are subsets ? ☐

(a) [1, 4] (b) [−1, 4] (c) [−2, 2] (d) [−2, 4]

(14) For the interval (−1, 1], which one is correct ? ☐

(a) −1 ∈ (−1, 1] (b) 0 ∈ (−1, 1]

(c) (−1, 1] = {−1, 1} (d) (−1, 1] = ∅

(15) Which one of the coloured portion in Venn diagrams represents A ∩ B ? ☐

(a)

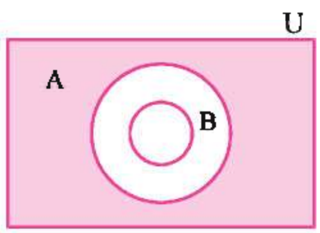


Figure 2.10

(b)

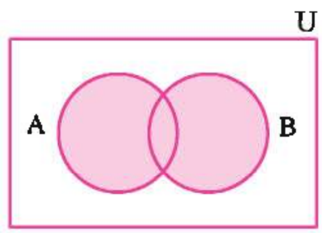


Figure 2.11

(c)

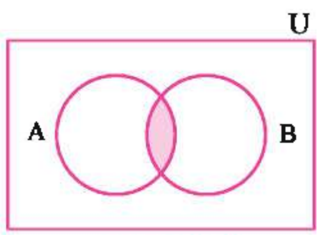


Figure 2.12

(d)

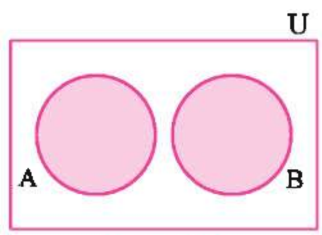


Figure 2.13

(16) From the Venn diagram 2.14, select the correct statement. ☐

(a) A = {1, 3, 4, 7}

(b) U = {1, 2,..., 7}

(c) A ∪ B = {4, 7, 2, 6}

(d) B = ∅

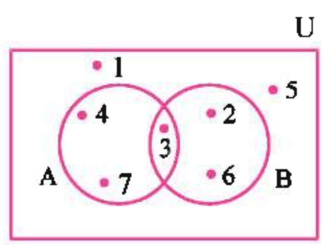


Figure 2.14

(17) Using Venn diagram 2.15, select the incorrect statement.

(a) A = {c, d, f, g}

(b) B = ∅

(c) U = {a, b, c, d, e, f, g}

(d) A ∪ B = {c, d, f, g}

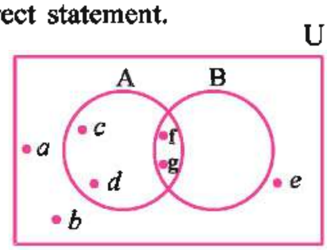


Figure 2.15

(18) A = {x | x is a natural number less than 8},
B = {x | x is a natural number greater than 5 and less than 18}, then

(a) A ∪ B = {x | x is a natural number less than 18}

(b) A ∪ B = {–1, –2, 1, 2, 0, 18}

(c) A ∪ B = ∅ (d) A ∩ B = {1}

(19) Which one of the following is correct ?

(a) A ∪ (B ∪ C) = A ∩ (B ∪ C)

(b) A ∪ (B ∪ C) = A ∪ (B ∩ C)

(c) A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

(d) A ∩ (B ∩ C) = A ∩ (B ∪ C)

(20) If A = {1, 3, 5, 7}, B = {2, 4, 6, 8}, then n(P(A ∩ B)) = ...

(a) 1 (b) 2^8 (c) 8 (d) 8^2

(21) Which one of the following is correct ? (A ≠ B)

(a) (A ∩ B) ⊂ A (b) (A ∪ B) ⊂ A

(c) (A ∪ B) ∩ B = A (d) (A ∪ B) ⊂ B

(22) If A ⊂ B, then ......

(a) A ∩ B = ∅ (b) A ∩ B = A (c) A ∩ B = B (d) A ∪ B = A

(23) U = {x | x ∈ N, x ≤ 10}, A = {1, 3, 5, 7, 9}, B = {2, 4, 6, 8, 10} then
(A ∪ B)' = ...

(a) U (b) {2} (c) ∅ (d) {1, 4, 7, 8}

(24) Taking the set R of real numbers as the universal set, Q' = ...

(a) N (b) Z (c) I (d) R

(25) Taking the set of natural numbers as the universal set and A = {x | x – 8 = 3},
A' = ...

(a) N (b) {5} (c) N – {5} (d) N – {11}

(26) Taking U = [1, 5], A = {x | $x \in$ N, $x^2 - 6x + 5 = 0$}, A' = ...

(a) {1, 5} (b) (1, 5) (c) [1, 5] (d) [−1, −5]

(27) Taking U = [1, 2], A = {x | $x \in$ N, $x^2 + x - 2 = 0$}, A' = ...

(a) (1, 2] (b) [1, 2] (c) {1, 2} (d) (1, 2)

(28) In which one of the following diagrams coloured region represents (A $\cap$ B)' ?

(a)

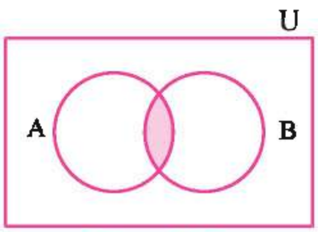


Figure 2.16

(b)

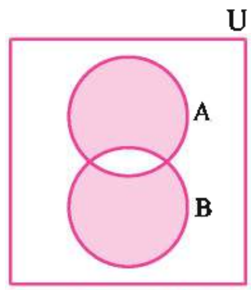


Figure 2.17

(c)

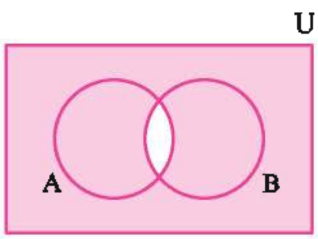


Figure 2.18

(d)

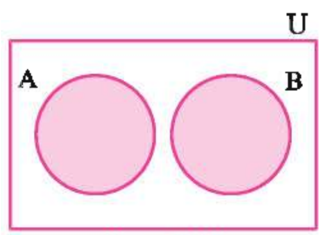


Figure 2.19

(29) In a residential colony, 50 families speak Gujarati, 30 families speak Hindi and 10 families can speak both Gujarati and Hindi. How many families speak at least one of these languages ?

(a) 80 (b) 90 (c) 70 (d) 60

(30) In a students hostel accomodating 200 students, 50 like Idli and 75 like Upma and 35 like both Idli and Upma. How many students like neither Idli nor Upma ?

(a) 75 (b) 110 (c) 200 (d) 90

(31) In a survey of students, 21 students liked Arts stream, 26 liked Commerce stream and 29 liked Science stream. 14 students liked Arts and Commerce, 10 students liked Arts and Science, 8 students liked Commerce and science and 6 liked all three streams. At least how many students like Arts or Commerce or Science stream ? Every one liked at least one stream.

(a) 76 (b) 82 (c) 50 (d) 110

Summary

We studied following points in this chapter :

1. Set is an undefined term.
2. Universal set
3. Subset
4. Equality of two sets.
5. Union set, Union operation and its properties
6. Intersection of two sets and intersection operation and its properties.
7. Complement of a set, complementation and its properties.
8. De Morgan's laws
9. Distributive laws
10. Difference of two sets and symmetric difference of two sets.
11. Cartesian product of two sets
12. Notation $n(A)$, formulae for $n(A \cup B)$ and $n(A \cup B \cup C)$

A Venn diagram of two sets contains four regions. A Venn diagram of three sets contain eight regions. How many regions will be there in a Venn diagram of four sets ? Can you draw it using circles only as is usual practice ?

Chapter 3

RELATIONS AND FUNCTIONS

3.1 Introduction

The concept of functions is very fundamental in modern mathematics. Many mathematicians have contributed to the formal development of idea of a function. Among them, French mathematician Dèscartes used the word **function** in the year 1637. At that time, he had only referred to x^n, $n \in$ N. James Gregory gave the definition of a function in 1667. He introduced functions to obtain a quantity by performing algebraic operations on some other quantities. In 1673, Liebnitz gave the definition of a function in the context of the coordinates, the slope of a tangent, and the slope of a normal at a point on a curve, as a quantity varying at every point. The definition of a function used now-a-days was given by Dirichlet. Georg Cantor gave definition of a function with the help of sets. In this chapter, we will study functions, their types and various operations on functions.

3.2 Relation

We are familiar with the Cartesian product of two non-empty sets. Accordingly if A = $\{a, b, c\}$, B = $\{c, d\}$, then A $\times$ B = $\{(a, c), (a, d), (b, c), (b, d), (c, c), (c, d)\}$.

Further for the set N of natural numbers 'The relation of being double', associates 1 with 2, 2 with 4, 3 with 6 and so on. In place of writing like this, we may write them as ordered pairs as (1, 2), (2, 4), (3, 6), (4, 8),.... Hence this relation can be expressed as a set {(1, 2), (2, 4), (3, 6),...}. We note that this is a subset of N $\times$ N.

In general, **for any non-empty sets A and B, a subset of A $\times$ B is called a relation from A to B.**

For the sets A and B discussed in the example above, $\{(a, c), (b, d)\}$ is a relation from A to B. Since n(A $\times$ B) = 6, the number of subsets of A $\times$ B is $2^6 = 64$, hence 64 relations are possible from A to B. A relation as a set is usually denoted by S. Thus above relation is written as S = $\{(a, c), (b, d)\}$. Further as we have discussed above any subset S of A $\times$ B is a relation from A to B. If a pair $(x, y) \in$ S, then we say that x is related to y by S. In the example above a is related to c and b is related to d. If a pair $(x, y) \in$ A $\times$ B, but $(x, y) \notin$ S, then we say that x is not related to y by S. For example, $(c, c) \notin$ S, hence c is not related to c by S.

If S is a relation from A to B, then **$\{a \mid (a, b) \in S\}$ is called the domain of S and $\{b \mid (a, b) \in S\}$ is called the range of S**. In the above example, the domain of S is $\{a, b\}$ and the range is $\{c, d\}$. It can be seen that for a relation from A to B, domain is a subset of A and range is a subset of B.

A relation S from A to B is said to be empty if $S = \emptyset$.

A relation S from A to B is said to be the universal relation if $S = A \times B$.

Further, if A = B i.e. if $S \subset (A \times A)$ then the relation S is called a relation on A.

In practice, the concept of relation occurs in many ways, in mathematics as well as in society. We define relation of 'being subset' on the power set P(A) of a set A. Thus, if $M \subset N$, then we say that M is related to N, by relation $\subset$. Similarly being 'less than' or being 'greater than' are relations on the set of numbers. Thus, if $a < b$ we say that a is related to b by the relation '$<$', or if $a > b$ we say that a is related to b by the relation '$>$'. In these examples, it can be noted that in a relation we take ordered pairs.

In society, if H is the set of all human beings, the relation 'being mother of' is a subset of $H \times H$, i.e. the relation is given by $M = \{(a, b) \mid a, b \in H, a \text{ is mother of } b\}$. In this case, we write aMb.

Example 1 : $A = \{1, 2,.., 10\}$, $B = \{3, 6, 9, 12\}$. Let the relation from A to B be given by $S = \{(a, b) \mid a \text{ is a multiple of } b\}$. Find the domain and the range of S.

Solution : Here $S = \{(3, 3), (6, 3), (9, 3), (6, 6), (9, 9)\}$ as the multiples of 3 belonging to B, in A are 3, 6, 9; multiple of 6 belonging to B, in A is 6 and multiple of 9 belonging to B, in A is 9. Thus, domain of S is $\{3, 6, 9\}$ and range is $\{3, 6, 9\}$.

Example 2 : Define a relation S on N by $S = \{(a, b) \mid a + 2b = 15\}$. Write S in roster form. Also find range and domain of S.

Solution :

For $a + 2b = 15$, $2b \leq 15$. Hence possible values of b are $b = 1, 2, 3, 4, 5, 6, 7$. Now for these values of b corresponding values of a (in N) are, 13, 11, 9, 7, 5, 3, 1.

Thus, $S = \{(13, 1), (11, 2), (9, 3), (7, 4), (5, 5), (3, 6), (1, 7)\}$

This gives domain of $S = \{1, 3, 5, 7, 9, 11, 13\}$

and range of $S = \{1, 2, 3, 4, 5, 6, 7\}$

Example 3 : Let $A = \{5, 7, 9\}$, $B = \{1, 3\}$ and let

$S = \{(a, b) \mid a \in A, b \in B, a - b \text{ is odd}\}$. Show that S is an empty relation.

Solution : Here, since elements in A and B are odd integers, the result of their subtraction is an even number. Thus. for no pair (a, b), $a - b$ is odd. This proves that S is an empty relation.

Example 4 : Let O be the set of odd natural numbers and E be the set of even natural numbers.

Let S = {(*a*, *b*) | $a + b$ is even}, T = {(*a*, *b*) | *ab* is even} be relations from O to E. Find S and T. Find domain and range of T.

Solution : If *x* is odd and *y* is even, then $x + y$ is always odd and *xy* is always even.

∴ $(x + y) \notin$ S for any $(x, y) \in$ O × E; and $(x, y) \in$ T for all $x \in$ O, $y \in$ E.

∴ S = ∅ and T = O × E

The domain and range of T are O and E respectively.

3.3 Visual Representation of Relations

We have seen that a relation from A to B is a subset of A × B. A relation can be described using Venn diagram and also in a tabular form. This is illustrated in the following example.

Let A = {1, 2, 5}, B = {2, 4, 6, 8, 10} and S = {(*a*, *b*) | *b* is divisible by *a*}. Then we have S = {(1, 2), (1, 4), (1, 6), (1, 8), (1, 10), (2, 2), (2, 4), (2, 6), (2, 8), (2, 10), (5, 10)}. This relation is represented in the figure 3.1 with the help of Venn diagram.

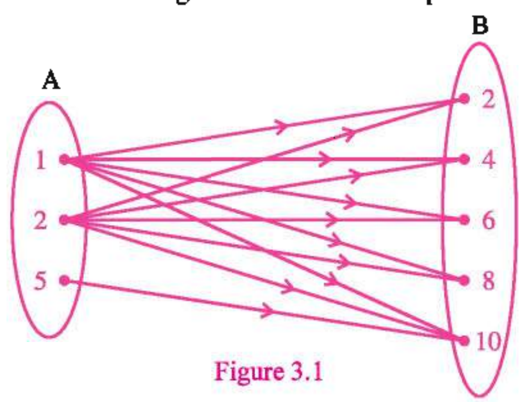


Figure 3.1

This can be understood as *a* is related to *b* if there is an 'arrow' from *a* to *b*. This is also called an **arrow diagram**.

Second Method (Tabular Method) :

B

	S	2	4	6	8	10
	1	1	1	1	1	1
A	2	1	1	1	1	1
	5	0	0	0	0	1

This table is formed by zeroes and ones. Here (1, 2) ∈ S, hence the entry, in the cell of intersection of the row containing 1 and the column containing 2, is 1. Further since (5, 2) $\notin$ S, so the entry in the table corresponding to these elements is 0. This table can also be written as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

☛ Note Such an arrangement is called an array or a matrix.

Exercise 3.1

1. Determine the domain and the range of the relation $S = \{(x, y) \mid x, y \in N, x + y = 8\}$.
2. Write the relation $S = \{(x, x^3) \mid x$ is a prime number less than $10\}$ in the roster form.
3. A = {1, 2, 3, 5}, B = {4, 6, 9}. A relation S is given by
   $S = \{(x, y) \mid$ the difference between x and y is odd; $x \in A, y \in B\}$.
   Write S in roster form.
4. A relation is shown in the figure 3.2 :
   Write this relation in roster form.

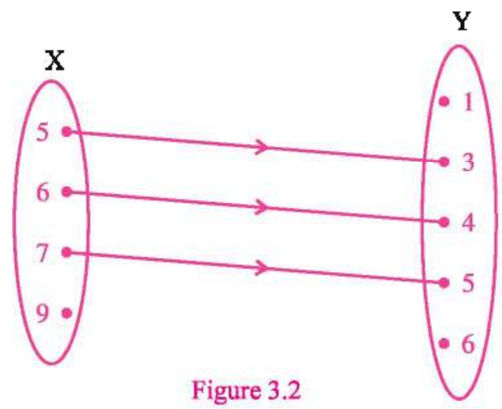


Figure 3.2

*

3.4 Functions

We shall now study a special type of relation, called a **function**. For non-empty sets A and B, if a non-empty relation f from A to B has domain A and for every $x \in A$, f contains one and only one (unique) ordered pair containing x, then f is called a function from A to B and is written as $f : A \to B$. Thus, the formal definition of a function is as follows.

Definition : Let A and B be two non-empty sets and $f \subset (A \times B)$ and $f \neq \emptyset$. Then $f : A \to B$ is said to be a function, if $\forall x \in A$, there corresponds a unique ordered pair $(x, y) \in f$. The set A is called the domain and B is called the codomain of the function $f : A \to B$ and the set of f of ordered pairs (x, y) is also called the graph of the function $f : A \to B$.

The set $\{y \mid (x, y) \in f\}$ is called the range of the function f. The domain and range of a function $f : A \to B$ are denoted by D_f and R_f respectively. For simplicity we refer to these sets as, "domain and range of f", instead of saying "domain and range of the function $f : A \to B$". We note that the range of a function is a subset of its codomain.

For a function $f : A \to B$; if $A \subset R$, then the function is called a function of real variable. If $B \subset R$, then $f : A \to B$ is called a real function and if $A \subset R$, $B \subset R$, then we say that $f : A \to B$ is a real function of real variable.

For a real function of real variable $f : A \to B$, the ordered pair $(x, y) \in R \times R$ and it can be represented as a point in the real plane. **The set $\{(x, y) \mid (x, y) \in f\}$ represents the graph of the function f in the plane.**

Consider the sets $A = \{1, 3, 5\}$ and $B = \{1, 3, 4, 5, 6, 7\}$.

Let $f = \{(1, 3), (3, 5), (5, 7)\}$. It can be seen that domain of f is entire A and to each element of A, there corresponds a unique element of B. Thus $f : A \to B$ is a function. Here domain of f is A and B is its codomain, whereas the range of f is $\{3, 5, 7\}$.

The function discussed above is represented by a Venn diagram 3.3 as follows.

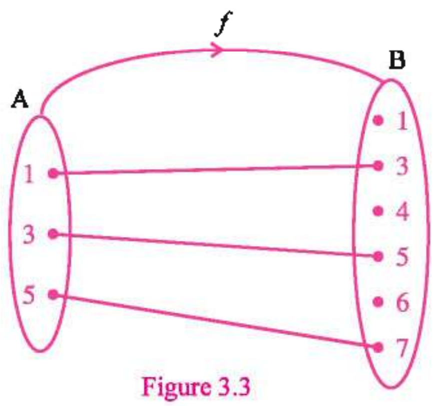


Figure 3.3

Here we note that function gives a correspondence between elements of one set and elements of another set. In the above example we write the function as correspondence $f(1) = 3$, $f(3) = 5$ and $f(5) = 7$. This means that $\forall (x, y) \in f$, we write $y = f(x)$. In the study of functions it is useful to recognize some pattern in the correspondence which may form a rule or a formula for the expression of the function. In the above example, we can write $f(x) = x + 2$, $\forall x \in A$. However, it is not necessary that every function can be expressed as a rule or a formula. Further, the function as a set depends on the domain and codomain of the function and not on the formula only.

The following illustrations help in understanding some of above ideas.

Let $f : N \to N$, $f(x) = x^2$. This function as a set of ordered pairs is written as $f = \{(1, 1), (2, 4), (3, 9), ...\}$. Now we define $g : Z \to Z$, $g(x) = x^2$, this function as a set is given by $g = \{..., (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), ..\}$. Thus, functions f and g are different functions even though their rules or formulae are same.

Consider, $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $C = \{1, 3, 5, 7, 9\}$. Now, define $f : A \to B$; $f(x) = 2x - 1$ and $g : A \to C$; $g(x) = 2x - 1$.

$f = \{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}$

$g = \{(1, 1), (2, 3), (3, 5), (4, 7), (5, 9)\}$

Here codomains are different. Thus f and g are not equal functions.

Now consider A = {1, 2, 3, 4}, B = {2, 3, 4, 5, 6} and C = {3, 5, 7, 9, 11}.

Define $f : A \rightarrow B$; $f(x) = x + 1$

$g : A \rightarrow C$; $g(x) = 2x + 1$

Here, codomains are different and formulae are also different. So f and g are not equal functions.

Finally, consider A = {1, 2, 3, 4}, B = {1, 3, 5} and

C = {x | x is a natural number less than 30}

Define $f : A \rightarrow C$; $f(x) = x^2$

$g : B \rightarrow C$; $g(x) = x^2$

Here, domains of f and g are different. So f and g are not equal functions.

Equal Functions : Two functions are said to be equal if their domains, codomains and graphs (set of ordered pairs) or formula (if any) are equal.

$f : A \rightarrow B$ and $g : C \rightarrow D$ are equal functions, if A = C, B = D and $f(x) = g(x)$ $\forall x \in$ A (or C).

For a function $f : A \rightarrow B$, $f(x)$ is said to be value of f at x or image of x under f; and x is called pre-image of $f(x)$. If $C \subset A$, then $\{y \mid y = f(x), x \in C\}$ is called the image of set C under f and it is denoted by $f(C)$. Thus, range of $f : A \rightarrow B$ is $f(A)$.

Example 5 : Let A = {1, 2, 3, 4}, B = {2, 4, 6, 8, 10, 12} and $f = \{(1, 2), (2, 4), (3, 6), (4, 10), (3, 12)\}$. Is f a function ?

Solution : No, because corresponding to $3 \in A$, there are two elements in f, namely 6 and 12 from the set B which give ordered pairs. **In a function, to every element of A, there should correspond a unique element of B.**

Example 6 : A = {1, 2, 3, 4, 5,}, B = {1, 3, 5, 7}. Define $f(x) = x - 2$. Is f a function from A to B ?

Solution : As a set we write if possible, $f = \{(1, -1), (2, 0), (3, 1), (4, 2), (5, 3)\}$, it can be seen that $f \not\subset (A \times B)$, thus f is not even a relation from A to B and hence not a function from A to B.

Example 7 : Define $f : N \rightarrow N$, $f(x) = 2x$. Is f a function ? Find range of f. If A = {1, 2, 4, 8, 16}, find $f(A)$. Also find the images and pre-images of 56 and 65, if they exist.

For every $x \in N$, there exists a unique $2x \in N$, thus $f : N \rightarrow N$ is a function.

Also, $f = \{(1, 2), (2, 4), (3, 6), (4, 8),...\} = \{(n, 2n) \mid n \in N\}$

$\therefore$ The range of f is $\{2, 4, 6, 8, 10,...\} = \{2n \mid n \in N\}$

Now, $f(1) = 2$, $f(2) = 4$, $f(4) = 8$, $f(8) = 16$, $f(16) = 32$,

hence $f(A) = \{2, 4, 8, 16, 32\}$.

Further, $f(56) = 112$ and $f(65) = 130$, hence the images of 56 and 65 and 112 and 130 respectively.

The pre-image of a number $x \in N$ is $\frac{x}{2}$, if x is an even number then this is a natural number. Thus pre-image of 56 exists and it is 28, whereas pre-image of 65 does not exist. In fact $f(28) = 56$ and $f(x) \neq 65$ for any $x \in N$.

Example 8 : Find the range of following real functions defined on R.

(1) $f : R \rightarrow R,\ f(x) = x^2 + 2x + 3$ (2) $f : R \rightarrow R,\ f(x) = x^4$

(3) $f : R \rightarrow R,\ f(x) = [x]$ (4) $f : R \rightarrow R,\ f(x) = 3x + 2$

Solution : (1) $f(x) = x^2 + 2x + 3$

$f(x) = x^2 + 2x + 1 + 2$

$= (x + 1)^2 + 2 \geq 2$ as $(x + 1)^2 \geq 0$

$\therefore$ $R_f \subset \{y \mid y \geq 2, y \in R\}$

Also if $y \in R$ and $y \geq 2$, let $\sqrt{y-2} - 1 = x$.

Then $(x + 1)^2 = y - 2$ or $x^2 + 2x + 3 = y$

$\therefore$ $\exists x \in R$ such that $y = x^2 + 2x + 3$, if $y \geq 2$.

$\therefore$ $y \in R_f$

$\therefore$ $\{y \mid y \geq 2, y \in R\} \subset R$

$\therefore$ $R_f = \{y \mid y \geq 2, y \in R\}$

(2) $x^4 \geq 0$. Thus $R_f \subset (R^+ \cup \{0\})$ **(i)**

Also if $y \geq 0$, let $\sqrt[4]{y} = x$, which exists as $y \geq 0$.

$\therefore$ $x^4 = y$

$\therefore$ $f(x) = y$

$\exists x$ such that $y = f(x)$ if $y \in R^+ \cup \{0\}$.

$\therefore$ $(R^+ \cup \{0\}) \subset R_f$ **(ii)**

$\therefore$ By (i) and (ii), $R_f = R^+ \cup \{0\}$.

(3) $[x]$ is the greatest integer not exceeding x.

$$\text{Thus, } [x] = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \\ 2 & 2 \leq x < 3 \end{cases}$$

For every real number x, $[x]$ is an integer.

$\therefore$ $f(x)$ is an integer.

$\therefore$ $R_f \subset Z$ **(i)**

Also for any n, $n \in Z$, $n = [n] = f(n)$

$\therefore$ $n \in R_f$

$\therefore$ $Z \subset R_f$ **(ii)**

$\therefore$ By (i) and (ii), $R_f = Z$

Note : Integers not exceeding 3 are 3, 2, 1, 0,...

The greatest of them is 3. Thus [3] = 3

If $0 \leq x < 1$ the integers not exceeding x are 0, –1, –2,.... 0 is the largest of them.

Thus for any real number x such that $0 \leq x < 1$, $[x] = 0$.

For any integer n, the integers not exceeding n are n, $n - 1$, $n - 2$,...; and among them n is the largest, hence $[n] = n$.

(4) If $x \in$ R, then $3x + 2 \in$ R.

$R_f \subset R$ **(i)**

Also if $y \in$ R, $\frac{y-2}{3} \in$ R. Let $x = \frac{y-2}{3} \in$ R

$\therefore \quad y = 3x + 2$

Thus for every $y \in$ R, there exists $x \in$ R such that $y = f(x)$

$\therefore \quad R \subset R_f$ **(ii)**

$\therefore$ By (i) and (ii), $R_f = R$

Example 9 : Draw the graph of the function

$f : R \rightarrow R, f(x) = 3x - 2$.

Here $f(0) = -2$, $f(1) = 1$, $f(2) = 4$, $f(10) = 28$, $f(0.5) = -0.5$ and so on. Hence $(0, -2) \in f$, $(1, 1) \in f$, $(2, 4) \in f$, $(-1, -5) \in f$...

These points when joined give a straight line as shown in the figure 3.4.

[A few of the points are plotted. But $x \in$ R. Thus a 'continuous' line is drawn.]

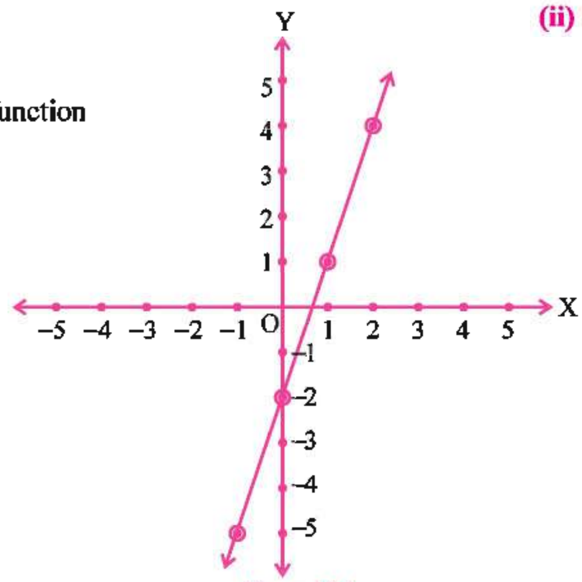


Figure 3.4

Example 10 : Let a function $f : R \rightarrow R$ be given by $f(x) = x^2$. Draw the graph of this function.

The graph of this function contains ordered pairs like (x, x^2), i.e. (–1, 1), (1, 1), (–2, 4), (2, 4) and so on. These points when connected give the curve in figure 3.5

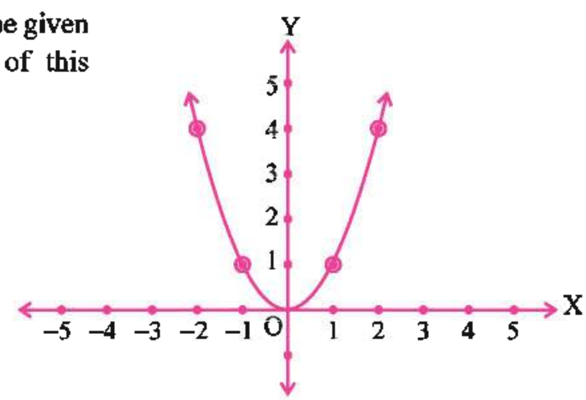


Figure 3.5

Exercise 3.2

1. Find the range of the following functions :

(1) $f : N \to N, f(x) = x + 2$ (2) $f : N \to N, f(x) = 2^x$

(3) $f : R \to R, f(x) = 5$ (4) $f : Z \to Z, f(x) = x - 1$

2. Draw the graph of :

(1) $f : R \to R, f(x) = x + 3$ (2) $f : R \to R, f(x) = 1 - x$

3. If $f(x) = x^2 + 4\sqrt{x} + 3$, then find $f(4), f(16)$.

4. Find the value of a, if $f(x) = \frac{1}{x} + ax$ and $f\left(\frac{1}{5}\right) = \frac{28}{5}$

*

3.5 Some Functions and Their Graphs

(1) Identity Function : Let A be a non-empty set. The function $f : A \to A$ defined by $f(x) = x, \forall x \in A$ is called the identity function on A. **Identity function on a set A is denoted by I_A.**

This function maps any element of A onto itself. For this function, the range is entire codomain. The graph of identity function on the set R of real numbers is given by the straight line $y = x$, as shown in the figure 3.6.

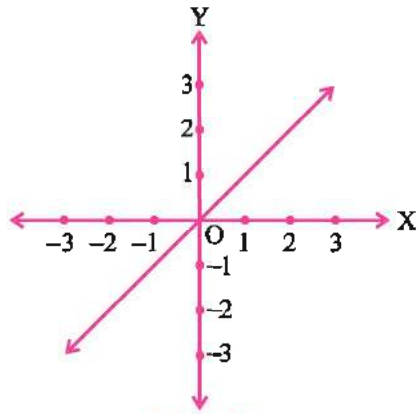


Figure 3.6

(2) Constant Function : A function whose range is singleton is called a constant function.

Thus, **the function $f : A \to B, f(x) = c$, where c is a fixed element of B, is a constant function and $\forall x \in A, f(x) = c$.**

$f : \{2, 4, 6, 8\} \to R, f(x) = 0$ is a constant function with $f(2) = 0, f(4) = 0, f(6) = 0, f(8) = 0$.

As an example, let x be the measure of an acute angle, then $x \in (0, 90)$ and also $sin^2x + cos^2x = 1$. Now we define $f : (0, 90) \to R, f(x) = sin^2x + cos^2x$, then $\forall x \in (0, 90), f(x) = 1$. Hence this is a constant function.

The graph of a constant function on the set of real numbers is a horizontal line $y = c$. In figure 3.7 the graph of $y = c$ $(c > 0)$ is shown.

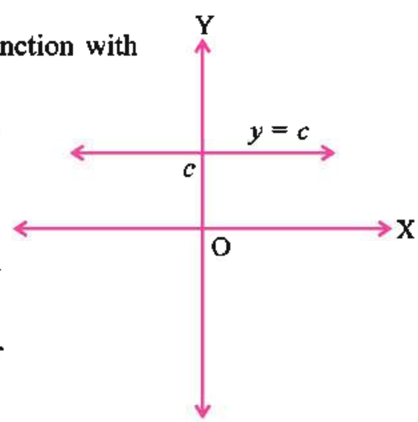


Figure 3.7

(3) Modulus Function : It is known that the modulus of a real number x is defined as

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

The function f : R $\rightarrow$ R defined by $f(x) = |x|$ is called modulus function or absolute value function. Since $|x| \geq 0$, $\forall x \in$ R, the range of this function is $R^+ \cup \{0\}$. To draw the graph of this function, we note that $f(1) = 1$, $f(-1) = 1$, $f(0) = 0$ and so on. Thus, the graph of this function is a union of two rays as represented pictorially in figure 3.8.

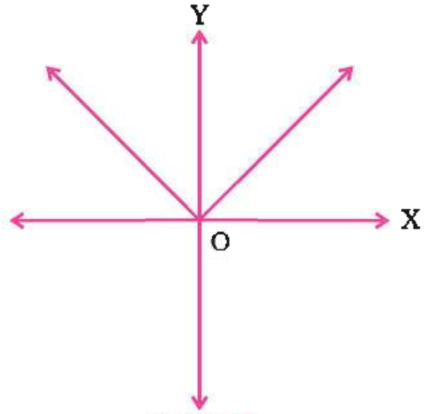


Figure 3.8

If the modulus function is defined on R^+, then it becomes identity function.

(4) Signum Function : The function f : R $\rightarrow$ R defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is called the signum function. This function has value 1 or -1 according as the number x is positive or negative and it is zero for $x = 0$. The domain of this function is entire R and range is the set $\{-1, 0, 1\}$. The graph of this function is given in figure 3.9.

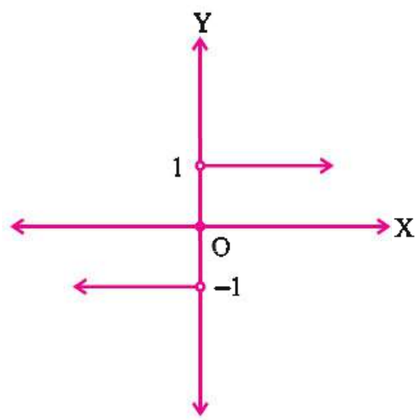


Figure 3.9

This function can also be defined as

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{|x|}{x} & \text{if } x \neq 0 \end{cases}$$

(5) Polynomial Function : Let a function g be defined as g : R $\rightarrow$ R

$g(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, $a_n \neq 0$. Then g is called a polynomial function of degree n, where n is a non-negative integer and $a_0, a_1, a_2, .., a_n$ are constant real numbers. We note that the constant function defined earlier is a special case of polynomial function with $n = 0$.

(6) Rational Function : A function $h(x)$, which can be expressed as $h(x) = \frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain where $g(x) \neq 0$, is called a rational function.

Thus h : R $- \{x \mid g(x) = 0\} \rightarrow$ R, $h(x) = \frac{f(x)}{g(x)}$ is a rational function if f and g are polynomial functions.

(7) Greatest Integer Function : The function $f : R \to R$ defined by $f(x) = [x]$, assumes the value of the greatest integer, less than or equal to x. $[x]$ is also the greatest integer not exceeding x. This function is called the greatest integer function. The domain of this function is R and range is the set of integers Z.

$[x] = -1$ for $-1 \le x < 0$

$[x] = 0$ for $0 \le x < 1$

$[x] = 1$ for $1 \le x < 2$ etc.

This function is also called **'floor' function**.

The graph of the function is as shown in the figure 3.10.

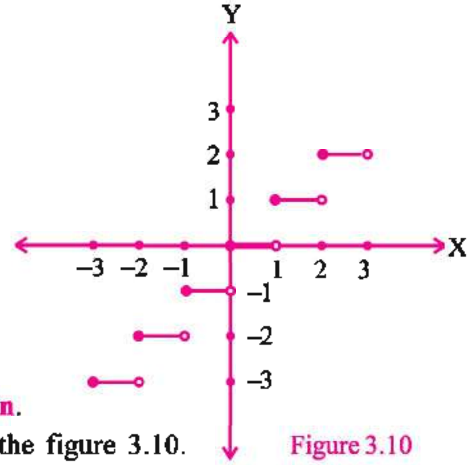


Figure 3.10

In the same manner least integer not less than x can be defined.

(8) Ceiling Function : $g : R \to R$ given by $g(x) = \lceil x \rceil$ = least integer not less than x. We get

$$g(x) = \begin{cases} -1 & \text{for } -2 < x \le -1 \\ 0 & \text{for } -1 < x \le 0 \\ 1 & \text{for } 0 < x \le 1 \text{ etc.} \end{cases}$$

This function is called **'celling' function**. The graph of this function is shown in the figure 3.11.

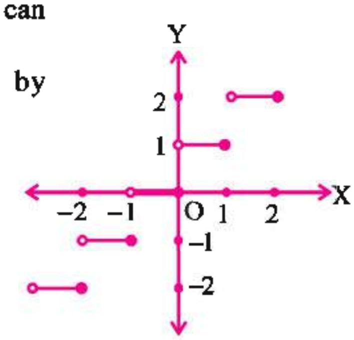


Figure 3.11

Exercise 3.3

1. Draw the graph of the following functions :

(1) $f : R \to R, f(x) = | x - 1 |$ (2) $g : R \to R, g(x) = [x + 1]$

(3) $h : N \to R, h(x) = x - [x]$

(4) $g : [-3, 3] \to Z$; $g(x)$ = least integer not less than x, $x \in [-3, 3]$.

2. Find the range of the following functions :

(1) $f : N \to R, \quad f(x) = \frac{1}{x}$ (2) $h : N \to R, \quad h(x) = x - [x]$

*

3.6 Algebra of Real Functions

We will study the addition, subtraction, multiplication and division of real two functions.

Here we take two functions $f : A \to R$ and $g : B \to R$, with $A \cap B \neq \emptyset$.

(1) Addition of two real functions : Let $f : A \to R$ and $g : B \to R$ be any two real functions, where $(A \cap B) \neq \emptyset$. Then their addition denoted by

$(f + g) : (A \cap B) \to R$ is defined as $(f + g)(x) = f(x) + g(x)$, $\forall x \in A \cap B$.

(2) Subtraction of two real functions : Let $f : A \rightarrow R$ and $g : B \rightarrow R$ be any two real functions. Then we define

$(f - g) : (A \cap B) \rightarrow R, (f - g)(x) = f(x) - g(x), \forall x \in A \cap B.$

(3) Multiplication by a real number : Let $X \subset R$. Let $f : X \rightarrow R$ be a real function, where X is a subset of R and let α be any real number. The product $(\alpha f) : X \rightarrow R$ is defined by $(\alpha f)(x) = \alpha f(x), \forall x \in X$. The real number α is called a scalar and hence such a product is referred to as multiplication of a function by a scalar.

(4) Multiplication of two real functions : Let $f : A \rightarrow R$ and $g : B \rightarrow R$ be any two real functions, then their product denoted by $(fg) : (A \cap B) \rightarrow R$ is defined as $(fg)(x) = f(x)\, g(x), \forall x \in A \cap B$.

(5) Quotient of two real functions : Let $f : A \rightarrow R$ and $g : B \rightarrow R$ be any two real functions, where $(A \cap B) - \{x \mid g(x) = 0\} \neq \emptyset$, then the quotient of f by g denoted as $\frac{f}{g}$ is the function given by $\left(\frac{f}{g}\right) : A \cap B - \{x \mid g(x) = 0\} \rightarrow R, \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

> ☛ **Note** Can you corelate multiplication by a scalar and multiplication of real functions ?

Example 11 : Let $f : R \rightarrow R$ and $g : R \rightarrow R$, be given by $f(x) = x^2$, $g(x) = 4x - 1$, find $f + g, f - g, fg$ and $\frac{f}{g}$.

Solution : $f + g : R \rightarrow R; (f + g)(x) = x^2 + 4x - 1,$

$f - g : R \rightarrow R; (f - g)(x) = x^2 - 4x + 1$

$fg : R \rightarrow R; (fg)(x) = x^2(4x - 1) = 4x^3 - x^2$

To define $\frac{f}{g}$, we must have $g(x) \neq 0$. Here $g(x) = 4x - 1$, which vanishes only at $x = \frac{1}{4}$. Thus, we remove this value from R and we get $\frac{f}{g} : R - \left\{\frac{1}{4}\right\} \rightarrow R; \left(\frac{f}{g}\right)(x) = \frac{x^2}{4x - 1}$.

3.7 Composition of Functions

Now we will study composition of functions. Let $f : A \rightarrow B$ be a function. Then to each x in A, there corresponds a unique element in B. Now, if $g : B \rightarrow C$ is a function, then there exists a unique element in C corresponding to each element in B. Thus, if the functions f and g are 'combined' then corresponding to each element in A, a unique element in C can be found. The composition of two functions is defined as below.

Definition : Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be two functions and $R_f \subset C$, then the composite function of f with g is given as a function $h : A \rightarrow D$ and is defined by $h(x) = g(f(x))\ \forall x \in A$. Such a function h is denoted by gof.

The composition of f with g is also written as $(gof) : A \rightarrow D$ and $(gof)(x) = g(f(x))$.

The Venn diagram (or Arrow diagram) of composition of two functions is shown in figure 3.12.

For the existence of composition of f with g i.e. gof it is necessary to have $R_f \subset D_g$.

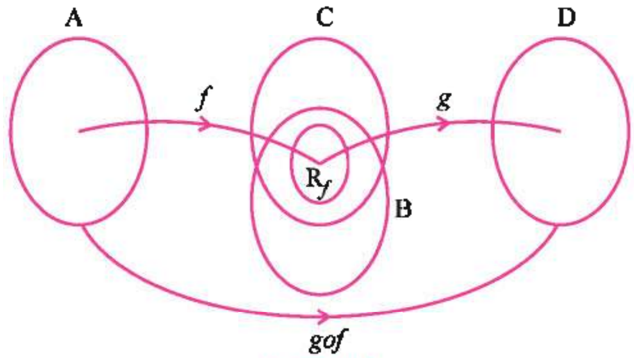


Figure 3.12

Also in order to define composition of g with f, fog it is necessary that $R_g \subset D_f$. If $f : A \rightarrow B$, $g : B \rightarrow C$ are functions, then it is a special composition, here $R_f \subset B = D_g$, hence $R_f \subset D_g$. gof is always possible. Further if $f : A \rightarrow B$, $g : B \rightarrow A$ are two functions, then fog and gof both are always possible.

Example 12 : Let $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{3, 7, 11, 15, 19, 23\}$. $f : A \rightarrow B$, $g : B \rightarrow C$ are given by $f(x) = 2x - 1$, $g(x) = 2x + 1$. Find fog or gof whichever is possible.

Solution : $R_f \subset B = D_g$. Hence gof is possible.

Now $(gof)(x) = g(f(x)) = g(2x - 1) = 2(2x - 1) + 1 = 4x - 2 + 1 = 4x - 1$

$\therefore$ $(gof)(1) = 3$, $(gof)(2) = 7$, $(gof)(3) = 11$, $(gof)(4) = 15$, $(gof)(5) = 19$

Thus, $gof : A \rightarrow C$, $gof = \{(1, 3), (2, 7), (3, 11), (4, 15), (5, 19)\}$.

Now, $g : B \rightarrow C$, $g = \{(1, 3), (3, 7), (5, 11), (7, 15), (9, 19)\}$

$\therefore$ $R_g = \{3, 7, 11, 15, 19\} \not\subset A = D_f$

$\therefore$ fog is not possible.

☛ **Note**	Here gof exists but fog does not exist.

Example 13 : $f : N \rightarrow N$, $f(x) = x^2$, $g : N \rightarrow N$, $g(x) = x^3$. Find fog and gof.

Solution : $fog : N \rightarrow N$; $(fog)(x) = f(g(x)) = f(x^3) = (x^3)^2 = x^6$

$gof : N \rightarrow N$; $(gof)(x) = g(f(x)) = g(x^2) = (x^2)^3 = x^6$.

☛ **Note**	Here $fog = gof$.

Example 14 : $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 4, 9, 16, 25\}$, $f : A \rightarrow B$, $f(x) = x^2$, $g : B \rightarrow A$, $g(x) = \sqrt{x}$. Find fog and gof.

Solution : $fog : B \rightarrow B$; $(fog)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$

$gof : A \rightarrow A$; $(gof)(x) = g(f(x)) = g(x^2) = \sqrt{x^2} = | x | = x$ as $x \in A$.

☛ **Note**	Here $gof = I_A$ and $fog = I_B$

Example 15 : $f : R \to R$, $f(x) = 3x + 2$, $g : R \to R$, $g(x) = 2x + 3$. Find *fog* and *gof*. Is $fog = gof$?

Solution : $R_f \subset R = D_g$ and $R_g \subset R = D_f$

$\therefore$ $fog : R \to R$ and $gof : R \to R$ exist.

$(fog)(x) = f(g(x)) = f(2x + 3) = 3(2x + 3) + 2 = 6x + 11$

$(gof)(x) = g(f(x)) = g(3x + 2) = 2(3x + 2) + 3 = 6x + 7$

$\therefore$ *gof* and *fog* have same domain and codomain, but $gof \neq fog$.

Theorem 3.1 : $f : A \to B$ **is a function. Then** $foI_A = f$, $I_Bof = f$.

Proof : Here, $I_A : A \to A$ and $f : A \to B$ and hence foI_A is well defined and also $foI_A : A \to B$ is a function.

Now, $\forall x \in A$, $(foI_A)(x) = f(I_A(x)) = f(x)$ **(I_A is identity function)**

$foI_A : A \to B$ and $f : A \to B$ and $(foI_A)(x) = f(x)$, $\forall x \in A$

$\therefore$ $foI_A = f$

Further, since $f : A \to B$ and $I_B : B \to B$, I_Bof is well defined and also $I_Bof : A \to B$ is a function and $f : A \to B$ is a function.

$(I_Bof)(x) = I_B(f(x)) = f(x)$, $\forall x \in B$

$\therefore$ $I_Bof = f$

Theorem 3.2 : Let $f : A \to B$, $g : B \to C$ and $h : C \to D$, be three functions. Then, $(hog)of = ho(gof)$

Proof : It can be seen that $hog : B \to D$ is a function and hence $(hog)of : A \to D$ is a function. Also since $gof : A \to C$ is a function, $ho(gof) : A \to D$ is a function. In other words $(hog)of$ and $ho(gof)$ are functions with same domain and codomain.

Now $\forall x \in A$, $((hog)of))(x) = (hog)(f(x))$

$= h(g(f(x)))$

$= h((gof)(x))$

$= (ho(gof))(x)$

$\therefore$ $(hog)of = ho(gof)$

Exercise 3

1. $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$; $f : A \to B$, $g : B \to A$ are given by

$f = \{(1, 5), (2, 6), (3, 4)\}$, $g = \{(5, 1), (6, 2), (4, 3)\}$. Find *fog* and *gof*, if they exist.

2. Let *f* and *g* be functions from R to R as follows. Find *fog*, *gof*, *fof*, *gog* :

(1) $f(x) = x + 1$ $\quad g(x) = 2x$

(2) $f(x) = x^2 + 2$ $\quad g(x) = 3x$

(3) $f(x) = x^2 + 3x + 1$ $\quad g(x) = 2x - 3$

(4) $f(x) = x + 1$ $\quad g(x) = x - 1$

(5) $f(x) = 2x^2 + 1$ $\quad g(x) = 3x$

3. For the relation $S = \{(x, y) \mid x, y \in N, x + y = 5\}$, find domain and range.

4. Write the relation $S = \{(x, x^2) \mid x \in N, x < 5\}$ in the roster form.

5. Represent the following relations graphically :

(1) $S = \{(x, y) \mid x, y \in N, x < 10, y < 10, \frac{x}{y} \text{ is integer}\}$

(2) $S = \{(x, y) \mid x \in N, y \in N, -3 < x < 2, y < 8, x + y = 8\}$

6. Find the range of the following functions defined on R :

(1) $f(x) = 2x$ $\quad$ (2) $f(x) = 2x^2$ $\quad$ (3) $f(x) = x - 2$

(4) $f(x) = 1000$ $\quad$ (5) $f(x) = |x|$

7. Draw the graphs of the following functions :

(1) $f : R \to R, f(x) = 1 + |x|$

(2) $f : R \to R, f(x) = x + 10$

(3) $f : N \to N, f(x) = x + 10$

8. If $f : R^+ \to R$ is given by $f(x) = x - \sqrt{x}$, find $f(9), f(2)$.

9. Let f and g be functions from R to R as follows. Find *fog*, *gof*, *fof*, *gog* :

(1) $f(x) = x^2$ $\quad g(x) = x - 1$

(2) $f(x) = x - 5$ $\quad g(x) = 5x$

(3) $f(x) = x^2 - 3$ $\quad g(x) = x^2 + 3$

10. Find *fog*, *gof*, *fof*, *gog* for $f : R^+ \to R^+, f(x) = x^2, g : R^+ \to R^+, g(x) = \sqrt{x}$.

11. (1) $f : R \to R, f(x) = |x|$, prove $fof = f$.

(2) $f : R - \{-1\} \to R - \{-1\}, f(x) = \frac{1-x}{1+x}$, prove $fof = I_{R - \{-1\}}$.

12. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Domain of a relation $S : A \to B$ is... ☐

(a) a subset of B (b) a subset of A (c) the universal set (d) empty set

(2) Range of a relation $S : A \to B$ is... ☐

(a) always empty (b) a subset of B (c) a subset of A (d) $A \times B$

(3) If for a relation $S : A \to B$, $S = A \times B$, then S is... ☐

(a) not defined (b) singleton (c) the universal relation (d) empty relation

(4) If $f : R \to R, f(x) = x - 2, g : R \to R, g(x) = x + 2; (f + g)(x) =$ ...... ☐

(a) x (b) $x^2 - 4$ (c) $2x$ (d) 4

(5) If $f : R - \{0\} \to R$, $f(x) = x$, $g : R - \{0\} \to R$, $g(x) = \frac{1}{x}$;
then, $fg : R - \{0\} \to R$, $(fg)(x) =$ ...... ☐

(a) x^2 (b) 1 (c) $\frac{1}{x^2}$ (d) x

(6) If $f : Z \to Z$, $f(x) = x - 3$, $g : Z \to Z$, $g(x) = x + 3$, then $fog : Z \to Z$, $(fog)(x) =$ ...... ☐

(a) $|x|$ (b) x (c) $x^2 - 9$ (d) $\frac{x+3}{x-3}$

(7) If $f : R \to R$, $f(x) = x^2 - 9$, $f(3) =$ ...... ☐

(a) -6 (b) 9 (c) 0 (d) 3

(8) If $f : R \to R$, $f(x) = x + 2$, $g : R \to R$, $g(x) = x - 2$, $fog : R \to R$, then fog is ...... ☐

(a) the identity function (b) a constant function
(c) not defined (d) the modulus function

(9) Graph of $I_R : R \to R$ identity function is... ☐

(a) a straight line (b) a fixed point (c) a circle (d) an interval

(10) The range of $f : R \to R$, $f(x) = x^2$ is... ☐

(a) R (b) Z (c) $R^+ \cup \{0\}$ (d) $R - \{0\}$

(11) If $f : \{x \mid |x| \le 1, x \in R\} \to R$, $f(x) = \sqrt{1 - x^2}$ and
$g : \{x \mid |x| \ge 1, x \in R\} \to R$, $g(x) = \sqrt{x^2 - 1}$, then $(f + g)(x) =$ ..... ☐

(a) I_R (b) $f + g : R \to R$, $(f + g)(x) = \sqrt{1 - x^2} + \sqrt{x^2 - 1}$
(c) does not exist (d) $f : \{1, -1\} \to R$, $(f + g)(x) = 0$

Summary

1. Relation, its domain and range
2. Void relation, universal relation and Venn diagram and arrow diagram.
3. Function, domain and range
4. Graphs of some special functions and other functions.
5. Algebraic operations on functions
6. Composite functions

— ❖ —

Chapter 4

TRIGONOMETRIC FUNCTIONS

4.1 Introduction

The study of Trigonometry initially started in India. The ancient Indian mathematicians Aryabhatta (476 AD), Brahmagupta (598 AD), Bhaskara I (600 AD) and Bhaskara II (1114 AD) got important results. All this knowledge first passed on from India to middle-east and from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions known as the *sine* of an angle and the introduction of the *sine* function represents the main contribution of the *siddhantas* (Sanskrit astronomical works) to the history of mathematics.

Bhaskara I (about 600 AD) gave formulae to find the values of *sine* functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* (period) contains a proof for the expansion of *sin*(A + B). Exact expression for *sines* or *cosines* of 18°, 36°, 54°, 72°, etc., are given by Bhaskara II.

The name of Thales (about 600 AD) is invariably associated with height and distance problems. He is credited with the determination of the height of great pyramid in Egypt by measuring shadows of the pyramid and an auxilliary staff (or gnomon) of known height and comparing the ratios :

$$\frac{H}{S} = \frac{h}{s} = \tan \text{ (sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on heights and distances using the similarity property are also found in ancient Indian works.

Trigonometry is very widely applied in Astronomy, Physics, Engineering and various branches of mathematics. The word 'trigonometry' is derived from two Greek words "trigono" and "metron". The word "trigono" means a "triangle" and the word "metron" mean "to measure". Hence the word trigonometry means 'measurement of a triangle'. In recent years, its application has been extended beyond the measurement of triangles. A class of functions called trigonometric functions forms the basis of the study of periodic phenomena like mechanical vibrations, motions of waves and so on. We obtained preliminary introduction to trigonometry in standard X. At that time we studied trigonometrical ratios like *sin*, *cos*, *tan* etc. for acute angles. This study was confined only to acute angles of a right angled triangle. Now we shall study trigonometric functions in a wider sense.

4.2 Trigonometric Point

In the coordinate plane, the circle whose centre is at the origin and whose radius is one unit is called the unit circle.

The unit circle intersects X-axis at A(1, 0) and A'(−1, 0) and Y-axis at B(0, 1) and B'(0, −1), as radius is one unit.

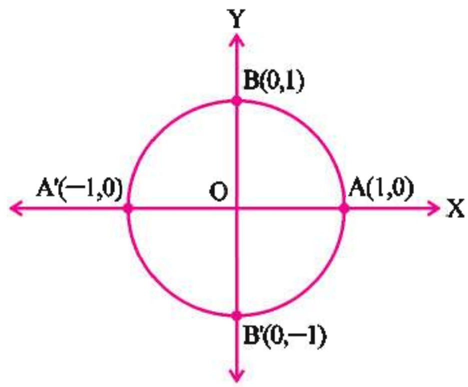


Figure 4.1

Let θ be any real number. Then we can obtain a unique point on the unit circle corresponding to this given real number θ.

If $\theta = 0$, we take the point corresponding to θ as A(1, 0). If $0 < \theta < 2\pi$, then a unique point P exists on the unit circle such that the length of $\widehat{AP}$ is θ. We measure the arc from A to P in anticlockwise direction. Again we assume the continuity of the arc and hence since circumference of unit circle is 2π, for every $\theta \in (0, 2\pi)$ there is an arc of length θ such that $l(\widehat{AP}) = \theta$. The point P thus obtained is called a trigonometric point. This point P is the point corresponding to $\theta \in [0, 2\pi)$ and is denoted by P(θ).

Now let $\theta \in$ R.

Let $\left[\frac{\theta}{2\pi}\right] = n$

Then n is an integer and

$$n \le \frac{\theta}{2\pi} < n + 1$$

$\therefore \quad 2n\pi \le \theta < 2n\pi + 2\pi$

$\therefore \quad 0 \le (\theta - 2n\pi) < 2\pi$

Let $\theta - 2n\pi = \alpha$. Then $0 \le \alpha < 2\pi$.

So as seen above, there is a unique point P(α) corresponding to α. For $\alpha \in [0, 2\pi)$ on the unit circle, we define P(θ) = P(α).

Hence, for every $\theta \in$ R, we have a unique point P(θ) on the unit circle. This point is called trigonometric point P(θ) or T-point P(θ).

We note that for each real number θ, there is a unique real number α such that $\theta = 2n\pi + \alpha$, for some integer n and $0 \le \alpha < 2\pi$. It is obvious that if $\theta \in [0, 2\pi)$, then $n = 0$ and $\theta = \alpha$.

Now, let us plot some trigonometrical points.

(1) $P\left(\frac{\pi}{4}\right)$

Here $\theta = \frac{\pi}{4}$ and $0 < \frac{\pi}{4} < 2\pi$

Since circumference of unit circle is 2π, length of $\overset{\frown}{AA'} = \frac{2\pi}{2} = \pi$ and length $\overset{\frown}{AB}$ is $\frac{2\pi}{4} = \frac{\pi}{2}$. So there would be a point P on $\overset{\frown}{AB}$ such that $l(\overset{\frown}{AP}) = \frac{\pi}{4}$.

This point P is $P\left(\frac{\pi}{4}\right)$.

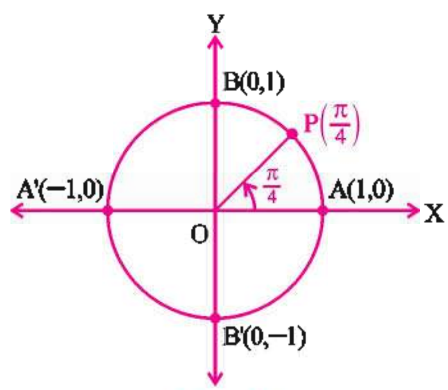


Figure 4.2

(2) $P\left(\frac{31\pi}{3}\right)$

Here $\theta = \frac{31\pi}{3}$ and $\theta \notin [0, 2\pi)$

$\therefore \quad n = \left[\frac{\theta}{2\pi}\right] = \left[\frac{31\pi}{3} \times \frac{1}{2\pi}\right]$

$= \left[\frac{31}{6}\right] = 5$

$\therefore \quad \alpha = \theta - 2n\pi$

$= \frac{31\pi}{3} - 10\pi = \frac{\pi}{3}$

Also P(θ) = P(α),

$\therefore \quad P\left(\frac{31\pi}{3}\right) = P\left(\frac{\pi}{3}\right)$.

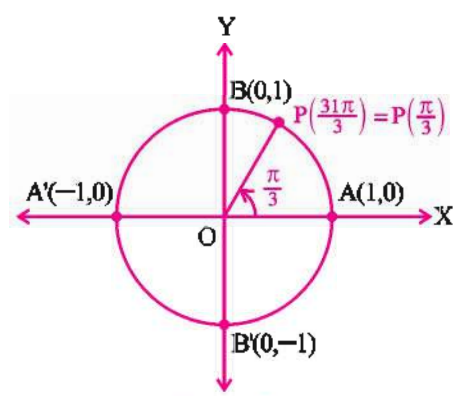


Figure 4.3

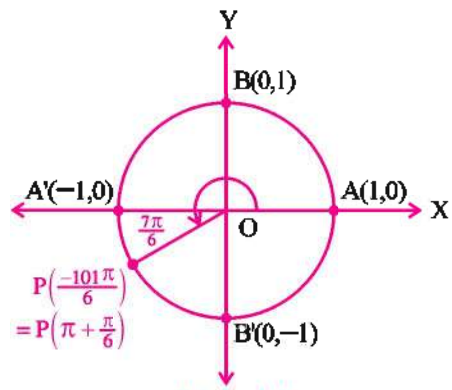


Figure 4.4

(3) $P\left(\frac{-101\pi}{6}\right)$

Here $\theta = -\frac{101\pi}{6}$, $\theta \notin [0, 2\pi)$

$\therefore \quad n = \left[-\frac{101\pi}{6} \times \frac{1}{2\pi}\right] = \left[\frac{-101}{12}\right] = -9$

$\therefore \quad \alpha = \theta - 2n\pi = -\frac{101\pi}{6} + 18\pi$

$= \frac{7\pi}{6} = \pi + \frac{\pi}{6}$

So, $P(\theta) = P(\alpha)$

$\therefore \quad P\left(\frac{-101\pi}{6}\right) = P\left(\frac{7\pi}{6}\right)$

$\therefore \quad P\left(\frac{7\pi}{6}\right)$ will be in the third quadrant.

4.3 Trigonometric Point Function and Period

We have already seen that corresponding to $\theta \in R$, a unique point $P(\theta) = P(x, y)$ exists on the unit circle. We express this information in the form of a function.

If we denote the unit circle by C, the function $f : R \to C, f(\theta) = P(\theta) = P(x, y)$ is called trigonometric point function or T-point function.

If for a function any two distinct elements of domain correspond to distinct elements of codomain, the function is said to be a one-one function. A function which is not one-one is many one.

Definition : Let $f : A \to B$ be a function of a real variable. $A \subset R$ and f is a non-constant function. Suppose there exists $p' \in R$, $p' \neq 0$ such that,

if $x \in A$, then $x + p' \in A$ and $f(x) = f(x + p')$, $\forall x \in A$,

then f is called a periodic function and p' is called a period of f. If p is the smallest positive number satisfying this property that is if $x \in A$ then $x + p \in A$ and $f(x) = f(x + p)$ $\forall x \in A$, then p is called the principal period of f.

Period : let $\theta_1, \theta_2 \in R$, such that $\theta_2 - \theta_1 = 2k\pi$, $k \in Z$ for some integer k.

Suppose $\theta_1 = 2m\pi + \alpha$, $0 \leq \alpha < 2\pi$, $m \in Z$, Then $P(\theta_1) = P(\alpha)$

Now, $\theta_2 = \theta_1 + 2k\pi = 2m\pi + \alpha + 2k\pi$

$\therefore \quad \theta_2 = 2(m + k)\pi + \alpha, \quad m + k \in Z, 0 \leq \alpha < 2\pi$

Let $m + k = n$, $n \in Z$

$\therefore \quad \theta_2 = 2n\pi + \alpha$

$\therefore \quad P(\theta_2) = P(\alpha)$

Thus, $P(\theta_1) = P(\theta_2) = P(\theta_1 + 2k\pi)$

$\therefore \quad f(\theta_1) = f(\theta_2) = f(\theta_1 + 2k\pi),\ k \in Z$

This proves that the values of f repeat after an interval of 2π. Thus, periods of this function are, ...$-6\pi, -4\pi, -2\pi, 2\pi, 4\pi$,...

If f has the principal period p, then $f(\theta) = f(\theta + p)$, $\forall\theta \in$ R, and if $0 < q < p$, then $f(\theta) \neq f(\theta + q)$ for some $\theta \in$ R.

Now, let us prove that 2π is the principal period of T-point function. We have seen that 2π is a period and $2\pi > 0$. Let q be the principal period of f. Let $0 < q < 2\pi$. Then we should have $f(\theta) = f(\theta + q)$ all $\theta \in$ R. If $\theta = 0$, then $f(0) = f(0 + q)$ which is impossible because $f(0)$ is A(1, 0) and $f(q)$ is the point Q such that $\overset{\frown}{AQ}$ has length q. As $0 < q < 2\pi$, Q can not be A as the whole circle has length 2π. Thus $f(0) \neq f(0 + q)$, so q is not a period of f.

So 2π is a period of f and any positive number less than 2π is not a period. Hence 2π is the principal periof of f.

4.4 Definition of *sine* and *cosine* Functions

We have defined the T-point function f which assigns to every real number θ, a unique point $P(\theta)$ on the unit circle. We now define two functions g and h from the unit circle C to R.

We define $g : C \rightarrow R$, $g(P(x, y)) = x$. The function g assigns to every point on the unit circle its x-coordinate, which is unique. The composite function of $f : R \rightarrow C$, $f(\theta) = P(\theta) = P(x, y)$ and $g : C \rightarrow R$, $g(P(x, y)) = x$ is called the *cosine* function. The composite function gof : R $\rightarrow$ R is called the *cosine* function and written in short as *cos* function.

$(gof)(\theta) = g(f(\theta)) = g(P(\theta)) = g(P(x, y)) = x$.

$\therefore \quad cos\theta = x$

This function can be understood from the figure 4.5.

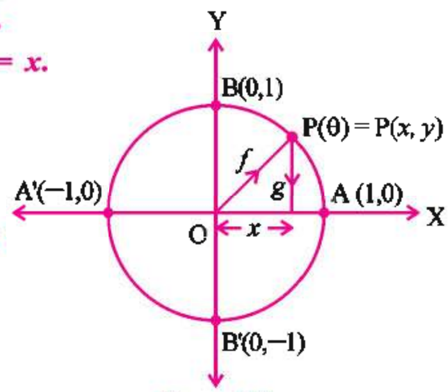


Figure 4.5

We write cos : R $\rightarrow$ R, $cos\theta = (gof)(\theta) = x$

Since T-point function is many-one, *cosine* function is also a many-one function. For example, as $cos\ 0$ is x-coordinate of A(1, 0), so $cos\ 0 = 1$ and since $P(2\pi) = P(0) = A$, $cos 2\pi = 1$

Let $h : C \rightarrow R$, $h(P(x, y)) = y$. The function h assigns to every point on the unit circle its y-coordinate, which is unique.

The composite function of $f : R \rightarrow C$, $f(\theta) = P(\theta) = P(x, y)$ and $h : C \rightarrow R$, $h(P(x, y)) = y$, $hof : R \rightarrow R$ is called *sine* function and is written in short as *sin* function.

$(hof)(\theta) = h(f(\theta)) = h(P(\theta)) = h(P(x, y)) = y$.

$\therefore$ $sin : R \rightarrow R$, $sin\theta = (hof)(\theta) = y$

This Composite function is called *sine* function. (figure 4.6).

We write $sin : R \rightarrow R$, $sin\theta = (hof)(\theta) = y$.

Following the arguments for *cos* function we can prove $sin 0 = sin 2\pi = 0$

Hence, *sine* function is a many-one function.

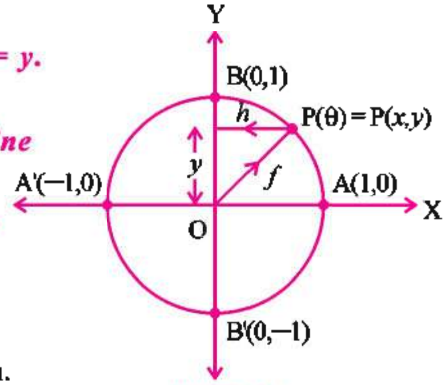


Figure 4.6

> **☛ Note** Using these definitions, it is clear that any point on unit circle has coordinates $(cos\theta, sin\theta)$ for some $\theta \in R$.

4.5 A Fundamental Identity

We have seen distance formula in standard X. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in coordinate plane. Then distance AB is given by the distance formula

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Let $P(\theta) = P(x, y)$ be a point on the unit circle.

Then $OP = 1$ **(radius of unit circle)**

$\therefore$ $OP^2 = 1$

$\therefore$ $(x - 0)^2 + (y - 0)^2 = 1$

$\therefore$ $x^2 + y^2 = 1$

but $x = cos\theta$ and $y = sin\theta$

$\therefore$ $(cos\theta)^2 + (sin\theta)^2 = 1$

It is customary to write $(cos\theta)^2$ as $cos^2\theta$ and $(sin\theta)^2$ as $sin^2\theta$.

Therefore for every $\theta \in R$, $cos^2\theta + sin^2\theta = 1$.

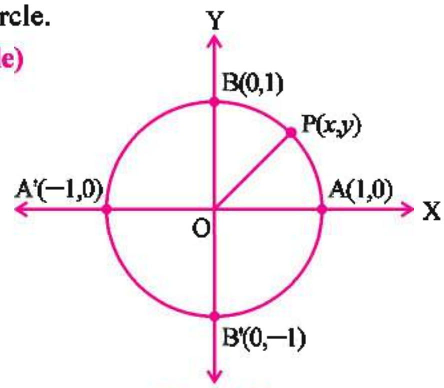


Figure 4.7

4.6 The Range of *cosine* and *sine*

For every $\theta \in R$, $cos^2\theta + sin^2\theta = 1$. Since square of any real number is non-negative and sum of the squares of two real numbers is 1,

$0 \le cos^2\theta \le 1$ and $0 \le sin^2\theta \le 1$.

$\therefore$ $| cos\theta | \le 1$, $| sin\theta | \le 1$

Thus, $cos\theta \in [-1, 1]$, $sin\theta \in [-1, 1]$, $\forall \theta \in R$

$\therefore$ Range of *cosine* and *sine* are subsets of $[-1, 1]$. In fact the range is the whole of $[-1, 1]$.

For all $p \in [-1, 1]$, the point $(p, \sqrt{1-p^2})$ is on the unit circle,

because $p^2 + (\sqrt{1-p^2})^2 = p^2 + 1 - p^2 = 1$

$\therefore$ For, the T-point function $f : R \to C$, there exists $\theta \in R$, such that

$f(\theta) = P(\theta) = (p, \sqrt{1-p^2})$.

Now, by definition of *cos* function, x-coordinate of $P(\theta)$ is $cos\theta$.

$\therefore$ $cos\theta = p$ and $p \in [-1, 1]$.

Thus for all $p \in [-1, 1]$, there exists $\theta \in R$ such that $cos\theta = p$.

$\therefore$ **The range of *cosine* function $[-1, 1]$.**

Similarly for all $p \in [-1, 1]$, the point $(\sqrt{1-p^2}, p)$ is on the unit circle. So there is $\theta \in R$, such that $sin\theta = p$.

$\therefore$ **The range of *sine* function is $[-1, 1]$.**

We know that $|x| = \begin{cases} x \text{ if } x \ge 0 \\ -x \text{ if } x < 0 \end{cases}$

So $|sin\theta| = \begin{cases} sin\theta \text{ if } sin\theta \ge 0 \\ -sin\theta \text{ if } sin\theta < 0 \end{cases}$

$\therefore$ $-1 \le sin\theta \le 1 \Leftrightarrow -1 \le sin\theta \le 0$ or $0 \le sin\theta \le 1$

$\Leftrightarrow 0 \le -sin\theta \le 1$ or $0 \le sin\theta \le 1$

$\Leftrightarrow |sin\theta| \le 1, \forall\theta \in R$

Similarly, $|cos\theta| \le 1, \forall\theta \in R$

$\therefore$ **The range of *cosine* and *sine* functions can also be expressed as $\{p \mid |p| \le 1, p \in R\}$.**

4.7 The Zeroes of *sine* and *cosine*

For any real function $f : A \to B$, the set $\{x \mid f(x) = 0, x \in A\}$ is called the set of zeroes of f.

Zeroes of *sine* : Suppose for some $\theta \in R$, $sin\theta = 0$.

$\therefore$ T-point $P(\theta)$ has y-coordinate 0.

$\therefore$ $P(\theta)$ is on X-axis.

$\therefore$ $P(\theta) = A(1, 0)$ or $A'(-1, 0)$

We know that A and A' correspond to $\alpha = 0$ and $\alpha = \pi$.

In general $\theta = 2n\pi + \alpha$, $n \in Z$.

If $\alpha = 0$, then $\theta = 2n\pi$ and if $\alpha = \pi$, $\theta = 2n\pi + \pi$, $n \in Z$.

$\therefore$ $\theta = 2n\pi$ or $\theta = (2n + 1)\pi$, $n \in Z$

As $2n\pi$ is an even multiple of π and $(2n + 1)\pi$ is an odd multiple of π,

we see that $sin\theta = 0 \Rightarrow \theta$ is an integral multiple of π. i.e. $\theta = k\pi$, $k \in Z$.

Conversely, if $\theta = k\pi$, $k \in Z$, then $P(\theta)$ is A or A' and hence $sin\theta = 0$

Thus the **set of zeroes of *sin* is $\{k\pi \mid k \in Z\}$.**

Zeroes of *cosine* : Suppose for some $\theta \in R$ *cosine* function has value zero, that is $cos\theta = 0$.

$\therefore$ T-point $P(\theta)$ has *x*-coordinate 0.

$\therefore$ $P(\theta)$ is on Y-axis.

$\therefore$ $P(\theta) = B(0, 1)$ or $B'(0, -1)$

We know that B and B' correspond to $\alpha = \frac{\pi}{2}$ and $\alpha = \frac{3\pi}{2}$.

In general $\theta = 2n\pi + \alpha$, $n \in Z$.

$\therefore$ $\theta = 2n\pi + \frac{\pi}{2}$ or $\theta = 2n\pi + \frac{3\pi}{2}$, $n \in Z$

Thus $\theta = (4n + 1)\frac{\pi}{2}$ or $\theta = (4n + 3)\frac{\pi}{2}$, $n \in Z$

$4n + 1 = 2(2n) + 1$, $4n + 3 = 2(2n + 1) + 1$

$\therefore$ $4n + 1$ and $4n + 3$ are of form $2k + 1$, $k \in Z$

$\therefore$ $(4n + 1)$ or $(4n + 3)$, $n \in Z$ is a form of odd integers, so we see that, θ is an odd multiple of $\frac{\pi}{2}$.

$\therefore$ $\theta = (2k - 1)\frac{\pi}{2}$ or $\theta = (2k + 1)\frac{\pi}{2}$, $k \in Z$

Conversely, it is clear that if $\theta = (2k + 1)\frac{\pi}{2}$, $k \in Z$.

then $\theta = (2(2n) + 1)\frac{\pi}{2}$ or $\theta = (2(2n + 1) + 1)\frac{\pi}{2}$ **(according as *k* is even or odd.)**

$\therefore$ $\theta = (4n + 1)\frac{\pi}{2}$ or $(4n + 3)\frac{\pi}{2}$

$\therefore$ $\theta = 2n\pi + \frac{\pi}{2}$ or $\theta = 2n\pi + \frac{3\pi}{2}$

$\therefore$ $\alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore$ $P(\theta) = P(\alpha) = B$ or B'

$\therefore$ *x*-coordinate of $P(\theta)$ is zero.

$\therefore$ $cos\theta = 0$

The *set* of zeroes of *cosine* function is

$\left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\}$ or $\left\{(2k-1)\frac{\pi}{2} \mid k \in Z\right\}$

4.8 Other Trigonometric Functions

Before defining other trigonometric functions, we recall the definition of quotient of two real functions of real variable. If $f : A \to R$, $g : B \to R$ are functions, where $A \subset R$, $B \subset R$ and $A \cap B - \{x \mid g(x) = 0\} \neq \emptyset$ then $\frac{f}{g} : A \cap B - \{x \mid g(x) = 0\} \to R$ is defined as $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Now we are ready to define another trigonometric function, called tangent (or *tan*). This function is the quotient of two functions *sin* : $R \to R$ and *cos* : $R \to R$.

Now, $\{x \mid cosx = 0\} = \left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\}$

We define tangent function as the quotient function of *sine* and *cosine* functions.

$tan : R - \left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\} \to R$, $tan\theta = \frac{sin\theta}{cos\theta}$.

As $tan0 = \frac{sin\,0}{cos\,0} = \frac{0}{1} = 0$ and $tan\,2\pi = \frac{sin2\pi}{cos2\pi} = 0$,

$tan0 = tan\,2\pi = 0$. Hence *tan* is a many-one function.

Range of *tan* function :

Let $p \in R$. Now $1 + p^2 \geq 1$ as $p^2 \geq 0$.

Let $x = \frac{1}{\sqrt{1+p^2}}$, $y = \frac{p}{\sqrt{1+p^2}}$

then, $x^2 + y^2 = \frac{1}{1+p^2} + \frac{p^2}{1+p^2} = \frac{1+p^2}{1+p^2} = 1$

Then, $(x, y) \in C$ (unit circle)

$\therefore$ There exists $\theta \in R$ such that $P(\theta) = P(x, y)$.

$\therefore$ $x = cos\theta$, $y = sin\theta$

$\therefore$ $cos\theta = \frac{1}{\sqrt{1+p^2}}$, $sin\theta = \frac{p}{\sqrt{1+p^2}}$

$\therefore$ $tan\theta = \frac{sin\theta}{cos\theta} = p$.

Again $cos\theta = x \neq 0$, because if $x = 0$, then $\frac{1}{\sqrt{1+p^2}} = 0$ and this is not possible.

$\therefore$ $cos\theta \neq 0 \Leftrightarrow \theta \neq (2k+1)\frac{\pi}{2}$, $k \in Z$

Hence, for every $p \in$ R, there is $\theta \in \mathrm{R} - \left\{(2k + 1)\frac{\pi}{2} \mid k \in \mathrm{Z}\right\}$ such that $tan\theta = p$

So the range of *tan* is R.

***cot* function :** The cotangent function is defined as the quotient of *cos* function and *sin* function.

sin : R $\rightarrow$ R and *cos* : R $\rightarrow$ R are functions.

The domain of *cot* is $\mathrm{R} \cap \mathrm{R} - \{x \mid sinx = 0\} = \mathrm{R} - \{k\pi \mid k \in \mathrm{Z}\}$

Therefore, $cot : \mathrm{R} - \{k\pi \mid k \in \mathrm{Z}\} \rightarrow \mathrm{R},\ cot\theta = \frac{cos\,\theta}{sin\,\theta}$.

cot function is also many-one because

$$cot\frac{\pi}{2} = \frac{cos\,\frac{\pi}{2}}{sin\,\frac{\pi}{2}} = \frac{0}{1} = 0, \quad cot\frac{3\pi}{2} = \frac{cos\frac{3\pi}{2}}{sin\frac{3\pi}{2}} = \frac{0}{-1} = 0$$

Like *tan* function, the range of *cot* function is also R.

For any given $p \in$ R, let $x = \frac{p}{\sqrt{1 + p^2}},\ y = \frac{1}{\sqrt{1 + p^2}}$. We see $x^2 + y^2 = 1$.

Thus $(x, y) \in$ C (unit circle).

As (x, y) is a point on the unit circle, therefore, there exists $\theta \in$ R such that $P(\theta) = P(x, y)$.

$$x = cos\theta,\ y = sin\theta,\ cot\theta = \frac{cos\,\theta}{sin\,\theta} = \frac{x}{y} = p$$

Again $sin\theta = \frac{1}{\sqrt{1 + p^2}} \neq 0 \Leftrightarrow \theta \neq k\pi$

Thus for any $p \in$ R, we can find $\theta \in \mathrm{R} - \{k\pi \mid k \in \mathrm{Z}\}$ such that $cot\,\theta = p$

So, range of *cot* is R.

***sec* function :** Let $f : \mathrm{R} \rightarrow \mathrm{R},\ f(x) = 1$ be a function. We know the function $cos : \mathrm{R} \rightarrow \mathrm{R}$. Quotient function of f and *cos* function is called *secant* function.

The domain of *sec* function $= (\mathrm{R} \cap \mathrm{R}) - \{\theta \mid cos\theta = 0\}$

$$= \mathrm{R} - \left\{(2k + 1)\,\frac{\pi}{2} \mid k \in \mathrm{Z}\right\}$$

$$\therefore \quad sec : \mathrm{R} - \left\{(2k + 1)\,\frac{\pi}{2} \mid k \in \mathrm{Z}\right\} \rightarrow \mathrm{R},\ sec\theta = \frac{1}{cos\,\theta}$$

sec function is many-one, because $sec0 = 1$ and $sec2\pi = 1$

Range of *sec* function :

We know that $sec\theta = \frac{1}{cos\theta}$.

$sec\theta$ is defined if $cos\theta \neq 0$.

Range of *cos* is $[-1, 1]$.

$-1 \le cos\theta \le 1, cos\theta \ne 0$

$\Leftrightarrow$ $-1 \le cos\theta < 0$ and $0 < cos\theta \le 1$

$\Leftrightarrow$ $-1 \ge \frac{1}{cos\theta}$ or $\frac{1}{cos\theta} \ge 1$

$\Leftrightarrow$ $-1 \ge sec\theta$ or $sec\theta \ge 1$

$\Leftrightarrow$ $sec\theta \le -1$ or $sec\theta \ge 1$

So range of *sec* function includes no real number between -1 and 1.

Hence $\forall\theta \in R - \left\{(2k+1)\frac{\pi}{2}\right\}$, $sec\theta \in R - (-1, 1)$.

Conversely let $p \in R - (-1, 1)$. So $p \ne 0$. Then $\frac{1}{p} \in [-1, 1]$.

So there is $\theta \in R - \left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\}$ such that $cos\,\theta = \frac{1}{p}$.

For this θ, $sec\theta = p$.

Thus, range of *sec* is $R - (-1, 1)$.

***cosec* function :**

The quotient function of $f : R \to R, f(x) = 1$ and *sin* : $R \to R$ is called *cosec* function.

The domain of *cosec* function $= (R \cap R) - \{\theta \mid sin\theta = 0\}$

$= R - \{k\pi \mid k \in Z\}$

$\therefore$ **$cosec : R - \{k\pi \mid k \in Z\} \to R$, $cosec\theta = \frac{1}{sin\theta}$.**

Like other trigonometric functions, *cosec* function is many-one because $cosec\frac{\pi}{2} = 1$ and $cosec\frac{5\pi}{2} = 1$.

Again since $-1 \le sin\theta \le 1$, $cosec\theta = \frac{1}{sin\theta}$ is defined if $sin\theta \ne 0$. We can prove exactly as in the case of *sec* that range of *cosec* function is $R - (-1, 1)$.

Since $|cos\theta| \le 1$, $\frac{1}{|cos\theta|} \ge 1$ i.e. $|sec\theta| \ge 1$

The range of *sec* function and *cosec* function can also be written as

$\{p \mid |p| \ge 1, p \in R\}$.

4.9 Other Identities

We have already seen that $sin^2\theta + cos^2\theta = 1, \forall\theta \in R$ **(i)**

$\theta \ne (2k+1)\frac{\pi}{2}, k \in Z \Leftrightarrow cos\theta \ne 0$

Dividing (i) by $cos^2\theta$ on both sides

$$\frac{cos^2\theta}{cos^2\theta} + \frac{sin^2\theta}{cos^2\theta} = \frac{1}{cos^2\theta}, \forall\theta \in R - \left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\}$$

$\therefore \quad 1 + tan^2\theta = sec^2\theta, \quad \forall \theta \in R - \left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\}$ (ii)

Similarly $\theta \neq k\pi, k \in Z \Leftrightarrow sin\theta \neq 0$.

Dividing (i) by $sin^2\theta$, we get

$$\frac{cos^2\theta}{sin^2\theta} + \frac{sin^2\theta}{sin^2\theta} = \frac{1}{sin^2\theta}, \forall \theta \in R - \{k\pi \mid k \in Z\}$$

$\therefore \quad cot^2\theta + 1 = cosec^2\theta, \quad \forall \theta \in R - \{k\pi \mid k \in Z\}$ (iii)

Example 1 : Find the set of zeroes of the following functions :

(1) $sin\ (x - 1)$ (2) $sin\ x - 1$ (3) $cos\ x + 1$ (4) $cot\ 5x$ (5) $sec\ 5x$

Solution : (1) $sin\ (x - 1) = 0 \Leftrightarrow (x - 1) = k\pi, k \in Z$

$\Leftrightarrow x = k\pi + 1, k \in Z$

$\therefore$ The set of zeroes of $sin(x - 1)$ is $\{k\pi + 1 \mid k \in Z\}$.

(2) Consider $sin\ x - 1$

$sin\ x - 1 = 0 \Leftrightarrow sin\ x = 1$

$\Leftrightarrow P(x) = B(0, 1)$

$\Leftrightarrow P(x) = P\left(\frac{\pi}{2}\right)$

$\Leftrightarrow x = 2k\pi + \frac{\pi}{2}, k \in Z$

$\Leftrightarrow x = \frac{4k\pi + \pi}{2}$

$\Leftrightarrow x = (4k + 1)\frac{\pi}{2}, k \in Z$

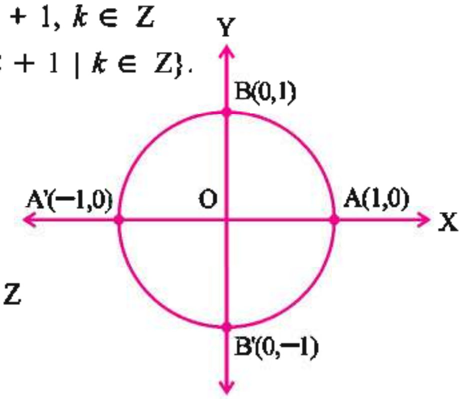


Figure 4.8

$\therefore$ The set of zeroes of $sin\,x - 1$ is $\left\{(4k + 1)\frac{\pi}{2} \mid k \in Z\right\}$.

(3) Consider $cos\ x + 1 = 0$

$cos\ x + 1 = 0 \Leftrightarrow cos\ x = -1$

$\Leftrightarrow P(x) = A'(-1, 0)$

$\Leftrightarrow P(x) = P(\pi)$

$\Leftrightarrow x = 2k\pi + \pi, k \in Z$ $(\alpha = \pi, \theta = 2k\pi + \alpha)$

$\Leftrightarrow x = (2k + 1)\pi, k \in Z$

$\therefore$ The set of zeroes of $cos\,x + 1$ is $\{(2k + 1)\pi \mid k \in Z\}$.

(4) Consider $cot\ 5x = 0$

$cot\ 5x = 0 \Leftrightarrow 5x = (2k + 1)\frac{\pi}{2}, k \in Z$

$\Leftrightarrow x = (2k + 1)\frac{\pi}{10}, k \in Z$

$\therefore$ The required set is $\left\{(2k + 1)\frac{\pi}{10} \mid k \in Z\right\}$.

(5) Let $sec\ 5x = 0$

Range of sec is R − (−1, 1), i.e. sec function takes no value between −1 and 1. Hence for any value of x, $sec\ 5x \neq 0$

$\therefore$ Set of zeroes of $sec\ 5x$ is $\emptyset$.

Example 2 : Find the range of the following functions :

(1) $5cos3x - 2$ (2) $6 - 7sin^2x$ (3) $3cosec^2x - 2$ (4) $2 + 3sec\ x$

(5) $| 3 - 7cos^2x |$ (6) $a\ cos^2(px + q) + b$

Solution : (1) $-1 \leq cos3x \leq 1$

$\Leftrightarrow -5 \leq 5cos3x \leq 5$

$\Leftrightarrow (-5 - 2) \leq (5cos3x - 2) \leq 5 - 2$

$\Leftrightarrow -7 \leq (5cos3x - 2) \leq 3$

$\therefore$ The range of $5cos3x - 2$ is $[-7, 3]$.

(2) $-1 \leq sin\ x \leq 1$

$\Leftrightarrow -1 \leq sin\ x \leq 0$ or $0 \leq sin\ x \leq 1$

$\Leftrightarrow 0 \leq sin^2x \leq 1$

$\Leftrightarrow 0 \geq -7sin^2x \geq -7$

$\Leftrightarrow -7 \leq -7sin^2x \leq 0$

$\Leftrightarrow -1 \leq 6 - 7sin^2x \leq 6$

$\therefore$ The range of $6 - 7sin^2x$ is $[-1, 6]$.

(3) $cosec^2x \geq 1 \Leftrightarrow 3cosec^2x \geq 3$

$\Leftrightarrow (3cosec^2x - 2) \geq 1$

$\therefore$ The range of $3cosec^2x - 2$ is $[1, \infty)$ or $\{p \mid p \geq 1, p \in R\}$

(4) $secx \leq -1$ or $secx \geq 1$

$\Leftrightarrow 3secx \leq -3$ or $3secx \geq 3$

$\Leftrightarrow 2 + 3secx \leq -1$ or $2 + 3secx \geq 5$

Figure 4.9

$\therefore$ The range of $2 + 3sec\ x$ is $R - (-1, 5)$ or $(-\infty, -1] \cup [5, \infty)$ or $\{p \mid p \leq -1$ or $p \geq 5, p \in R\}$

(5) $0 \le cos^2 x \le 1$

$\Leftrightarrow 0 \ge -7\ cos^2 x \ge -7$

$\Leftrightarrow -7 \le -7\ cos^2 x \le 0$

$\Leftrightarrow -4 \le (3 - 7\ cos^2 x) \le 3$

$\Leftrightarrow 3 - 7cos^2 x \in [-4, 0] \cup [0, 3]$

$\Leftrightarrow |\ 3 - 7cos^2 x\ | \in [0, 4] \cup [0, 3]$

$\therefore$ The range of $|\ 3 - 7cos^2 x\ |$ is $[0, 4]$

(6) $a\ cos^2(px + q) + b$

Case 1 : $a > 0$:

$0 \le cos^2\ (px + q) \le 1 \Leftrightarrow 0 \le a\ cos^2\ (px + q) \le a$

$\Leftrightarrow b \le a\ cos^2\ (px + q) + b \le a + b$

Case 2 : $a < 0$:

(ii) $0 \le cos^2\ (px + q) \le 1 \Leftrightarrow 0 \ge a\ cos^2\ (px + q) \ge a$

$\Leftrightarrow a \le a\ cos^2\ (px + q) \le 0$

$\Leftrightarrow a + b \le a\ cos^2\ (px + q) + b \le a$

$\therefore$ If $a > 0$, then $a\ cos^2\ (px + q) + b$ has range $[b, a + b]$ and if $a < 0$, then the range is $[\ a + b, b]$

Example 3 : Prove that $1 + \dfrac{4tan^2\theta}{(1 - tan^2\theta)^2} = \dfrac{1}{1 - 4sin^2\theta cos^2\theta}$

Solution : L.H.S. $= 1 + \dfrac{4tan^2\theta}{(1 - tan^2\theta)^2}$

$$= \frac{(1 - tan^2\theta)^2 + 4tan^2\theta}{(1 - tan^2\theta)^2}$$

$$= \frac{(1 + tan^2\theta)^2}{(1 - tan^2\theta)^2} \qquad ((a - b)^2 + 4ab = (a + b)^2)$$

$$= \frac{\left(1 + \frac{sin^2\theta}{cos^2\theta}\right)^2}{\left(1 - \frac{sin^2\theta}{cos^2\theta}\right)^2}$$

$$= \frac{(sin^2\theta + cos^2\theta)^2}{(cos^2\theta - sin^2\theta)^2}$$

$$= \frac{1}{(cos^2\theta + sin^2\theta)^2 - 4sin^2\theta\ cos^2\theta} \qquad ((a - b)^2 = (a + b)^2 - 4ab)$$

$$= \frac{1}{1 - 4sin^2\theta\ cos^2\theta} = \text{R.H.S}$$

Example 4 : Prove that $\frac{tan\theta + sec\theta - 1}{tan\theta - sec\theta + 1} = \frac{cos\theta}{1 - sin\theta} = \frac{1 + sin\theta}{cos\theta}$

Solution : L.H.S. $= \frac{tan\theta + sec\theta - 1}{tan\theta - sec\theta + 1}$

$$= \frac{(tan\theta + sec\theta) - (sec^2\theta - tan^2\theta)}{(tan\theta - sec\theta + 1)}$$

$$= \frac{(tan\theta + sec\theta) - (sec\theta - tan\theta)(sec\theta + tan\theta)}{(tan\theta - sec\theta + 1)}$$

$$= \frac{(tan\theta + sec\theta)(1 - (sec\theta - tan\theta))}{(tan\theta - sec\theta + 1)}$$

$$= \frac{(tan\theta + sec\theta)(1 - sec\theta + tan\theta)}{(tan\theta - sec\theta + 1)}$$

$$= (tan\theta + sec\theta)$$

$$= \frac{sin\theta}{cos\theta} + \frac{1}{cos\theta} = \frac{sin\theta + 1}{cos\theta} \qquad \text{(i)}$$

Now $\frac{sin\theta + 1}{cos\theta} = \frac{1 + sin\theta}{cos\theta} \times \frac{1 - sin\theta}{1 - sin\theta}$

$$= \frac{1 - sin^2\theta}{cos\theta(1 - sin\theta)}$$

$$= \frac{cos^2\theta}{cos\theta(1 - sin\theta)} = \frac{cos\theta}{1 - sin\theta} \qquad \text{(ii)}$$

From (i) and (ii) the result follows.

Example 5 : Prove that $sec^2\theta + cosec^2\theta \geq 4$

Solution : $sec^2\theta + cosec^2\theta = 1 + tan^2\theta + 1 + cot^2\theta$

$= 2 + tan^2\theta + cot^2\theta$

$= 2 + tan^2\theta + cot^2\theta - 2tan\theta\, cot\theta + 2cot\theta\, tan\theta$

$= 2 + (tan\theta - cot\theta)^2 + 2$

$= 4 + (tan\theta - cot\theta)^2$

$\geq 4 \qquad (tan\theta - cot\theta)^2 \geq 0$

Exercise 4.1

1. Find the zeroes of the following functions :

(1) $tan\,(2\theta + 1)$ (2) $cos\,(3x + 2)$ (3) $sinx + 1$

(4) $cosx - 1$ (5) $cot\,3x$ (6) $cosec\,5x$

2. Find the range of the given functions :

(1) $5\ sin 7x + 3$ (2) $3 - sec^2 x$ (3) $3sin^2\ x - 4$

(4) $|\ 2 - 5\ sin^2 x|$ (5) $|\ 3 - 4sec^2 x\ |$ (6) $3\ cosec\ x - 2$

3. Prove :

(1) If $tan\theta + cot\theta = 2$, then $tan^4\theta + cot^4\theta = 2$

(2) If $sin\theta + cosec\theta = 2$, then $sin^n\theta + cosec^n\theta = 2$

(3) $2\ sec^2\theta - sec^4\theta - 2cosec^2\theta + cosec^4\theta = cot^4\theta - tan^4\theta$

(4) $\dfrac{1 + tan^2\theta}{1 + cot^2\theta} = \left(\dfrac{1 - tan\theta}{1 - cot\theta}\right)^2$

(5) $\left(\dfrac{1 + sin\theta - cos\theta}{1 + sin\theta + cos\theta}\right)^2 = \dfrac{1 - cos\theta}{1 + cos\theta}$

(6) $\dfrac{1}{cosec\theta - cot\theta} - \dfrac{1}{sin\theta} = \dfrac{1}{sin\theta} - \dfrac{1}{cosec\theta + cot\theta}$

(7) $\dfrac{sin\theta + 1 - cos\theta}{sin\theta + cos\theta - 1} = \dfrac{1 + sin\theta}{cos\theta}$

(8) $\dfrac{sinA}{secA + tanA - 1} + \dfrac{cosA}{cosecA + cotA - 1} = 1$

(9) $\left(\dfrac{1}{sec^2A - cos^2A} + \dfrac{1}{cosec^2A - sin^2A}\right) sin^2A\ cos^2A = \dfrac{1 - sin^2A\,cos^2A}{2 + sin^2A\,cos^2A}$

4. If $tan\theta = \dfrac{2x(x+1)}{2x+1}$, then find the value of $sin\theta$; $0 < \theta < \frac{\pi}{2}$.

5. If $10sin^4\theta + 15cos^4\theta = 6$, find the value of $27cosec^2\theta + 8\ sec^2\theta$.

6. If $cosec\theta + cot\theta = \frac{3}{2}$, find $cos\theta$.

7. Prove $tan^2\theta - sin^2\theta = tan^2\theta \cdot sin^2\theta$. Hence, deduce that $tan^2\theta \geq sin^2\theta$.

*

4.10 Periods of Trigonometric Functions

We know that the principal period of the trigonometric point function is 2π.

$\therefore$ $P(2\pi + \theta) = P(\theta)$

So, if we take one complete revolution from the point P, we again come back to the same point P. Thus if θ increases or decreases by an integral multiple of 2π, the values of *sine* and *cosine* functions do not change.

Thus, $sin(2k\pi + \theta) = sin\theta$, $cos(2k\pi + \theta) = cos\theta$, $k \in Z$.

In the same way, as *sec* and *cosec* are just reciprocals of *cos* and *sin* respectively,

$sec(2k\pi + \theta) = sec\theta$ and $cosec(2k\pi + \theta) = cosec\theta$.

For the time being we shall assume that values of *tan* and *cot* functions do not change if θ increases or decreases by an integral multiple of π.

Thus principal period of *sin, cos, sec, cosec* is 2π and *tan* and *cot* have principal period π.

Thus, $tan(k\pi + \theta) = tan\theta$, $cot(k\pi + \theta) = cot\theta$, $k \in Z$.

Example 6 : Prove $sin\left(\frac{37\pi}{6}\right) = sin\frac{\pi}{6}$.

Solution : We know $sin(2k\pi + \theta) = sin\theta$, $k \in Z$

$\therefore \quad sin\left(\frac{37\pi}{6}\right) = sin\left(6\pi + \frac{\pi}{6}\right)$

$= sin\frac{\pi}{6}$ **($6\pi = 3(2\pi)$ is a period of *sin* function)**

Example 7 : Prove $cos\left(\frac{-101\pi}{3}\right) = cos\frac{\pi}{3}$.

Solution : $cos\left(\frac{-101\pi}{3}\right) = cos\left(-34\pi + \frac{\pi}{3}\right)$

$= cos\frac{\pi}{3}$ **(-34π is a period of *cos* function)**

Example 8 : Prove $tan\ \frac{22\pi}{3} = tan\frac{\pi}{3}$

Solution : We know $tan(k\pi + \theta) = tan\theta$, $k \in Z$.

$\therefore \quad tan\ \frac{22\pi}{3} = tan\left(7\pi + \frac{\pi}{3}\right)$

$= tan\frac{\pi}{3}$ **(7π is a period of *tan* function)**

4.11 Increasing and Decreasing Functions

We know that the graph of the function $f : R \to R, f(x) = 2x + 1$ is a straight line. This is a graph which is ascending that means as x increases the value of $f(x)$ increases.

For $f : N \to N$, $f(x) = x^2$, $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, $f(4) = 16$. This shows that as x increases the value of $f(x)$ also increases.

i.e. $4 > 3 \Rightarrow f(4) > f(3)$. Such a function is called an increasing function.

For $f : N \to R$, $f(x) = \frac{1}{x}$, $f(1) = 1$, $f(2) = \frac{1}{2}$, $f(3) = \frac{1}{3}$, $f(4) = \frac{1}{4}$, This shows that as x increases the value of $f(x)$ decreases. i.e. $4 > 2 \Rightarrow f(4) < f(2)$, Such a function is called a decreasing function.

In general **let $A \subset R$, $B \subset R$ and $f : A \to B$ be a function and if $\forall x_1, x_2 \in A$, $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$, then f is called an (strictly) increasing function, and an increasing function is indicated by the symbol $f \uparrow$.**

Let $A \subset R$, $B \subset R$ and $f : A \to B$ be a function and if $\forall x_1, x_2 \in A$, $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$, then f is called a (strictly) decreasing function. A decreasing function is indicated by the symbol $f \downarrow$.

All trigonometric functions are strictly increasing or decreasing in a quadrant. Henceforth we will call them increasing functions or decreasing functions.

Let us now think about the trigonometric functions. If $0 < \theta < \frac{\pi}{2}$, then as θ increases, the point P(θ) continuously moves from A to B on the circle in anticlockwise direction. So, its x-coordinate decreases from 1 to 0 and y-coordinate increases from 0 to 1. This means that for $\theta \in \left(0, \frac{\pi}{2}\right)$, *cos*$\theta$ decreases and *sin*θ increases.

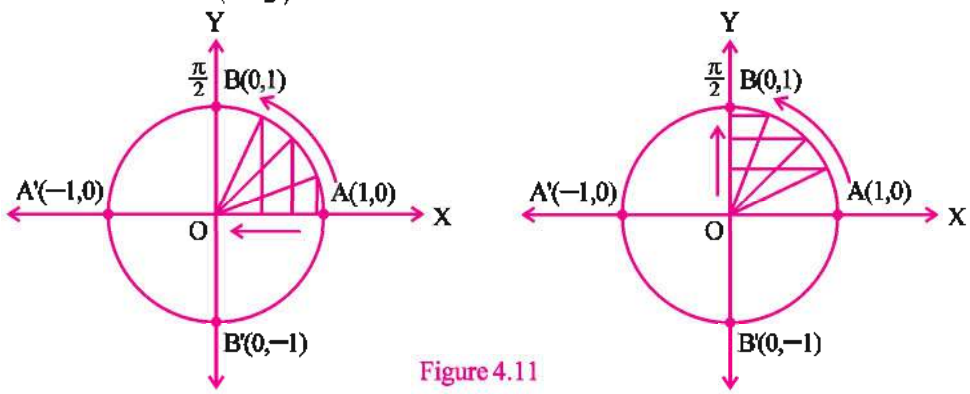


Figure 4.11

In the second quadrant $\frac{\pi}{2} < \theta < \pi$. As θ increases from $\frac{\pi}{2}$ to π, P(θ) descends along the $\overset{\frown}{BA'}$. So *sin*θ decreases from 1 to 0 and *cos*θ also decreases from 0 to -1. In the third quadrant $\pi < \theta < \frac{3\pi}{2}$. As θ increases from π to $\frac{3\pi}{2}$, P(θ) descends along the $\overset{\frown}{A'B'}$. So *sin*θ decreases from 0 to -1 and *cos*θ increases from -1 to 0. In the fourth quadrant, $\frac{3\pi}{2} < \theta < 2\pi$. *cos*$\theta$ increases from 0 to 1 and *sin*θ increases from -1 to 0. This information is summarised in the following table.

Thus,

Function \ Quadrant	First	Second	Third	Fourth
	$\left(0, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \pi\right)$	$\left(\pi, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
sin	↑	↓	↓	↑
cos	↓	↓	↑	↑

As *cosec* and *sec* are reciprocals of *sin* and *cos*, it is clear that when *sin*θ increases, *cosec*θ decreases and when *sin*θ decreases, *cosec*θ increases. The results are summarised in the following table. Similar statements apply for *cos* and *sec* functions.

Function \ Quadrant	First	Second	Third	Fourth
	$\left(0, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \pi\right)$	$\left(\pi, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, 2\pi\right)$
sec	↑	↑	↓	↓
cosec	↓	↑	↑	↓

We assume that *tan* is increasing and *cot* is decreasing in all quadrants.

We note that the discussion of a decreasing or an increasing function is confined only to a particular quadrant. *tan* function is increasing in every quadrant but we cannot say that $30^\circ < 150^\circ \Rightarrow tan30^\circ < tan150^\circ$ as $tan30^\circ$ is positive and $tan150^\circ$ is negative. (Why ?)

Example 9 : Examine the truth of the following :

$$\frac{\pi}{2} < \theta < \pi \Rightarrow cos\frac{\pi}{2} < cos\theta < cos\pi$$

Solution : *cos* function is ↓ in the second quadrant.

$\therefore \quad \frac{\pi}{2} < \theta < \pi \Rightarrow cos\frac{\pi}{2} > cos\theta > cos\pi$

$\therefore$ The statement is false.

> **☛ Note** $cos\frac{\pi}{2} < cos\theta < cos\pi$ would mean $0 < cos\theta < -1$ which is clearly false.

Example 10 : Examine the truth : $sin\theta_1 > sin\theta_2 \Leftrightarrow cos\theta_1 > cos\theta_2$, $0 < \theta < \frac{\pi}{2}$.

Solution : In the first quadrant *sin* is an increasing function and *cos* is a decreasing function.

$\therefore \quad sin\theta_1 > sin\theta_2 \Leftrightarrow \theta_1 > \theta_2$

$\qquad\qquad\qquad\quad \Leftrightarrow cos\theta_1 < cos\theta_2$

$\therefore$ The statement is false.

Example 11 : Prove that $cot\theta$ is a decreasing function in $\left(0, \frac{\pi}{2}\right)$

Solution : As, $0 < \theta < \frac{\pi}{2}$, $sin\theta > 0$ and $cos\theta > 0$.

Now sin is an increasing function in $\left(0, \frac{\pi}{2}\right)$.

Let $\theta_1, \theta_2 \in \left(0, \frac{\pi}{2}\right)$ and $\theta_1 < \theta_2$

$\therefore \quad \theta_1 < \theta_2 \Rightarrow sin\theta_1 < sin\theta_2 \Rightarrow \frac{1}{sin\theta_1} > \frac{1}{sin\theta_2}$ **(i)**

Now cos is a decreasing function in $\left(0, \frac{\pi}{2}\right)$

$\therefore \quad \theta_1 < \theta_2 \Rightarrow cos\theta_1 > cos\theta_2$ **(ii)**

From (i) and (ii) $\frac{1}{sin\theta_1} \times cos\theta_1 > \frac{1}{sin\theta_2} \times cos\theta_2$

$\therefore \quad cot\theta_1 > cot\theta_2$

So, $\theta_1 < \theta_2 \Rightarrow cot\theta_1 > cot\theta_2$

Hence, $cot\theta$ is a decreasing function in $\left(0, \frac{\pi}{2}\right)$.

4.12 Measures of Angles

We will learn about two methods of measuring an angle in this section.

Degree Measure : In this system a right angle is divided into ninety congruent parts. Each part is said to have measure one degree, written as 1°. Thus, one degree is one-ninetieth part of the measure of a right angle. A degree is further divided in 60 equal parts and each part is called a minute. The symbol $1'$ is used to denote one minute. One minute is further divided in 60 equal parts, each part is called a second. The symbol $1''$ is used to denote one second.

Thus, $1^\circ = 60' = 60$ minutes.

$1' = 60'' = 60$ seconds.

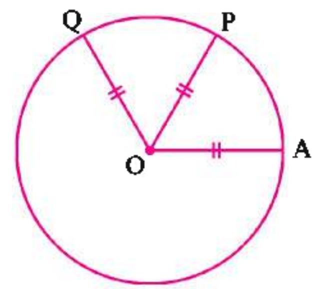


Figure 4.11

Radian Measure : Radian measure is another unit for measurement of an angle. One radian is the measure of the angle subtended at the centre of a circle by an arc of length equal to the radius of the circle. It is denoted by 1^c. (c for circular measure.) It is also denoted by 1^R.

Consider a circle of radius r having centre O. Let A be a point on the circle. Now cut off an arc AP whose length is equal to the radius r of the circle. i.e. $l(\widehat{AP}) = r$. Then the measure of $\angle AOP$ is 1 radian $(= 1^c)$. If $l(\widehat{AQ}) = 2r$, then the measure of $\angle AOQ$ is 2 radian $(= 2^c)$. Since a radian is the unit of measurement of an angle, it should be a constant quantity not depending upon the radius of the circle.

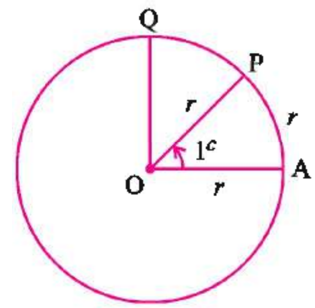


Figure 4.12

Consider a circle with centre O and radius r. Take a point A on the circle and cut off an $\widehat{AP}$ whose length is equal to the radius r. Draw $\overline{OA}$ and $\overline{OP}$ and draw $\overline{OQ} \perp \overline{OA}$. Now by definition, $m\angle AOP = 1^c$ and $\angle AOQ$ is a right angle.

Since in a circle, the angles at the centre of a circle have measures proportional to the lengths of arcs subtending them,

$$\frac{m\angle AOP}{m\angle AOQ} = \frac{l(\widehat{AP})}{l(\widehat{AQ})}$$

$$\therefore \quad \frac{m\angle AOP}{m\angle AOQ} = \frac{r}{\frac{1}{2}(\pi r)} \qquad \left(l(\widehat{AQ}) = \frac{1}{4}(2\pi r) = \frac{1}{2}\pi r\right)$$

$$\therefore \quad \frac{1^c}{m\angle AOQ} = \frac{2}{\pi}$$

$$\therefore \quad m\angle AOQ = \frac{\pi}{2} \text{ radian}$$

$\therefore$ Radian measure of a right angle is $\frac{\pi}{2}$.

$\therefore$ Radian is a constant angle and Radian measure of a right angle $= \frac{\pi}{2}$

Degree measure of a right angle = 90

$\therefore \quad \frac{\pi^c}{2} = 90^\circ$

$\therefore \quad \pi^c = 180^\circ$

1 radian = $\frac{180}{\pi}$ degree = $57^\circ\, 16'\, 22''$ or 1 degree = $\frac{\pi}{180}$ radian = 0.01746

Thus, the degree measure of an angle whose radian measure is α is $\frac{180\alpha}{\pi}$. The radian measure of an angle whose degree measure is α is $\frac{\pi\alpha}{180}$.

Radian Measure = $\frac{\pi}{180} \times$ degree measure

Degree measure = $\frac{180}{\pi} \times$ radian measure

For instance, if the radian measure of an angle is $\frac{\pi}{6}$, its degree measure is $\frac{180}{\pi} \times \frac{\pi}{6} = 30^\circ$.

An angle whose degree measure is 120 has radian measure equal to

$$120 \times \frac{\pi}{180} = \frac{2\pi^c}{3}.$$

If in a circle of radius r, an $\widehat{AP}$ of length l subtends an angle of measure θ radian at the centre, we, have

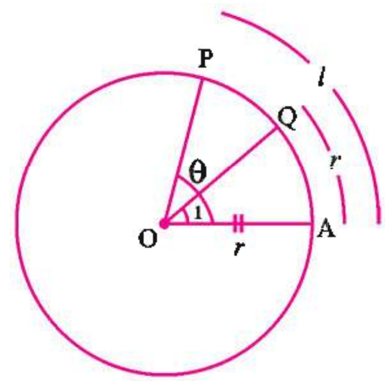


Figure 4.13

$l(\widehat{AQ}) = r$

$m\angle AOP = \theta,\ m\angle AOQ = 1$

$$\therefore \quad \frac{\text{radian measure } \angle AOP}{\text{radian measure } \angle AOQ} = \frac{l(\widehat{AP})}{l(\widehat{AQ})}$$

$$\therefore \quad \frac{\theta}{1} = \frac{l}{r}$$

$$\therefore \quad \theta = \frac{l}{r}$$

$$\text{Radian measure of an angle} = \frac{\text{Length of the arc}}{\text{Radius}}$$

☛ Note Till standard X, we have learnt about the degree measure of an angle only. For an angle A with the measure 60°, we used to write $m\angle A = 60$ but now we have another way of measuring angles, the radian measure. So for an angle A with the degree measure 60, we shall write $m\angle A = 60^\circ$ from now onwards. If A is a right angle, then we shall write $m\angle A = 90^\circ$ or $m\angle A = \frac{\pi}{2} = \frac{\pi^c}{2}$. If A is an angle of an equilateral triangle we should not write $m\angle A = 60$ but we should write $m\angle A = 60^\circ$ or $m\angle A = \frac{\pi}{3}$ because $\frac{\pi}{3}$ radians = 60°.

Hence $m\angle A = x^\circ$ means $\angle A$ has degree measure x and $m\angle A = x$ means $\angle A$ has radian measure x.

Since degree measure of an angle lies in (0, 180) and $0^o = 0^c$ and $180^o = \pi^c$, the radian measure of an angle lies in $(0, \pi)$.

Example 12 : Convert $47^o\ 30'$ into radian measure.

Solution : We know that $1^o = 60'$

$\therefore\quad 30' = \left(\frac{30}{60}\right)^o = \left(\frac{1}{2}\right)^o$

$47^o\ 30' = \left(47\frac{1}{2}\right)^o = \left(\frac{95}{2}\right)^o$

We know that $180^o = \pi^c$

$\therefore\quad \left(\frac{95}{2}\right)^o = \left(\frac{\pi}{180} \times \frac{95}{2}\right)^c = \left(\frac{19\pi}{72}\right)^c = \frac{19\pi}{72}$

Hence, radian measure of the angle with degree measure $47^o\ 30'$ is $\left(\frac{19\pi}{72}\right)^c$ or $\frac{19\pi}{72}$.

Example 13 : Convert $39^o\ 22'\ 30''$ into radian measure.

Solution : $60'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$

$\therefore\quad 22'\ 30'' = \left(22\frac{1}{2}\right)' = \left(\frac{45}{2}\right)'$

$\left(\frac{45}{2}\right)' = \left(\frac{45}{2} \times \frac{1}{60}\right)^o = \left(\frac{3}{8}\right)^o$

$\therefore\quad 39^o\ 22'\ 30'' = \left(39\frac{3}{8}\right)^o = \left(\frac{315}{8}\right)^o$

$\left(\frac{315}{8}\right)^o = \left(\frac{315}{8} \times \frac{\pi}{180}\right)^c = \left(\frac{7\pi}{32}\right)^c$

Hence, radian measure of the angle with degree measure $39^o\ 22'\ 30''$ is $\left(\frac{7\pi}{32}\right)^c = \frac{7\pi}{32}$.

Example 14 : Convert 2 radian into degree measure.

Solution : We know that $\pi^c = 180^o$

$\therefore\quad 2^c = \left(\frac{180}{\pi} \times 2\right)^o = \left(\frac{180 \times 7 \times 2}{22}\right)^o$

$= \left(114\frac{6}{11}\right)^o = 114^o + \left(\frac{6}{11} \times 60\right)'$ **(as $1^o = 60'$)**

$= 114^o + \left(32\frac{8}{11}\right)' = 114^o + 32' + \left(\frac{8}{11} \times 60\right)''$ **(as $1' = 60''$)**

$= 114^o + 32' + 44''$

Hence, 2 radian $= 114^o\ 32'\ 44''$

Example 15 : Find in degree measure, the measure of the angle subtended at the centre of a circle of radius 25 *cm* by an arc of length 55 *cm*.

Solution : Here, $r = 25$ *cm*, $l = 55$ *cm*

We have, $\theta = \left(\frac{l}{r}\right)^c$

$\therefore\quad \theta = \left(\frac{55}{25}\right)^c = \left(\frac{11}{5} \times \frac{180}{\pi}\right)^o = \left(\frac{11 \times 36 \times 7}{22}\right)$

$\therefore\quad \theta = 126^o$

Example 16 : Find in degree and radian, measure of the angle between the hour hand and the minute hand of a clock at half past four.

Solution : We know that the hour hand completes one rotation in 12 hours, while the minute hand completes one rotation in 60 minutes.

The measure of the angle formed by the hour hand in 12 hours = 360^o

$\therefore$ The measure of the angle formed by the hour hand in 4 hours 30 minutes i.e. $\frac{9}{2}$ hours is $\left(\frac{360}{12} \times \frac{9}{2}\right)^o = 135^o$

The measure of the angle formed by the minute hand in 1 hour = 360^o

$\therefore$ The measure of the angle formed by the minute hand in 30 minutes = $\left(\frac{360}{60} \times 30\right)^o = 180^o$

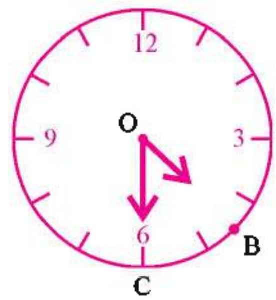


Figure 4.14

Hence, required $m\angle BOC = 180^o - 135^o = 45^o$

$$= \left(45 \times \frac{\pi}{180}\right) = \frac{\pi}{4} \text{ radian}$$

Example 17 : A circular wire of radius 4 *cm* is cut and bent so as to lie along the circumference of a loop whose radius is 64 *cm*. Find the measure of the angle in degree which is subtended at the centre of the loop.

Solution : Radius of the circular wire is 4 *cm*.

$\therefore$ Length of wire = circumference

$= 2\pi r$

$= 2\pi \times 4$

$= 8\pi$

Now, it is being placed along a circular loop of radius 64 *cm*.

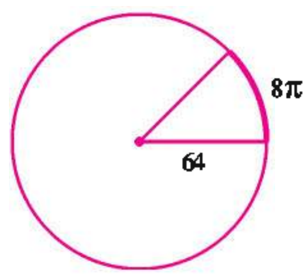


Figure 4.15

Hear, $l = 8\pi$, $r = 64$ *cm*

Hence, $\theta = \frac{l}{r} = \frac{8\pi}{64} = \frac{\pi}{8} = \left(\frac{\pi}{8} \times \frac{180}{\pi}\right)^o = 22^o\ 30'$

Example 18 : Find in degrees, measure of the angle through which a pendulum swings if its length is 40 *cm* and it swings in an arc of length 8 *cm*.

Solution : Here, $r = 40$ *cm*, $l = 8$ *cm*

Hence, $\theta = \frac{l}{r} = \frac{8}{40} = \frac{1}{5}$ radian $= \left(\frac{1}{5} \times \frac{180}{\pi}\right)$ degree

$$= \left(\frac{36 \times 7}{22}\right)^{o} = \left(11\frac{5}{11}\right)^{o}$$

$$= 11^{\circ} + \left(\frac{5}{11} \times 60\right)'$$

$$= 11^{\circ} + \left(27\frac{3}{11}\right)'$$

$$= 11^{\circ} + 27' + 16'' = 11^{\circ}\ 27'\ 16''$$

Example 19 : In two circles, if the arcs of the same lengths subtend angles having measures 60° and 75° at the centre, find the ratio of their radii.

Solution : Let r_1 and r_2 be the radii of the two circles. Also let the length of the arc be l.

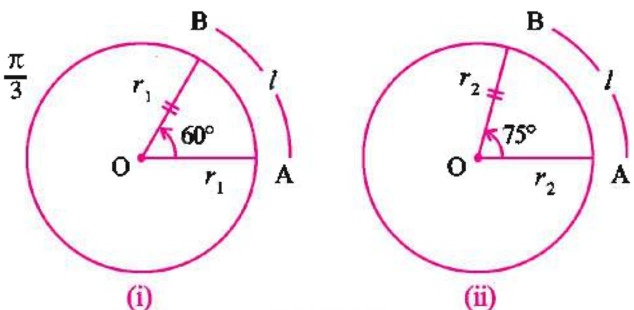


Figure 4.16

For the first circle,

$$\theta = 60^{\circ} = 60 \times \frac{\pi}{180} = \frac{\pi}{3}$$

We know that $\theta = \frac{l}{r}$

$$\therefore \quad r = \frac{l}{\theta}$$

$$\therefore \quad r_1 = \frac{l}{\frac{\pi}{3}}$$

$$\therefore \quad r_1 = \frac{3l}{\pi} \qquad \text{(i)}$$

For the second circle,

$$\theta = 75^{\circ} = 75 \times \frac{\pi}{180} = \frac{5\pi}{12}$$

$$\therefore \quad r_2 = \frac{12l}{5\pi} \qquad \left(r = \frac{l}{\theta}\right) \text{(ii)}$$

Dividing (i) by (ii)

$$\frac{r_1}{r_2} = \frac{\frac{3l}{\pi}}{\frac{12l}{5\pi}}$$

$$\therefore \quad \frac{r_1}{r_2} = \frac{15}{12} = \frac{5}{4}$$

Hence $r_1 : r_2 = 5 : 4$

Example 20 : If $\frac{\sin A}{\sin B} = \sqrt{2}$, $\frac{\tan A}{\tan B} = \sqrt{3}$, find $\tan A$ and $\tan B$; A, B $\in \left(0, \frac{\pi}{2}\right)$

Solution : Here $\frac{1}{\sin B} = \frac{\sqrt{2}}{\sin A}$, $\frac{1}{\tan B} = \frac{\sqrt{3}}{\tan A}$

$$\therefore \quad \operatorname{cosec} B = \sqrt{2}\operatorname{cosec} A,\ \cot B = \sqrt{3}\cot A$$

Now $cosec^2B - cot^2B = 1$

$\therefore \quad 2cosec^2A - 3cot^2A = 1$

$\therefore \quad 2(1 + cot^2A) - 3cot^2A = 1$

$\therefore \quad 2 + 2cot^2A - 3cot^2A = 1$

$\therefore \quad cot^2A = 1$

$\therefore \quad cotA = \pm 1$

$\therefore \quad tanA = \pm 1$

But $0 < A < \frac{\pi}{2}$

$\therefore \quad tanA = 1$

Now $cotB = \sqrt{3}cot\ A = \sqrt{3}$

$\therefore \quad tanB = \frac{1}{\sqrt{3}}$

Exercise 4.2

1. Find the radian measures corresponding to the following degree measures :

(1) $240°$ (2) $75°$ (3) $40°\ 20'$ (4) $110°\ 30'$

2. Find the degree measures corresponding to the following radian measures ($\pi = \frac{22}{7}$).

(1) $\frac{\pi}{15}$ (2) 8 (3) $\frac{\pi}{32}$ (4) $\frac{1}{4}$

3. Find the measure of the angle in degree subtended at the centre of a circle of radius 3 *cm* by an arc of length 1 *cm*.

4. In a circle of diameter 60 *cm*, the length of a chord is 30 *cm*. Find the length of the minor arc corresponding to the chord.

5. Find the measure of the angle in degree through which a pendulum swings if its length is 75 *cm* and its tip describes an arc of length 21 *cm*.

6. If in two circles, arcs of the same length subtend angles of measures $65°$ and $110°$ at the centre, find the ratio of their radii.

7. Find in degree and radian, measure of the angle between the hour hand and the minute hand of a clock at half past two.

8. The measures of the angles of a triangle are in A.P. and the angle with largest measure has radian measure $\frac{5\pi}{12}$. Then find the measure of the angle with smallest measure.

9. If $\frac{sin\ A}{sin\ B} = \sqrt{3}$, $\frac{tan\ A}{tan\ B} = 3$, find sin^2A.

*

4.13 Odd and Even Functions

Let $A \subset R$, $B \subset R$ and $f : A \to B$ be a function and

if $\forall x, -x \in A, f(-x) = f(x)$, then f is called an even function.

If $\forall x, -x \in A, f(-x) = -f(x)$, then f is called an odd function.

For instance, $f : R \to R, f(x) = x^2$ satisfies $f(-x) = f(x)$ for all $x \in R$ because $(-x)^2 = x^2$, so f is an even function.

Similarly, for $f : R \to R, f(x) = x^3, f(-x) = -f(x)$ for all $x \in R$ as

$(-x)^3 = -x^3$. So f is an odd function.

We will now prove that *sin* is an odd function and *cos* is an even function.

First suppose $0 < \alpha < 2\pi$. Now we will see four possible situations for the points $P(\alpha)$ and $P(2\pi - \alpha)$ in the following figures on the unit circle.

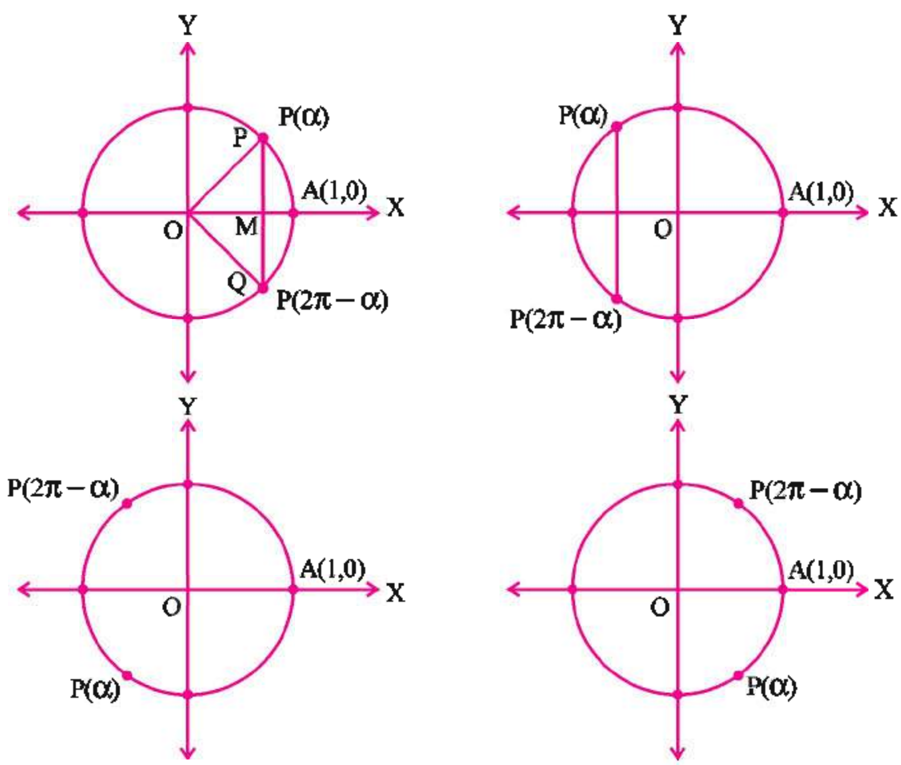


Figure 4.17

If $0 < \alpha < \frac{\pi}{2}$, then $P(\alpha)$ is in the first quadrant and $P(2\pi - \alpha)$ is in the fourth quadrant.

The lengths of $\overset{\frown}{AP}$ and $\overset{\frown}{AQ}$ are α. So, $m\angle AOP = m\angle AOQ$

Now, join P and Q. Suppose $\overline{PQ}$ intersects X-axis at M.

P(α) and P($2\pi - \alpha$) are represented by P and Q respectively.

$\Delta POM \cong \Delta QOM$

So $\overline{PQ}$ is perpendicular to X-axis.

Now, P has coordinates (x, y) and Q has coordinates $(x, -y)$.

P(α) is $(cos\alpha, sin\alpha)$ and P($2\pi - \alpha$) is $(cos(2\pi - \alpha), sin(2\pi - \alpha))$.

$\therefore$ $(x, y) = (cos\alpha, sin\alpha)$ and $(x, -y) = (cos(2\pi - \alpha), sin(2\pi - \alpha))$

$\therefore$ $x = cos\alpha$, $y = sin\alpha$ and $x = cos(2\pi - \alpha)$, $-y = sin(2\pi - \alpha)$

This means $cos\alpha = cos(2\pi - \alpha) = x$, $sin\alpha = y$, $sin(2\pi - \alpha) = -y$

As 2π is a period of *sin* and *cos*, we get

$cos\alpha = x$, $cos(-\alpha) = x$, $sin\alpha = y$, $sin(-\alpha) = -y$

Thus, $cos\alpha = cos(-\alpha) = x$, $sin(-\alpha) = -y = -sin\alpha$

Same thing can be observed and verified in every other quadrant.

Next, suppose $\theta \in R$ and

let $\theta = 2n\pi + \alpha \qquad 0 < \alpha < 2\pi$

$\therefore$ $-\theta = -2n\pi - \alpha$

$\therefore$ $-\theta = -2n\pi - \alpha + 2\pi - 2\pi$

$\therefore$ $-\theta = 2\pi(-1 - n) + (2\pi - \alpha)$ and $0 < (2\pi - \alpha) < 2\pi$

As 2π is a period of *cos*,

$cos\theta = cos\alpha$ and $cos(-\theta) = cos(2\pi - \alpha) = cos(-\alpha)$ (*cos* has period 2π)

But $cos\alpha = cos(-\alpha)$

$\therefore$ $cos\theta = cos(-\theta)$ and similarly we can see that $sin(-\theta) = -sin\theta$,

Thus, $\forall\ \theta \in R$, $cos(-\theta) = cos\theta$, $sin(-\theta) = -sin\theta$

Therefore, *sin* is an odd function and *cos* is an even function.

We have studied trigonometric functions of real numbers.

If the radian measure of an angle is θ, we define $sin\theta$, $cos\theta$ etc. for real number θ. Since radian measure of an angle $\theta \in (0, \pi)$, restrictions of trigonometric functions of real numbers for domain $(0, \pi)$ are trigonometric functions of angles.

4.14 Value of Trigonometric Functions Obtained from a Right Triangle

Let ΔMOP be a right angle triangle with right angle at M in coordinate plane.

Let unit circle with center at O intersect $\overrightarrow{OP}$ at P' and let $\overline{P'M'}$ and $\overline{PM}$ be perpendicular to X-axis. Thus $\overleftrightarrow{PM} \parallel \overleftrightarrow{P'M'}$.

$\therefore \quad \Delta POM \sim \Delta P'OM'$

$\therefore \quad \frac{OP}{OP'} = \frac{OM}{OM'} = \frac{PM}{P'M'}$

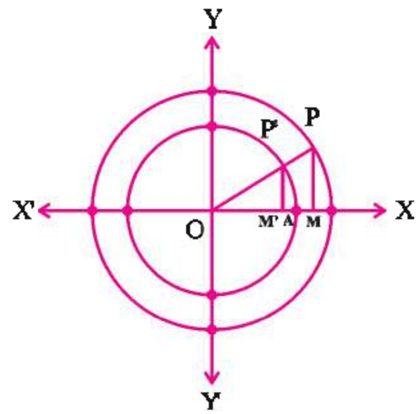


Figure 4.18

If $l(\widehat{AP'}) = \theta$, then P' is the trigonometric point corresponding to θ on the unit circle. So P' $(x, y) = (cos\theta, sin\theta)$. If OM' and M'P' are x and y coordinates of P' respectively, then OM' = $cos\theta$, M'P' = $sin\theta$ and OP' = 1.

$\therefore \quad \frac{OP}{1} = \frac{OM}{cos\theta} = \frac{PM}{sin\theta}$

$\therefore \quad cos\theta = \frac{OM}{OP}, \; sin\theta = \frac{PM}{OP}$

Now the radian measure of $\angle P'OM'$ is θ. So the radian measure of $\angle POM$ is also θ.

$\therefore$ In ΔPOM, $cos\theta = \frac{OM}{OP} = \frac{\text{Adjacent side}}{\text{Hypotenuse}}$

and $sin\theta = \frac{PM}{OP} = \frac{\text{Opposite side}}{\text{Hypotenuse}}$

$\therefore \quad cos\left(\frac{180\theta}{\pi}\right)^{\circ} = \frac{\text{Adjacent side}}{\text{Hypotenuse}}, \quad sin\left(\frac{180\theta}{\pi}\right)^{\circ} = \frac{\text{Opposite side}}{\text{Hypotenuse}}$

Taking $\left(\frac{180\theta}{\pi}\right)^{\circ} = \alpha$

$cos\alpha = \frac{\text{Adjacent side}}{\text{Hypotenuse}}, \; sin\alpha = \frac{\text{Opposite side}}{\text{Hypotenuse}}$

Thus, trigonometrical functions defined on a unit circle are not different from those defined on the basis of a right angled triangle. But trigonometrical ratios on the basis of right angled triangle had limitations, since the measures of angles were confined between (0, 90). Now trigonometric functions are defined for real numbers.

4.15 Values of Trigonometric Functions in Each Quadrant

We know that for each real number θ, there is a unique point $P(\theta) = P(x, y)$ on the unit circle. Now, by definition of *cos* and *sin*, $x = cos\theta$, $y = sin\theta$.

If $0 < \theta < \frac{\pi}{2}$, then $P(\theta) = (x, y)$ is in the first quadrant and has $x > 0$ and $y > 0$. So, $x = cos\theta > 0$, $y = sin\theta > 0$.

If $\frac{\pi}{2} < \theta < \pi$, then $P(\theta) = P(x, y)$ is in the 2nd quadrant. In the second quadrant, $x < 0$ and $y > 0$. So, $x = cos\theta < 0$, $y = sin\theta > 0$.

If $\pi < \theta < \frac{3\pi}{2}$, then $P(\theta) = P(x, y)$ is in the 3rd quadrant and in the third quadrant $x < 0$, $y < 0$. So, $x = cos\theta < 0$ and $y = sin\theta < 0$.

If $\frac{3\pi}{2} < \theta < 2\pi$, then $P(\theta) = P(x, y)$ is in the 4th quadrant. In the 4th quadrant $x > 0$, $y < 0$. So, $x = \cos\theta > 0$ and $y = sin\theta < 0$.

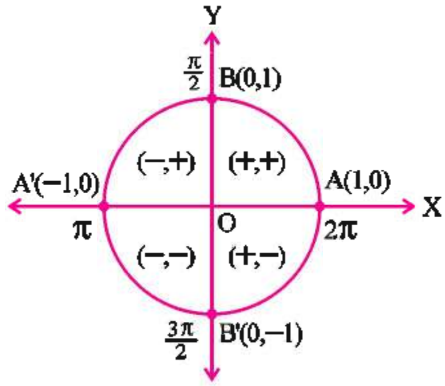


Figure 4.19

As $tan\theta = \frac{sin\theta}{cos\theta}$, $cot\theta = \frac{cos\theta}{sin\theta}$, $cosec\theta = \frac{1}{sin\theta}$ and $sec\theta = \frac{1}{cos\theta}$, we can find the signs of other trigonometric functions in different quadrants.

Function \ Quadrant	First	Second	Third	Fourth
sin	+	+	−	−
cos	+	−	−	+
tan	+	−	+	−
cot	+	−	+	−
cosec	+	+	−	−
sec	+	−	−	+

Example 21 : If $cot\theta = \frac{-5}{12}$, and $P(\theta)$ lies in the second quadrant, find the value of other trigonometric functions.

Solution : Since $cot\theta = \frac{-5}{12}$, we have $tan\theta = \frac{-12}{5}$

Now, $sec^2\theta = 1 + tan^2\theta$

$= 1 + \frac{144}{25} = \frac{169}{25}$

$\therefore \quad sec\theta = \pm\frac{13}{5}$

Since $P(\theta)$ is in the second quadrant, $sec\theta$ will be negative.

$\therefore \quad sec\theta = -\frac{13}{5}$ and $cos\theta = \frac{-5}{13}$

Further, we know that $sin\theta = cos\theta \cdot tan\theta = \left(\frac{-5}{13}\right)\left(\frac{-12}{5}\right) = \frac{12}{13}$

$\therefore \quad cosec\theta = \frac{13}{12}$

Example 22 : If $cos\theta = \frac{(a+b)^2}{4ab}$, $ab > 0$, then prove that $a = b$

Solution : $| cos\theta | \leq 1$

$\therefore \quad \frac{(a+b)^2}{4|ab|} \leq 1$

$\therefore \quad (a + b)^2 \leq 4ab$ $\qquad (ab > 0)$

$\therefore \quad (a - b)^2 \leq 0$

But $(a - b)^2 \nless 0$

$\therefore \quad a - b = 0$

$\therefore \quad a = b$

Example 23 : If $tan\theta = 2 - \sqrt{3}$, then find $cos\theta$. $0 < \theta < \frac{\pi}{2}$

Solution : $sec^2\theta = 1 + tan^2\theta$

$= 1 + (2 - \sqrt{3})^2$

$= 1 + (4 - 4\sqrt{3} + 3)$

$= 8 - 4\sqrt{3}$

$= 8 - 2\sqrt{12}$

$= (\sqrt{6} - \sqrt{2})^2$

$\therefore \quad sec\theta = \sqrt{6} - \sqrt{2}$ $\qquad \left(0 < \theta < \frac{\pi}{2}\right)$

$\therefore \quad cos\theta = \frac{1}{\sqrt{6}-\sqrt{2}} = \frac{1}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$

$\therefore \quad cos\theta = \frac{\sqrt{6}+\sqrt{2}}{4}$

Example 24 : If $tan\theta = x - \frac{1}{4x}$, find $sec\theta + tan\theta$

Solution : We have $tan\theta = x - \frac{1}{4x}$

$\therefore \quad tan^2\theta = x^2 - \frac{1}{2} + \frac{1}{16x^2}$

Now, $sec^2\theta = 1 + tan^2\theta$

$= 1 + \left(x^2 - \frac{1}{2} + \frac{1}{16x^2}\right)$

$= x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$

$\therefore \quad sec\theta = x + \frac{1}{4x}$ or $-x - \frac{1}{4x}$

$\therefore \quad sec\theta + tan\theta = x + \frac{1}{4x} + x - \frac{1}{4x} = 2x$

or $\quad sec\theta + tan\theta = -x - \frac{1}{4x} + x - \frac{1}{4x} = -\frac{1}{2x}$

Exercise 4.3

1. If $cosec\theta = \frac{5}{3}$ and $P(\theta)$ lies in the second quadrant, then find the values of other trigonometric functions.
2. Find the values of all trigonometric functions, if $sin\theta = -\frac{2\sqrt{6}}{5}$ and $\theta \in \left(\pi, \frac{3\pi}{2}\right)$
3. If $cos\theta = \frac{-1}{2}$, $\pi < \theta < \frac{3\pi}{2}$, find the value of $5\,tan^2\theta - 6\,cosec^2\theta$
4. If $sin\theta = \frac{3}{5}$, $tan\alpha = \frac{1}{2}$ and $\frac{\pi}{2} < \theta < \pi$, $\pi < \alpha < \frac{3\pi}{2}$, find $4\,tan\theta - \sqrt{5}\,sec\alpha$.
5. If $sec\theta + tan\theta = p$, obtain the value of $sec\theta$, $tan\theta$ and $sin\theta$ in terms of p.
6. If $sin\theta = \frac{4}{5}$, find the value of $\frac{5cos\theta + 4cosec\theta + 3tan\theta}{4cot\theta + 3sec\theta + 5sin\theta}$. $\frac{\pi}{2} < \theta < \pi$.
7. Prove that : $\sqrt{\frac{1 - sin\theta}{1 + sin\theta}} = \begin{cases} sec\theta - tan\theta, & \text{if } 0 < \theta < \frac{\pi}{2} \\ -sec\theta + tan\theta, & \text{if } \frac{\pi}{2} < \theta < \pi \end{cases}$

*

Miscellaneous Examples

Example 25 : If $cos\alpha + sin\alpha = \sqrt{2}\,cos\alpha$, then find $cos\alpha - sin\alpha$. $(0 < \alpha < \frac{\pi}{4})$

Solution : $cos\alpha + sin\alpha = \sqrt{2}\,cos\alpha$

$\therefore \quad (cos\alpha + sin\alpha)^2 = 2cos^2\alpha$

$\therefore \quad cos^2\alpha + 2sin\alpha\,cos\alpha + sin^2\alpha = 2cos^2\alpha$

$\therefore \quad 1 + 2sin\alpha\,cos\alpha = 2(1 - sin^2\alpha)$

$\therefore \quad 2sin^2\alpha = 1 - 2sin\alpha\,cos\alpha$

$= sin^2\alpha + cos^2\alpha - 2sin\alpha\,cos\alpha$

$= (cos\alpha - sin\alpha)^2$

Now. $sin\alpha > 0$ for $0 < \alpha < \frac{\pi}{4}$ and $cos\alpha > sin\alpha$ for $0 < \alpha < \frac{\pi}{4}$

$\therefore \quad \sqrt{2}\,sin\alpha = cos\alpha - sin\alpha$

$\therefore \quad cos\alpha - sin\alpha = \sqrt{2}\,sin\alpha$

Another method : we have $cos\alpha + sin\alpha = \sqrt{2}\,cos\alpha$

$\therefore \quad sin\alpha = (\sqrt{2} - 1)\,cos\alpha$

$\therefore \quad cos\alpha = \frac{sin\alpha}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$

$\therefore \quad \cos\alpha = (\sqrt{2} + 1)\sin\alpha \qquad ((\sqrt{2} - 1)(\sqrt{2} + 1) = (\sqrt{2})^2 - 1 = 1)$

$\therefore \quad \cos\alpha = \sqrt{2}\sin\alpha + \sin\alpha$

$\therefore \quad \cos\alpha - \sin\alpha = \sqrt{2}\sin\alpha$

Example 26 : If $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$

Prove : (a) $\sin^4\alpha + \sin^4\beta = 2\sin^2\alpha\ \sin^2\beta$ (b) $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = 1$

Solution : We have, $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$

$\therefore \quad \dfrac{(1-\sin^2\alpha)^2}{1-\sin^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$

Let $\sin^2\alpha = m$ and $\sin^2\beta = n$

$\therefore \quad \dfrac{(1-m)^2}{1-n} + \dfrac{m^2}{n} = 1$

$\therefore \quad n(1 - m)^2 + m^2(1 - n) = n(1 - n)$

$\therefore \quad n(1 - 2m + m^2) + m^2(1 - n) = n - n^2$

$\therefore \quad n - 2mn + m^2n + m^2 - m^2n = n - n^2$

$\therefore \quad n^2 - 2mn + m^2 = 0$

$\therefore \quad (n - m)^2 = 0$

$\therefore \quad n = m$

$\therefore \quad \sin^2\alpha = \sin^2\beta \qquad$ (i)

$\therefore \quad 1 - \cos^2\alpha = 1 - \cos^2\beta$

$\therefore \quad \cos^2\alpha = \cos^2\beta \qquad$ (ii)

(a) $\sin^4\alpha + \sin^4\beta = (\sin^2\alpha - \sin^2\beta)^2 + 2\sin^2\alpha\ \sin^2\beta \qquad (a^2 + b^2 = (a - b)^2 + 2ab)$

$= (\sin^2\alpha - \sin^2\alpha)^2 + 2\sin^2\alpha\ \sin^2\beta$

$= 2\sin^2\alpha\ \sin^2\beta$

(b) $\dfrac{\cos^4\beta}{\cos^2\alpha} + \dfrac{\sin^4\beta}{\sin^2\alpha} = \dfrac{\cos^4\beta}{\cos^2\beta} + \dfrac{\sin^4\beta}{\sin^2\beta} \qquad$ (i) and (ii)

$= \cos^2\beta + \sin^2\beta = 1$

Example 27 : If $\tan^2\theta = 1 - a^2$, prove that $\sec\theta + \tan^3\theta\ \mathrm{cosec}\theta = (2 - a^2)^{\frac{3}{2}}$.

$(0 < \theta < \frac{\pi}{2})$

Solution : L.H.S. $= \sec\theta + \tan^3\theta\ \mathrm{cosec}\theta$

$= \dfrac{1}{\cos\theta} + \dfrac{\sin^3\theta}{\cos^3\theta} \times \dfrac{1}{\sin\theta}$

$= \frac{1}{cos\theta} + \frac{sin^2\theta}{cos^3\theta}$

$= \frac{cos^2\theta + sin^2\theta}{cos^3\theta}$

$= \frac{1}{cos^3\theta}$

$= sec^3\theta$

$= (sec^2\theta)^{\frac{3}{2}}$ $\left(0 < \theta < \frac{\pi}{2}\right)$

$= (1 + tan^2\theta)^{\frac{3}{2}}$

$= (1 + 1 - a^2)^{\frac{3}{2}}$

$= (2 - a^2)^{\frac{3}{2}}$ = R.H.S.

Example 28 : If x is any non-zero real number, then show that $cos\theta$ and $sin\theta$ can never be equal to $x + \frac{1}{x}$.

Solution : As $x \in R - \{0\}$, we have two cases.

Case 1 : $x > 0$

We have $x + \frac{1}{x} = (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x} \times \frac{1}{\sqrt{x}} + 2\sqrt{x} \times \frac{1}{\sqrt{x}}$

$= \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2$

$\therefore \quad x + \frac{1}{x} \geq 2$ $\left(\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0\right)$

Case 2 : $x < 0$. Let $x = -y$. Then $y > 0$

$\therefore \quad x + \frac{1}{x} = -y - \frac{1}{y} = -\left(y + \frac{1}{y}\right)$

As $y > 0$, $y + \frac{1}{y} \geq 2$ from case 1.

$\therefore \quad \left(y + \frac{1}{y}\right) \geq 2$

$-\left(y + \frac{1}{y}\right) \leq -2$

$\therefore \quad \left(x + \frac{1}{x}\right) \leq -2$

$\therefore \quad \left(x + \frac{1}{x}\right) \geq 2$ for $x > 0$ and $\left(x + \frac{1}{x}\right) \leq -2$ for $x < 0$

But $-1 \leq sin\theta \leq 1$ and $-1 \leq cos\theta \leq 1$ for all $\theta \in R$.

Hence, $sin\theta$ and $cos\theta$ can not be equal to $x + \frac{1}{x}$ for any non-zero x.

Exercise 4

1. If $a, b, c \in R^+$ and $a \neq b$, $b \neq c$, $c \neq a$, then prove that $sec\theta = \frac{ab + bc + ca}{a^2 + b^2 + c^2}$ is not possible for any $\theta \in R$.

2. Prove : $\frac{sin\alpha}{cos\alpha + sin\beta} + \frac{sin\beta}{cos\beta - sin\alpha} = \frac{sin\alpha}{cos\alpha - sin\beta} + \frac{sin\beta}{cos\beta + sin\alpha}$

3. If $f(n) = cos^n\theta + sin^n\theta$, then prove that $2f(6) - 3f(4) + 1 = 0$

4. If $m\, cos\alpha - n\, sin\alpha = p$, then prove that $m\, sin\alpha + n\, cos\alpha = \pm\sqrt{m^2 + n^2 - p^2}$.

5. If $a\, cos^3x + 3a\, cos\, x\, sin^2 x = m$ and $a\, sin^3x + 3a\, cos^2x\, sin\, x = n$, then prove that $(m + n)^{\frac{2}{3}} + (m - n)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$

6. If $sin\theta + cos\theta = m$, then prove that $sin^6\theta + cos^6\theta = \frac{4 - 3(m^2 - 1)^2}{4}$

7. $\frac{1 + sec^2A\, cot^2B}{1 + sec^2C\, cot^2B} = \frac{1 + tan^2A\, cos^2B}{1 + tan^2C\, cos^2B}$

8. Prove that $\frac{2 - 3sin\theta + sin^3\theta}{sin\theta + 2} = 2sin\theta\,(sin\theta - 1) + cos^2\theta$

9. Prove that $cot^2x - cos^2x = cos^2x\, cot^2x$. Hence deduce that $cot^2x \geq cos^2x$.

10. If $0 < \theta < \frac{\pi}{2}$, then show that $sin\theta + cos\theta + tan\theta + cot\theta > sec\theta + cosec\theta$

11. Prove that $2sec^2\theta - sec^4\theta - 2cosec^2\theta + cosec^4\theta = \frac{1 - tan^8\theta}{tan^4\theta}$

12. Prove that $\frac{tan^2\theta\,(cosec\,\theta - 1)}{1 + cos\theta} = \frac{(1 - cos\theta)cosec^2\theta}{cosec\,\theta + 1}$

13. If $a^2\, sec^2\alpha - b^2\, tan^2\alpha = c^2$, prove $sin^2\alpha = \frac{c^2 - a^2}{c^2 - b^2}$

14. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If $sin\theta + cosec\theta = 2$, then $sin^6\theta + cosec^6\theta$ is... ☐

(a) 1 (b) 64 (c) 2 (d) 16

(2) If $f(x) = cos^2x + sec^2x$, then ☐

(a) $f(x) < 1$ (b) $f(x) = 1$ (c) $0 < f(x) < 1$ (d) $f(x) \geq 2$

(3) Which of the following is not correct for some $\theta \in R$? ☐

(a) $sin\theta = -\frac{1}{5}$ (b) $cos\theta = 1$ (c) $sec\theta = \frac{1}{2}$ (d) $tan\theta = 40$

(4) If $tan\theta = 3$ and $P(\theta)$ lies in the third quadrant, the value of $sin\theta$ is ☐

(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{-1}{\sqrt{10}}$ (c) $\frac{-3}{\sqrt{10}}$ (d) $\frac{3}{\sqrt{10}}$

(5) Which of the following is correct ?

(a) $sin\ 1^{o} > sin\ 1$ (b) $sin\ 1^{o} < sin\ 1$

(c) $sin\ 1^{o} = sin\ 1$ (d) $sin\ 1^{o} = \frac{\pi}{180}\ sin\ 1$

(6) The length of an arc of a circle of radius 28 *cm* that subtends an angle of measure 45° at the centre is...

(a) 12 *cm* (b) 16 *cm* (c) 22 *cm* (d) 24 *cm*

(7) The angle between the minute hand and the hour hand of a clock at 8:30 has degree measure ......

(a) 80° (b) 75° (c) 60° (d) 105°

(8) A circular wire of radius 7 *cm* is cut and bent again into an arc of a circle of radius 12 *cm*. The angle subtended by the arc at the centre has degree measure ......

(a) 50° (b) 210° (c) 100° (d) 60°

(9) The radius of the circle whose arc has length 15π *cm* and which subtends an angle having radian measure $\frac{3\pi}{4}$ at the centre is...

(a) 10 *cm* (b) 20 *cm* (c) $11\frac{1}{4}$ *cm* (d) $22\frac{1}{2}$ *cm*

(10) If $tan\theta = x - \frac{1}{4x}$, then $sec\theta - tan\theta$ is...

(a) $-2x$ or $\frac{1}{2x}$ (b) $\frac{-1}{2x}$ or $2x$ (c) $2x$ (d) $\frac{-1}{2x}$

(11) If $\frac{cosA}{3} = \frac{cosB}{4} = \frac{1}{5}$, $\frac{-\pi}{2} < A < 0$, $\frac{-\pi}{2} < B < 0$, then the value of $2sinA + 4sinB$ is...

(a) –4 (b) 0 (c) 2 (d) 4

(12) If $\pi < \theta < 2\pi$, then $\sqrt{\frac{1+cos\theta}{1-cos\theta}}$ is equal to...

(a) $cosec\theta + cot\theta$ (b) $cosec\theta - cot\theta$

(c) $-cosec\theta + cot\theta$ (d) $-cosec\theta - cot\theta$

(13) If $cosec\theta - cot\theta = 2$, $\frac{\pi}{2} < \theta < \pi$, then $cos\theta$ is...

(a) $\frac{-3}{5}$ (b) $\frac{-5}{3}$ (c) $\frac{5}{3}$ (d) $\frac{3}{5}$

(14) If $sec\theta = m$, $tan\theta = n$, then $\frac{1}{m}\left\{(m+n)+\frac{1}{m+n}\right\} =$ ......

(a) 2 (b) mn (c) $2m$ (d) $2n$

(15) The value of the expression $\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta$ is ......

(a) 0 (b) 1 (c) 2 (d) greater than 3

(16) The expression $\tan^2\alpha + \cot^2\alpha$ is...

(a) ≥ -2 (b) ≥ 2 (c) ≤ 2 (d) ≤ -2

(17) If $\mathrm{cosec}\,\theta + \cot\theta = \frac{5}{2}$, then the value of $\tan\theta$

(a) $\frac{14}{24}$ (b) $\frac{20}{21}$ (c) $\frac{21}{20}$ (d) $\frac{15}{16}$

(18) $1 - \frac{\sin^2\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} - \frac{\sin\theta}{1-\cos\theta}$ equals...

(a) 0 (b) 1 (c) $\sin\theta$ (d) $\cos\theta$

(19) If $\sec\theta = \sqrt{2}$, $\frac{3\pi}{2} < \theta < 2\pi$, then $\frac{1+\tan\theta+\mathrm{cosec}\,\theta}{1+\cot\theta-\mathrm{cosec}\,\theta}$ is...

(a) $-\sqrt{2}$ (b) -1 (c) $\frac{1}{\sqrt{2}}$ (d) 0

(20) If $p = a\cos^2\theta\sin\theta$ and $q = a\sin^2\theta\cos\theta$, then $\frac{(p^2+q^2)^3}{p^2q^2}$ is...

(a) $\frac{1}{a}$ (b) a^2 (c) a (d) a^3

(21) If $\sec A - \tan A = \frac{a+1}{a-1}$, then $\cos A$ is...

(a) $\frac{2a}{a^2-1}$ (b) $\frac{2a}{a^2+1}$ (c) $\frac{a^2+1}{a^2-1}$ (d) $\frac{a^2-1}{a^2+1}$

Summary

1. Trigonometric point, Trigonometric point function, Period
2. *sine* function, *cosine* function, their zeroes and range, fundamental identity
3. Other trigonometric functions, their ranges, identities
4. Increasing and decreasing functions
5. Degree measure and radian measure
6. Even and odd functions
7. Right angled triangle and related trigonometric functions
8. Values of trigonometric functions in each quadrant.

Chapter 5

SPECIAL VALUES AND GRAPHS OF TRIGONOMETRIC FUNCTIONS

5.1 Introduction

We have already studied about trigonometric functions, their domains, range, zeroes and periods in the previous chapter. Now we shall obtain values of these functions at some special points and draw graphs of these trigonometric functions.

5.2 Values of T-functions at T-points on the Axes

We know that $\forall\theta \in R$, there is a unique point $P(\theta)$ on the unit circle. Unit circle intersects X-axis at A(1, 0) and A'(−1, 0) and Y-axis at B(0, 1) and B'(0, −1) respectively. We also know that x-coordinate of $P(\theta)$ is $cos\theta$ and y-coordinate of $P(\theta)$ is $sin\theta$.

Corresponding to real number 0, there is unique T-point P(0) = A(1, 0) on unit circle.

We have $cos0 = 1$ and $sin0 = 0$.

Corresponding to real number $\frac{\pi}{2}$, there is unique T-point $P\left(\frac{\pi}{2}\right)$ = B(0, 1) on unit circle.

$cos\frac{\pi}{2} = 0$, $sin\frac{\pi}{2} = 1$.

Similarly, $P(\pi)$ is A'(−1, 0). $cos\pi = -1$, $sin\pi = 0$.

Similarly, $P\left(\frac{3\pi}{2}\right)$ is B'(0, −1). $cos\frac{3\pi}{2} = 0$, $sin\frac{3\pi}{2} = -1$.

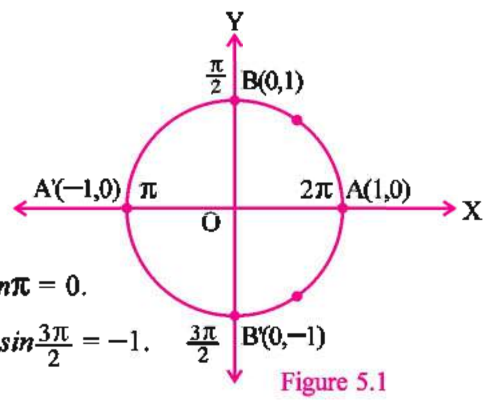


Figure 5.1

We know that A(1, 0) is $P(2\pi)$.

So, $cos\ 2\pi = 1$, $sin\ 2\pi = 0$. We tabulate these values as follows :

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
cos	1	0	−1	0	1
sin	0	1	0	−1	0
tan	0	$\frac{\pi}{2}$ not in domain	0	$\frac{3\pi}{2}$ not in domain	0
cot	0 not in domain	0	π not in domain	0	2π not in domain
sec	1	$\frac{\pi}{2}$ not in domain	−1	$\frac{3\pi}{2}$ not in domain	1
cosec	0 not in domain	1	π not in domain	−1	2π not in domain

5.3 Coordinates of $P\left(\frac{\pi}{4}\right)$:

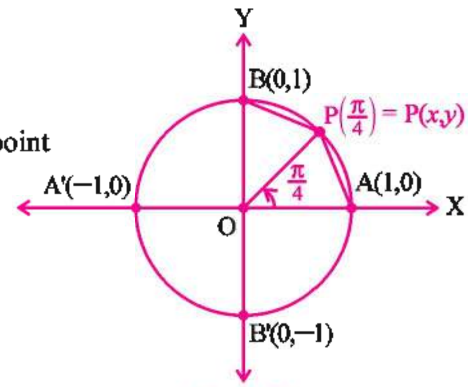


Figure 5.2

Let trigonometric point $P\left(\frac{\pi}{4}\right)$ be $P(x, y)$.

Now length of minor $\overparen{AB}$ is $\frac{\pi}{2}$. If P is the point on $\overparen{AB}$ such that $\overparen{AP} \cong \overparen{PB}$, then $l(\overparen{AP}) = \frac{\pi}{4}$.

In a given circle, if arcs are congruent, then corresponding chords are also congruent.

$\therefore$ AP = PB

$\therefore$ $AP^2 = PB^2$

Again A is (1, 0), P is (x, y) and B is (0, 1).

$\therefore$ $(x - 1)^2 + (y - 0)^2 = (x - 0)^2 + (y - 1)^2$

$\therefore$ $x^2 - 2x + 1 + y^2 = x^2 + y^2 - 2y + 1$

$\therefore$ $-2x = -2y$

$\therefore$ $x = y$ (i)

Now, $P(x, y)$ is on the unit circle.

$\therefore$ $x^2 + y^2 = 1$

$\therefore$ by (i), $x^2 + x^2 = 1$

$\therefore$ $2x^2 = 1$

$\therefore$ $x^2 = \frac{1}{2}$

$\therefore$ $x = \pm\frac{1}{\sqrt{2}}$

Now, as $P\left(\frac{\pi}{4}\right)$ is in the first quadrant, $x > 0$, $y > 0$.

$\therefore$ $x = \frac{1}{\sqrt{2}}$

$\therefore$ From (i), $x = y = \frac{1}{\sqrt{2}}$

$\therefore$ By definition of *cos* and *sin* functions, coordinates of $P\left(\frac{\pi}{4}\right)$ are

$$(x, y) = \left(cos\frac{\pi}{4}, sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Thus $cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

$\therefore$ $tan\frac{\pi}{4} = 1$, $cot\frac{\pi}{4} = 1$, $sec\frac{\pi}{4} = \sqrt{2}$, $cosec\frac{\pi}{4} = \sqrt{2}$.

5.4 Coordinates of $P\left(\frac{\pi}{3}\right)$:

Let the coordinates of $P\left(\frac{\pi}{3}\right)$ be (x, y).

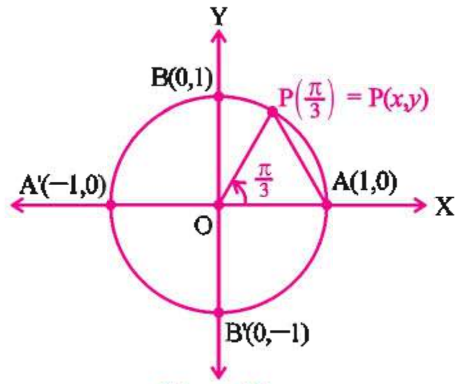


Figure 5.3

Length of minor $\widehat{AP}$ is $\frac{\pi}{3}$.

$\therefore \quad m\angle AOP = \frac{\pi}{3} = 60^\circ$

In ΔOAP, OA = OP **(Radii)**

$\therefore \quad m\angle OPA = m\angle OAP$ **(i)**

As $m\angle AOP = 60^\circ$,

$\quad m\angle OPA + m\angle OAP = 120^\circ$

$\therefore \quad$ From (i), $m\angle OPA = m\angle OAP = 60^\circ$

$\therefore \quad \Delta OAP$ is an equilateral triangle.

Again OA = OP = 1 **(Radii of unit circle)**

$\therefore \quad AP = 1$

$\therefore \quad AP^2 = 1$

Now, since P(x, y) and A(1, 0),

$\therefore \quad (x - 1)^2 + (y - 0)^2 = 1$

$\therefore \quad x^2 - 2x + 1 + y^2 = 1$

but $x^2 + y^2 = 1$

$\therefore \quad 2x = 1$

$\therefore \quad x = \frac{1}{2}$

Again $x^2 + y^2 = 1$

$\therefore \quad \frac{1}{4} + y^2 = 1$

$\therefore \quad y^2 = \frac{3}{4}$

$\therefore \quad y = \frac{\sqrt{3}}{2}$ **$\left(P\left(\frac{\pi}{3}\right)\right.$ is in the first quadrant, $\left.y > 0\right)$**

$\therefore \quad$ Coordinates of $P\left(\frac{\pi}{3}\right)$ are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$\therefore \quad \cos\frac{\pi}{3} = \frac{1}{2}, \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. So, $\tan\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$.

$\therefore \quad \sec\frac{\pi}{3} = 2, \operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}, \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$.

5.5 Coordinates of $P\left(\frac{\pi}{6}\right)$:

Suppose the coordinates of $P\left(\frac{\pi}{6}\right)$ are (x, y).

Length of minor $\widehat{AP}$ is $\frac{\pi}{6}$.

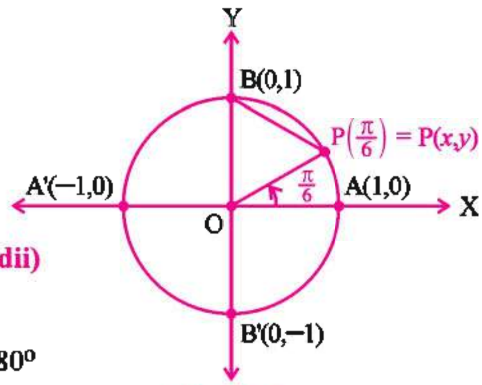


Figure 5.4

$l(\overset{\frown}{AP}) = \frac{\pi}{6}$. **(Minor arc)**

$m\angle AOP = \frac{\pi}{6} = 30^{o}$

Since P is in the interior of $\angle AOB$,

$m\angle POB = 90^{o} - 30^{o} = 60^{o}$

Now, In ΔOPB, OB = OP **(Radii)**

Hence $\angle OBP \cong \angle OPB$ and

$m\angle OBP + m\angle OPB + m\angle POB = 180^{o}$

$m\angle OBP + m\angle OPB = 120^{o}$

($m\angle POB = 60^{o}$)

As $\angle OBP \cong \angle OPB$,

$m\angle OBP = m\angle OPB = 60^{o}$

$\angle OBP \cong \angle OPB \cong \angle POB$

$\therefore$ ΔPOB is an equilateral triangle.

OP = OB = PB = 1

$\therefore$ $PB^2 = 1$

$\therefore$ $(x - 0)^2 + (y - 1)^2 = 1$ **(P(x, y) and B(0, 1))**

$\therefore$ $x^2 + y^2 - 2y + 1 = 1$

but $x^2 + y^2 = 1$ as P(x, y) is on unit circle.

$\therefore$ $2y = 1$

$\therefore$ $y = \frac{1}{2}$

Also $x^2 + y^2 = 1$

$\therefore$ $x^2 + \frac{1}{4} = 1$.

So, $x = \frac{\sqrt{3}}{2}$ **$\left(P\left(\frac{\pi}{6}\right)\right.$ is in the first quadrant, so $\left.x > 0\right)$**

Hence, by the definition of *cos* and *sin*,

$$(x, y) = \left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$\therefore$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{6} = \frac{1}{2}$. Hence $\tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$.

$\therefore$ $\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$, $\operatorname{cosec}\frac{\pi}{6} = 2$, $\cot\frac{\pi}{6} = \sqrt{3}$.

Example 1 : Evaluate : $3\,cos^2\frac{\pi}{4} - sec\frac{\pi}{3} + 5\,tan^2\frac{\pi}{3}$.

Solution : We know that $cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $sec\frac{\pi}{3} = 2$, $tan\frac{\pi}{3} = \sqrt{3}$

$$3cos^2\frac{\pi}{4} - sec\frac{\pi}{3} + 5\,tan^2\frac{\pi}{3} = 3\left(\frac{1}{\sqrt{2}}\right)^2 - 2 + 5(\sqrt{3})^2$$

$$= 3 \times \frac{1}{2} - 2 + 5 \times 3$$

$$= \frac{3}{2} - 2 + 15 = \frac{29}{2}$$

Example 2 : Evaluate : $4\,tan^2\frac{\pi}{6} - 5\,cosec^2\frac{\pi}{4} - \frac{1}{3}\,sin\frac{\pi}{6}$.

Solution : We know that $tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, $cosec\frac{\pi}{4} = \sqrt{2}$, $sin\frac{\pi}{6} = \frac{1}{2}$.

$$4\,tan^2\frac{\pi}{6} - 5\,cosec^2\frac{\pi}{4} - \frac{1}{3}\,sin\frac{\pi}{6} = 4 \times \left(\frac{1}{\sqrt{3}}\right)^2 - 5(\sqrt{2})^2 - \frac{1}{3}\left(\frac{1}{2}\right)$$

$$= \frac{4}{3} - 10 - \frac{1}{6}$$

$$= \frac{8 - 60 - 1}{6} = \frac{-53}{6}$$

Example 3 : Prove that : $\frac{4}{3}\,cot^2\frac{\pi}{6} + 3\,sin^2\frac{\pi}{3} - 2\,cosec^2\frac{\pi}{3} - \frac{3}{4}\,tan^2\frac{\pi}{6} = 3\frac{1}{3}$.

Solution : We know that $cot\frac{\pi}{6} = \sqrt{3}$, $sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $cosec\frac{\pi}{3} = \frac{2}{\sqrt{3}}$, $tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

L.H.S. $= \frac{4}{3}\,cot^2\frac{\pi}{6} + 3\,sin^2\frac{\pi}{3} - 2\,cosec^2\frac{\pi}{3} - \frac{3}{4}\,tan^2\frac{\pi}{6}$

$$= \frac{4}{3}(\sqrt{3})^2 + 3\left(\frac{\sqrt{3}}{2}\right)^2 - 2\left(\frac{2}{\sqrt{3}}\right)^2 - \frac{3}{4}\left(\frac{1}{\sqrt{3}}\right)^2$$

$$= \frac{4}{3} \times 3 + 3 \times \frac{3}{4} - 2 \times \frac{4}{3} - \frac{3}{4} \times \frac{1}{3}$$

$$= 4 + \frac{9}{4} - \frac{8}{3} - \frac{1}{4}$$

$$= \frac{48 + 27 - 32 - 3}{12}$$

$$= \frac{40}{12} = \frac{10}{3} = 3\frac{1}{3} = \text{R.H.S.}$$

Example 4 : Prove that $\dfrac{tan\frac{\pi}{4} - tan\frac{\pi}{6}}{1 + tan\frac{\pi}{4}\,tan\frac{\pi}{6}} = 2 - \sqrt{3}$

Solution : We know that $tan\frac{\pi}{4} = 1$, $tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

$$\text{L.H.S.} = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= \frac{3 - 2\sqrt{3} + 1}{3 - 1}$$

$$= \frac{2(2 - \sqrt{3})}{2} = 2 - \sqrt{3} = \text{R.H.S.}$$

EXERCISE 5.1

1. Evaluate : $\sec\frac{\pi}{6}\ \tan\frac{\pi}{3} + \sin\frac{\pi}{4}\ \operatorname{cosec}\frac{\pi}{4} + \cos\frac{\pi}{6}\ \cot\frac{\pi}{3}$.
2. Evaluate : $\cot^2\frac{\pi}{6} + \operatorname{cosec}\frac{\pi}{6} + 3\tan^2\frac{\pi}{6}$.
3. Evaluate : $2\sin^2\frac{\pi}{4} + 2\cos^2\frac{\pi}{4} + \sec^2\frac{\pi}{3}$.
4. Prove that : $\left(3\cos\frac{\pi}{3}\ \sec\frac{\pi}{3} - 4\sin\frac{\pi}{6}\ \tan\frac{\pi}{4}\right)\cos 2\pi = 1$.
5. Evaluate : $\left(\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right)\left(\sin^2\frac{\pi}{6} - \sin\frac{\pi}{6}\cos\frac{\pi}{6} + \cos^2\frac{\pi}{6}\right)$
6. Evaluate : $\dfrac{5\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{4} - 4\tan\frac{\pi}{6}}{2\sin\frac{\pi}{6}\cos\frac{\pi}{6} + \tan\frac{\pi}{4}}$
7. Prove that : $\cos\frac{\pi}{3}\ \cos\frac{\pi}{4} + \sin\frac{\pi}{3}\ \sin\frac{\pi}{4} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

*

5.6 Graphs of Trigonometric Functions

Graph of $y = \sin x$

Values of $\sin x$ for some values of x are given in the following table :

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	0
	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

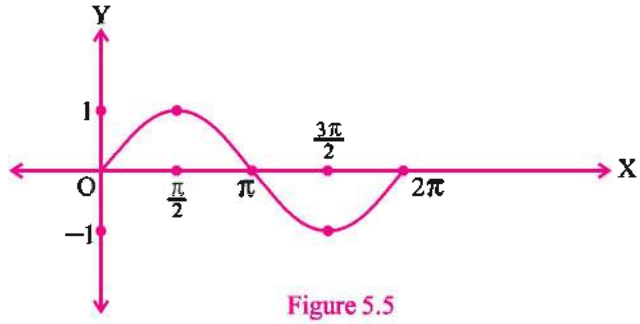


Figure 5.5

Since *sin* is a periodic function with period 2π, we will draw the graph of $y = \sin x$ in the interval [0, 2π] (Fig. 5.5). Once it is drawn in one such interval, it can be repeated at the interval of length 2π. (Fig. 5.6)

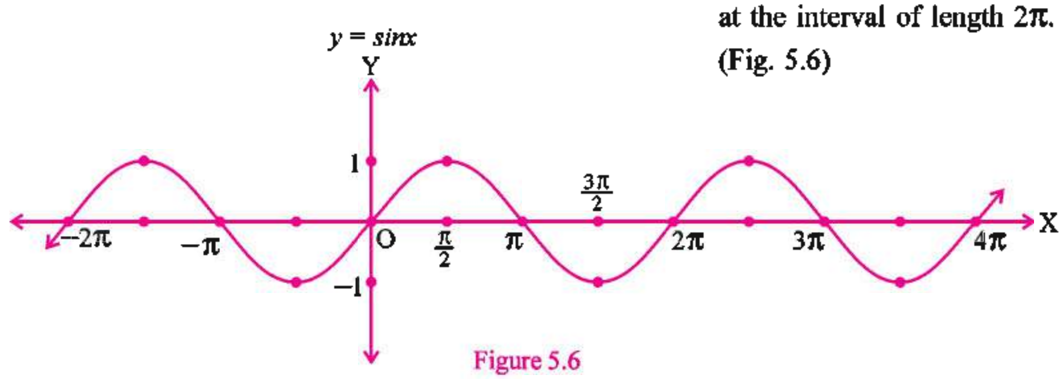


Figure 5.6

It is clear from the graph that,

(1) The graph of $y = \sin x$ intersects the X-axis at many points, say 0, ±π, ±2π, ±3π,... At these points $\sin x = 0$.

(2) The graph of $y = \sin x$ intersects the X-axis at points 0, ±π, ±2π, ±3π,... The set of zeroes of *sin* is $\{k\pi \mid k \in Z\}$.

(3) The maximum and minimum values of $y = \sin x$ are 1 and −1 respectively and *sin* assumes all values between −1 and 1.

(4) In the first quadrant, that is in $\left(0, \frac{\pi}{2}\right)$, the graph 'is ascending', so the function is increasing. Similarly it is decreasing in second and third quadrants and increasing in the fourth quadrant.

(5) In the restricted domains such as $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$,... *sin* is one-one.

(6) The graph of $y = \sin x$ repeats after an interval of length 2π because 2π is the period of *sin*.

If $y = f(x)$ is a periodic function with period T and amplitude k, then it is sufficient to draw its graph in an interval of length T only. Because once it is drawn in one such interval, it can be easily repeated over the intervals of length T. The amplitude of a function is defined as the greatest numerical value which it can attain.

If period of $y = f(x)$ is 2π and amplitude is k, then period of $y = c \cdot f(ax + b)$, for $a > 0$ is $\frac{2\pi}{a}$ and amplitude is $| c | k$. Now using this discussion we shall draw the graphs of $c \sin ax$, $c \cos ax$, $c \tan ax$.

Graph of $g(x) = c \sin ax$ $(a > 0)$

First, we draw the graph of $y = \sin x$ and mark the numbers where it crosses X-axis. Then if these points are P(x), we divide x by a. The points of intersection of graph of $y = c \sin ax$ with X-axis are $0, \frac{\pi}{a}, \frac{2\pi}{a}, ...$ etc. Its period is $\frac{2\pi}{a}$. The points on Y-axis -1, 1 are replaced by $-| c |$ and $| c |$. The highest and lowest point on the graph have x-coordinates $\frac{\pi}{2a}$ and $\frac{3\pi}{2a}$ in $\left[0, \frac{2\pi}{a}\right]$. The range of this function is $[-| c |, | c |]$. We mark maximum and minimum values of $y = c \sin ax$ as $| c |$ and $-| c |$ respectively on Y-axis. The graph of $y = c \sin ax$ is confined to $[-| c |, | c |]$.

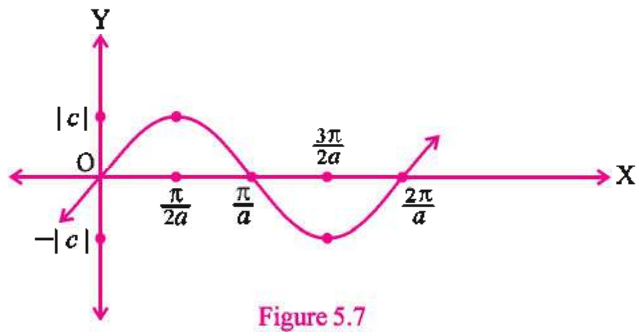


Figure 5.7

Example 5 : Draw the graph of $y = 3\sin 2x$.

Solution : Compare $y = 3 \sin 2x$ with $y = c \sin ax$.

$\therefore$ $a = 2$ and $c = 3$. Its period is $\frac{2\pi}{2} = \pi$ and range is $[-3, 3]$.

Its graph is same as that of $y = \sin x$ with points on X-axis where it intersects X-axis as $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$,... etc replacing π, 2π, 3π,... etc and using the fact that range is $[-3, 3]$, it lies between lines $y = -3$ and $y = 3$.

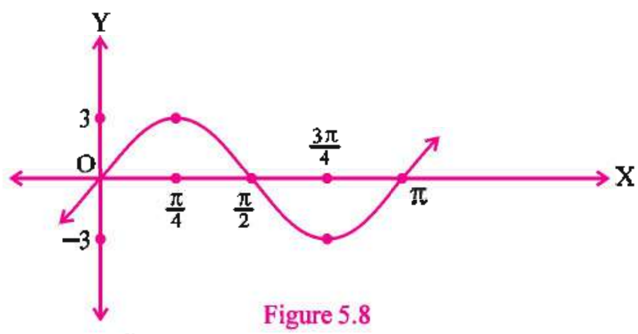


Figure 5.8

Example 6 : Draw the graph of $y = 2\sin \frac{x}{2}$.

Solution : Here $a = \frac{1}{2}$ and $c = 2$

$\therefore$ Period is 4π and range is $[-2, 2]$

Points on X-axis where the graph intersects X-axis are 2π, 4π,...etc (replacing π, 2π, ...etc.)

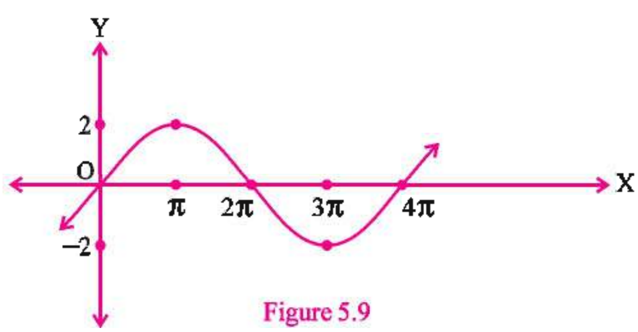


Figure 5.9

Graph of $f(x) = \cos x$, $0 \le x \le 2\pi$

Values of $\cos x$ for some values of x are given in the following table :

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87	1

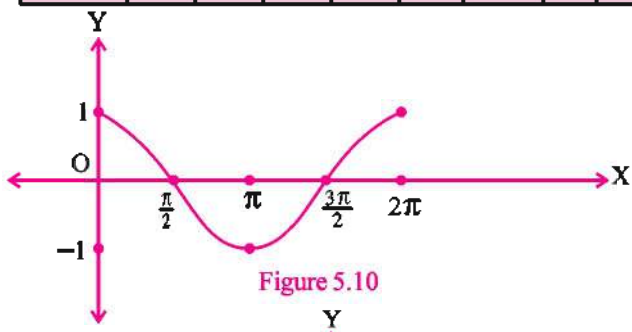


Figure 5.10

cos is also a periodic function with principal period 2π. So after the graph is drawn in an interval of length 2π, it can be repeated over intervals of length 2π. (Fig. 5.11)

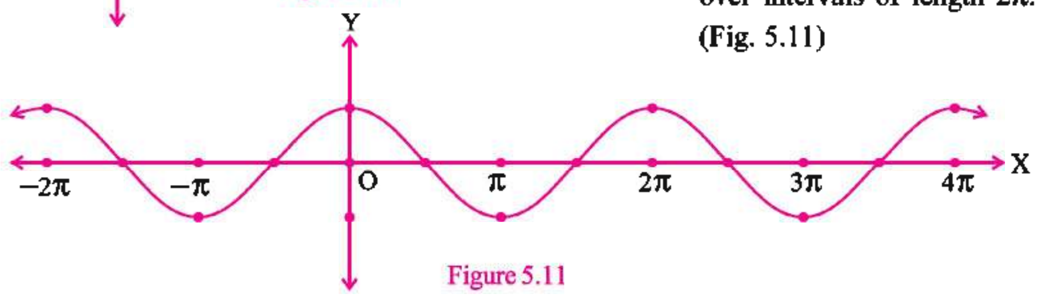


Figure 5.11

It is clear from the graph that :

(1) The graph of $y = \cos x$ intersects X-axis at more than one point, say $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$,...

(2) $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$,... are the points of intersection of the graph of $y = \cos x$ with X-axis. The set of zeroes of *cos* is $\left\{(2k+1)\frac{\pi}{2} \mid k \in Z\right\}$.

(3) The maximum and minimum values of $y = \cos x$ are 1 and -1 respectively and *cos* assumes all values between -1 and 1.

(4) In the first quadrant, that is in $\left(0, \frac{\pi}{2}\right)$, as x increases, the graph descends. So in the first quadrant, *cos* is a decreasing function. It is also seen from the graph that in $\left(\frac{\pi}{2}, \pi\right)$, that is in the second quadrant also, *cos* decreases. But in $\left(\pi, \frac{3\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$ that is in the third and in the fourth quadrant, *cos* is an increasing function and its graph ascends.

(5) In the restricted domains such as $[0, \pi]$, $[\pi, 2\pi]$,... *cos* is one-one.

(6) The graph of $y = \cos x$ repeats after an interval of length 2π, because 2π is the period of $\cos x$.

Graph of $y = c \cos ax$ $(a > 0)$

Draw the graph of $y = \cos x$ and divide the x-coordinates of the points of intersection with X-axis by a. Its period is $\frac{2\pi}{a}$. Range of this function is $[-|c|, |c|]$. We mark maximum and minimum values of $y = c \cos ax$ respectively as $|c|$ and $-|c|$ on Y-axis.

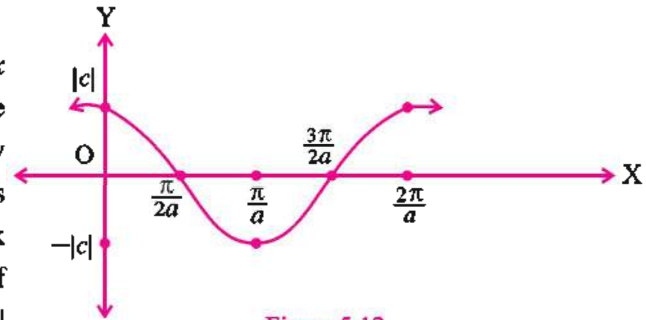


Figure 5.12

Example 7 : Draw the graph of $y = 2 \cos 3x$.

Solution : Compare $y = 2 \cos 3x$ with $y = c \cos ax$.
$a = 3$, $c = 2$

Its principal period is $\frac{2\pi}{3}$ and range is $[-2, 2]$.

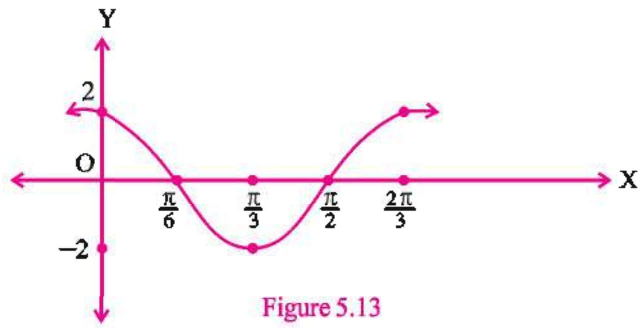


Figure 5.13

Example 8 : Draw the graph of $y = 3 \cos \frac{x}{3}$.

Solution : $a = \frac{1}{3}$, $c = 3$

Period is 6π.

Range is $[-3, 3]$.

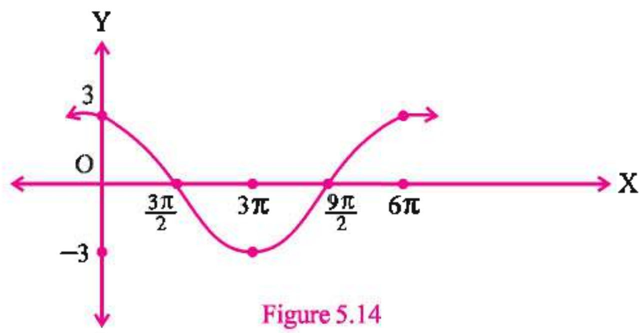


Figure 5.14

The graph of $y = \tan x$, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Values of $\tan x$ for some values of x are given in the following table :

x	$\frac{-\pi}{3}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan x$	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$
	−1.74	−1	−0.57	0	0.57	1	1.74

Since *tan* is a periodic function with principal period π, we draw the graph of $y = tan\, x$ as in figure 5.15 and it is repeated again in every interval of length π.

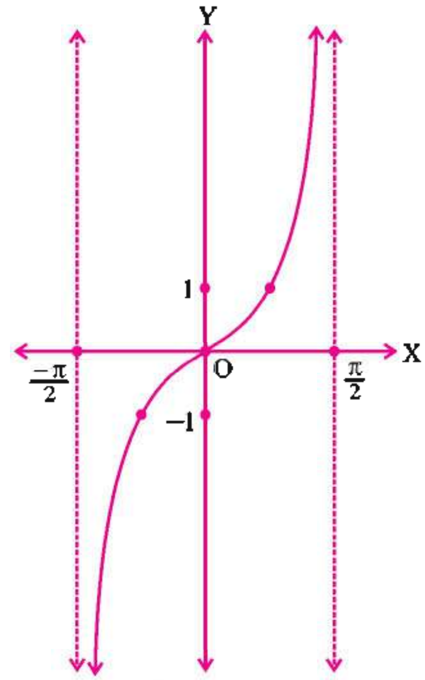


Figure 5.15

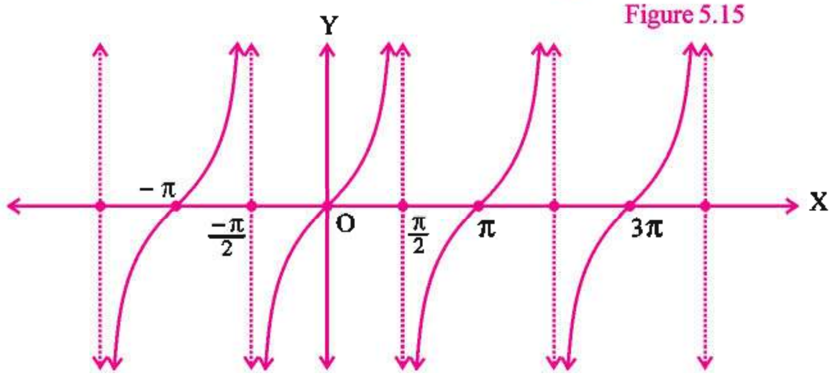


Figure 5.16

We can see the following from the graph :

(1) The graph of $y = tan\, x$, intersects X-axis at more than one point, say, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$,... .

(2) $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$,... are the points of intersection of the graph with X-axis. We see that the set the zeroes of *tan* is $\{k\pi \mid k \in Z\}$.

(3) Range of *tan* is R.

(4) In any quadrant, as x increases, the graph rises. $y = tan\, x$ is an increasing function in every quadrant.

(5) The graph repeats after an interval of π. The principal period of *tan* is π.

(6) *tan* becomes one-one if its domain is confined to $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$,...

Graph of $y = c \tan ax$ $(a > 0)$

First we draw the graph of $y = \tan x$ and mark the numbers where it crosses X-axis. Then we divide these numbers by a. Its principal period is $\frac{\pi}{a}$. The range of $y = \tan x$ is R. So, the range of $y = c \tan ax$ is also R.

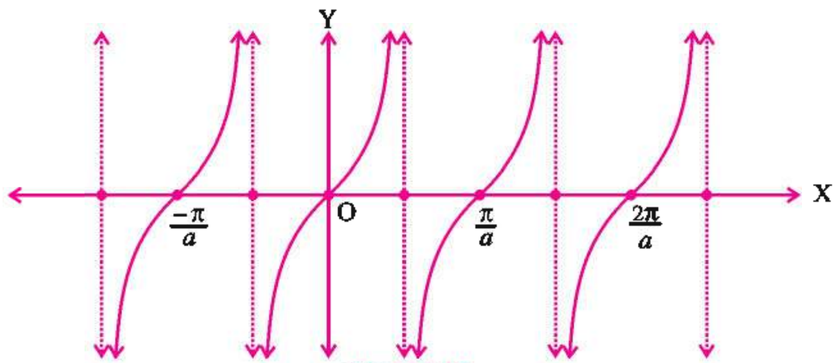


Figure 5.17

Example 9 : Draw the graph of

$$y = 3 \tan 2x,\ x \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right).$$

Solution : $a = 2$, $c = 3$

$\therefore$ Principal period is $\frac{\pi}{2}$,

Range is R.

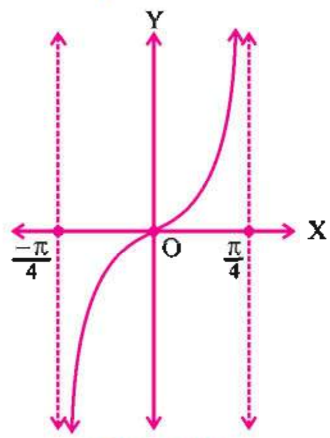


Figure 5.18

Example 10 : Draw the graph of

$$y = \tan \frac{x}{2},\ x \in (-\pi, \pi).$$

Solution : $a = 1$, $c = \frac{1}{2}$

$\therefore$ Principal period is 2π,

Range is R.

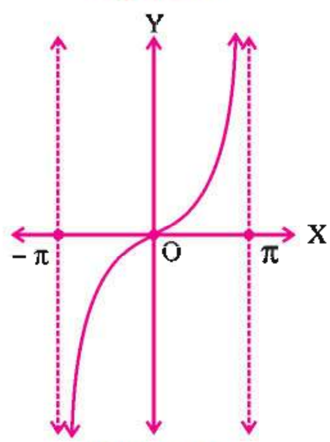


Figure 5.19

EXERCISE 5.2

1. Draw the graph of $y = 2\sin\frac{x}{2}$, $0 \le x \le 6\pi$.
2. Draw the graph of $y = 2\sin 3x$, $0 \le x \le \frac{2\pi}{3}$.
3. Draw the graph of $y = 3\cos\frac{x}{2}$, $0 \le x \le 4\pi$.
4. Draw the graph of $y = \sin 2x$, $0 \le x \le \pi$.
5. Draw the graph of $y = \tan\frac{x}{3}$, $x \in \left(\frac{-3\pi}{2}, \frac{3\pi}{2}\right)$.
6. Draw the graph of $y = 2\tan x$, $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

*

5.7 Angles with General Measure and Their Trigonometric Functions

We already know about the measure of an angle. We have assumed that every angle has a measure and degree measure of an angle is a real number between 0 to 180, that is, between 0 to π radian. But we have defined trigonometric functions over the whole of **R** and not just over the interval $(0, \pi)$.

So now we wish to extend our definition of the measure of an angle from 0 to 180 and allow angles to assume any real number as measure. For this we are using the idea of a ray revolving around another ray.

We will assume anti-clockwise rotation as positive direction of rotation.

Let Q be point on $\overrightarrow{OX}$. We shall consider $\overrightarrow{OQ}$ as a variable ray. $\overrightarrow{OQ}$ rotates from its initial position $\overrightarrow{OA}$ about $\overrightarrow{OX}$. Initially $\overrightarrow{OQ} = \overrightarrow{OA}$. If the ray $\overrightarrow{OQ}$ starts rotating and takes its final position $\overrightarrow{OP}$, then $\angle AOP$ is the angle formed by rotation of $\overrightarrow{OQ}$. If $\overrightarrow{OQ}$ does not rotate at all, then $\overrightarrow{OQ} = \overrightarrow{OA}$. Then we say that $\overrightarrow{OA} \cup \overrightarrow{OQ}$ describes an angle of general measure 0°. Thus, when there is no rotation, $\overrightarrow{OQ}$ coincides with $\overrightarrow{OA}$. $\overrightarrow{OA} \cup \overrightarrow{OQ}$ represents an angle of general measure 0°. If $\overrightarrow{OQ}$ starts rotating from A and without passing through A again takes its final position at $\overrightarrow{OA'}$, then we say that $\overrightarrow{OA} \cup \overrightarrow{OA'}$ makes an angle of general measure 180°.

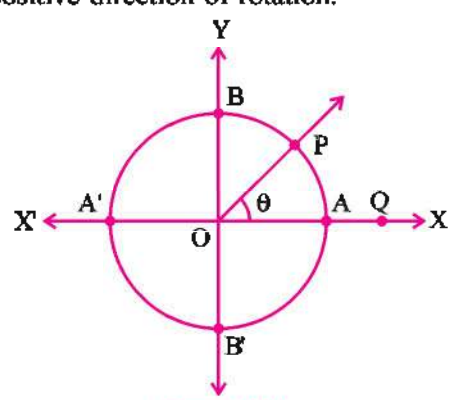


Figure 5.20

If $0 < \theta < 180$ and if $\overrightarrow{OQ}$ rotates anti-clockwise and takes the position $\overrightarrow{OP}$ without passing through A again in the half plane above X-axis, we get $\angle AOP$ as an angle having general measure θ° obtained by rotation of $\overrightarrow{OQ}$.

If $180 < \theta < 360$, then $-180 < \theta - 360 < 0$.

$\therefore \quad 0 < 360 - \theta < 180$

Hence, $\overrightarrow{OQ}$ rotates in the half-plane below X-axis in clockwise direction and without passing through A again takes the position of $\overrightarrow{OP}$. We get $m\angle AOP = 360 - \theta$ and we get angle of general measure θ as shown in 5.21. Thus if $\theta = 210$, $360 - \theta = 360 - 210 = 150$. $\angle AOP$ with general measure 210° is shown in figure 5.21.

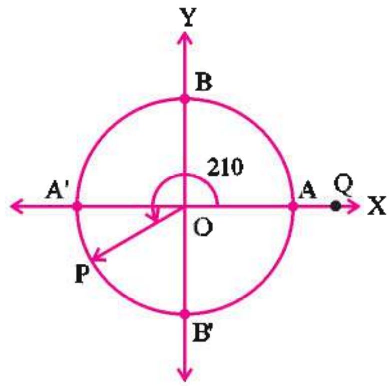


Figure 5.21

If $\theta \notin [0, 360)$ and $\theta > 0$, we can write $\theta = 360n + \alpha$, where $n = \left[\frac{\theta}{360}\right]$. $n \in N$ and $0 \le \alpha < 360$. We can get angle with general measure α as described earlier. n is the number of rotations of $\overrightarrow{OQ}$ before $\overrightarrow{OQ}$ coincides with $\overrightarrow{OP}$ and this rotation is anti-clockwise and $n > 0$.

Example 11 : For $\theta = 760$, describe the angle with general measure θ°.

Solution : $\left[\frac{\theta}{360}\right] = \left[\frac{760}{360}\right] = 2$ and $760 = 360 \cdot 2 + 40$

$\therefore \quad \alpha = 40$

Thus $\overrightarrow{OQ}$ must complete 2 rotations anti-clockwise and then we get $\overrightarrow{OP}$ in upper half plane of $\overleftrightarrow{OA}$, so that $m\angle AOP = 40$. The angle $\angle AOP$ thus, generated by rotation of $\overrightarrow{OQ}$ has general measure 760°.

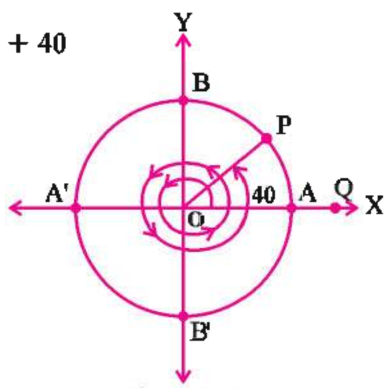


Figure 5.22

Suppose $\theta < 0$. If $-180 < \theta < 0$, then we can find a point on the unit circle below $\overleftrightarrow{AA'}$ such that $m\angle AOP = |\theta| = -\theta$ as $0 < -\theta < 180$.

If $\overrightarrow{OQ}$ rotates clockwise and without passing through A again takes position of $\overrightarrow{OP}$, we get $\angle AOP$ as angle having general measure θ°.

So if $\theta = -60$, then we shall have P in the lower half of the unit circle such that $m\angle AOP = 60$. Thus, $\overrightarrow{OQ}$ after rotating clockwise takes position such that $\angle AOP$ has general degree measure -60.

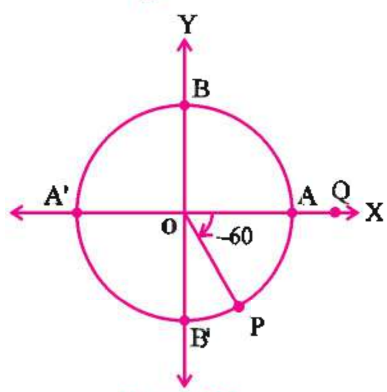


Figure 5.23

If $-360 < \theta < -180$, then $0 < 360 + \theta < 180$.

So we can find P in the upper half-plane of $\overleftrightarrow{AA'}$ on the unit circle such that $m\angle AOP = (360 + \theta)$. Thus, $\overrightarrow{OQ}$ rotates clockwise and takes position of $\overrightarrow{OP}$ without passing through A, so that $m\angle AOP = (360 + \theta)$. $\angle AOP$ is the angle having general degree measure θ.

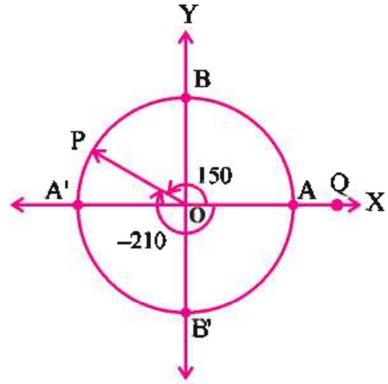


Figure 5.24

If $\theta = -210$, then $360 + \theta = 150$. We get $\angle AOP$ of general measure -210°, as we have shown in figure 5.24.

In general, if $\theta < 0$, then we can take

$|\theta| = 360n + \alpha,\ 0 \le \alpha < 360$

Thus $-\theta = 360n + \alpha$ $\quad (|\theta| = -\theta)$

$\therefore\quad \theta = -360n - \alpha \quad (-360 < -\alpha \le 0)$

Thus, angle of general degree measure θ is angle of general degree measure $-\alpha$ obtained by n complete rotations of $\overrightarrow{OQ}$ in clockwise direction.

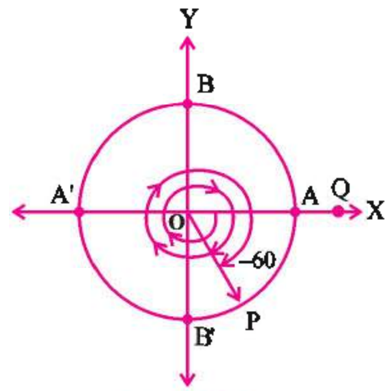


Figure 5.25

Thus if $\theta = -780$, then

$780 = 360 \times 2 + 60$

$\therefore\quad -780 = -360 \times 2 - 60$

Thus, angle of general degree measure -780 is obtained by two complete rotations in clockwise direction and further taking $\angle AOP$ of general degree measure -60.

Example 12 : For $\theta = -1110$, obtain the number of rotations n, α and then draw the angle.

Solution : $\left[\frac{-\theta}{360}\right] = \left[\frac{1110}{360}\right] = 3$

$\therefore\quad -1110 = (-360)3 + \alpha,\ \ \alpha = -30$

$= (-360)3 + (-30)$

Thus, angle of general degree measure -1110 is obtained by three complete rotations in clockwise direction and further taking $\angle AOP$ with general degree measure -30.

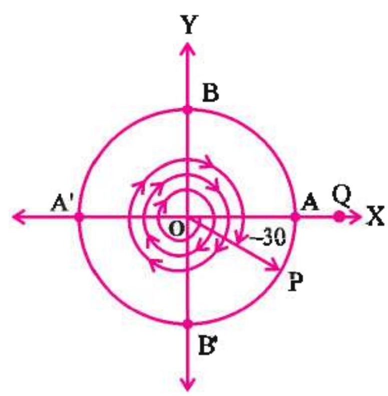


Figure 5.26

5.8 Trigonometric Functions of General Angles

We know that *sine* and *cosine* functions are from R to R. So, $\sin\theta$ and $\cos\theta$ are defined for each $\theta \in$ R.

Now if general degree measure of an angle ∠AOP is θ; we define $\sin\theta^\circ$ as $\sin\frac{\pi\theta}{180}$. Here $\frac{\pi\theta}{180} \in$ R and *sin* is a function from R to R. So $\sin\frac{\pi\theta}{180}$ is a real number. Thus, $\sin 18^\circ = \sin\frac{18\pi}{180} = \sin\frac{\pi}{10}$. We note that $\sin 18^\circ$ must not be confused with $\sin 18$, because $\sin 18^\circ = \sin\frac{18\pi}{180}$ and this is different from $\sin 18$. If we write $\sin 18$ then it is *sin* of real number 18 (or 18 radians).

For *sin* and *cosine* of angles in degree measure, we must write $\sin\theta^\circ$ and $\cos\theta^\circ$ to avoid confusion.

Miscellaneous Examples :

Example 13 : For $\theta = -960$, draw an angle having general degree measure θ.

Solution : $n = \left[\frac{-\theta}{360}\right] = \left[\frac{960}{360}\right] = 2$

$\therefore \quad -960 = (-360)2 + \alpha, \quad \alpha = -240^\circ$

$= (-360)2 + (-240)$

$n = 2, \alpha = -240^\circ, -360 < \alpha \le 0$

Thus, two complete rotations in clockwise direction and further taking ∠AOP with general degree measure -240 gives angle of general measure -960°.

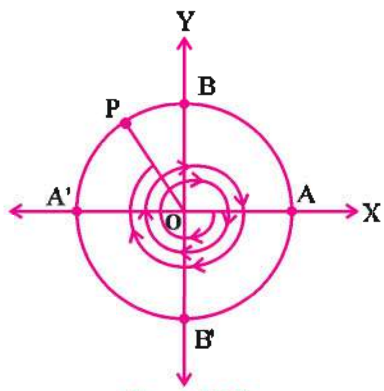


Figure 5.27

Example 14 : Draw the graph of $y = \sin x$ and $y = 2\sin\frac{x}{2}$ on the same set of coordinate axes. $x \in [0, 2\pi]$

Solution : For, $y = \sin x$, Range $= [-1, 1]$ and Period is 2π.

For $y = 2\sin\frac{x}{2}$, $c = 2$ and $a = \frac{1}{2}$. So range is $[-2, 2]$ and Period is 4π.

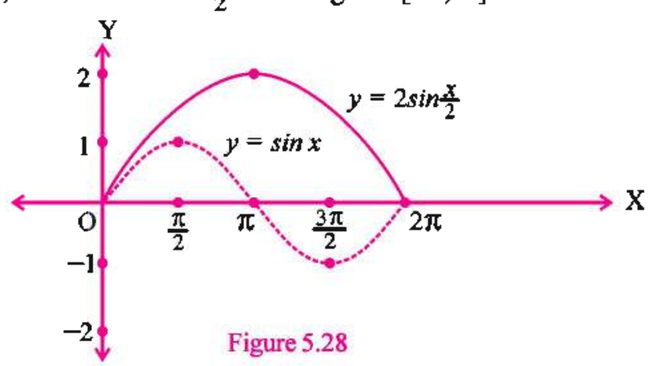


Figure 5.28

EXERCISE 5.3

1. For the following find the complete number of rotations n and α.

   (1) 750° (2) 1125° (3) 1485°

2. Obtain the number of rotations n, α and then draw such an angle having general measure.

   (1) 840° (2) -765° (3) -1470°

EXERCISE 5

1. Plot the graph of $y = \sin x$ and $y = \cos x$ on the same set of coordinate axes.
2. Draw the graph of $y = 3\sin 2x$.
3. Draw the graph of $y = 2\cos 3x$.
4. Obtain the number of rotations n, α and then draw such an angle having general measure.

   (1) -1320° (2) -2000° (3) -540°

5. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

   (1) Value of $\tan\left(\frac{19\pi}{3}\right)$ is... ☐

   (a) $\sqrt{3}$ (b) $-\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{-1}{\sqrt{3}}$

   (2) Value of $\cot\left(\frac{-15\pi}{4}\right)$ is... ☐

   (a) 1 (b) -1 (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{-1}{\sqrt{3}}$

   (3) If $\sec\theta + \tan\theta = \sqrt{3}$, $0 < \theta < \pi$, then θ is equal to... ☐

   (a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{-\pi}{3}$

   (4) If $\tan\theta = -\frac{1}{\sqrt{5}}$ and P(θ) lies in the 4th quadrant, then the value of $\cos\theta$ is ... ☐

   (a) $\frac{\sqrt{5}}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{6}}$

   (5) If $x \cdot \sin 45^\circ \cos^2 60^\circ = \frac{\tan^2 60^\circ \operatorname{cosec}^2 30^\circ}{\sec 45^\circ \cot^2 30^\circ}$, then $x =$ .... ☐

   (a) 16 (b) 1 (c) $8\sqrt{2}$ (d) $\frac{16}{3}$

(6) Value of $cot^2\frac{\pi}{4} + sec^2\frac{\pi}{4} - 4cos\frac{\pi}{3}$ is... ☐

(a) 1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) $\frac{3}{2}$

(7) Value of $2sin^2\frac{\pi}{6} - cosec\frac{\pi}{6} \cdot cos^2\frac{\pi}{3}$ is... ☐

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

(8) For $\theta = -1470$ the number of complete rotations is .... ☐

(a) -3 (b) 3 (c) -4 (d) 4

(9) For $\theta = 750$, the angle of general measure θ^o is drawn, then $P(\theta)$ lies in ...... quadrant. ☐

(a) first (b) second (c) third (d) fourth

(10) $cos\left(\frac{65\pi}{4}\right)$ is... ☐

(a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\pm\frac{1}{\sqrt{2}}$ (d) None of these

Summary

1. Value of trigonometric functions for points $P(\theta)$ on axes.
2. $P\left(\frac{\pi}{4}\right) = \left(cos\frac{\pi}{4}, sin\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
3. $P\left(\frac{\pi}{3}\right) = \left(cos\frac{\pi}{3}, sin\frac{\pi}{3}\right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
4. $P\left(\frac{\pi}{6}\right) = \left(cos\frac{\pi}{6}, sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
5. Graphs of Trigonometric functions,
   $y = sinx$, $y = cosx$ and $y = tanx$.
6. Trigonometric functions having general degree measure.

Chapter 6

STRAIGHT LINES

6.1 Introduction

In 1637, the French mathematician René Descartes published his book 'La Géométrie'. He was the first mathematician who used algebra for the study of geometry. He used ordered pairs of real numbers to represent points in a plane. The ordered pairs are known as cartesian coordinates. Using cartesian coordinates, he represented lines and curves by algebraic equations. In coordinate geometry the methods of algebra are used to solve geometrical problems. So it is mainly a combination of algebra and geometry. Of course, long before this the Arab mathematician *al-khwarizmi* used geometric figures as aids for solving problems in algebra. Our Indian mathematician Bhaskara also gave major contribution in this context.

6.2 Recall

Now let us have a recall of coordinate geometry which we studied in earlier classes. We studied about coordinate axes, coordinate plane, plotting of points in a plane, distance formula, division formula, area of a triangle etc.

There is one-one correspondence between R $\times$ R and all the points of a plane. In the XY plane, the X-axis and Y-axis are represented respectively by

$$\{P(x, 0) \mid x \in R\} \text{ and } \{P(0, y) \mid y \in R\}$$

The points of the plane that are not on either of the two axes get distributed into four disjoint sets. These sets are called first, second, third and fourth quadrant. The quadrants are represented by

First quadrant $= \{P(x, y) \mid x > 0, y > 0\}$

Second quadrant $= \{P(x, y) \mid x < 0, y > 0\}$

Third quadrant $= \{P(x, y) \mid x < 0, y < 0\}$

Fourth quadrant $= \{P(x, y) \mid x > 0, y < 0\}$

The positions of points P(1, 2), Q (−1, 4), R(−2, 0) and S(3, −2) in the XY plane are shown in the figure 6.1. In any ordered pair absolute value of x-coordinate indicates the distance of the point $(x, y)$from Y-axis and absolute value of y-coordinate indicates the distance of point from X-axis. P(1, 2) is at unit distance from Y-axis and 2 units distance from X-axis. But Q(−1, 4) is at unit distance from Y-axis but it is in second quadrant as $x < 0$.

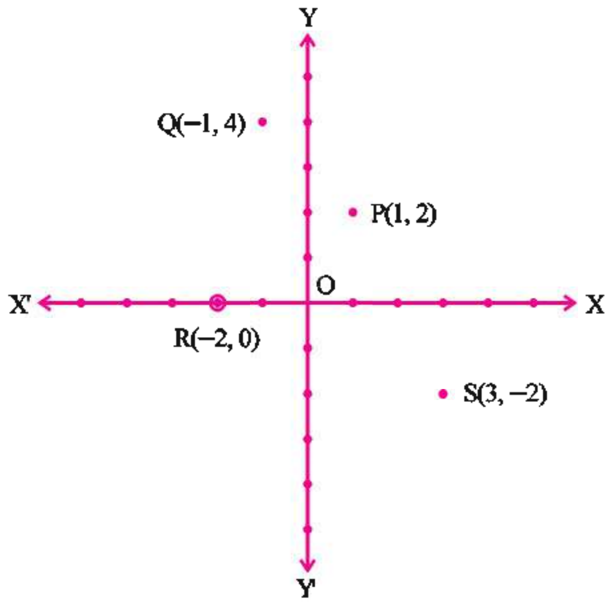


Figure 6.1

We also studied the following formulae.

(1) Distance formula : Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two given points in the plane. The distance between A and B is denoted by $d(A, B)$ or AB and given by

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

For example, the distance between A(9, 8) and B(6, 4) is

$$AB = \sqrt{(9-6)^2 + (8-4)^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

(2) Division formula : $A(x_1, y_1)$ and $B(x_2, y_2)$ are given points in the plane. Let $P(x, y)$ be the point which divides $\overline{AB}$ internally in the ratio λ from A, $\lambda \in R - \{0, -1\}$.

Then $\lambda = \frac{AP}{PB}$.

$$P(x, y) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$$

If the ratio λ is in the form $m : n$, then

$$P(x, y) = \left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right)$$

For example,

Let A(2, 3) and B(4, 8) be points in XY plane. The coordinates of the point that divides $\overline{AB}$ internally from A in the ratio 3 : 2 are

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}\right) = \left(\frac{3 \cdot 4 + 2 \cdot 2}{3 + 2}, \frac{3 \cdot 8 + 2 \cdot 3}{3 + 2}\right) = \left(\frac{16}{5}, 6\right)$$

(3) The Mid-point of a Line-segment : The mid-point of the $\overline{AB}$ is equidistant from the end-points $A(x_1, y_1)$ and $B(x_2, y_2)$ and lies on $\overline{AB}$. The midpoint of the line-segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ divides $\overline{AB}$ in the ratio 1 : 1 i.e. $m = 1, n = 1$.

The coordinates of the mid-point of $\overline{AB}$ $= \left(\frac{1 \cdot x_2 + 1 \cdot x_1}{1+1}, \frac{1 \cdot y_2 + 1 \cdot y_1}{1+1}\right)$

$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

(4) Area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2} \mid x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \mid \quad \text{(i)}$$

For example : Let A(3, 2), B(11, 8) and C(8, 12) be the vertices of a triangle.

Area of $\Delta ABC = \frac{1}{2} \mid 3(8 - 12) + 11(12 - 2) + 8(2 - 8) \mid$

$= \frac{1}{2} \mid 3(-4) + 11(10) + 8(-6) \mid$

$= \frac{1}{2} \mid -12 + 110 - 48 \mid$

$= \frac{1}{2} \mid 50 \mid$

$= 25$

> ☛ **Note** Area of a triangle is always positive. If the expression (i) is zero, then a triangle is not possible. Thus the points are collinear.

In this chapter, we shall continue the study of coordinate geometry. For the study of the simplest geometric figure, straight line, the division formula is very useful.

6.3 Shifting of Origin

As we know, in the coordinate plane all the points have a fixed position. The corresponding ordered pairs with the points on the plane depend on the position of axes and origin.

Select any pair of perpendicular lines in a plane. The point of intersection O of the lines is called origin. We consider one of the lines as X-axis and the other as Y-axis.

Suppose P is a point in the XOY plane. The coordinates of P are (x, y).

Let O' (h, k) be any point in the same plane P $\neq$ O'.

Now consider two lines $\overleftrightarrow{O'X'}$ and $\overleftrightarrow{O'Y'}$ such that $\overleftrightarrow{O'X'} \parallel \overleftrightarrow{OX}$ and $\overleftrightarrow{O'Y'} \parallel \overleftrightarrow{OY}$.

Also direction $\overrightarrow{OX}$ = direction $\overrightarrow{O'X'}$ and direction $\overrightarrow{OY}$ = direction $\overrightarrow{O'Y'}$.

Suppose coordinates of P are (x', y') *w.r.t.* new axes $\overleftrightarrow{O'X'}$ and $\overleftrightarrow{O'Y'}$.

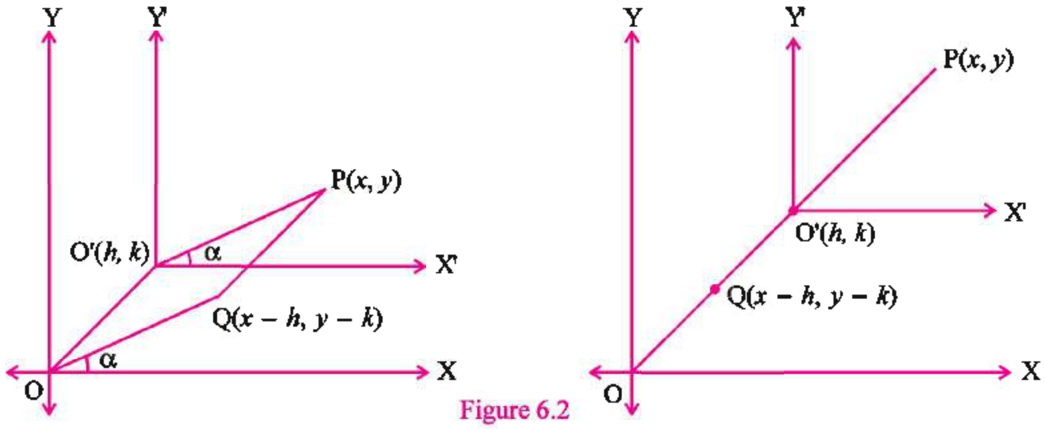


Figure 6.2

Let Q be the point with coordinates $(x - h, y - k)$ *w.r.t.* old system of axes $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$.

Now, $O'P = \sqrt{(x-h)^2 + (y-k)^2}$

$OQ = \sqrt{(x-h)^2 + (y-k)^2}$

$\therefore \quad O'P = OQ$

Similarly $OO' = PQ = \sqrt{h^2 + k^2}$

$P \neq O'$. So O, O', P and Q form a parallelogram or they are collinear.

Suppose $m\angle PO'X' = \alpha$ **$(0 < \alpha < 2\pi)$**

So $m\angle QOX = \alpha$

$\therefore \quad (x', y') = (O'P \cos\alpha, O'P \sin\alpha)$

and $(x - h, y - k) = (OQ \cos\alpha, OQ \sin\alpha)$

$= (O'P \cos\alpha, O'P \sin\alpha)$

$\therefore \quad (x', y') = (x - h, y - k)$

$\therefore \quad x' = x - h, \qquad y' = y - k$

$x = x' + h, \qquad y = y' + k$

So when we shift the origin to (h, k) the new coordinates of $P(x, y)$ are $(x - h, y - k)$.

Using this we can find the coordinates of P(6, 8) after shifting the origin at (3, 5).

Here $(h, k) = (3, 5)$, $(x, y) = (6, 8)$

Suppose new coordinates of P are (x', y').

So $x' = x - h$ and $y' = y - k$

$= 6 - 3 \qquad = 8 - 5$

$= 3 \qquad = 3$

$\therefore$ The new coordinates of P are (3, 3).

6.4 Division of a Line-segment

Suppose A and B are distinct points of the plane. They determine unique line $\overleftrightarrow{AB}$.

Let P(x, y) be any point on $\overleftrightarrow{AB}$ other than A or B. Then there are three possibilities for the position of point P on the line. The possibilities are (1) A–P–B (2) A–B–P (3) P–A–B.

For the knowledge of exact location of point P, the ratio $\frac{AP}{PB}$ is useful.

Definition : (1) If A–P–B, we say that P divides $\overline{AB}$ in the ratio $\lambda = \frac{AP}{PB}$ from the side of A. Here $\lambda > 0$ and we say that P divides $\overline{AB}$ internally.

(2) If P–A–B or A–B–P, we say that P divides $\overline{AB}$ in the ratio $\lambda = -\frac{AP}{PB}$ from the side of A. We say P divides $\overline{AB}$ externally. Here $\lambda < 0$.

In all cases $\lambda \in R - \{0, -1\}$.

Thus suppose P divides $\overline{AB}$ in the ratio λ ($\lambda \neq 0$ and $\lambda \neq -1$) from the side of A.

Then (1) If A–P–B, then $\lambda > 0$. **(Internal division)**

(2) If P–A–B, then $-1 < \lambda < 0$. **(External division)**

(3) If A–B–P, then $\lambda < -1$. **(External division)**

☛ Note Can you prove $-1 < \lambda < 0$ in (2) and $\lambda < -1$ in (3) ?

Coordinates of the Point of Division :

A(x_1, y_1) and B(x_2, y_2) are given points in the plane. We wish to find the coordinates of the point dividing $\overline{AB}$ internally in the ratio λ from A ($\lambda > 0$).

Suppose P(x, y) divides $\overline{AB}$ internally in the ratio λ from A, $\lambda = \frac{AP}{PB}$.

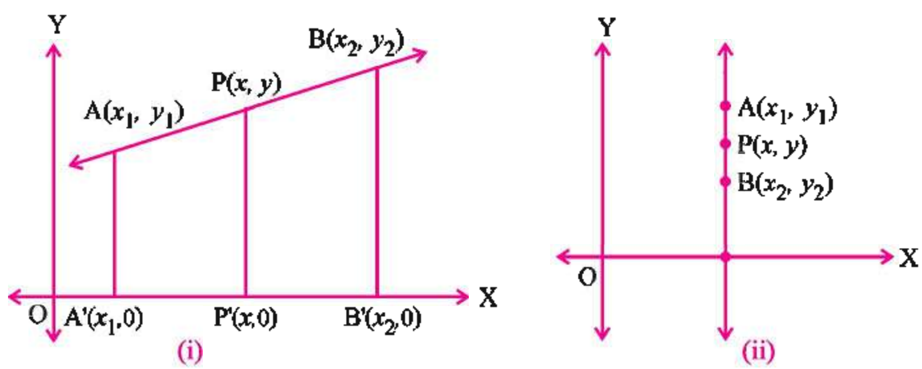


Figure 6.3

If $\overleftrightarrow{AB}$ is not vertical, let A', P', B' be the feet of perpendiculars from A, P and B respectively to X-axis.

As shown in figure 6.3(i) $\overleftrightarrow{AA'} \parallel \overleftrightarrow{PP'} \parallel \overleftrightarrow{BB'}$ and X-axis as well as $\overleftrightarrow{AB}$ are their transversals.

$\therefore \quad \frac{AP}{PB} = \frac{A'P'}{P'B'} = \frac{|x - x_1|}{|x_2 - x|}$

Since A'–P'–B'; $x - x_1 > 0$ and $x_2 - x > 0$ or $x - x_1 < 0$ and $x_2 - x < 0$

$\therefore \quad \lambda = \frac{x - x_1}{x_2 - x}$ $\left(\lambda = \frac{AP}{PB}\right)$

$\therefore \quad \lambda x_2 - \lambda x = x - x_1$

$\therefore \quad \lambda x_2 + x_1 = \lambda x + x$

$\therefore \quad x = \frac{\lambda x_2 + x_1}{\lambda + 1}$

If $\overleftrightarrow{AB}$ is perpendicular to X-axis, then $x = x_1 = x_2$. So $\frac{\lambda x_2 + x_1}{\lambda + 1} = \frac{\lambda x + x}{\lambda + 1} = x$.

So, in both the cases we have, $x = \frac{\lambda x_2 + x_1}{\lambda + 1}$, if $\lambda > 0$.

Similarly, we can prove $y = \frac{\lambda y_2 + y_1}{\lambda + 1}$ by considering the feet of perpendiculars from A, P, B to Y-axis, if $\overleftrightarrow{AB}$ is not horizontal and also even if $\overleftrightarrow{AB}$ is horizontal as in case of x.

Hence if $A(x_1, y_1)$, $B(x_2, y_2)$ are given and P divides $\overline{AB}$ internally from A in the ratio λ ($\lambda > 0$), then the coordinates of P are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$.

Conversely if $A(x_1, y_1)$, $B(x_2, y_2)$ and $P\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$, $\lambda > 0$ are given, we shall prove that A–P–B and $\frac{AP}{PB} = \lambda$.

Now, $AP^2 = \left(\frac{\lambda x_2 + x_1}{\lambda + 1} - x_1\right)^2 + \left(\frac{\lambda y_2 + y_1}{\lambda + 1} - y_1\right)^2$

$= \left(\frac{\lambda x_2 - \lambda x_1}{\lambda + 1}\right)^2 + \left(\frac{\lambda y_2 - \lambda y_1}{\lambda + 1}\right)^2$

$= \frac{\lambda^2 (x_2 - x_1)^2}{(\lambda + 1)^2} + \frac{\lambda^2 (y_2 - y_1)^2}{(\lambda + 1)^2}$

$= \frac{\lambda^2}{(\lambda + 1)^2} [(x_2 - x_1)^2 + (y_2 - y_1)^2]$

$= \left(\frac{\lambda}{\lambda + 1}\right)^2 AB^2$

$\therefore \quad AP = \left|\frac{\lambda}{\lambda+1}\right| \cdot AB$

$\therefore \quad AP = \frac{\lambda}{\lambda+1} \cdot AB$ **($\lambda > 0$ and $\lambda + 1 > 0$)**

Similarly, $PB = \left|\frac{1}{\lambda+1}\right| \cdot AB = \frac{1}{\lambda+1} \cdot AB$

$\therefore \quad \frac{AP}{PB} = \frac{\frac{\lambda}{\lambda+1}}{\frac{1}{\lambda+1}} \cdot \frac{AB}{AB}$ **(A and B are distinct, so $AB \neq 0$)**

$\frac{AP}{PB} = \lambda$

Also $AP + PB = \lambda PB + PB$

$= (\lambda + 1)\, PB$

$= (\lambda + 1) \frac{1}{\lambda+1} \cdot AB$

$= AB$

$AP + PB = AB$

$\therefore \quad$ A–P–B

Thus, A–P–B and $\frac{AP}{PB} = \lambda$.

So, for $\lambda > 0$, if $P\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}\right)$ is given, then we get A–P–B and $\frac{AP}{PB} = \lambda$.

$\therefore$ If P divides $\overline{AB}$ internally in ratio $\lambda > 0$, then $P\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}\right)$ and conversely if $P\left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}\right)$ $\lambda > 0$, then P divides $\overline{AB}$ internally in ratio $\frac{AP}{PB}$.

6.5 Coordinates of the Point Dividing a Line-segment Externally

$A(x_1, y_1)$ and $B(x_2, y_2)$ are given points in the plane. We wish to find the coordinates of the point dividing $\overline{AB}$ externally in the ratio λ from A ($\lambda < 0$).

Let P divide $\overline{AB}$ externally from A in the ratio λ.

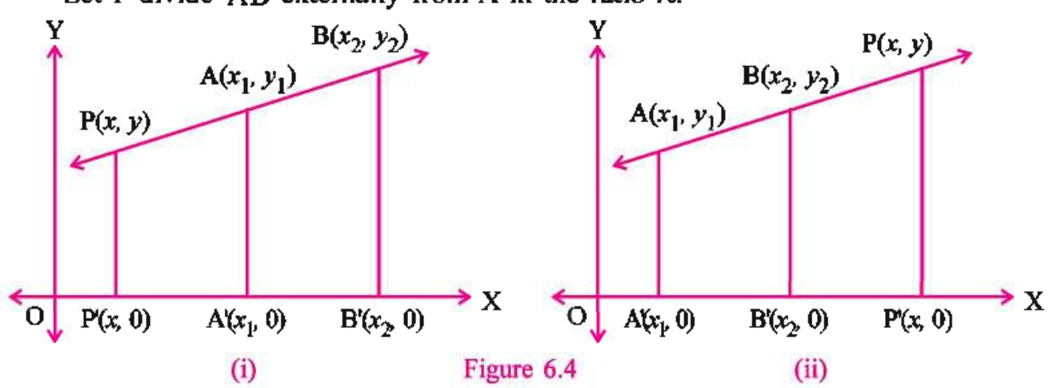


Figure 6.4

Proceeding exactly as we have done in case of internal division, $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'} \parallel \overleftrightarrow{PP'}$.

X-axis and $\overleftrightarrow{AB}$ are their transversals.

$\therefore \quad \frac{AP}{PB} = \frac{A'P'}{P'B'}$

Now, $-\frac{AP}{PB} = -\frac{A'P'}{P'B'} = -\frac{|x - x_1|}{|x_2 - x|} = -\frac{x - x_1}{x - x_2}$ as $x_1 - x$ and $x_2 - x$ have same sign.

$\therefore \quad \lambda = \frac{x - x_1}{x_2 - x}$ $\qquad (-\frac{AP}{PB} = \lambda)$

$\therefore \quad \lambda x_2 - \lambda x = x - x_1$

$\therefore \quad \lambda x + x = \lambda x_2 + x_1$

$\therefore \quad (\lambda + 1)x = \lambda x_2 + x_1$

$\therefore \quad x = \frac{\lambda x_2 + x_1}{\lambda + 1}$ $\qquad (\lambda \neq -1)$

If $\overleftrightarrow{AB}$ is perpendicular to X-axis, then $x = x_1 = x_2$. So $\frac{\lambda x_2 + x_1}{\lambda + 1} = \frac{\lambda x + x}{\lambda + 1} = x$.

Similarly, we can prove $y = \frac{\lambda y_2 + y_1}{\lambda + 1}$

The coordinates of the point dividing $\overline{AB}$ externally in the ratio λ from A are

$\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ where $\lambda < 0, \lambda \neq -1$

Conversely if $\lambda < 0, \lambda \neq -1$, let $A(x_1, y_1)$, $B(x_2, y_2)$ and $P\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ be given points.

We shall prove $\frac{AP}{PB} = -\lambda$ and either P–A–B or A–B–P

The same calculations as in case of internal division give,

$$AP = \left|\frac{\lambda}{\lambda + 1}\right| \cdot AB \text{ and } PB = \left|\frac{1}{\lambda + 1}\right| \cdot AB$$

If $-1 < \lambda < 0$, then $\lambda + 1 > 0$ and $|\lambda| = -\lambda$

$\therefore \quad AP = \frac{-\lambda}{\lambda + 1} \cdot AB$ and $PB = \frac{1}{\lambda + 1} \cdot AB$ $\qquad$ (i)

So, $\frac{AP}{PB} = \frac{\frac{-\lambda}{\lambda + 1} \cdot AB}{\frac{1}{\lambda + 1} \cdot AB} = -\lambda$

From (i) $-AP + PB = \frac{\lambda}{\lambda + 1} AB + \frac{1}{\lambda + 1} AB = AB$

$\therefore \quad AP + AB = PB$

$\therefore \quad$ P–A–B

If $\lambda < -1$, $\lambda + 1 < 0$

$\therefore \quad |\lambda + 1| = -(\lambda + 1)$ and $|\lambda| = -\lambda$

$AP = \left|\frac{\lambda}{\lambda+1}\right| \cdot AB = \frac{-\lambda}{-(\lambda+1)} \cdot AB = \frac{\lambda}{\lambda+1} \cdot AB$

$\therefore$ $AP = \frac{\lambda}{\lambda+1} \cdot AB$ and $PB = \frac{1}{-(\lambda+1)} \cdot AB$ **(ii)**

$\therefore$ $\frac{AP}{PB} = -\lambda$

From (ii) AP − PB = AB

$\therefore$ AP = AB + PB

$\therefore$ A–B–P

So, for $\lambda < 0$, $\lambda \neq -1$, P is the point on $\overleftrightarrow{AB}$ such that P–A–B or A–B–P, and $\lambda = -\frac{AP}{PB}$.

If $\frac{AP}{PB} = -\lambda$, $\lambda < 0$, $\lambda \neq -1$ and P–A–B or A–B–P,

$P(x, y) = \left(\frac{\lambda x_2 + x_1}{\lambda+1}, \frac{\lambda y_2 + y_1}{\lambda+1}\right)$ and conversely,

From internal and external division of $\overline{AB}$, we can say that there is one-one correspondence between all the points on $\overleftrightarrow{AB}$ except A and B and real numbers λ, $\forall \lambda \in R - \{0, -1\}$.

> ☛ **Note 1** $A(x_1, y_1)$ and $B(x_2, y_2)$ are given points. If P divides $\overline{AB}$ in the ratio λ from B, then the coordinates of P are $\left(\frac{\lambda x_1 + x_2}{\lambda+1}, \frac{\lambda y_1 + y_2}{\lambda+1}\right)$.

> ☛ **Note 2** If $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ at P, then we say that $\overline{AB}$ is divided by $\overleftrightarrow{CD}$ internally in the ratio $\frac{AP}{PB}$, if A–P–B and in the ratio $-\frac{AP}{PB}$ if P–A–B or A–B–P.

Example 1 : For A(8, 4), B(−3, 1), find the point P which divides $\overline{AB}$ from B's side in the ratio −1 : 2.

Solution : $A(x_1, y_1) = (8, 4)$ and $B(x_2, y_2) = (-3, 1)$

Let P(x, y) be the point which divides $\overline{AB}$ from the side of B in the ratio $\lambda = \frac{-1}{2}$.

So $P(x, y) = \left(\frac{\lambda x_1 + x_2}{\lambda+1}, \frac{\lambda y_1 + y_2}{\lambda+1}\right)$ **(P divides from B)**

$= \left(\frac{-\frac{1}{2}(8) + (-3)}{-\frac{1}{2}+1}, \frac{-\frac{1}{2}(4) + 1}{-\frac{1}{2}+1}\right)$

$= \left(\frac{-4 + (-3)}{\frac{1}{2}}, \frac{-2 + 1}{\frac{1}{2}}\right) = (-14, -2)$

$\therefore$ The required point is P(−14, −2).

Example 2 : Find the points of trisection of the line segment joining the points P(3, 5) and Q(12, 14).

Solution :

Figure 6.5

Let R and S be points of trisection. Point R divides $\overline{PQ}$ from P in the ratio 1:2.

$$R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right) = R\left(\frac{\frac{1}{2}(12) + 3}{\frac{1}{2} + 1}, \frac{\frac{1}{2}(14) + 5}{\frac{1}{2} + 1}\right)$$

$$= R\left(\frac{12 + 6}{1 + 2}, \frac{14 + 10}{1 + 2}\right) = R(6, 8)$$

Point S is the midpoint of $\overline{RQ}$.

$\therefore$ Coordinates of S $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$$= \left(\frac{6 + 12}{2}, \frac{8 + 14}{2}\right) = (9, 11)$$

$\therefore$ R(6, 8) and S(9, 11) are points of trisection of $\overline{PQ}$.

> **☛ Note** See that x-coordinates of P, R, S, Q namely 3, 6, 9, 12 form an A.P. and same applies for y-coordinates. Similarly, if A(x_1, y_1) and B(x_2, y_2) are distinct points and $\overline{AB}$ is divided into n congruent parts, let $d = \frac{x_2 - x_1}{n}$ and $d' = \frac{y_2 - y_1}{n}$. Then coordinates of dividing points are
>
> $(x_1 + d, y_1 + d'), (x_1 + 2d, y_1 + 2d'), ..., (x_1 + (n - 1)d, y_1 + + (n - 1)d')$

Example 3 : A(3, −2) and B(0, 7) are given points. Find P $\in \overleftrightarrow{AB}$ such that AP = 4AB.

Solution : (Method 1) : Suppose the coordinates of P are (x, y)

Here P $\in \overleftrightarrow{AB}$ and AP = 4AB

$\therefore$ $\frac{AP}{4} = \frac{AB}{1} = k$ (say)

$\therefore$ AP = 4k and AB = k

Case 1 : A−B−P

Figure 6.6

P divides $\overline{AB}$ from A in the ratio $\lambda = \frac{-AP}{PB} = \frac{-4k}{3k} = \frac{-4}{3}$.

$\therefore$ P$(x, y) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$

$$= \left(\frac{\frac{-4}{3}(0) + 3}{\frac{-4}{3} + 1}, \frac{\frac{-4}{3}(7) + (-2)}{\frac{-4}{3} + 1} \right)$$

$$= \left(\frac{-4(0) + 3(3)}{-4 + 3}, \frac{-4(7) + 3(-2)}{-4 + 3} \right)$$

$$= \left(\frac{9}{-1}, \frac{-28 - 6}{-1} \right) = (-9, 34)$$

∴ The coordinates of P are (–9, 34).

Case 2 : P–A–B

Figure 6.7

P divides $\overline{AB}$ from A in the ratio $\lambda = \frac{-AP}{PB} = \frac{-4}{5}$.

$$\therefore \quad P(x, y) = \left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1} \right)$$

$$= \left(\frac{\frac{-4}{5}(0) + 3}{\frac{-4}{5} + 1}, \frac{\frac{-4}{5}(7) + (-2)}{\frac{-4}{5} + 1} \right)$$

$$= \left(\frac{-4(0) + 3(5)}{-4 + 5}, \frac{-4(7) + (-2)5}{-4 + 5} \right)$$

$$= (15, -38)$$

∴ The coordinates of P are (15, –38).

Case 3 : Here AP > AB. So A–P–B is not possible.

So the coordinates of P are (–9, 34) or (15, –38).

Method 2 : Suppose coordinates of P are (x, y).

Here $P \in \overleftrightarrow{AB}$ and AP = 4AB i.e. $\frac{AP}{AB} = 4$

∴ Point A divides $\overline{PB}$ from P in the ratio $\lambda = -4:1$ or $\lambda = 4:1$

(1) Let $\lambda = -4:1$

$$\therefore \quad (3, -2) = \left(\frac{-4(0) + 1(x)}{-4 + 1}, \frac{-4(7) + 1(y)}{-4 + 1} \right)$$

$$\therefore \quad 3 = \frac{-4(0) + 1(x)}{-4 + 1}, \quad -2 = \frac{-4(7) + 1(y)}{-4 + 1}$$

$$\therefore \quad 3 = \frac{x}{-3}, \quad -2 = \frac{-28 + y}{-3}$$

$$\therefore \quad x = -9, \quad y = 34$$

∴ The coordinates of P are (–9, 34).

Let $\lambda = 4:1$

$$(3, -2) = \left(\frac{4(0) + 1(x)}{4 + 1}, \frac{4(7) + 1(y)}{4 + 1} \right)$$

$\therefore \quad (3, -2) = \left(\frac{x}{5}, \frac{28+y}{5}\right)$

$\therefore \quad 3 = \frac{x}{5} \quad$ and $\quad -2 = \frac{28+y}{5}$

$\therefore \quad x = 15 \quad$ and $\quad y = -38$

$\therefore$ The coordinates of P are (15, −38).

So the coordinates of P are (−9, 34) or (15, −38).

EXERCISE 6.1

1. A(3, −5) and B(2, 3) are given points. Find the point dividing $\overline{AB}$ from A in the ratio 2 : 3.
2. A(2, 0) and B(2, 6). Find the point dividing $\overline{AB}$ from B in the ratio −3 : 5.
3. For A(−7, 8) and B(−3, −5), find the ratio in which X-axis divides $\overline{AB}$ from A.
4. Find the points of trisection of the line-segment joining the points A(1, 2) and B(7, 8).
5. A(1, 2) and B(6, 3) are given points. Find the point P ∈ $\overleftrightarrow{AB}$ such that 3AB = 2PB.
6. A(1, 2) and B(0, 3) are given points and P(10, −7) ∈ $\overleftrightarrow{AB}$. In which ratio does P divide $\overline{AB}$ from A's side ?

*

6.6 Parametric Equations of a Line

As we have seen for $\forall \lambda \in R - \{0, -1\}$ all the points on $\overleftrightarrow{AB}$ other than A and B are obtained by $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$. where A($x_1$, y_1) and B(x_2, y_2).

So, $\overleftrightarrow{AB} = \left\{(x, y) \,\middle|\, x = \frac{\lambda x_2 + x_1}{\lambda + 1}, y = \frac{\lambda y_2 + y_1}{\lambda + 1}, \lambda \in R - \{0, -1\}\right\} \cup \{A, B\}$

Let $t = \frac{\lambda}{\lambda + 1}$. So $1 - t = \frac{1}{\lambda + 1}$.

Now, $x = \frac{\lambda x_2 + x_1}{\lambda + 1}$ and $y = \frac{\lambda y_2 + y_1}{\lambda + 1}$

$\therefore \quad x = \frac{\lambda}{\lambda + 1} x_2 + \frac{1}{\lambda + 1} x_1$ and $y = \frac{\lambda}{\lambda + 1} y_2 + \frac{1}{\lambda + 1} y_1$

$\therefore \quad x = tx_2 + (1 - t)x_1$ and $y = ty_2 + (1 - t)y_1$

Also for every $\lambda \in R - \{0, -1\}$ there exists unique $t \in R$ and for every $t \in R - \{0, 1\}$ there exists unique $\lambda \in R$.

Also $\lambda \neq 0 \Leftrightarrow t \neq 0$ and $\lambda \neq -1 \Leftrightarrow t \neq 1$

$\therefore$ For every P(x, y) ∈ $\overleftrightarrow{AB} - \{A, B\}$

$(x, y) = (tx_2 + (1 - t)x_1, ty_2 + (1 - t)y_1), t \neq 0, 1$

$$\overleftrightarrow{AB} = \left\{(x, y) \left| \begin{array}{l} x = tx_2 + (1-t)x_1 \\ y = ty_2 + (1-t)y_1 \end{array} ; t \in R - \{0, 1\} \right.\right\} \cup \{A, B\}$$

Now observe this, $x = tx_2 + (1 - t)x_1$, $y = ty_2 + (1 - t)y_1$

Substituting $t = 0$ we get $x = x_1$, $y = y_1$. So, $(x, y) = (x_1, y_1)$ for $t = 0$.

Substituting $t = 1$ we get $x = x_2$, $y = y_2$. So, $(x, y) = (x_2, y_2)$ for $t = 1$.

Hence if we allow t to be 0 or 1 also i.e. let $t \in R$,

$$\therefore \quad \overleftrightarrow{AB} = \left\{(x, y) \left| \begin{array}{l} x = tx_2 + (1-t)x_1 \\ y = ty_2 + (1-t)y_1 \end{array} ; t \in R \right.\right\}$$

The equations $x = tx_2 + (1 - t)x_1$ and $y = ty_2 + (1 - t)y_1$, $t \in R$ are called the parametric equations of the line passing through (x_1, y_1) and (x_2, y_2). The variable t is called a parameter.

If we restrict t to take values only in certain subsets of R, we get corresponding subsets of $\overleftrightarrow{AB}$.

We can describe some subsets of $\overleftrightarrow{AB}$ with the appropriate restrictions on t.

- $$\overline{AB} = \left\{(x, y) \left| \begin{array}{l} x = tx_2 + (1-t)x_1 \\ y = ty_2 + (1-t)y_1 \end{array} ; t \in [0, 1] \right.\right\}$$

Figure 6.8

- $$\overrightarrow{AB} = \left\{(x, y) \left| \begin{array}{l} x = tx_2 + (1-t)x_1 \\ y = ty_2 + (1-t)y_1 \end{array} ; t \geq 0 \right.\right\}$$

Figure 6.9

- $$\overleftrightarrow{AB} - \overline{AB} = \left\{(x, y) \left| \begin{array}{l} x = tx_2 + (1-t)x_1 \\ y = ty_2 + (1-t)y_1 \end{array} ; t \in R - [0, 1] \right.\right\}$$

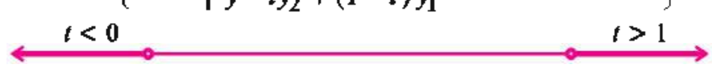


Figure 6.10

- $$\overrightarrow{BA} = \left\{(x, y) \left| \begin{array}{l} x = tx_2 + (1-t)x_1 \\ y = ty_2 + (1-t)y_1 \end{array} ; t \leq 1 \right.\right\}$$

Figure 6.11

- $\overrightarrow{AB} - \overline{AB} = \left\{ (x, y) \,\middle|\, \begin{array}{l} x = tx_2 + (1-t)x_1 \\ y = ty_2 + (1-t)y_1 \end{array} ; t > 1 \right\}$

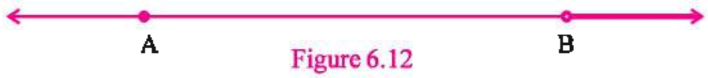


Figure 6.12

Parametric equations of a line are not unique

Let A(1, 2), B(3, 5) be two points.

Parametric equations of $\overleftrightarrow{AB}$ are,

$x = 3t + 1(1 - t) = 2t + 1$ and $y = 5t + 2(1 - t) = 3t + 2$, $t \in R$

i.e. $x = 2t + 1$, $y = 3t + 2$, $t \in R$

See that these equations gives A(1, 2) for $t = 0$ and B(3, 5) for $t = 1$.

Taking $t = 3$ and $t = 4$ respectively, we see that P(7, 11), Q(9, 14) are on $\overleftrightarrow{AB}$.

Parametric equations of $\overleftrightarrow{PQ}$ are,

$x = 9t' + (1 - t')7 = 2t' + 7$, $y = 14t' + (1 - t')11 = 3t' + 11$

See that $t' = 0$ and 1 respectively will give P(7, 11) and Q(9, 14).

$\overleftrightarrow{PQ} = \overleftrightarrow{AB}$, but they have different parametric equations.

This is because values 0 and 1 of parameter give points using which equations are derived.

A change of variable $t' = t - 3$ will give parametric equations of $\overleftrightarrow{PQ}$ as

$x = 2(t - 3) + 7 = 2t + 1$, $y = 3(t - 3) + 11 = 3t + 2$ same as parametric equations of $\overleftrightarrow{AB}$. Thus same line may have different parametric equations but a linear transformation will transform them into each other.

6.7 Equation of a Line Perpendicular to X-axis

Suppose A(a, y_1) and B(a, y_2) are distinct points on $\overleftrightarrow{AB}$.

Here the parametric equations of $\overleftrightarrow{AB}$ are

$x = tx_2 + (1 - t)x_1$ and $y = ty_2 + (1 - t)y_1$, $t \in R$

$= ta + (1 - t)a$

$= a$

$\therefore \quad x = a$

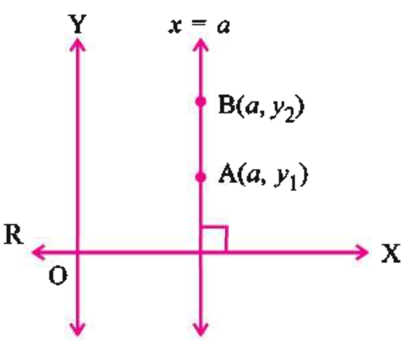


Figure 6.13

$\therefore$ For all the points on $\overleftrightarrow{AB}$, the x-coordinate is a and y-coordinate can be any real number.

This is a typical property of lines perpendicular to X-axis.

So, the equation of vertical $\overleftrightarrow{AB}$ is taken as $x = a$, $a \in R$.

For any point (x, y) on $\overleftrightarrow{AB}$, $x = a$ and $y \in R$ is arbitrary.

The equation of Y-axis is $x = 0$ and the equation of all the lines parallel to Y-axis i.e. perpendicular to X-axis is $x = a$ $(a \neq 0)$.

If two points on a line have same x-coordinate, then all points on the line have same x-coordinate and the line is vertical or perpendicular to X-axis.

6.8 Equation of a Line Perpendicular to Y-axis

Suppose $A(x_1, b)$ and $B(x_2, b)$ are distinct points on $\overleftrightarrow{AB}$. As we have seen in 6.7, **the points on a line perpendicular to Y-axis have same y-coordinate i.e. $y = b$.**

So the equation of $\overleftrightarrow{AB}$ is $y = b$, $b \in R$.

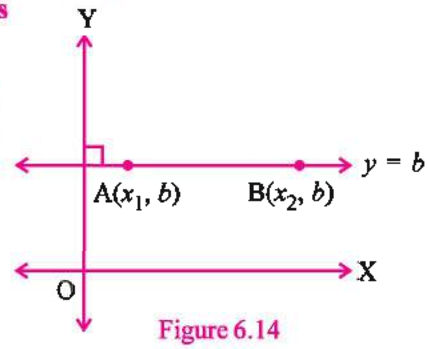


Figure 6.14

For any point (x, y) on line perpendicular to Y-axis, x-coordinate is arbitrary and $y = b$, b constant. so we write its equation as $y = b$ using its typical property. **Equation of X-axis is $y = 0$ and the equation of all lines parallel to X-axis is $y = b$ $(b \neq 0)$.**

Thus if two points on a line have same y-coordinate, then all points on the line have same y-coordinate and the line is horizontal or perpendicular to Y-axis.

6.9 The Cartesian Equation of a Line

Eliminating t from the parametric equations of a line, we get cartesian equation of a line.

We have $x = tx_2 + (1 - t)x_1$, $y = ty_2 + (1 - t)y_1$; $t \in R$ as parametric equations of $\overleftrightarrow{AB}$ passing through $A(x_1, y_1)$ and $B(x_2, y_2)$.

Suppose $\overleftrightarrow{AB}$ is not perpendicular to any axis.

$\therefore$ $x_1 \neq x_2$ and $y_1 \neq y_2$.

Also for any $P(x, y) \in \overline{AB}$, $x \neq x_1$, $y \neq y_1$, $x \neq x_2$, $y \neq y_2$. **$(P \neq A, P \neq B)$**

Now $x = tx_2 + (1 - t)\, x_1 = tx_2 + x_1 - tx_1$

$\therefore$ $x - x_1 = t(x_2 - x_1)$ and similarly $y - y_1 = t(y_2 - y_1)$.

$\therefore$ $t = \dfrac{x - x_1}{x_2 - x_1}$ and $t = \dfrac{y - y_1}{y_2 - y_1}$

Obviously converse is also true.

If $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = t$ (say)

$x = tx_2 + (1 - t)\, x_1$, $y = ty_2 + (1 - t)\, y_1$

$\therefore$ $P(x, y) \in \overleftrightarrow{AB}$.

$\therefore$ Cartesian equation of line is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$.

Example 4 : Obtain the parametric and cartesian equations of lines passing through the following pairs of points.

(1) (1, 2), (3, 5) (2) (5, 6), (5, −1) (3) (1, 3), (2, 0)

Solution : (1) Parametric equations of the line are,

$$\left.\begin{aligned} x &= tx_2 + (1 - t)x_1 \\ y &= ty_2 + (1 - t)y_1 \end{aligned}\right\} ; t \in R$$

Here $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (3, 5)$

$$\left.\begin{aligned} x &= t \cdot 3 + (1 - t) \cdot 1 \quad \text{and} \quad & y &= t \cdot 5 + (1 - t) \cdot 2 \\ &= 3t + 1 - t & &= 5t + 2 - 2t \\ &= 2t + 1 & &= 3t + 2 \end{aligned}\right\} t \in R$$

$\therefore$ Parametric equations of the line are $x = 2t + 1$, $y = 3t + 2$, $t \in R$.

$\frac{x - 1}{2} = t$ and $\frac{y - 2}{3} = t$

$\therefore$ $\frac{x - 1}{2} = \frac{y - 2}{3}$

$\therefore$ $3x - 2y + 1 = 0$ is the cartesian equation of the line passing through (1, 2) and (3, 5).

(2) The parametric equations of the line passing through (5, 6) and (5, −1) are,

$$\left.\begin{aligned} x &= tx_2 + (1 - t)x_1 \quad \text{and} \quad & y &= ty_2 + (1 - t)y_1 \\ &= 5t + (1 - t)5 & &= t(-1) + (1 - t)6 \\ x &= 5 & &= 6 - 7t \end{aligned}\right\} t \in R$$

Parametric equations of the line are $x = 5$, $y = 6 - 7t$, $t \in R$

The line is perpendicular to X-axis and its cartesian equation is $x = 5$.

(3) The parametric equations of line passing through (1, 3) and (2, 0) are,

$$\left.\begin{aligned} x &= tx_2 + (1 - t)x_1 \quad \text{and} \quad & y &= ty_2 + (1 - t)y_1 \\ &= 2t + (1 - t)1 & &= t(0) + (1 - t)3 \\ &= t + 1 & &= 3 - 3t \end{aligned}\right\} t \in R$$

Parametric equations of the line are $x = t + 1$, $y = 3 - 3t$, $t \in R$

We have $t = x - 1$ and $t = \frac{y - 3}{-3}$

$\therefore$ Eliminating t, $x - 1 = \frac{y - 3}{-3}$.

$\therefore$ $-3x + 3 = y - 3$

$\therefore$ $3x + y - 6 = 0$ is the cartesian equation of the line.

Example 5 : Obtain the equation of the median of ΔABC through A where A(6, 2), B(5, −1) and C(1, 7).

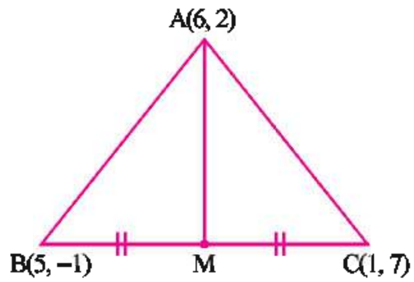


Figure 6.15

Solution : Mid-point of $\overline{BC}$

$$= \left(\frac{5+1}{2}, \frac{-1+7}{2}\right) = (3, 3)$$

$\overline{AM}$ is the median.

Parametric equations of $\overline{AM}$ are

$$\left.\begin{array}{l} x = tx_2 + (1-t)x_1 \\ y = ty_2 + (1-t)y_1 \end{array}\right\} ; t \in [0, 1]$$

$$\left.\begin{array}{l} x = t \cdot 3 + (1-t) \cdot 6 = 6 - 3t \\ y = t \cdot 3 + (1-t) \cdot 2 = 2 + t \end{array}\right\} ; t \in [0, 1]$$

$\therefore$ The median $\overline{AM} = \left\{(x, y) \,\middle|\, \begin{array}{l} x = 6 - 3t \\ y = 2 + t \end{array} ; t \in [0, 1]\right\}$

Example 6 : The parametric equations of a line are $x = 2t - 1$ and $y = 6 - 5t$, $t \in R$. If the x-coordinate of a point A on this line is 7, find the y-coordinate of A.

Solution : Here $x = 2t - 1$. The x-coordinate is 7.

$\therefore$ $7 = 2t - 1$

$\therefore$ $2t = 8$

$\therefore$ $t = 4$

Now substitute $t = 4$ in the equation, $y = 6 - 5t$

$y = 6 - 5(4) = 6 - 20 = -14$

$\therefore$ The y-coordinate of A is −14.

Example 7 : If A is (3, 2) and B is (4, −3). $P(x, y) \in \overline{AB}$. Find the maximum and minimum value of $3x - y$.

Solution : The parametric equations of $\overline{AB}$ are

$x = tx_2 + (1-t)x_1$ and $y = ty_2 + (1-t)y_1$ $t \in [0, 1]$

$\quad = t \cdot 4 + (1-t) \cdot 3$ $\quad = t(-3) + (1-t) \cdot 2$ $t \in [0, 1]$

$x = t + 3$ $\quad = -5t + 2$ $t \in [0, 1]$

Now, $\overline{AB} = \{(x, y) \mid x = t + 3, y = -5t + 2; t \in [0, 1]\}$

Now, $3x - y = 3(t + 3) - (-5t + 2) = 3t + 9 + 5t - 2 = 8t + 7$

Now, $P(x, y) \in \overline{AB}$. So $0 \le t \le 1$

$\therefore$ $0 \le 8t \le 8$

$\therefore$ $0 + 7 \le 8t + 7 \le 8 + 7$

$\therefore$ $7 \le 3x - y \le 15$ $\qquad (3x - y = 8t + 7)$

$\therefore$ Maximum value of $3x - y$ is 15 and minimum value is 7.

Example 8 : A(1, 2) and B(−1, 0) are given points. $P(x, y) \in \overline{AB}$. Find the maximum and minimum value of $10y - 2x$.

Solution : Parametric equations of $\overleftrightarrow{AB}$ are

$$x = tx_2 + (1 - t)x_1$$
$$y = ty_2 + (1 - t)y_1 \qquad t \in R$$

Here A(1, 2) and B(−1, 0) are given points.

$\therefore \quad x = t(-1) + (1 - t)1 = -t + 1 - t = 1 - 2t,\ t \in R$

$\quad y = t(0) + (1 - t)2 = 2 - 2t$

Now $10y - 2x = 10(2 - 2t) - 2(1 - 2t)$

$= 20 - 20t - 2 + 4t$

$= 18 - 16t$

As $P(x, y) \in \overline{AB}$. So $0 \le t \le 1$

$\therefore \quad 0 \ge -16t \ge -16$

$\therefore \quad 18 + 0 \ge 18 - 16t \ge 18 - 16$

$\therefore \quad 18 \ge 10y - 2x \ge 2$ **$(10y - 2x = 18 - 16t)$**

$\therefore \quad 2 \le 10y - 2x \le 18$

$\therefore$ Maximum value of $10y - 2x$ is 18 and minimum value of $10y - 2x$ is 2.

Example 9 : A(3, 5) and B(−2, 1) are given points. Find the ratio in which Y-axis divides $\overline{AB}$ from A ? Find the dividing point.

Solution : Let P(0, y) be the required point on the Y-axis as required.

Suppose P divides $\overline{AB}$ from A in the ratio λ.

$\therefore \quad (0, y) = \left(\frac{\lambda(-2) + 3}{\lambda + 1}, \frac{\lambda(1) + 5}{\lambda + 1}\right)$

$\therefore \quad \frac{-2\lambda + 3}{\lambda + 1} = 0$

$\therefore \quad -2\lambda + 3 = 0$

$\therefore \quad \lambda = \frac{3}{2}$

Now, $y = \frac{\lambda + 5}{\lambda + 1}$

$\therefore \quad y = \frac{\frac{3}{2} + 5}{\frac{3}{2} + 1} = \frac{3 + 10}{3 + 2} = \frac{13}{5}$

$\therefore$ Y-axis divides $\overline{AB}$ in the ratio $\lambda = \frac{3}{2}$ from A and the dividing point is $\left(0, \frac{13}{5}\right)$.

EXERCISE 6.2

1. Find a point P(a, b) on the line $x = 2t + 5$, $y = 3 - 3t$, $t \in$ R such that $a + b = 7$.
2. Let A(1, −5), B(5, −1). If P(x, y) $\in \overline{AB}$, find the maximum and minimum values of $2x - 5y$.
3. The parametric equations of $\overleftrightarrow{AB}$ are $x = 7t - 1$, $y = 4t + 7$, $t \in$ R. If the y-coordinate of a point P on the $\overleftrightarrow{AB}$ is 11, find the x-coordinate.
4. A(3, 2) and B(−10, 0) are given points in the plane. Express $\overleftrightarrow{AB}$, $\overrightarrow{AB}$, $\overline{AB}$, $\overleftrightarrow{AB} - \overline{AB}$ as sets.
5. For A(2, 5) and B(6, 5), show that (3, −9) is not on $\overleftrightarrow{AB}$.
6. Obtain the cartesian equation of the line passing through (−1, 1) and (2, −3).

*

6.10 Slope of a Line

Suppose a line is not perpendicular to Y-axis. Then the line l intersects X-axis in unique point P and let A $\in l$ and A is in upper semi-plane of X-axis. Let B be any point on $\overrightarrow{PX}$.

$\angle$APB is said to be the angle made by the line l with the positive direction of X-axis. If the line is perpendicular to Y-axis, we say it makes angle of general radian measure 0 with the positive direction of X-axis. Let $m\angle APB = \theta$. Obviously $0 < \theta < \pi$.

Slope : The trigonometrical tangent of the angle that a line not perpendicular to X-axis makes with positive direction of the X-axis in anticlock-wise direction is called the slope of the line. It is denoted by the symbol *m*.

If the line is perpendicular to Y-axis, the slope is defined to be $tan0 = 0$.

If the line is perpendicular to X-axis, its slope is not defined.

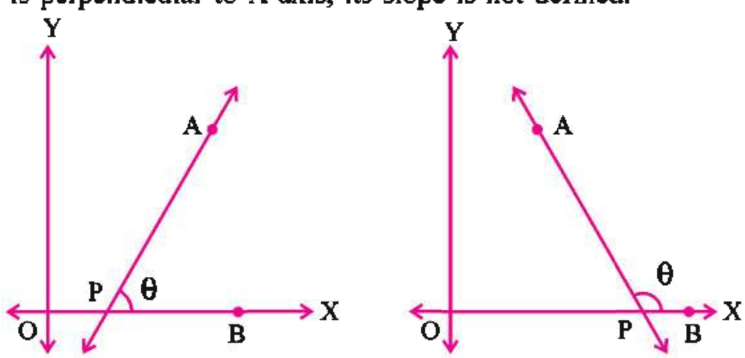


Figure 6.16

If θ is the measure of the angle made by the line with the positive direction of X-axis, then $tan\theta$ is called the slope or gradient of the line.

If the measure of the angle made by the line with the positive direction of X-axis in anticlock-wise direction is θ, then $0 < \theta < \pi$.

The slope of the line which makes an angle of measure $\frac{\pi}{2}$ with the positive direction of X-axis is not defined. i.e. slope of a vertical line is not defined.

Thus $m = tan\theta \quad 0 < \theta < \pi, \quad \theta \neq \frac{\pi}{2}$.

The slope of the horizontal line i.e. a line perpendicular to Y-axis is 0. Thus summing up slope of a line not perpendicular to X-axis is $m = tan\theta$

$0 \leq \theta < \pi, \theta \neq \frac{\pi}{2}$.

Expression for the slope of the line when coordinates of any two distinct points on the line are given.

Slope of a segment :

We will define slope of a segment and prove that all segments on a line have same slope. This constant slope of segments on a line is the slope of the line.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two distinct points in plane. Let $x_1 \neq x_2$, so that the $\overleftrightarrow{AB}$ is not vertical.

We define slope of $\overline{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ $\quad (x_1 \neq x_2)$

Now we will prove that this is a constant for any pair of distinct points on the line.

Let $C(x_3, y_3)$ and $D(x_4, y_4)$ be any two points on $\overleftrightarrow{AB}$. Since $\overleftrightarrow{AB}$ is not vertical, $x_3 \neq x_4$. Let perpendicular from A to Y-axis and perpendicular from B to X-axis intersect in M. Similarly N is obtained as the point of intersection of perpendiculars from C to Y-axis and D to X-axis respectively.

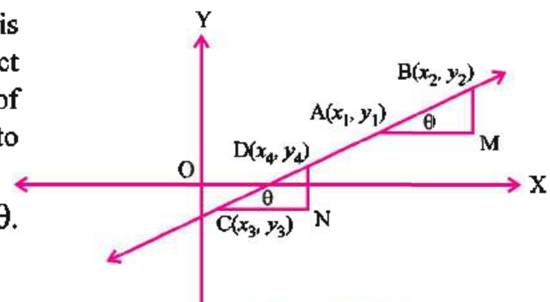


Figure 6.17(i)

Let $m\angle BAM = \theta$. Then $m\angle DCN = \theta$.

Also $0 < \theta < \frac{\pi}{2}$ in figure 6.17(i).

Also θ is the measure of the angle made by $\overleftrightarrow{AB}$ with the +ve direction of X-axis.

Now, $tan\theta = \frac{BM}{AM} = \frac{y_2 - y_1}{x_2 - x_1}$

Also $tan\theta = \frac{DN}{CN} = \frac{y_4 - y_3}{x_4 - x_3}$

$\therefore \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$

$\therefore \quad$ Slope of $\overline{AB}$ = Slope of $\overline{CD}$

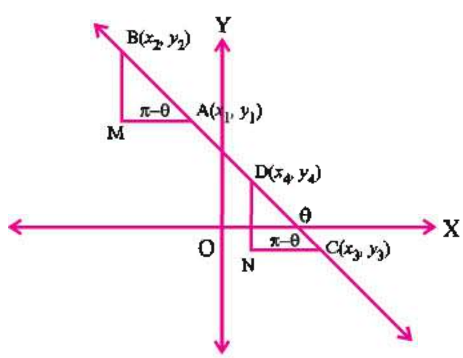


Figure 6.17(ii)

In figure 6.17 (ii) $\overleftrightarrow{AB}$ makes angle of measure θ with the +ve direction of X-axis where $\frac{\pi}{2} < \theta < \pi$.

$$tan(\pi - \theta) = \frac{y_2 - y_1}{x_1 - x_2} = -\frac{y_2 - y_1}{x_2 - x_1}$$

Now, $tan(\pi - \theta) = tan(-\theta)$ **(Period of *tan* is π)**

$= -tan\theta$ **(*tan* is an odd function)**

$\therefore \quad tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

Similarly $tan\theta = \dfrac{y_4 - y_3}{x_4 - x_3}$

$\therefore \quad tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_4 - y_3}{x_4 - x_3}$

Thus the slopes of any two segments on a line are same and this constant is the slope of the line.

$\therefore$ If $A(x_1, y_1)$, $B(x_2, y_2)$ lie on a non-vertical line, its slope is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

If the line is horizontal i.e. perpendicular to Y-axis, so, $y_1 = y_2$, slope of line $m = 0$.

Hence in all the cases slope of non-vertical $\overleftrightarrow{AB} = m = tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$.

6.11 Theorem

Every first degree equation in x, y represents a straight line.

$S = \{(x, y) \mid ax + by + c = 0, a^2 + b^2 \neq 0, a, b, c \in R\}$ represents a line.

Proof : Let $ax + by + c = 0$ be a first degree equation. $a, b, c \in R$, $a^2 + b^2 \neq 0$.

If $a \neq 0$, $b = 0$, then $S = \left\{(x, y) \mid x = \frac{-c}{a}\right\}$ and S represents a vertical line.

If $a = 0$, $b \neq 0$ then $S = \left\{(x, y) \mid y = \frac{-c}{b}\right\}$ and S represents a horizontal line.

Let $a \neq 0$, $b \neq 0$.

$\left(\frac{-c}{a}, 0\right)$ and $\left(-\frac{c+b}{a}, 1\right)$ are distinct members of S.

Let $A(x_1, y_1) \in S$, $B(x_2, y_2) \in S$

$\therefore \quad A(x_1, y_1)$ and $B(x_2, y_2)$ satisfy $ax + by + c = 0$.

$\therefore \quad ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c_2 = 0$

Now parametric equations of $\overleftrightarrow{AB}$ are

$$\left.\begin{array}{l} x = tx_2 + (1 - t)x_1 \\ y = ty_2 + (1 - t)y_1 \end{array}\right\} ; t \in R$$

Let $C(x_3, y_3)$ be any point on $\overleftrightarrow{AB}$.

$$\therefore \quad \left.\begin{array}{l} x_3 = tx_2 + (1 - t)x_1 \\ y_3 = ty_2 + (1 - t)y_1 \end{array}\right\} ; \text{for some } t \in R$$

Now $ax_3 + by_3 + c$

$= a\,[tx_2 + (1-t)\,x_1] + b\,[ty_2 + (1-t)\,y_1] + c(t + 1 - t)$

$= t\,(ax_2 + by_2 + c) + (1-t)(ax_1 + by_1 + c)$

$= t \cdot 0 + (1-t) \cdot 0 = 0$

$\therefore$ $C(x_3, y_3)$ satisfies $ax + by + c = 0$.

Thus every point on the line $\overleftrightarrow{AB}$ satisfies $ax + by + c = 0$.

$\therefore$ $\overleftrightarrow{AB} \subset S$

Now $A(x_1, y_1) \in S$, $B(x_2, y_2) \in S$. Let $C(x_3, y_3) \in S$.

A, B, C satisfy the equation $ax + by + c = 0$, $a, b, c \in R$, $a^2 + b^2 \neq 0$

$ax_1 + by_1 + c = 0$ **(i)**

$ax_2 + by_2 + c = 0$ **(ii)**

$ax_3 + by_3 + c = 0$ **(iii)**

Now, $a \neq 0$, $b \neq 0$

$\therefore$ $y_1 \neq y_2, x_1 \neq x_2, y_2 \neq y_3, x_2 \neq x_3, y_3 \neq y_1, x_3 \neq x_1$,

From (i) and (ii), $ax_1 + by_1 = ax_2 + by_2$

$\therefore$ $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-a}{b}$

Similarly from (ii) and (iii) we get,

$\dfrac{y_3 - y_1}{x_3 - x_1} = \dfrac{-a}{b}$

$\therefore$ $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_3 - y_1}{x_3 - x_1}$

$\therefore$ $\dfrac{y_3 - y_1}{y_2 - y_1} = \dfrac{x_3 - x_1}{x_2 - x_1} = t$ (say)

$\therefore$ $\dfrac{y_3 - y_1}{y_2 - y_1} = t$ and $\dfrac{x_3 - x_1}{x_2 - x_1} = t$

$\therefore$ $\left.\begin{array}{l} y_3 = ty_2 + (1-t)\,y_1 \\ x_3 = tx_2 + (1-t)\,x_1 \end{array}\right\}$; $t \in R$

$\therefore$ $C(x_3, y_3)$ satisfies the equation of $\overleftrightarrow{AB}$.

$\therefore$ $S \subset \overleftrightarrow{AB}$.

$\therefore$ S represents a line.

$\therefore$ $S = \overleftrightarrow{AB}$.

Remember :

- If $0 < \theta < \frac{\pi}{2}$, then the slope of $\overleftrightarrow{AB}$ is positive.
- If $\frac{\pi}{2} < \theta < \pi$, then the slope of $\overleftrightarrow{AB}$ is negative.
- If $\overleftrightarrow{AB}$ is X-axis or parallel to X-axis, then slope of $\overleftrightarrow{AB}$ is 0.
- If $\overleftrightarrow{AB}$ is Y-axis or parallel to Y-axis, then slope of $\overleftrightarrow{AB}$ is not defined.

Example 10 : Find the slope of the line which passes through A(1, 2) and B(3, 6).

Solution : The slope of $\overleftrightarrow{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{3-1} = \frac{4}{2} = 2$

6.12 Necessary and Sufficient Condition for Two Distinct Lines to be Parallel

Suppose two distinct lines l_1 and l_2 are neither horizontal nor vertical.

Let m_1 and m_2 be the slopes of the lines l_1 and l_2 respectively. So $m_1 \neq 0$, $m_2 \neq 0$.

Let θ_1 and θ_2 be the measures of angles made by the lines l_1 and l_2 respectively with the positive direction of the X-axis.

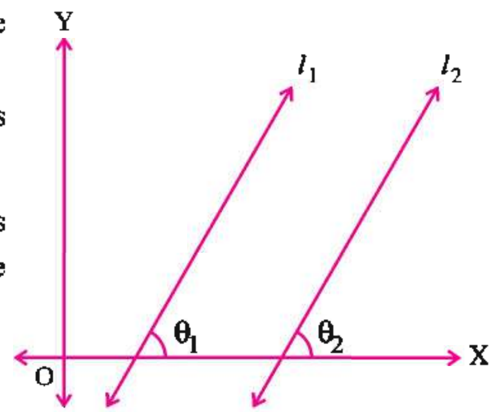


Figure 6.18

We have

$0 < \theta_1, \theta_2 < \pi$ and $\theta_1 \neq \frac{\pi}{2}, \theta_2 \neq \frac{\pi}{2}$

$\therefore \quad m_1 = tan\theta_1$ and $m_2 = tan\theta_2$

Now $l_1 \parallel l_2 \Leftrightarrow \theta_1 = \theta_2$

$\Leftrightarrow tan\theta_1 = tan\theta_2$ $\left(tan \text{ is one-one function for } \theta \in (0, \pi) - \left\{\frac{\pi}{2}\right\}\right)$

$\Leftrightarrow m_1 = m_2$

If lines l_1 and l_2 both are perpendicular to Y-axis, then $m_1 = 0$ and $m_2 = 0$.

So $m_1 = m_2$.

Conversely if $m_1 = m_2 = 0$, then lines are horizontal.

$\therefore$ They are parallel.

If lines l_1 and l_2 both are perpendicular to X-axis, then slope of both the lines are undefined. In this case we can also say if the slopes of lines l_1 and l_2 are not defined, then they must be perpendicular to X-axis. So they are parallel.

Thus $l_1 \parallel l_2 \Leftrightarrow$ they have the same slopes or both of them have no slope.

Example 11 : Find k if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ where A(2, k), B(−1, 3), C(1, 7) and D(3, 2) are given points.

Solution : The slope of $\overleftrightarrow{AB} = \frac{3-k}{-1-2} = \frac{3-k}{-3}$. Slope of $\overleftrightarrow{CD} = \frac{2-7}{3-1} = \frac{-5}{2}$.

Here $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

$\therefore$ Slope of $\overleftrightarrow{AB}$ = slope of $\overleftrightarrow{CD}$.

$\therefore \frac{3-k}{-3} = \frac{-5}{2}$

$\therefore 6 - 2k = 15$

$\therefore 2k = -9$

$\therefore k = \frac{-9}{2}$

Also B, C, D are not collinear. (Verify !)

This means $\overleftrightarrow{AB} \neq \overleftrightarrow{CD}$.

$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \Rightarrow k = \frac{-9}{2}$

6.13 Necessary and Sufficient Condition for Two Distinct Lines to be Perpendicular to Each Other.

In coordinate geometry we take two mutually perpendicular lines. We consider one of them as X-axis and the other as Y-axis. So obviously one line parallel to X-axis and other parallel to Y-axis are perpendicular to each other.

Condition that Two Lines are Perpendicular to Each Other

Let two lines l_1 and l_2, none of them perpendicular to any axis, intersect X-axis in B(x_2, 0) and C(x_3, 0) respectively. Let $l_1 \nparallel l_2$ and they intersect in A(x_1, y_1).

Slope of line l_1, $m_1 = \frac{y_1}{x_1 - x_2}$.

Similarly slope of line l_2, $m_2 = \frac{y_1}{x_1 - x_3}$.

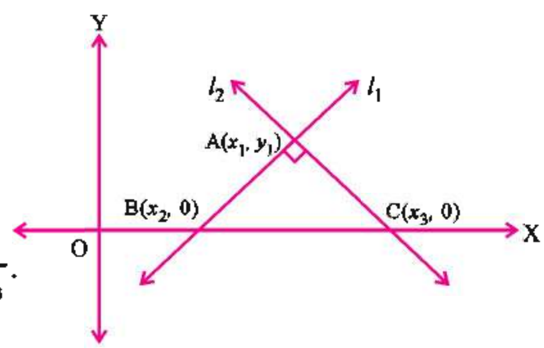


Figure 6.19

$$l_1 \perp l_2 \Leftrightarrow m\angle BAC = \frac{\pi}{2}$$

$$\Leftrightarrow AB^2 + AC^2 = BC^2$$

$$\Leftrightarrow (x_1 - x_2)^2 + y_1^2 + (x_1 - x_3)^2 + y_1^2 = (x_2 - x_3)^2$$

$$\Leftrightarrow x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 + x_1^2 - 2x_1x_3 + x_3^2 + y_1^2 = x_2^2 - 2x_2x_3 + x_3^2$$

$$\Leftrightarrow y_1^2 = x_1x_2 - x_2x_3 + x_1x_3 - x_1^2$$

$$\Leftrightarrow\ y_1{}^2 = x_2(x_1 - x_3) - x_1(x_1 - x_3)$$

$$\Leftrightarrow\ y_1{}^2 = -(x_1 - x_3)(x_1 - x_2)$$

$$\Leftrightarrow\ \left(\frac{y_1}{x_1 - x_3}\right)\left(\frac{y_1}{x_1 - x_2}\right) = -1$$

$$\Leftrightarrow\ m_1 m_2 = -1$$

Thus $l_1 \perp l_2 \Leftrightarrow m_1 m_2 = -1$

Note : Since lines are not vertical, $x_1 \neq x_3$, $x_1 \neq x_2$.

Also if one of the lines is perpendicular to X-axis and the other is perpendicular Y-axis, they are perpendicular. Here $m_2 = 0$ and m_1 does not exist.

So we can say that lines having non-zero slopes m_1 and m_2 are perpendicular to each other if and only if $m_1 m_2 = -1$.

Example 12 : Prove that A(2, 1), B(1, 2) and C(3, 4) are vertices of a right triangle.

Solution : Slope of $\overleftrightarrow{AB} = \frac{2-1}{1-2} = \frac{1}{-1} = -1$

Slope of $\overleftrightarrow{AC} = \frac{4-1}{3-2} = 3$

Slope of $\overleftrightarrow{BC} = \frac{4-2}{3-1} = \frac{2}{2} = 1$

$\therefore$ Slopes of all the lines are distinct.

So they form a triangle.

Slope of $\overleftrightarrow{AB}$ × slope of $\overleftrightarrow{BC} = (-1)(1) = -1$

$\therefore$ $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ and $\angle$B is a right angle.

$\therefore$ A, B, C are vertices of a right angle triangle.

6.14 Angle Between Two Intersecting Lines

Two distinct lines in R^2 intersect each other, if they are not parallel.

Now if lines are perpendicular to each other, then the angle between them has radian measure $\frac{\pi}{2}$. We will assume following formula from trigonometry. We will study it in detail in semester II.

$$tan(\theta_1 - \theta_2) = \frac{tan\theta_1 - tan\theta_2}{1 + tan\theta_1 tan\theta_2},\ \text{if } tan\theta_1\, tan\theta_2 \neq -1.$$

If two intersecting distinct lines are not perpendicular to each other, then two pairs of congruent vertically opposite angles are formed at their point of intersection. One of these pairs is a pair of congruent obtuse angles and the other is a pair of congruent acute angles.

The radian measure of this acute angle is called the measure of the angle between the two lines. If α is the measure of the angle between the lines, then $0 < \alpha < \frac{\pi}{2}$.

(1) Measure of the angle between two lines, when one line is vertical line and the other has slope *m*.

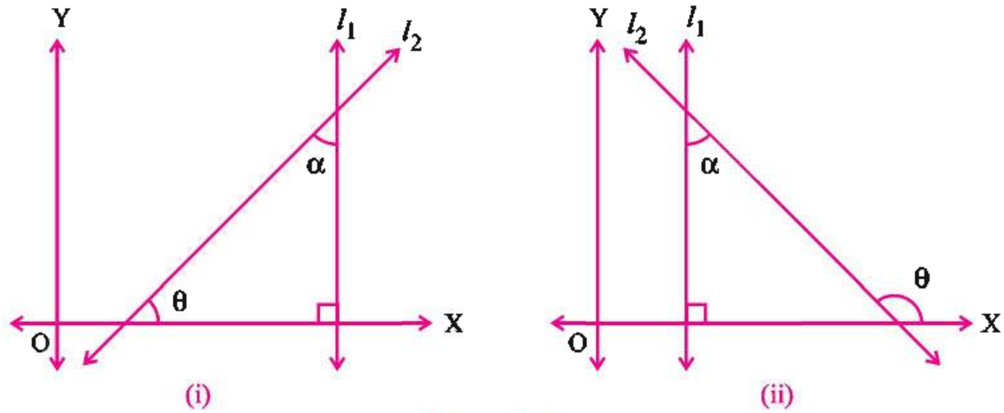


Figure 6.20

l_1 is a vertical line. Slope of line l_1 is not defined. If l_2 makes an angle of measure θ with the positive direction of X-axis, then $m = \tan\theta$. $0 < \theta < \pi$, $\theta \neq \frac{\pi}{2}$.

Suppose α is the measure of the angle between the lines.

As in figure 6.20(i) $0 < \theta < \frac{\pi}{2}$ and $\alpha = \frac{\pi}{2} - \theta$

Also $\frac{\pi}{2} - \theta > 0$ and $\left|\frac{\pi}{2} - \theta\right| = \frac{\pi}{2} - \theta$

As in figure 6.20(ii), $\frac{\pi}{2} < \theta < \pi$ and $\alpha = \theta - \frac{\pi}{2} = -\left(\frac{\pi}{2} - \theta\right)$

Also $\frac{\pi}{2} - \theta < 0$ and $\left|\frac{\pi}{2} - \theta\right| = -\left(\frac{\pi}{2} - \theta\right)$

So in any case $\alpha = \left|\frac{\pi}{2} - \theta\right|$

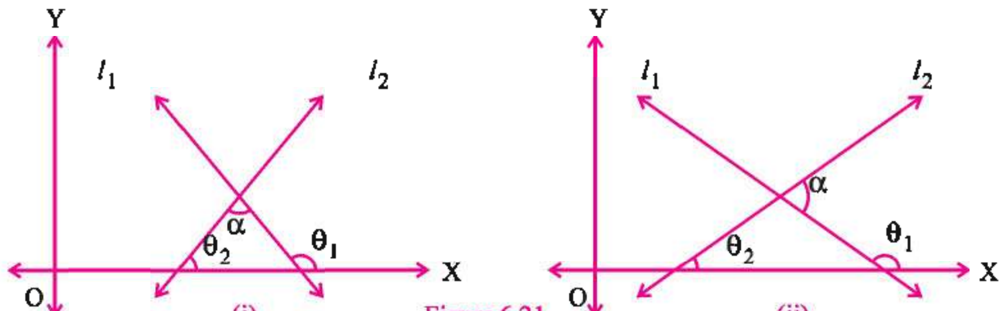
Thus if one of the line is vertical and the other makes an angle of measure θ with the positive direction of X-axis, the measure of the angle α between the lines is given by $\alpha = \left|\frac{\pi}{2} - \theta\right|$.

(2) Angle Between Two Lines; Neither of them is Perpendicular to any Axis

Let l_1 and l_2 be two non-vertical lines with slopes m_1 and m_2 respectively.

If θ_1 and θ_2 are the inclinations of lines l_1 and l_2 respectively,

then $m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$, $0 < \theta_1, \theta_2 < \pi$, $\theta_1 \neq \frac{\pi}{2}$, $\theta_2 \neq \frac{\pi}{2}$



(i) Figure 6.21 (ii)

Case 1 : As shown in figure 6.21(i), $\theta_1 = \alpha + \theta_2$

$$\therefore \quad \alpha = \theta_1 - \theta_2$$

$$\therefore \quad \tan\alpha = \tan(\theta_1 - \theta_2)$$

$$\therefore \quad \tan\alpha = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\therefore \quad \tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$$

Case 2 : As shown in figure 6.21(ii), $\theta_1 = (\pi - \alpha) + \theta_2$

$$\therefore \quad \alpha = \pi - (\theta_1 - \theta_2)$$

$$\therefore \quad \tan\alpha = \tan(\pi - (\theta_1 - \theta_2))$$

$$= -\tan(\theta_1 - \theta_2) \qquad \textbf{(as } \tan(\pi - \theta) = \tan(-\theta) = -\tan\theta\textbf{)}$$

$$= -\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= -\frac{m_1 - m_2}{1 + m_1 m_2}$$

So from both the cases, we find that the measure of the angle α between two lines having slopes m_1 and m_2 is given by

$$\tan\alpha = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

But $0 < \alpha < \frac{\pi}{2}$. Hence $\tan\alpha > 0$.

$$\therefore \quad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Thus if none of the lines is vertical, the measure of the angle α between the two lines is obtained by the formula $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, where $0 < \alpha < \frac{\pi}{2}$.

Example 13 : A(1, 2), B(3, 5) and C(6, 1) are given points. Find $\tan\alpha$, where α is the measure of the angle between lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

Solution : Slope of $\overleftrightarrow{AB} = \frac{5-2}{3-1} = \frac{3}{2}$.

Slope of $\overleftrightarrow{AC} = \frac{1-2}{6-1} = \frac{-1}{5}$.

Suppose the angle between the lines has measure α.

$$\therefore \quad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\therefore \quad \tan\alpha = \left|\frac{\frac{3}{2} - \left(-\frac{1}{5}\right)}{1 + \frac{3}{2}\left(-\frac{1}{5}\right)}\right|$$

$$= \left|\frac{\frac{3}{2} + \frac{1}{5}}{1 - \frac{3}{10}}\right|$$

$$= \left|\frac{15+2}{7}\right|$$

$$= \frac{17}{7}$$

> **Note** l_1 and l_2 are two lines with slopes m_1 and m_2 respectively.
>
> If lines are parallel, then angle between them has general measure 0°. i.e. $\alpha = 0$.
>
> $\therefore \quad \tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = 0$ as $m_1 = m_2$.

6.15 The Slope of the Line Represented by $ax + by + c = 0$, $a^2 + b^2 \neq 0$, $a, b, c \in R$

Suppose a line is neither vertical nor horizontal. So that $a \neq 0$, $b \neq 0$. Then the line intersects X-axis at $A\left(\frac{-c}{a}, 0\right)$ and Y-axis at $B\left(0, \frac{-c}{b}\right)$. Let $c \neq 0$. So that $A \neq B$.

$$\therefore \quad \text{Slope of line} = \frac{0 - \left(\frac{-c}{b}\right)}{\frac{-c}{a} - 0} = \frac{-a}{b}$$

If $a = 0$, then the line is a horizontal line, so slope of the line is $m = 0$.

If $b = 0$, then the line is a vertical line, so slope of the line is not defined.

If $c = 0$, i.e. if the line passes through (0, 0) then A = B.

In that case we can take A(0, 0) and $B\left(\frac{-b}{a}, 1\right)$.

Again $m = \dfrac{1-0}{\frac{-b}{a}-0} = \dfrac{-a}{b}$.

If $b \neq 0$, slope of the line $ax + by + c = 0$ $(a^2 + b^2 \neq 0)$ is given by $m = \dfrac{-a}{b}$ and if $b = 0$, the line has no slope.

Example 14 : Find the measure of the angle between the lines $\sqrt{3}x + y = 5$ and $x + \sqrt{3}y + 7 = 0$.

Solution : Slope of the line $\sqrt{3}x + y = 5$ is $m_1 = -\sqrt{3}$

Slope of the line $x + \sqrt{3}y + 7 = 0$ is $m_2 = \dfrac{-1}{\sqrt{3}}$

If α is the measure of the angle between the lines,

$$\therefore \quad tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{-\sqrt{3} - \left(-\frac{1}{\sqrt{3}}\right)}{1 + (-\sqrt{3})\left(-\frac{1}{\sqrt{3}}\right)}\right|$$

$$= \left|\frac{-\sqrt{3} + \frac{1}{\sqrt{3}}}{1+1}\right|$$

$$= \left|\frac{-3+1}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$$

Since $0 < \alpha < \frac{\pi}{2}$, $\alpha = \frac{\pi}{6}$.

Hence measure of the angle between the lines is $\frac{\pi}{6}$.

Example 15 : Find the measure of angle between the lines $x - 6 = 0$ and $\sqrt{3}x - y + 5 = 0$.

Solution : $x - 6 = 0$ is a vertical line.

$\therefore$ Its slope is not defined.

$\therefore$ The slope of $\sqrt{3}x - y + 5 = 0$ is $m = tan\theta = \sqrt{3}$.

$\therefore$ $\theta = \frac{\pi}{3}$ $\qquad$ **$(0 < \theta < \frac{\pi}{2})$**

$\therefore$ Measure of the angle between the lines $\alpha = \left|\frac{\pi}{2} - \theta\right|$

$$= \left|\frac{\pi}{2} - \frac{\pi}{3}\right|$$

$$= \frac{\pi}{6}$$

Example 16 : Find the measure of the angle between the lines $3x + y + 5 = 0$ and $x + 2y + 7 = 0$.

Solution : Slope of the line $3x + y + 5 = 0$ is $m_1 = -3$

$\therefore$ Slope of the line $x + 2y + 7 = 0$ is $m_2 = -\frac{1}{2}$

If α is the measure of the angle between given lines,

$$tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{-3-\left(-\frac{1}{2}\right)}{1+(-3)\left(-\frac{1}{2}\right)}\right|$$

$$= \left|\frac{-3+\frac{1}{2}}{1+\frac{3}{2}}\right|$$

$$= \left|\frac{-6+1}{3+2}\right| = |-1| = 1$$

$\therefore$ $\alpha = \frac{\pi}{4}$ as $0 < \alpha < \frac{\pi}{2}$

$\therefore$ The measure of the angle between the lines is $\frac{\pi}{4}$.

EXERCISE 6.3

1. The measure of the angle between two lines is $\frac{\pi}{4}$ and one of the lines has slope $\frac{1}{3}$. Find the slope of the other line.
2. A(2, 1), B(3, −1) and C(−3, 4) are the mid-points of the sides of ΔPQR. Find the slope of the line containing each side.
3. A(k, 3), B(2, −1), C(0, 5), D(6, 7) are given points. Find k if $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ and find k also if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.
4. P(−1, 2), Q(7, 6), R(−1, 5), S(0, 3) are given points. Prove that $\overleftrightarrow{PQ} \perp \overleftrightarrow{RS}$.
5. Find the measure of the angle between the lines $y - 5 = 0$ and $x + y + 3 = 0$.
6. Prove using slopes of the lines that the line segment joining the mid-points of $\overline{AB}$ and $\overline{AC}$ in ΔABC is parallel to $\overline{BC}$.
7. Prove using slopes that if the mid-points of the sides of a quadrilateral are joined in order, we get a parallelogram.
8. Using slopes of lines, show that (−1, 4), (2, 3) and (8, 1) are collinear.
9. Find the slope of the perpendicular bisector of $\overline{AB}$ where, A is (2, 6) and B is (0, −1).

10. $A(x_1, y_1)$, $B(x_2, y_2)$ are given points. If $P(h, k) \in \overleftrightarrow{AB}$,
prove that $(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1)$

11. Find the slope of the line which passes through the origin and the mid-point of the line segment joining the points A(3, 1) and B(1, 5).

12. If three points (2, 0), (0, 3) and (a, b) are on the same line, show that $\frac{a}{2} + \frac{b}{3} = 1$.

13. The slope of a line is double than that of another line. If the measure of the angle between them is α and $\tan\alpha = \frac{1}{3}$, find the slopes of the lines.

*

6.16 Intercepts of a Line on the Axes

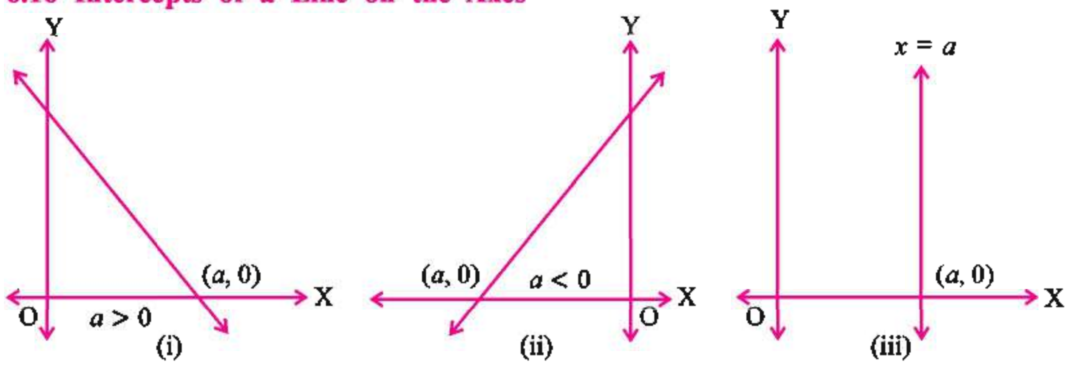


Figure 6.22

X intercept : If a line intersects X-axis in unique point $(a, 0)$, then a is called the X-intercept of the line. A line perpendicular to Y-axis has no X-intercept.

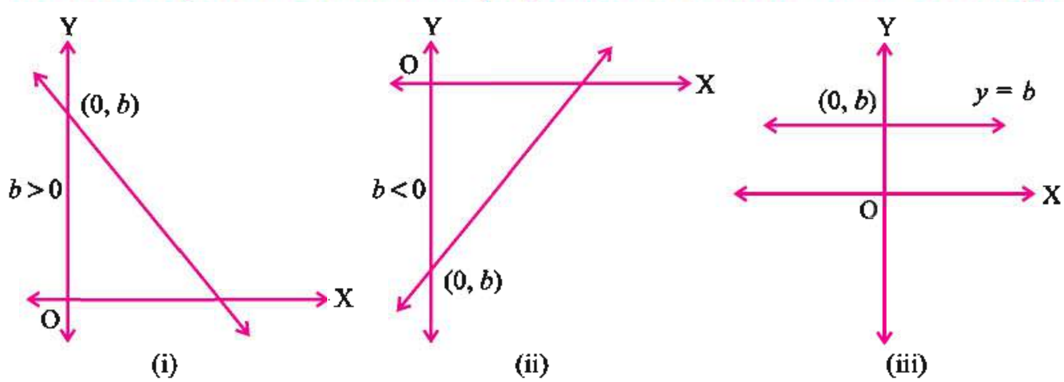


Figure 6.23

Y intercept : If a line intersects Y-axis in a unique point $(0, b)$, then b is called the Y-intercept of the line. A line perpendicular to X-axis has no Y-intercept.

Now let us obtain the intercepts of the line $ax + by + c = 0$, $a^2 + b^2 \neq 0$.

If $a \neq 0$, then line $ax + by + c = 0$ is not perpendicular to Y-axis.

To find X-intercept, substitute $y = 0$ in the equation and we get $x = \frac{-c}{a}$.

Thus the line intersects X-axis in $\left(\frac{-c}{a}, 0\right)$.

$\therefore$ Its X-intercept is $\frac{-c}{a}$

If $b \neq 0$, then line $ax + by + c = 0$ is not perpendicular to X-axis.

To find Y-intercept, substitute $x = 0$ and we get $y = \frac{-c}{b}$.

Thus the line intersects Y-axis in $\left(0, \frac{-c}{b}\right)$.

$\therefore$ Its Y-intercept is $\frac{-c}{b}$

If $a = 0$, then the line is $by + c = 0$. It does not intersect X-axis, if $c \neq 0$. So its X-intercept is not defined. If $c = 0$, the line is X-axis and it has no X-intersect. (Why ?)

If $b = 0$, then the line is $ax + c = 0$. It does not intersect Y-axis, if $c \neq 0$. So its Y-intercept is not defined. If $c = 0$, the line is Y-axis and has no Y-intersect.

If $c = 0$ and $a \neq 0$, $b \neq 0$, then the line is $ax + by = 0$. Both the intercepts are zero.

Example 17 : Obtain the intercepts on the axes of the following lines, if they exist.

(1) $2x + 3y - 6 = 0$ (2) $2x - 7y = 0$ (3) $2x - 8 = 0$ (4) $2y + 1 = 0$

Solution : (1) For the line $2x + 3y - 6 = 0$

$a = 2$, $b = 3$, $c = -6$

X-intercept $= \frac{-c}{a} = -\frac{(-6)}{2} = 3$, Y-intercept $= \frac{-c}{b} = -\frac{(-6)}{3} = 2$

(2) For the line $2x - 7y = 0$,

line passes through origin and it is not any axis.

$a = 2$, $b = -7$, $c = 0$

X-intercept $= \frac{-c}{a} = 0$, Y-intercept $= \frac{-c}{b} = 0$

(3) For the line $2x - 8 = 0$,

$a = 2$, $b = 0$, $c = -8$

Since $b = 0$, the line is a vertical line.

$\therefore$ Y-intercept is not defined.

X-intercept $= \frac{-c}{a} = -\frac{(-8)}{2} = 4$

(4) For the line $2y + 1 = 0$

$a = 0$, $b = 2$, $c = 1$

Here $a = 0$. So the line is a horizontal line.

$\therefore$ X-intercept is not defined.

Y-intercept $= \frac{-c}{b} = -\frac{1}{2}$

6.17 Various Forms of Equations of a Line

(1) Point Slope Form :

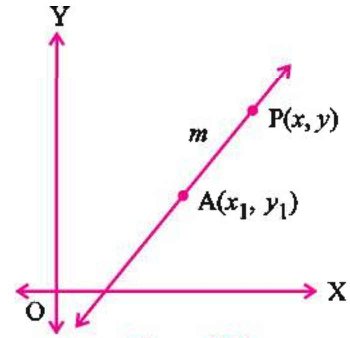


Figure 6.24

Suppose l is a non-vertical line with the slope m. Since line is not vertical, no two points on it have same x-coordinate.

Let $P(x, y)$ be any other point on the line l. $(x \neq x_1)$

So by the definition, the slope of the line l is given by

$$m = \frac{y - y_1}{x - x_1}$$

i.e. $y - y_1 = m(x - x_1)$

Also every point satisfying this equation is on the line.

Hence the equation of the non-vertical line, passing through (x_1, y_1) and having slope m is

$$y - y_1 = m(x - x_1)$$

> **☛ Note** See that $A(x_1, y_1)$ also satisfies the equation.

Example 18 : Find the equation of the line passing through (1, 2) and having slope $\frac{1}{2}$.

Solution : Equation of line is $y - y_1 = m(x - x_1)$

$\therefore \quad y - 2 = \frac{1}{2}(x - 1)$

$\therefore \quad 2y - 4 = x - 1$

$\therefore \quad x - 2y + 3 = 0$

(2) Two Point Form :

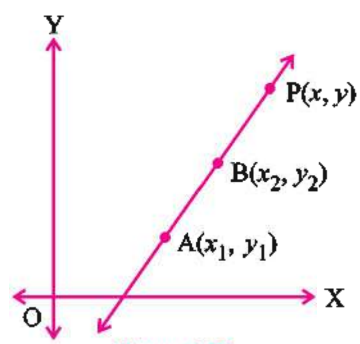


Figure 6.25

$A(x_1, y_1)$ and $B(x_2, y_2)$ are given distinct points.

Suppose $\overleftrightarrow{AB}$ is not perpendicular to any axis.

$\therefore \quad x_1 \neq x_2, y_1 \neq y_2$

Let $P(x, y)$ be any point other than A or B on the $\overleftrightarrow{AB}$.

$\therefore \quad$ A, P, B are collinear points.

Since the line is not perpendicular to any of the axes,

$x \neq x_1, y \neq y_1, x \neq x_2, y \neq y_2$

Slope of $\overleftrightarrow{AP}$ = Slope of $\overleftrightarrow{AB}$

$\therefore \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

$\therefore \quad \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

Also any point satisfying this equation is on $\overleftrightarrow{AB}$.

If $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} = t$ (say) for any P(x, y), it is easy to see that

$$\left.\begin{aligned} x &= tx_2 + (1 - t)x_1 \\ y &= ty_2 + (1 - t)y_1 \end{aligned}\right\} ; t \in R$$

$\therefore \quad P(x, y) \in \overleftrightarrow{AB}$.

Hence the equation of the line not perpendicular to any axis and passing through A(x_1, y_1) and B(x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

☛ Note See that A and B also satisfy the equation.

(3) Slope-intercept Form :

Suppose a line is not perpendicular to X-axis and has Y-intercept c. Hence it passes through (0, c).

Let the slope of the line be m.

$\therefore$ The equation of the line passing through (0, c) and having slope m is

$$y - c = m(x - 0)$$

$\therefore \quad y = mx + c$

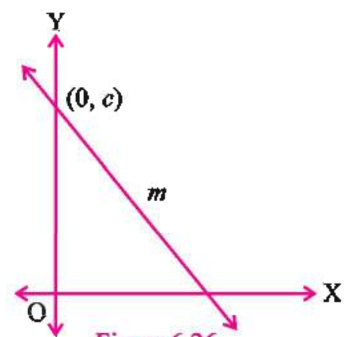


Figure 6.26

☛ Note If the line l with slope m makes X-intercept d, then equation of the line is $y = m(x - d)$.

Example 19 : Find the equation of the line making an angle of measure $\frac{\pi}{3}$ with positive direction of X-axis and having Y-intercept 3.

Solution : Given line makes an angle of measure $\frac{\pi}{3}$ with positive direction of X-axis.

$\therefore \quad m = \tan\frac{\pi}{3} = \sqrt{3}$

Also Y-intercept is 3.

Equation of the line is $y = mx + c$

$\therefore \quad y = \sqrt{3}x + 3$

(4) Intercept form of the Equation of a Line :

Suppose a line l makes X-intercept a and Y-intercept b on axes, where $a \neq 0$, $b \neq 0$.

$\therefore$ Line l is passing through A(a, 0) and B(0, b)

Its equation is $\frac{y-b}{0-b} = \frac{x-0}{a-0}$ **(Two point form)**

$\therefore$ $ay - ab = -bx$

$\therefore$ $bx + ay = ab$

$\therefore$ $\frac{x}{a} + \frac{y}{b} = 1$ **($a \neq 0$, $b \neq 0$)**

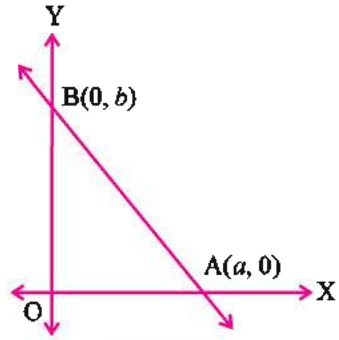


Figure 6.27

Thus the equation of the line making intercepts a and b on X-axis and Y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$.

☛ Note This form is valid only for lines not perpendicular to any axis and not passing through origin.

Example 20 : Find the equation of the line passing through (1, 2) and having sum of intercepts 6.

Solution : Suppose X-intercept and Y-intercept of the line are a and b respectively.

The equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

The line is passing through (1, 2), so $\frac{1}{a} + \frac{2}{b} = 1$ (i)

Now sum of intercepts is 6

$a + b = 6$

$b = 6 - a$

$\therefore$ From (i) $\frac{1}{a} + \frac{2}{6-a} = 1$

$6 - a + 2a = 6a - a^2$

$a^2 - 5a + 6 = 0$

$a = 3$ or $a = 2$

$a = 3 \Rightarrow b = 6 - a = 3$

$\therefore$ The equation of the line is $\frac{x}{3} + \frac{y}{3} = 1$ or $x + y = 3$.

$a = 2 \Rightarrow b = 6 - a = 4$

$\therefore$ The equation of the line is $\frac{x}{2} + \frac{y}{4} = 1$ or $2x + y = 4$.

$\therefore$ There are two lines with equations $x + y = 3$ and $2x + y = 4$ as required.

(5) The $p - \alpha$ Form :

(1) Suppose a line l does not pass through the origin.

Let M be the foot of the perpendicular from the origin to line l. Let OM = p.

p is the distance of the line from the origin.

Suppose $\overrightarrow{OM}$ makes an angle of measure α with positive direction of X-axis. $\alpha \in (-\pi, \pi]$.

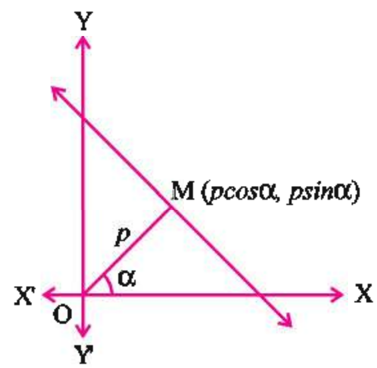


Figure 6.28

Let $\alpha \neq 0, \pm\frac{\pi}{2}, \pi$

$\therefore$ Slope of $\overleftrightarrow{OM} = \frac{p\sin\alpha - 0}{p\cos\alpha - 0} = \tan\alpha$

Since $\overleftrightarrow{OM} \perp$ line l,

slope of line $l = -\frac{1}{\tan\alpha} = -\cot\alpha$ **($m_1m_2 = -1$ for perpendicular lines)**

Line l passes through ($p\cos\alpha$, $p\sin\alpha$) and has the slope $-\cot\alpha$.

$\therefore$ The equation of the line l is $y - p\sin\alpha = -\cot\alpha\,(x - p\cos\alpha)$

$\therefore$ $y - p\sin\alpha = \frac{-\cos\alpha}{\sin\alpha}(x - p\cos\alpha)$

$y\sin\alpha - p\sin^2\alpha = -x\cos\alpha + p\cos^2\alpha$

$\therefore$ $x\cos\alpha + y\sin\alpha = p(\cos^2\alpha + \sin^2\alpha)$

$\therefore$ $x\cos\alpha + y\sin\alpha = p$

$\therefore$ **The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle of measure α with positive X-axis is ($\alpha \neq 0, \pm\frac{\pi}{2}, \pi$)**

$$x\cos\alpha + y\sin\alpha = p \quad (\alpha \in (-\pi, \pi])$$

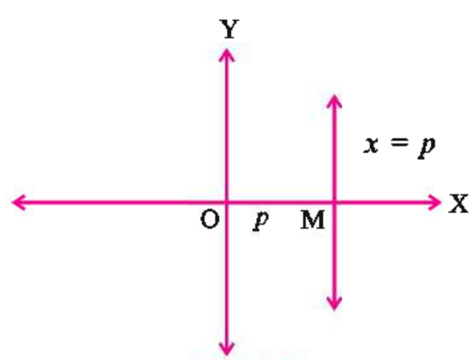


Figure 6.29

If $\alpha = 0$, the line is perpendicular to X-axis and its equation is $x = p$.

Also $cos0 = 1$, $sin0 = 0$.

$x = p$ is same as $xcos0 + ysin0 = p$

If $\alpha = \pi$, the equation of the line is $x = -p$.

Also $cos\pi = -1$, $sin\pi = 0$

$\therefore \quad xcos\pi + ysin\pi = p$

becomes $-x = p$ or $x = -p$.

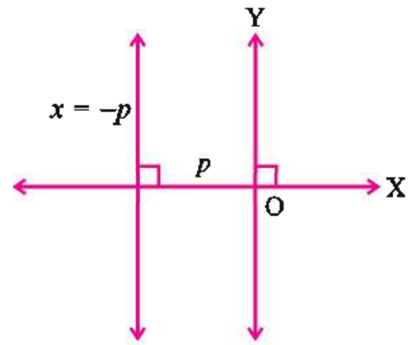


Figure 6.30

If $\alpha = \pm \frac{\pi}{2}$, then equation of the line is

$y = p$ or $y = -p$

according as $\alpha = \frac{\pi}{2}$ or $\alpha = -\frac{\pi}{2}$ respectively.

$xcos\frac{\pi}{2} + ysin\frac{\pi}{2} = p$ is $y = p$

$xcos\left(-\frac{\pi}{2}\right) + ysin\left(-\frac{\pi}{2}\right) = p$ is $y = -p$

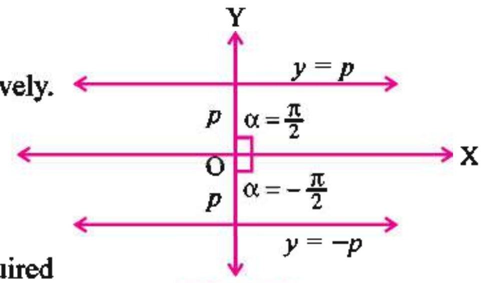


Figure 6.31

Thus in all cases the equation of the required line is $xcos\alpha + ysin\alpha = p$.

Example 21 : The length of the perpendicular from the origin to a line is 7 and the angle made by the perpendicular to the line from the origin with the positive direction of X-axis has measure $\frac{\pi}{6}$. Find the equation of the line.

Solution : Here $p = 7$, $\alpha = \frac{\pi}{6}$

$\therefore$ The equation of the line is $xcos\alpha + ysin\alpha = p$

$\therefore \quad xcos\frac{\pi}{6} + ysin\frac{\pi}{6} = 7$

$\therefore \quad \frac{\sqrt{3}x}{2} + \frac{y}{2} = 7$

$\therefore \quad \sqrt{3}x + y = 14$

To find the distance of the line from the origin p and α, where $\alpha \in (-\pi, \pi]$ and $p \neq 0$, if the ray starting from the origin and perpendicular to the line $ax + by + c = 0$, $a, b, c \in R$, $a^2 + b^2 \neq 0$ intersects the unit circle in $P(\alpha)$.

The $p - \alpha$ form of the equation of the line is $xcos\alpha + ysin\alpha = p$

The cartesian equation of the same line is $ax + by + c = 0$.

The line does not pass through origin, so $c \neq 0$ and $p \neq 0$.

Let $a \neq 0$, $b \neq 0$.

Hence the line is not perpendicular to any axis.

So we have $\alpha \neq 0, \pi, \frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Both the equations are same.

$\therefore \quad \frac{a}{\cos\alpha} = \frac{b}{\sin\alpha} = -\frac{c}{p}$

$\therefore \quad \cos\alpha = \frac{-ap}{c}, \sin\alpha = \frac{-bp}{c}$

Now, $\cos^2\alpha + \sin^2\alpha = 1$

$\therefore \quad \left(\frac{-ap}{c}\right)^2 + \left(\frac{-bp}{c}\right)^2 = 1$

$\therefore \quad \frac{a^2p^2}{c^2} + \frac{b^2p^2}{c^2} = 1$

$\therefore \quad \frac{p^2}{c^2}(a^2 + b^2) = 1$

$\therefore \quad p^2 = \frac{c^2}{a^2+b^2}$

$\therefore \quad p = \pm \frac{c}{\sqrt{a^2+b^2}}$

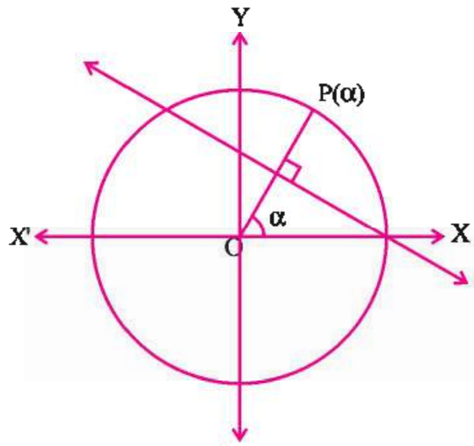


Figure 6.32

Hence $\cos\alpha = \mp \frac{a}{\sqrt{a^2+b^2}}, \sin\alpha = \mp \frac{b}{\sqrt{a^2+b^2}}, p = \pm \frac{c}{\sqrt{a^2+b^2}}$

If $c < 0$, $p = \frac{-c}{\sqrt{a^2+b^2}} = \frac{|c|}{\sqrt{a^2+b^2}}$

$\cos\alpha = \frac{-ap}{c} = \frac{a}{\sqrt{a^2+b^2}}, \sin\alpha = \frac{-bp}{c} = \frac{b}{\sqrt{a^2+b^2}}, \quad \alpha \in (-\pi, \pi]$

If $c > 0$, $p = \frac{c}{\sqrt{a^2+b^2}} = \frac{|c|}{\sqrt{a^2+b^2}}$

$\cos\alpha = \frac{-ap}{c} = \frac{-a}{\sqrt{a^2+b^2}}, \sin\alpha = \frac{-b}{\sqrt{a^2+b^2}}, \quad \alpha \in (-\pi, \pi]$

If p is the distance of the line $ax + by + c = 0$ from origin and perpendicular ray from the origin to the line meets unit circle in P(α), then

$$p = \frac{|c|}{\sqrt{a^2+b^2}},$$

and $\cos\alpha = \frac{-a}{\sqrt{a^2+b^2}}, \sin\alpha = \frac{-b}{\sqrt{a^2+b^2}}$ if $c > 0$

and $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$ if $c < 0$.

If the line is perpendicular to X-axis, then equation of the line is $ax + c = 0$.

Equation of the line can be written as $x = -\frac{c}{a}$. Hence $p = \left|\frac{c}{a}\right|$.

If $\frac{c}{a} < 0$, then $\alpha = 0$ and if $\frac{c}{a} > 0$, then $\alpha = \pi$.

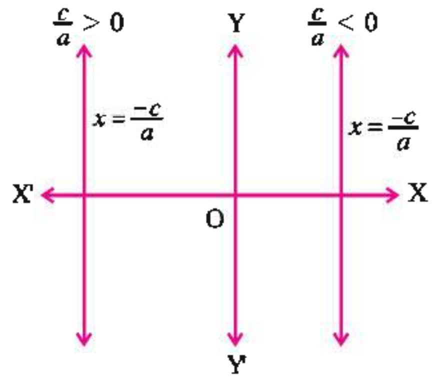


Figure 6.33

If the line is perpendicular to Y-axis, then its equation is $by + c = 0$.

Equation of the line can be written as $y = -\frac{c}{b}$. Hence $p = \left|\frac{c}{b}\right|$.

If $\frac{c}{b} < 0$, then $\alpha = \frac{\pi}{2}$ and if $\frac{c}{b} > 0$, then $\alpha = -\frac{\pi}{2}$.

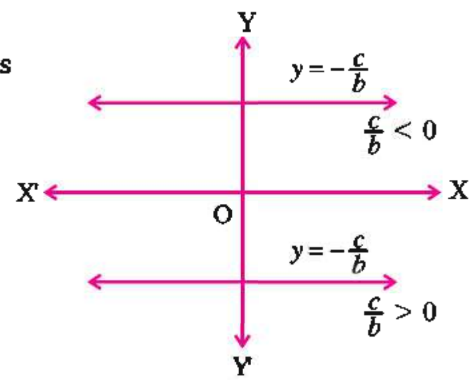


Figure 6.34

Example 22 : Transform the equation $\sqrt{3}x + y - 10 = 0$ to $p - \alpha$ form. Also find the value of p and α.

Solution : Given equation is $\sqrt{3}x + y - 10 = 0$

$$a = \sqrt{3},\ b = 1,\ c = -10$$

$$\sqrt{a^2 + b^2} = \sqrt{3+1} = 2$$

Since $c < 0$, $cos\alpha = \frac{a}{\sqrt{a^2+b^2}} = \frac{\sqrt{3}}{2} > 0$, $sin\alpha = \frac{b}{\sqrt{a^2+b^2}} = \frac{1}{2} > 0$

$\therefore \quad \alpha = \frac{\pi}{6}$ and $p = \frac{|c|}{\sqrt{a^2+b^2}} = \frac{10}{2} = 5$

$\therefore \quad p - \alpha$ form of the equation of the line is $xcos\frac{\pi}{6} + ysin\frac{\pi}{6} = 5$

6.18 Family of Lines parallel and perpendicular to $ax + by + c = 0$, $a^2 + b^2 \neq 0$

(i) Let $ax + by + c = 0$ be a line not perpendicular to X-axis.

$\therefore \quad b \neq 0$ and its slope is $\frac{-a}{b}$.

Slope of $ax + by + k = 0$, $(k \in R - \{c\})$ is also $\frac{-a}{b}$ and therefore the lines $ax + by + c = 0$ and $ax + by + k = 0$ are parallel $(k \neq c)$.

Also any line parallel to $ax + by + c = 0$ must have slope $\frac{-a}{b}$ and if it passes through (x_1, y_1) its equation is $(y - y_1) = \frac{-a}{b}(x - x_1)$.

$\therefore \quad ax + by - ax_1 - by_1 = 0.$

Setting $-ax_1 - by_1 = k$, the equation of any line parallel to $ax + by + c = 0$ is $ax + by + k = 0$ $(k \in R - \{c\})$.

If $b = 0$, then $ax + c = 0$ is perpendicular to X-axis and $ax + k = 0$ $(k \in R - \{c\})$ is also perpendicular to X-axis. Hence they are parallel.

Note : (x_1, y_1) does not lie on $\{(x, y) \mid ax + by + c = 0, a^2 + b^2 \neq 0\}$ as lines are parallel.

Hence $ax_1 + by_1 + c \neq 0$

$\therefore \quad -k + c \neq 0$

$\therefore \quad k \neq c$

Hence for $a, b \in R$, $a^2 + b^2 \neq 0$, the family of lines parallel to $ax + by + c = 0$ is $ax + by + k = 0$ $(k \in R - \{c\})$.

(ii) Let $ax + by + c = 0$ be a line with $a \neq 0$, $b \neq 0$.

The line is not perpendicular to any axis.

Its slope is $\frac{-a}{b}$ and any line perpendicular to it has slope $\frac{b}{a}$. $(m_1 m_2 = -1)$

If the perpendicular line passes through (x_1, y_1), its equation is $y - y_1 = \frac{b}{a}(x - x_1)$.

$\therefore \quad bx - ay - bx_1 + ay_1 = 0.$

Let $ay_1 - bx_1 = k$

$\therefore \quad bx - ay + k = 0$ represents a line perpendicular to line $ax + by + c = 0$.

Also any two lines $ax + by + c = 0$ and $bx - ay + k = 0$ are perpendicular as $m_1 = \frac{-a}{b}$, $m_2 = \frac{b}{a}$ and $m_1 m_2 = -1$.

If $a = 0$ or $b = 0$, obviously $ax + c = 0$ is perpendicular to $-ay + k = 0$ and $by + c = 0$ is perpendicular to $bx + k = 0$.

Also any line perpendicular to $ax + c = 0$ has equation of the form $-ay + k = 0$ and any line perpendicular to $bx + c = 0$ has equation of the form $by + k = 0$.

Hence family of lines perpendicular to $ax + by + c = 0$ $(a^2 + b^2 \neq 0)$ is $bx - ay + k = 0$, $k \in R$.

EXERCISE 6.4

1. Obtain the equation of the line passing through (8, 7) and (–2, 5).

2. A(1, 2), B(3, 6) and C(–2, 1) are vertices of ΔABC. Find the equation of the perpendicular bisector of $\overline{BC}$.

3. Find the equations of the lines inclined at an angle of measure $\frac{\pi}{4}$ with X-axis and passing through (1, 2).
4. Prove that the line passing through the point (x_1, y_1) and parallel to the line $ax + by + c = 0$ has equation $a(x - x_1) + b(y - y_1) = 0$.
5. Find the equation of the line perpendicular to $2x + 3y + 10 = 0$ and whose Y-intercept exceeds X-intercept by 2.
6. Prove that the line perpendicular to $x\sec\theta + y\operatorname{cosec}\theta = a$ and passing through $(a\cos^3\theta, a\sin^3\theta)$ is $x\cos\theta - y\sin\theta = a(\cos^2\theta - \sin^2\theta)$.
7. Obtain the equation of the line perpendicular to $\sqrt{3}x - y + 5 = 0$, given that its X-intercept is 3.
8. If the distance of a line from the origin is 2 and if the ray from the origin perpendicular to the line intersects the unit circle in $P\left(\frac{\pi}{6}\right)$, obtain the equation of the line.
9. The equation of a line containing one of the sides of an equilateral triangle is $2x + 2y - 5 = 0$ and one of the vertices of the triangle is (1, 2). Find the equations of lines containing the remaining sides of the triangle.
10. A line passing through (3, 1) makes a triangle of area 8 in the first quadrant with two axes. Find the equation of this line.
11. Find the equation of the line passing through $(\sqrt{3}, -1)$, if its perpendicular distance from the origin is $\sqrt{2}$.
12. One side of a rectangle lies along the line $4x + 7y + 5 = 0$. Two of its opposite vertices are (–3, 1) and (1, 1). Find the equations of the lines containing other three sides.
13. The straight lines $3x + 4y + 5 = 0$ and $4x - 3y - 10 = 0$ intersect at point A. Point B on line $3x + 4y + 5 = 0$ and point C on line $4x - 3y - 10 = 0$ are such that AB = AC. Find the equation of the line $\overleftrightarrow{BC}$ passing through the point (1, 2).

*

6.19 Distance of an outside Point from a Line

The line $ax + by + c = 0$ which is not perpendicular to any axis meets X-axis and Y-axis at A and B respectively. Then $A\left(-\frac{c}{a}, 0\right)$ and $B\left(0, -\frac{c}{b}\right)$.

Let $P(x_1, y_1)$ be any point not on $\overleftrightarrow{AB}$. Draw $\overline{PM} \perp \overleftrightarrow{AB}$. $M \in \overleftrightarrow{AB}$.

Now area of $\Delta PAB = \frac{1}{2}\left| x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0\left(y_1 - 0\right) \right|$

$$= \frac{1}{2}\left|\frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab}\right|$$

$$= \frac{1}{2}\left| (ax_1 + by_1 + c)\frac{c}{ab} \right| \quad \text{(i)}$$

Also area of $\Delta PAB = \frac{1}{2} AB \cdot PM$

$$= \frac{1}{2}\sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \cdot PM$$

$$= \frac{1}{2}\left|\frac{c}{ab}\right|\sqrt{a^2 + b^2} \cdot PM \qquad \text{(ii)}$$

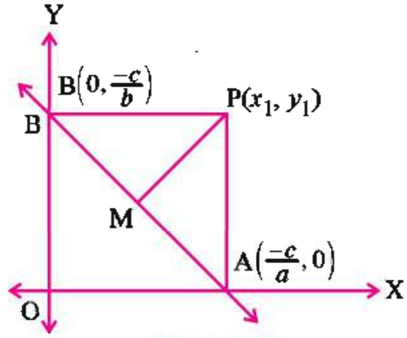


Figure 6.35

From (i) and (ii)

$$\frac{1}{2}\left|(ax_1 + by_1 + c)\frac{c}{ab}\right| = \frac{1}{2}\left|\frac{c}{ab}\right|\sqrt{a^2 + b^2} \cdot PM$$

$$\therefore \quad PM = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Perpendicular distance p of any point $P(x_1, y_1)$ from line $ax + by + c = 0$ is given by $p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

Another Method : Distance of a Point from a Line :

We know that the perpendicular distance of line $ax + by + c = 0$ from the origin is given by,

$$p = \frac{|c|}{\sqrt{a^2 + b^2}}$$

To find distance of the line $ax + by + c = 0$ from any outside point (x_1, y_1) in the plane, shift the origin to (x_1, y_1), so that $x = x' + x_1$, $y = y' + y_1$ in usual notation.

Now the equation of the line relative to new axes becomes

$$a(x' + x_1) + b(y' + y_1) + c = 0$$

$$\therefore \quad ax' + by' + ax_1 + by_1 + c = 0$$

Now (x_1, y_1) has become new origin.

$\therefore$ Distance of (x_1, y_1) from $ax + by + c = 0$ is equal to distance of new origin from $ax' + by' + ax_1 + by_1 + c = 0$.

$$\therefore \quad p = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \qquad \left(p = \frac{|c|}{\sqrt{a^2 + b^2}}\right)$$

6.20 Distance Between Two Parallel Lines

Suppose $ax + by + c = 0$ and $ax + by + c' = 0$, $a^2 + b^2 \neq 0$, $c \neq c'$ are two parallel lines.

If $a \neq 0$, then $A\left(-\frac{c}{a}, 0\right)$ is a point on the line $ax + by + c = 0$.

Now perpendicular distance between two parallel lines is equal to the perpendicular distance of either of these lines from any point on the other line.

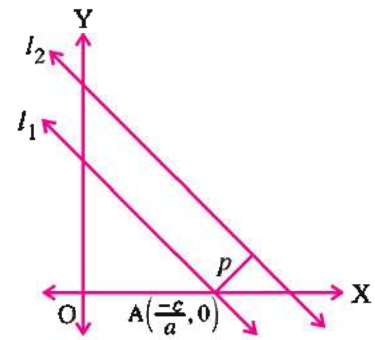


Figure 6.36

So the distance of line $ax + by + c' = 0$ from the point $A\left(-\frac{c}{a}, 0\right)$ is the distance between the lines.

Suppose p is the distance between the lines

$$p = \frac{\left|a\left(-\frac{c}{a}\right) + b(0) + c'\right|}{\sqrt{a^2 + b^2}}$$

$$p = \frac{|c - c'|}{\sqrt{a^2 + b^2}}$$

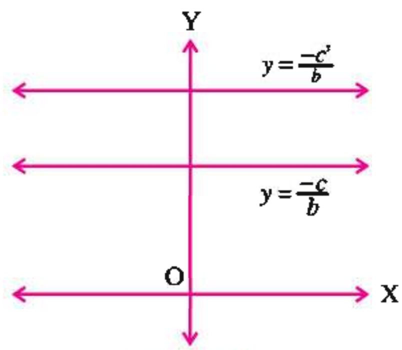


Figure 6.37

If $a = 0$, then $b \neq 0$. So the equations of two parallel lines will be $by + c = 0$ and $by + c' = 0$.

$\therefore$ The equations of the lines are $y = -\frac{c}{b}$ and $y = -\frac{c'}{b}$.

$$\text{Distance between the lines} = \left|\frac{-c}{b} - \left(\frac{-c}{b'}\right)\right|$$

$$= \frac{|c - c'|}{|b|}$$

$$= \frac{|c - c'|}{\sqrt{a^2 + b^2}} \qquad (a = 0)$$

In all cases, the perpendicular distance between two parallel lines $ax + by + c = 0$ and $ax + by + c' = 0$ is given by $p = \frac{|c - c'|}{\sqrt{a^2 + b^2}}$.

6.21 Intersection of Distinct Non-Parallel Lines

Now we will determine the condition that distinct non-parallel lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ intersect in a point.

Since the given lines are distinct non-parallel, they will intersect in a point. First of all suppose none of them is perpendicular to X-axis.

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel

$\Leftrightarrow$ they have the same slope

$\Leftrightarrow$ $m_1 = m_2$ where $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$

$\Leftrightarrow$ $\frac{a_1}{b_1} = \frac{a_2}{b_2}$

$\Leftrightarrow$ $a_1b_2 - a_2b_1 = 0$

$\therefore$ $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will intersect in a point

$\Leftrightarrow$ $a_1b_2 - a_2b_1 \neq 0$

If none of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is perpendicular to X-axis, they will intersect in a point if and only if $a_1b_2 - a_2b_1 \neq 0$.

If they are not parallel only one of them can be vertical.

Say $a_1x + b_1y + c_1 = 0$ is vertical and $a_2x + b_2y + c_2 = 0$ is not vertical.

$\therefore$ $b_1 = 0$ and $b_2 \neq 0$

$\therefore$ Since $b_1 = 0$, $a_1 \neq 0$

$\therefore$ $a_1b_2 - a_2b_1 = a_1b_2 \neq 0$ **($a_1 \neq 0$, $b_2 \neq 0$)**

Since $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ are not parallel, they will intersect if $a_1b_2 - a_2b_1 = a_1b_2 \neq 0$..

$\therefore$ $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ will intersect in a unique point if and only if $a_1b_2 - a_2b_1 \neq 0$.

6.22 Family of Lines Passing Through the Point of Intersection of Given Lines

Theorem 1 : If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two lines intersecting in a point, then $l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) = 0$ ($l^2 + m^2 \neq 0$) represents a line through the point of intersection of given lines $a_ix + b_iy + c_i = 0$ ($a_i^2 + b_i^2 \neq 0$, $i = 1, 2$)

Proof : First of all we will assert that, $(la_1 + ma_2)x + (lb_1 + mb_2)y + (lc_1 + mc_2) = 0$ is a linear equation i.e. $(la_1 + ma_2)^2 + (lb_1 + mb_2)^2 \neq 0$.

Let if possible $(la_1 + ma_2)^2 + (lb_1 + mb_2)^2 = 0$ **(i)**

Since $(la_1 + ma_2)^2 \geq 0$ and $(lb_1 + mb_2)^2 \geq 0$, (i) implies

$$la_1 + ma_2 = 0$$

$$lb_1 + mb_2 = 0$$

$\therefore$ $b_2(la_1 + ma_2) - a_2(lb_1 + mb_2) = 0$

$\therefore$ $l\,(a_1b_2 - a_2b_1) = 0$

Similarly, $m\,(a_1b_2 - a_2b_1) = 0$

But since lines intersect in a point, $a_1b_2 - a_1b_1 \neq 0$

$\therefore$ $l = 0 = m$

$\therefore$ $l^2 + m^2 = 0$

But we are given that $l^2 + m^2 \neq 0$

$\therefore$ We come to a contradiction.

$\therefore$ $(la_1 + ma_2)^2 + (lb_1 + mb_2)^2 \neq 0$

$\therefore$ $(la_1 + ma_2)x + (lb_1 + mb_2)y + (lc_1 + mc_2) = 0$ represents a line.

If (h, k) is the point of intersection of $a_ix + b_iy + c_i = 0$ $(i = 1, 2)$, then

$$a_1h + b_1k + c_1 = 0$$

$$a_2h + b_2k + c_2 = 0$$

$\therefore$ $l\,(a_1h + b_1k + c_1) + m(a_2h + b_2k + c_2) = 0$

$\therefore$ The line represented by $l\,(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) = 0$ passes through (h, k), the point of intersection of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

The converse is also true. But we will not prove it.

Theorem 2 : If lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ intersect in a point, then the equation of any line passing through their point of intersection is of the form $l\,(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) = 0$ for some $l, m \in$ R, not both zero.

We assume this theorem without proof.

☛ Note If $l = 0$, then $m \neq 0$.

$\therefore$ The equation $l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) = 0$ becomes $m(a_2x + b_2y + c_2) = 0$ i.e. $a_2x + b_2y + c_2 = 0$.

Similarly if $m = 0$, then $l \neq 0$ and the equation represents $a_1x + b_1y + c_1 = 0$.

Thus if we require a line other than $a_2x + b_2y + c_2 = 0$, then $l \neq 0$. The equation

$\therefore$ $l(a_1x + b_1y + c_1) + m(a_2x + b_2y + c_2) = 0$ is equivalent to

$$(a_1x + b_1y + c_1) + \frac{m}{l}(a_2x + b_2y + c_2) = 0$$

Let $\lambda = \frac{m}{l}$.

Then we can take the equation of any line through the point of intersection of given lines as $a_1x + b_1y + c_1 + \lambda(a_2x + b_2y + c_2) = 0$, if the required line is not $a_2x + b_2y + c_2 = 0$.

Miscellaneous Examples :

Example 23 : Obtain the equation of the line passing through the point of intersection of lines $x + y + 4 = 0$ and $3x - y - 8 = 0$ and also through $(2, -3)$ (without finding the point of intersection).

Solution : It is clear that $(2, -3)$ does not satisfy $3x - y - 8 = 0$. Hence it is not the required line.

$\therefore$ Let the equation of required line be $(x + y + 4) + \lambda(3x - y - 8) = 0$

$\therefore$ $(1 + 3\lambda)x + (1 - \lambda)y + (4 - 8\lambda) = 0$

This line passes through $(2, -3)$

$\therefore$ $(1 + 3\lambda)2 + (1 - \lambda)(-3) + (4 - 8\lambda) = 0$

Simplifying we get $\lambda = -3$

The equation of the required line is

$\therefore$ $(x + y + 4) + (-3)(3x - y - 8) = 0$

$\therefore$ $-8x + 4y + 28 = 0$

$\therefore$ $2x - y - 7 = 0$ is the equation of the required line.

Example 24 : A line passes through $(4, -2)$ and the length of perpendicular segment from the origin to this line has length 2. Find the equation of the line.

Solution : Let the equation of the required line be $x\cos\alpha + y\sin\alpha = p$.

Here $p = 2$. Also the line passes through $(4, -2)$.

$\therefore \quad 4cos\alpha - 2sin\alpha = 2$

$\therefore \quad -2sin\alpha = 2 - 4cos\alpha$

$\therefore \quad sin^2\alpha = (1 - 2cos\alpha)^2$

$\therefore \quad sin^2\alpha = 1 - 4cos\alpha + 4cos^2\alpha$

$\therefore \quad 1 - cos^2\alpha = 1 - 4cos\alpha + 4cos^2\alpha$

$\therefore \quad 5cos^2\alpha - 4cos\alpha = 0$

$\therefore \quad cos\alpha\,(5cos\alpha - 4) = 0$

$\therefore \quad cos\alpha = 0$ or $cos\alpha = \frac{4}{5}$

$\therefore \quad cos\alpha = 0 \Rightarrow sin\alpha = -1$ and $cos\alpha = \frac{4}{5} \Rightarrow sin\alpha = \frac{3}{5}$ **($4cos\alpha - 2sin\alpha = 2$)**

(1) Let $cos\alpha = 0$, $sin\alpha = -1$ and $p = 2$

$\therefore$ The equation of the line $xcos\alpha + ysin\alpha = p$ becomes

$$x \cdot 0 + y(-1) = 2$$

$\therefore \quad y = -2$

$\therefore \quad y + 2 = 0$ is one of the equations required.

(2) Let $cos\alpha = \frac{4}{5}$, $sin\alpha = \frac{3}{5}$ and $p = 2$

$\therefore \quad x\left(\frac{4}{5}\right) + y\left(\frac{3}{5}\right) = 2$

$\therefore \quad 4x + 3y = 10$ is the equation of the other line as required.

Example 25 : Find the equation of the line which makes a triangle of area $\frac{50}{\sqrt{3}}$ units with coordinates axes and perpendicular from the origin to this line makes angle of measure $\frac{\pi}{6}$ with positive direction of X-axis.

Solution : Let $xcos\alpha + ysin\alpha = p$ be the equation of required line.

Here $\alpha = \frac{\pi}{6}$

$\therefore$ The equation of the line is

$$x\ cos\frac{\pi}{6} + y\ sin\frac{\pi}{6} = p$$

$$\frac{\sqrt{3}x}{2} + \frac{y}{2} = p$$

$$\sqrt{3}x + y = 2p$$

The line intersects X-axis at A and Y-axis at B.

We have $A\left(\frac{2p}{\sqrt{3}}, 0\right)$, $B(0, 2p)$.

Area of $\Delta OAB = \frac{50}{\sqrt{3}}$

$\therefore \quad \frac{1}{2}OA \cdot OB = \frac{50}{\sqrt{3}}$

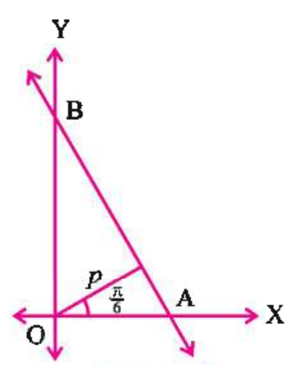


Figure 6.38

$\therefore \quad \frac{1}{2} \frac{2p}{\sqrt{3}} \cdot 2p = \frac{50}{\sqrt{3}}$

$\therefore \quad 2p^2 = 50$

$\therefore \quad p^2 = 25$

$\therefore \quad p = 5$

$\therefore$ The equation of the line is $\sqrt{3}x + y = 10$.

EXERCISE 6

1. Prove $A(2t^2, 4t)$, $S(2, 0)$ and $B\left(\frac{2}{t^2}, \frac{-4}{t}\right)$ are collinear points.
2. Prove that the product of the perpendicular distances from the point $(\pm\sqrt{a^2 - b^2}, 0)$ to the line $\frac{x}{a} cos\theta + \frac{y}{b} sin\theta = 1$ is b^2.
3. A line intersects X-axis at A and Y-axis at B. Given that AB = 15 and that $\overleftrightarrow{AB}$ makes a triangle of area 54 with the two axes, find the equation of the line.
4. If the lines $3x + y + 4 = 0$, $3x + 4y = 20$ and $24x - 7y + 5 = 0$ contain sides of a triangle, prove that this triangle is isosceles.
5. Prove that the four lines $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{b} + \frac{y}{a} = 1$, $\frac{x}{a} + \frac{y}{b} = 2$, $\frac{x}{b} + \frac{y}{a} = 2$ form a rhombus.
6. Which point on Y-axis is at distance 5 from the line $3x + 4y + 5 = 0$?
7. Find the equation of the line passing through (2, 3) and intercepting a line segment of length $\frac{2\sqrt{2}}{3}$ between the lines $2x + y = 3$ and $2x + y = 5$.
8. The lines $5x + 3y - 4 = 0$ and $3x + 8y + 13 = 0$ contain two of the altitudes of ΔABC. Find the coordinates of B and C. Vertex A of ΔABC has coordinates (–4, –5).
9. Find the equation of the line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).
10. A line intersects X-axis and Y-axis at A and B respectively. (2, 2) divides $\overline{AB}$ in the ratio 2:1 from A. Find the equation of the line.
11. If p is the length of the perpendicular from the origin to the line $bx + ay + ab = 0$, prove that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}$.
12. Find the equations of lines cutting off intercepts on axes whose sum and product are 7 and 12 respectively.
13. Find the points on X-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 4.

14. Find the equation of the line passing through the point of intersection of the lines $x + y + 1 = 0$ and $x - y + 1 = 0$ and whose distance from the origin is 1.

15. The foot of the perpendicular from the origin to a straight-line is (1, 2). Find the equation of line.

16. Find the equation of the line parallel to the line $4x - 2y - 1 = 0$ and passing through the point of intersection of the lines $5x + y + 4 = 0$ and $2x + 3y - 1 = 0$.

17. Find the equation of the line that passes through the point of intersection of $3x - 4y + 1 = 0$ and $5x + y - 1 = 0$ and that cuts off intercepts of equal magnitude on the two axes.

18. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) A straight line through the origin O meets the parallel lines $2x + y = 5$ and $2x + y = 3$ at points P and Q respectively. Then the points O divides the segment $\overline{PQ}$ in the ratio ...... from P. ☐

(a) $\frac{5}{3}$ (b) $\frac{2}{3}$ (c) $\frac{2}{\sqrt{5}}$ (d) $\frac{-5}{3}$

(2) Let $\overline{AD}$ be the median of the triangle with vertices A(1, 2), B(6, −1), C(7, 3). The equation of the line passing through (1, 1) and parallel to $\overline{AD}$ is... ☐

(a) $2x + 11y = 13$ (b) $2x + 11y = 5$ (c) $2x + 11y = 18$ (d) $11x - 2y = 13$

(3) The equation of the line parallel to X-axis and passing through $A\left(2, \frac{-3}{2}\right)$ is... ☐

(a) $x = 2$ (b) $2x - 3 = 0$ (c) $2y - 3$ (d) $2y + 3 = 0$

(4) The line $x + y = 4$ divides the line segment joining A(−2, 3) and B(1, 5) in the ratio $1 : \lambda$ from A. Then the value of λ = ... ☐

(a) 3 : 2 (b) 2 : 3 (c) 1 : 3 (d) −2 : 3

(5) $A(x_1, y_1)$ and $B(x_2, y_2)$ are end-points of $\overline{AB}$.

$P(tx_2 + (1 - t)x_1, ty_2 + (1 - t)y_1)$, $t < 0$, then P divides $\overline{AB}$ from A in ratio .... ☐

(a) $1 - t$ (b) $\frac{t-1}{t}$ (c) $\frac{1}{1-t}$ (d) $\frac{t}{1-t}$

(6) The slope of the line $\{(x, y) \mid x = 3t + 1, y = 2t + 6, t \in R\}$ is... ☐

(a) $-\frac{2}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

(7) The perpendicular distance of the line $3x + 4y + 10 = 0$ from the origin is... ☐

(a) −2 (b) $\frac{2}{5}$ (c) $\frac{1}{5}$ (d) 2

(8) If p and p' are the lengths of perpendiculars from the origin to $x\cos\alpha + y\sin\alpha = \sec\alpha$ and $x\sin\alpha - y\cos\alpha = \tan\alpha$ then $p^2 - p'^2 = $ ...... ☐

(a) 1 (b) 2 (c) $\cos^2\alpha$ (d) $\sec^2\alpha \cdot \tan^2\alpha$

(9) The distance between parallel lines $3x + 4y - 5 = 0$ and $6x + 8y - 15 = 0$ is... ☐

(a) 1 (b) $\frac{1}{2}$ (c) $\frac{25}{10}$ (d) 2

(10) If the lines $x\cos\alpha + y\sin\alpha = p$ and $x - \sqrt{3}y + 1 = 0$ are perpendicular to each other, then $\alpha = $ ...... $(0 < \alpha < \frac{\pi}{2})$ ☐

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

(11) p–α form of the equation of the line $x + \sqrt{3}y - 4 = 0$ is ...... ☐

(a) $x\cos\frac{\pi}{6} + y\sin\frac{\pi}{6} = 2$ (b) $x\cos\frac{\pi}{3} + y\sin\frac{\pi}{3} = 2$

(c) $x\cos\left(-\frac{\pi}{3}\right) + y\sin\left(-\frac{\pi}{3}\right) = 2$ (d) $x\cos\left(-\frac{\pi}{6}\right) + y\sin\left(-\frac{\pi}{6}\right) = 2$

Summary

1. Division of a line segment
2. Parametric equations of a line
3. Lines perpendicular to axes
4. Cartesian equation of a line
5. Slope of a line
6. Condition for two lines to be perpendicular or parallel to each other.
7. Angle between two lines
8. Intercepts of a line on axes
9. Different forms of equations of a line
11. Length of perpendicular from origin to a line.
12. Family of lines passing through the point of intersection of two lines.

Chapter 7

PERMUTATIONS AND COMBINATIONS

7.1 Introduction

Sometimes we come across problems of counting and choosing as follows :

We want to form three digit numbers using digits 1, 2 and 3 without repetition of any digit in any number. How many such numbers can be formed ?

We can choose first digit as 1, 2 or 3. Since we do not wish to repeat any digit, the second digit can be chosen in two ways from 1, 2, 3 excluding the digit chosen. The digit left is the third digit. Look at the tree in figure 7.1.

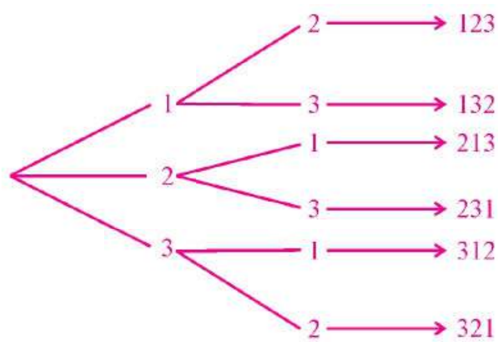


Figure 7.1

Six numbers 123, 132, 213, 231, 312, 321 can be formed.

Rucha has two tops and three matching pants and two pairs of shoes. She wants to go to a party. In how many ways can she dress herself ?

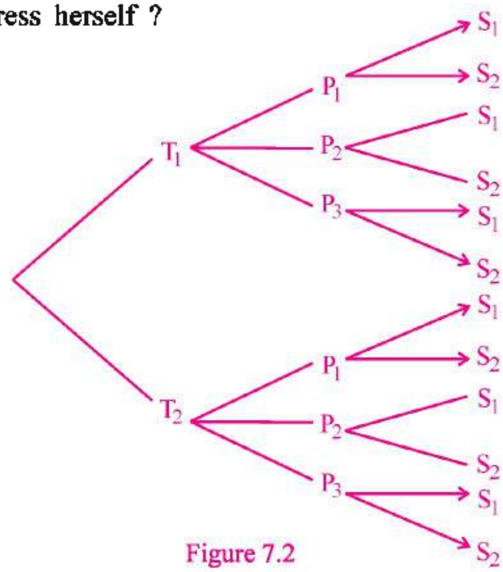


Figure 7.2

Let the tops be called T_1 and T_2. Let the pants be called P_1, P_2, P_3 and let pairs of shoes be called S_1 and S_2. Consider the tree in figure 7.2.

So the possible combinations are $T_1P_1S_1$, $T_1P_1S_2$, $T_1P_2S_1$, $T_1P_2S_2$, $T_1P_3S_1$, $T_1P_3S_2$, $T_2P_1S_1$, $T_2P_1S_2$, $T_2P_2S_1$, $T_2P_2S_2$, $T_2P_3S_1$, $T_2P_3S_2$ i.e. 12 in all.

Dev has three school bags, two lunchboxes and two ballpens. In how many ways can he choose to go to school with one item of each type ? Let the bags be called B_1, B_2, B_3 the lunchboxes be called C_1, C_2 and the pens be called P_1, P_2.

Possible combinations are $B_1C_1P_1$, $B_1C_1P_2$, $B_1C_2P_1$, $B_1C_2P_2$, $B_2C_1P_1$, $B_2C_1P_2$, $B_2C_2P_1$, $B_2C_2P_2$, $B_3C_1P_1$, $B_3C_1P_2$, $B_3C_2P_1$, $B_3C_2P_2$. Again in 12 ways !

Now this type of listing is tedious and not always possible !

We wish to try a password to open file. The password consists of six different digits. How many trials will be required ? $10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151200$!

This type of counting problems are solved by fundamental principle of counting or multiplication principle.

Fundamental Principle of Counting : If an event can occur in m ways and corresponding to each way another event can occur in p ways, the total number of occurrences of the events is mp.

In fact if A is the set of elements of occurrences of the first event and B is the set of elements of occurrences of the second event, we know $n(A) = m$, $n(B) = p$ and therefore $n(A \times B) = mp$.

Similarly if one event can occur in p ways and corresponding to it a second event can occur in q ways and corresponding to them a third event can occur in r ways, then the three events together can occur in pqr ways.

Now in earlier example first event (selection of top by Rucha) can occur in 2 ways, second event (selection of pants by Rucha) can occur in 3 ways and selection of shoes can occur in 2 ways. Hence Rucha can dress herself in $2 \times 3 \times 2 = 12$ ways.

Similarly Dev can prepare schoolbag in $3 \times 2 \times 2 = 12$ ways. Thus fundamental principle of counting gives precise results without elaborate listing.

Example 1 : How many four digit numbers can be formed using 1, 2, 4, 6, 8 ? (Repetition of digits is not allowed.)

Solution : We have to fill four places : unit place, tens place, hundred place and thousand place using 5 digits.

Th	H	Ten	U

First place can be filled in 5 ways. Since repetition is not allowed, so second, third and fourth place can be filled in corresponding 4, 3 and 2 ways respectively. Hence according to the fundamental principle of counting, the number of four digit numbers formed using 1, 2, 4, 6, 8 is $5 \times 4 \times 3 \times 2 = 120$.

Example 2 : How many four letter words (with or without meaning or dictionary occurrence and without repetition of letter) can be formed using letters of the word KENY ? How many of them will have first letter E ?

Solution : Four letters words using K, E, N, Y are $4 \times 3 \times 2 \times 1$ in number. The number of words is 24. If the first letter is E, consider the box $\boxed{E}\boxed{\ }\boxed{\ }\boxed{\ }$. We have to fill three places with three letters K, N, Y. Hence their number is $3 \cdot 2 \cdot 1 = 6$.

Example 3 : How many three digit even numbers can be formed using digits 0, 1, 2,..., 9 ? (No repetition of digits)

Solution : Last digit has to be 0, 2, 4, 6 or 8 to form an even number.

		0

		2

Consider the above array. If the last digit is zero, other two digits can be chosen in $9 \times 8 = 72$ ways.

If the last digit is 2, first digit can be choosen in 8 ways (it cannot be zero) and second also in 8 ways (now zero is allowed). Thus $8 \times 8 = 64$ numbers with each of 2, 4, 6, 8 as last digit are formed. Therefore the total number is $64 \times 4 + 72 = 328$.

(**Think :** How many odd ? How many total ?)

Example 4 : How many three digit numbers are there which do not contain 2 at all ? Which contain 2 at least once ? Which contain 2 at most once ? (See that repetitions is allowed.)

Solution : First digit can be any one of 1, 3, 4, 5,..., 9. Second digit can be any one of 0, 1, 3, 4, 5,...,9. Third digit can be any one of 0, 1, 3, 4, 5,..., 9.

$\therefore$ Total number of three digit numbers not having 2 at all is $8 \times 9 \times 9 = 648$ **(i)**

Now, total number of three digit numbers is $9 \times 10 \times 10 = 900$ (First digit is non-zero)

$\therefore$ $900 - 648 = 252$ contain 2 at least once. **(ii)**

2 at most once means numbers occuring not having 2 at all and numbers having 2 once. Their number is,

900 (total) − (numbers having 2 in all the places + numbers having 2 in exactly two places)

Numbers having 2 in all the places is 222 (one only).

Numbers having 2 in two places (but not all) are [2][2][ ], [ ][2][2], [2][ ][2].

The blank space can be chosen in $9 + 8 + 9 = 26$ ways.

$\therefore$ Total number of numbers having 2 in at least two places is 27.

$\therefore$ $900 - 27 = 873$ numbers contain 2 in at most one place. **(iii)**

Example 5 : In how many ways can small triangles in figure 7.3 (ΔABP, ΔBQC, ΔBPQ, ΔPQR) be coloured using fundamental colours RBY (Red, Blue, Yellow) so that no two adjacent regions are coloured using the same colour ?

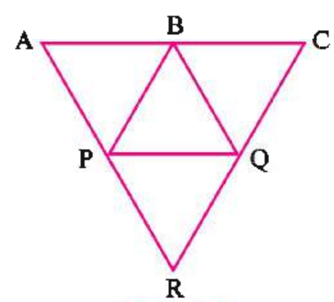


Figure 7.3

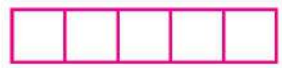
Solution : ΔBPQ has boundary touching all regions.

It can be coloured using 3 colours in 3 ways. Other three regions can be coloured in $2 \times 2 \times 2 = 8$ ways.

The four regions can be coloured in $3 \times 8 = 24$ ways.

Example 6 : How many five letter passwords can be generated using first three letters as any of the English alphabets and last two being any digit from 0 to 9 ? (See repetition is allowed.)

Solution : First three letters can be chosen in $26 \times 26 \times 26$ ways and last two in 10×10 ways.



Hence the possible number of passwords is $26 \times 26 \times 26 \times 10 \times 10 = 1757600$.

EXERCISE 7.1

1. How many signals can be formed using five different flags of different colours lying in a row on vertical masts ? Each signal may contain 2 or more flags of different colours.
2. How many odd numbers of four digits are there ? (No repetition of digits.)
3. How many words can be formed using all alphabates of the word TULSI ? (No repetitions) How many start with T ? How many end in I ?
4. How many three digit numbers are there which are multiples of 5 ? (Without repetition of digits)
5. How many cars are there with numbers like GJ-X-AB-*abcd* where X is any digit from 1 to 9; A is H; B is any one of alphabets in English and *abcd* is a four digit number ? (*a* can be zero)
6. How many numbers are there between 99 and 1000 (i) having last digit 0, (ii) having last digit 5, (iii) divisible by 4, (iv) divisible by 2 but not by 4 ?
7. Dev wants to create a password with five letters for his e-mail account with following properties.
   (1) First three letters should be any of the English alphabets so that none of the alphabets of his name occur in any order.
   (2) The remaining two should be any digits from 0 to 9 but they do not represent his age. In how many ways can he do this ? Dev is 12 years old.

*

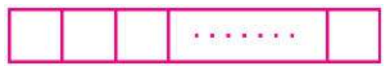
7.2 Permutations

The problems we studied in earlier section deal with arrangement of objects in definite order. To determine the number of three digit numbers using 1, 2, 3, 4 involves arrangements like 123, 124, 234,...etc. Here we are counting number of 'permutations' of 4 digits taken 3 at a time. Their number is $4 \times 3 \times 2 = 24$. (Fundamental principle of counting).

Definition : A permutation is an arrangement in a definite order of a number of distinct objects taking some or all at a time. The number of linear permutations of n different objects taking r at a time where $1 \le r \le n$, $r \in N$ and $n \in N$ is denoted by the symbol ${}_nP_r$, if repetitions of objects is not allowed and arrangement is linear. The arrangement also called a linear permutation.

Theorem 1 : ${}_nP_r = n(n - 1)(n - 2)...(n - r + 1)$

The number of permutations is the number of ways in which r places can be filled with n different objects in the following vacant places. (without repetition)



r places

The first place can be filled with any of n objects and this is possible in n ways. Since there is no repetition, correspondingly second place can be filled in $(n - 1)$ ways, third place in $(n - 2)$ ways etc. The last rth place can be filled in $n - (r - 1)$ ways. (Earlier $(r - 1)$ places are filled.)

$\therefore$ By the fundamental principle of counting.

$${}_nP_r = n(n - 1)(n - 2)....(n - r + 1)$$

Start with n, decreasing by 1 each time, write r integers and multiply all of them.

For example ${}_7P_3 = 7 \times 6 \times 5 = 210$

$$\begin{aligned} {}_nP_n &= n(n - 1)(n - 2)...(n - n + 1) \\ &= n(n - 1)(n - 2)...1 \end{aligned}$$

There is a special symbol for this product $n(n - 1)(n - 2)...1$ and it is called factorial n and is denoted by $n!$ (read 'n-factorial') or $\underline{|n}$.

Thus, ${}_nP_n = n!$

$1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ etc.

$$\begin{aligned} \text{Now, } n! &= n(n - 1)(n - 2)...1 \\ &= n(n - 1)! \\ &= n(n - 1)(n - 2)! \\ &= n(n - 1)...(n - r + 1)(n - r)! \end{aligned}$$

$$\therefore n(n - 1)...(n - r + 1) = \frac{n!}{(n - r)!} \qquad 1 \le r < n$$

Now using, ${}_nP_r = n(n - 1)...(n - r + 1)$

$$\therefore \quad {}_nP_r = \frac{n!}{(n - r)!} \qquad 1 \le r < n$$

But if $r = n$, ${}_nP_n = n! = \lfloor n$.

Definition : $0! = 1$

Thus we define $0!$ so that

$${}_nP_n = n! = \frac{n!}{0!}$$

$$\therefore \quad {}_nP_r = \frac{n!}{(n-r)!} \qquad\qquad 1 \le r \le n$$

Theorem 2 : Number of permutations of n distinct objects taken r at a time with repetitions allowed is n^r.

Proof : Each of the r places can be filled with n objects in n ways. Hence number of such permutations is $n \times n \times n .....r$ times $= n^r$.

1	2	3				r
n	n	n	n	n	. . . .	n

Example 7 : If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, find x.

Solution : $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$

$$\therefore \quad \frac{1}{8!}\left(1 + \frac{1}{9}\right) = \frac{x}{10!} \qquad\qquad (9! = 9 \cdot 8!)$$

$$\therefore \quad \frac{1}{8!}\left(\frac{10}{9}\right) = \frac{x}{10!}$$

$$\therefore \quad x = \frac{10(10!)}{9 \cdot 8!}$$

$$= \frac{10 \cdot 10 \cdot 9!}{9!} \qquad\qquad (10! = 10 \cdot 9!,\ 9! = 9 \cdot 8!)$$

$$\therefore \quad x = 100$$

Example 8 : Solve for n, $\frac{{}_{n-1}P_3}{{}_nP_4} = \frac{1}{9}$.

Solution : $\frac{{}_{n-1}P_3}{{}_nP_4} = \frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)} = \frac{1}{9}$

$\therefore \quad \frac{1}{n} = \frac{1}{9}$. So $n = 9$

Example 9 : Find r, if $5\ {}_4P_r = 6\ {}_5P_{r-1}$

Solution : $5\ {}_4P_r = 6\ {}_5P_{r-1}$

$$\therefore \quad \frac{5(4!)}{(4-r)!} = \frac{6(5!)}{(5-r+1)!}$$

$$\therefore \quad \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4-r)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(6-r)!}$$

$\therefore \quad \frac{(6-r)!}{(4-r)!} = 6$

$\therefore \quad \frac{(6-r)(5-r)(4-r)!}{(4-r)!} = 6$ $\quad [n! = n(n-1)(n-2)!]$

$\therefore \quad (6-r)(5-r) = 6$

$\therefore \quad r^2 - 11r + 30 = 6$

$\therefore \quad r^2 - 11r + 24 = 0$

$\therefore \quad r = 8$ or 3

But $r \neq 8$ as $1 \leq r \leq 4$ and $1 \leq r - 1 \leq 5$

$\therefore \quad r = 3$

Example 10 : Find r, if ${}_5P_r = {}_6P_{r-1}$.

Solution : ${}_5P_r = {}_6P_{r-1}$.

$\therefore \quad \frac{5!}{(5-r)!} = \frac{6!}{(7-r)!}$ $\quad (6 - r + 1 = 7 - r)$

$\therefore \quad \frac{(7-r)!}{(5-r)!} = \frac{6(5!)}{5!} = 6$

$\therefore \quad (7-r)(6-r)\frac{(5-r)!}{(5-r)!} = 6$

$\therefore \quad r^2 - 13r + 42 = 6$

$\therefore \quad r^2 - 13r + 36 = 0$

$\therefore \quad r = 9$ or 4

But $1 \leq r \leq 5$ and $1 \leq r - 1 \leq 6$

$\therefore \quad r = 4$

Example 11 : How many three digit numbers (i.e. between 100 and 999 including both) can be formed using digit 0, 1, 2,..., 9 if repetitions of the digits is not allowed ?

Solution : We can arrange 10 digits in 3 places in ${}_{10}P_3 = 10 \cdot 9 \cdot 8 = 720$ ways.

0		

But 0 can not occur in the first place in a number.

So we have to remove ${}_9P_2 = 9 \cdot 8 = 72$ numbers from this list.

$\therefore \quad 720 - 72 = 648$ three digit numbers can be formed using all digits.

Example 12 : From a managing committee of 10 persons one president, one vice-president and one secretary are to be elected. No person can hold two posts at a time. In how many ways can this happen ?

Solution : This is a problem of arranging 10 persons in 3 places (posts) without repetition.

This is possible in ${}_{10}P_3 = 10 \cdot 9 \cdot 8 = 720$ ways.

Here all posts are different. Hence it is a case of arrangement.

Example 13 : How many new permutations of all letters of the word TUESDAY are possible ? How many will begin with T and end in Y ?

Solution : The number of arrangements is $_7P_7 = 7!$.

Now, $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$

Hence, 5040 arrangements exist and TUESDAY is one of them. Thus 5039 new arrangements are possible.

Keeping T and Y in their places remaining 5 letters can be arranged in $5! = 120$ ways.

Example 14 : In permutations of all letters in TABLE in dictionary order (no meaning necessary), what is the position of the word TABLE ? Which is the last 'word' ?

Solution : The number of words beginning with A are $4! = 24$. They will precede TABLE in dictionary order. Similarly the words starting with B, E, L each are 24 in number and the word TABLE will follow them in dictionary order.

Now begin with T. Using A, B, L and E, TABEL will precede TABLE. Thus occurrence of the word TABLE will be $(24 + 24 + 24 + 24 + 1 + 1)$th i.e. 98th (see that there are $5! = 120$ in all.)

After 96 words starting with A, B, E, L the words with T will start. 6 words each with TA, TB, TE will then follow. Hence after 114 words, the words with TL will follow. They are TLABE, TLAEB, TLBAE, TLBEA, TLEAB and TLEBA. Hence the last word is TLEBA.

[In fact the last word would be TLEBA writing alphabets in reverse order !]

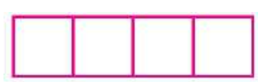
Example 15 : Dev participates in chess, 100 metres race, shooting and disc throw. Each event carries three medals, gold, silver and bronze. In how many way can he get the medals ?

Solution : There are four games or sports. To each three medals are assigned. Each of space W_1, W_2, W_3, W_4 can be filled in three ways (repetition allowed). Hence Dev can get a medal in $3^4 = 81$ ways.

W_1	W_2	W_3	W_4

Example 16 : How many four digit numbers can be formed using 5, 2, 3, 7, 8 ?

Solution : There are four places and each can be filled in 5 ways with 5, 2, 3, 7, or 8.



$5^4 = 625$ numbers can be formed.

(See $n = 5$ things to be arranged in $r = 4$ places in $n^r = 5^4 = 625$ ways.)

☛ **Note** If we want six digits numbers, it is possible in $5^6 = 15625$ ways. Here $r > n$ is possible unlike in $_nP_r$, $r \le n$.

Example 17 : How many permutations of all alphabets of the word DAUGHTER is possible without repetition ? In how many of them vowels and consonants will be in their positions only ?

Solution : There are 8 different letters in the given word.

Hence there are $8! = 40320$ permutations. But if the vowels A, U, E and consonants D, G, H, T, R remain in their places but can be mutually rearranged then the number of permutations is $3! \times 5! = 6 \times 120 = 720$.

Example 18 : If $n(A) = m$ and $n(B) = n$ $(m, n \in N)$, then how many functions from A to B are there ?

Solution : Let $A = \{x_1, x_2, x_3, ... x_m\}$

$B = \{y_1, y_2, y_3, ... y_n\}$

Then $f = \{(x_i, y_j)\}$, where $x_i \in A$, $y_j \in B$ so that no x_i is repeated and no x_i from A is left out without occurrence in some pair in f.

Thus, each x_i can correspond to some y_j in n ways.

$\therefore$ Thus there are $n \times n \times n....m$ times choices for set f.

$\therefore$ Number of functions is n^m.

☛ Note Let $A = \{1, 2, 3\}$, $B = \{a, b\}$

$f_1 = \{(1, a), (2, a), (3, a)\}$ $\quad$ $f_2 = \{(1, b), (2, b), (3, b)\}$

$f_3 = \{(1, a), (2, a), (3, b)\}$ $\quad$ $f_4 = \{(1, b), (2, b), (3, a)\}$

$f_5 = \{(1, a), (2, b), (3, a)\}$ $\quad$ $f_6 = \{(1, b), (2, a), (3, b)\}$

$f_7 = \{(1, b), (2, a), (3, a)\}$ $\quad$ $f_8 = \{(1, a), (2, b), (3, b)\}$

$\therefore$ $2^3 = 8$ functions $f : A \to B$ exists.

EXERCISE 7.2

1. Evaluate : (1) ${}_8P_4$ (2) ${}_9P_3$ (3) ${}_6P_6$

2. Evaluate : (1) $6!$ (2) $\frac{8!}{2!}$ (3) $\frac{9!}{7!}$

3. Prove ${}_nP_r = {}_{n-1}P_r + r({}_{n-1}P_{r-1})$

4. Find r : (1) $\frac{{}_{15}P_r}{{}_{16}P_{r-1}} = \frac{3}{4}$ (2) ${}_7P_r = 7\ {}_6P_r$

5. Find n : $7\,{}_nP_3 = 20\ {}_{n+1}P_2$

6. If $\frac{{}_{56}P_{r+6}}{{}_{54}P_{r+3}} = 30800$, find r.

7. Prove ${}_nP_r = n\ {}_{n-1}P_{r-1}$

8. Find n, if $(n+1)! = 12(n-1)!$

9. If $\frac{n!}{2!(n-2)!} : \frac{n!}{4!(n-4)!} = 2$, find n.

10. How many three digit even numbers can be formed using digits of 2468 ? (With or without repetition.)

11. In how many ways can a true or false test containing n questions be answered ?
12. There are 10 M.C.Q. questions each containing 4 options. In how many ways can the test be answered ?
13. A coin is tossed four times and the result head (H) or tail (T) is noted. How many results are possible ?
14. A number lock on a suitcase has four wheels. To open the suitcase a code is required which contains an English alphabet on each of first two wheels and any digit from 0 to 9 on last two wheels. How many such codes are possible if (i) repetition is allowed (ii) repetition is not allowed ?
15. In how many ways can 6 letters be posted with 3 courier agencies available ?
16. m men and n women $(m > n)$ are seated in a row so that no two women occupy adjacent positions. In how many ways is it possible ?
17. In how many ways can n children be arranged in a queue so that (i) Seeta and Geeta always occupy adjacent positions ? (ii) Seeta and Geeta never occupy adjacent positions ?
18. Find the number of four digit numbers divisible by 4 obtained using 1, 2, 3, 4, 5, 6 without repetition.
19. Six girls are to be arranged in a sequence to offer a bouquet to six guests on dias. General Secretary Rani will be first to offer the bouquet to the guest of honour. Fifth position is reserved for Ria. Aishwarya and Isha will be consecutive in any order. Remaining girls Sneha and Smruti will occupy other two places in any order. In how many ways can this arrangement be made ?
20. Find the number of ways in which four boys and four girls can be arranged so that (i) no two girls are adjacent (ii) all the boys are together and all the girls are together.
21. Six girls and six boys stand in a queue alternating so that the queue starts with a girl. In how many ways is this possible ?
22. In how many ways can alphabets of the word MONDAY be arranged taking (i) two at a time (ii) four at a time without repetition.
23. How many permutations of all the letters of the word ZERO are there ? (No repetition) In dictionary order what is the position of the word ZERO ?
24. How many four digit numbers divisible by 5 can be formed using 0, 1, 2, 3, 4, 5 only once ? (No repetition)
25. How many 4 digit numbers can be formed using digits of 2745 only once ? How many are divisible 3 ? How many are divisible by 9 ?
26. How many words having four alphabets can be formed using alphabets from the word VOWEL in which vowels occupy the place of vowels ?

*

Permutations when Some of the Objects are Identical

Let us try to find permutations of the word TREE. Here E occurs twice. Temporarily call the occurrences of E as E_1, E_2.

Permutations with same E	Permutations with E_1, E_2
TREE	TRE_1E_2
	TRE_2E_1
RTEE	RTE_1E_2
	RTE_2E_1
TERE	TE_1RE_2
	TE_2RE_1
RETE	RE_1TE_2
	RE_2TE_1
TEER	TE_1E_2R
	TE_2E_1R
REET	RE_1E_2T
	RE_2E_1T
ERTE	E_1RTE_2
	E_2RTE_1
ETRE	E_1TRE_2
	E_2TRE_1
ERET	E_1RE_2T
	E_2RE_1T
ETER	E_1TE_2R
	E_2TE_1R
EETR	E_1E_2TR
	E_2E_1TR
EERT	E_1E_2RT
	E_2E_1RT

Figure 7.4

Hence, if we consider E_1, E_2 different, as before there are ${}_4P_4 = 4!$ permutations. But E_1 and E_2 are same and this gives us $12 = \frac{24}{2} = \frac{24}{2!}$ permutations.

Let us consider the problem more closely taking another illustration. How many four digit numbers can be formed using digits of 1112 without repetition ?

Here 1 occurs thrice. Call these occurrences as 1_a, 1_b, 1_c.

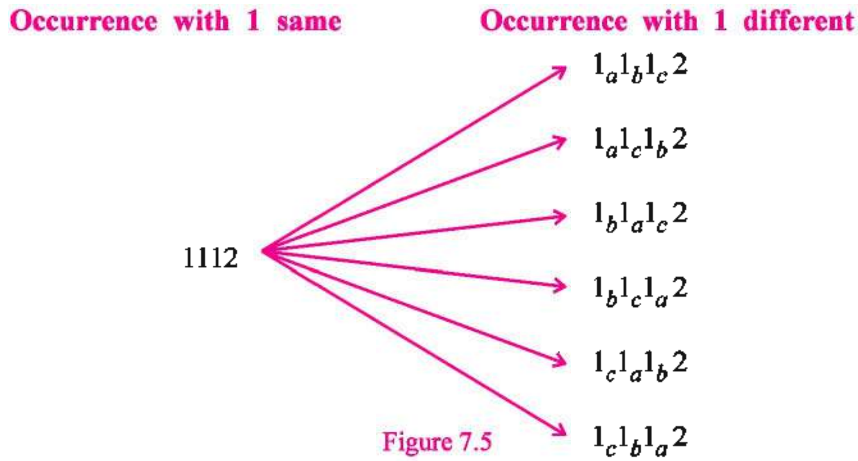


Figure 7.5

Similarly 1121, 1211, 2111 all give six occurrences with $1_a 1_b 1_c$.

Hence we get ${}_4P_4 = 24$ permutations with 1's different but four permutations only 1112, 1121, 1211, 2111 with 1's same. In fact each of them gives $3! = 6$ arrangements considering 1_a, 1_b, 1_c.

Hence we get $24 = 4 \times 6$ arrangements.

In fact number of arrangements is $4 = \frac{24}{6} = \frac{{}_4P_4}{3!}$. Thus we have following theorem.

Theorem 3 : If p_1 objects are alike, p_2 objects are alike different from earlier ones,..., p_k objects are alike different from earlier ones and $n = p_1 + p_2 + ... + p_k$ then the number of arrangements of n things is

$$\frac{n!}{p_1!\, p_2!\, p_3! \dots p_k!}$$

Proof : Consider objects as $a_1, a_2, ..., a_{p_1}$; $b_1, b_2, ..., b_{p_2}$; ... $m_1, m_2, ..., m_{p_k}$. All n objects are now different and they can be arranged in $n!$ ways.

But mutually alike objects can be arranged in $p_1!$, $p_2!$, ..., $p_k!$ ways considering them different.

$\therefore$ (Total number of distinct arrangements) $\times\, p_1! \times p_2! \times ... \times p_k! = n!$

$\therefore$ Total number of distinct arrangements $= \frac{n!}{p_1!\, p_2! \dots p_k!}$

Thus in the above examples answers are $\frac{4!}{2!} = 12$ and $\frac{4!}{3!} = 4$.

Example 19 : In how many ways can the letters of the word PERMUTATIONS be permuted ? Also find,

(i) How many start with P and end in S ?

(ii) In how many of them vowels are together ?

Solution : There are 12 letters containing 2 T's.

Hence the number of permutations is $\frac{12!}{2!}$.

(i) If they start with P and end in S, remaining 10 letters contain 2 T's.

$\therefore$ Required number is $\frac{10!}{2!} = 1814400$

(ii) There are five vowels A, E, I, O, U (all !). Consider them as one group. Hence, we have 8 words (7 + 1 group) with 2 T's. 5 vowels can be arranged in 5! ways.

$\therefore$ The required number of permutations is $\frac{8!}{2!} \times 5! = 2419200$

Example 20 : How many permutations can be made using letters of the word MATHEMATICS ? In how many of them vowels occur together ?

Solution : There are 11 letters containing 2 M's, 2 T's, 2 A's.

$\therefore$ Number of arrangements is $\frac{11!}{2!\,2!\,2!} = 4989600$

A, E, I are vowels occurring. (A twice)

Consonants remaining are M, T, C, S, H (M twice, T twice)

Considering seven consonants and group AEAI as one letter the number of permutations is $\frac{8!}{2!\,2!}$ and vowels A, E, A, I can be arranged in $\frac{4!}{2!}$ ways.

$\therefore$ The total number of arrangements $= \frac{8!}{2!\,2!} \times \frac{4!}{2!}$

$= 10080 \times 12 = 120960$

Example 21 : In how many ways can seven digit numbers greater than 10,00,000 be formed using digits 1, 2, 0, 2, 4, 2, 4 ?

Solution : There are seven digits, three 2's and two 4's.

First digit has to be 1, 2 or 4 according to condition.

$\therefore$ The numbers beginning with 2 are $\frac{6!}{2!\,2!}$ (One of the 2's is fixed)

$\therefore$ The number is $\frac{720}{4} = 180$

The numbers beginning with 4 are $\frac{6!}{3!} = \frac{720}{6} = 120$

The numbers beginning with 1 are $\frac{6!}{3!\,2!} = 60$

∴ The total number of seven digit numbers is 360 and obviously each of these seven digit numbers is greater than 100000.

[or consider seven digit numbers $\frac{7!}{3!\,2!} = \frac{5040}{12} = 420$

The numbers with 0 leading are $\frac{6!}{3!\,2!} = \frac{720}{12} = 60$

∴ Required number is 420 – 60 = 360]

Example 22 : Find the number of 'words' that can be formed using all the letters of the word ALLAHABAD.

(i) In how many of them vowels are in even positions ?

(ii) In how many of them 2 L's do not occur together ?

Solution : There are 9 letters and 4 A's and 2 L's in the word.

∴ The number of permutations is $\frac{9!}{4!\,2!} = 7560$

(i) Four vowels, all A, can occur in even positions 2, 4, 6, 8 in only one way. $\left(\frac{4!}{4!} = 1\right)$

Remaining 5 letters with 2L's can be permuted in $\frac{5!}{2!} = 60$ ways

Hence the total number of permutations is 60.

(ii) Suppose two L's are combined to form a group. Now we have 8 letters with 4 A's.

∴ The number of arrangements in which both L occur together = $\frac{8!}{4!} = 1680$

∴ The number of arrangements in which L's do not occur together = total number of arrangements – the number of arrangements in which L's occur together = 7560 – 1680 = 5880

Example 23 : If all the letters of the word AGAIN are permuted, what is the fiftieth word in the dictionary order ?

Solution : A dictionary starts with A. Words which begin with A,

A				

, are 4! = 24 in number. (G, A, I, N used).

Next numbers beginning with G are $\frac{4!}{2!} = 12$ in number **(two A's)**

Similarly, numbers beginning with I are $\frac{4!}{2!} = 12$ in number.

After these 48 words, words will begin with NAA and have GI and IG to follow. Hence 50th word is NAAIG.

☛ **Note** What is the last word ? Write without calculating !

EXERCISE 7.3

1. In arrangements of all letters of the word BOOK, what is the rank of the word BOOK ?

2. Arranging all letters of AGAIN in the dictionary order, what is the last word ? What is its rank ?

3. There are 7 mercury lamps in a hall. Each one of them can be switched on or off independently. In how many ways will the hall be lit ?

4. Determine the number of positive integers less than 10000 with all digits distinct.

5. Find the sum of all the 4 digit numbers formed using digits of 2468 only once.

6. How many permutations of all letters of the word TRIANGLE contain T and E in the end positions in any order ?

7. How many arrangements of all the letters of word ARROW do not contain two R's together ?

8. Find the number of permutations of letters of the word EXERCISES in which vowels are together.

9. How many numbers between 10,000 and 20,000 can be made using all digits of 12234 ? How many of them are less than 11,000 ?

10. Read the given sentence : 'LOOK AND GO'.

How many such sentences with 3 words on the same pattern and using the same letters (first 4 letter word, 3 letter word, 2 letter word) can be formed ? Meaning not attached.

11. How many permutations of all the letters of the word REKHA begin with R ? What is the rank of the word REKHA in dictionary order arrangement ?

12. In how many ways can the product $2^2\ 3^3\ 5^4$ be written in a product form like $2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 3 \cdot 5$?

13. How many permutations of all the letters of the word INDEPENDENCE are there ?

14. m is the number of arrangements of x things taking all at a time. n is the number of arrangements of $x - 2$ things taking 4 at a time. p is the number of arrangements of $x - 6$ things taking all at a time and if $m = 30np$, then find x.

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Circular Permutation :

Let 4 people a, b, c, d be arranged around a round dining table. In how many ways is this possible ?

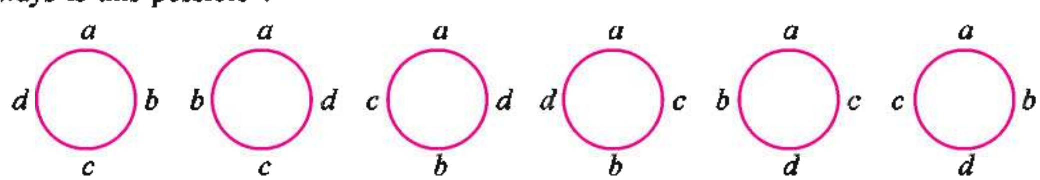


Figure 7.6

abcd, adcb, adbc, acbd, acdb, abdc are six possible arrangements. We would expect 4! = 24 as answer. But $\frac{24}{4} = 6$ is the number of arrangements, as *abcd, bcda, cdab, dabc* have relatively in the same position on a circle.

Definition : The number of ways of arranging n different objects on a circle is called the number of circular permutation of n objects.

Theorem 4 : The number of circular permutations of n different things is $(n - 1)!$

Proof : Let the objects be called $a_1, a_2, a_3, ..., a_n$. In a linear order their permutations amount to ${}_nP_n = n!$

But on a circle $a_1, a_2, ..., a_n$; $a_2, a_3, ..., a_n, a_1$; $a_3, a_4, ..., a_n, a_1, a_2$; $a_n, a_1, a_2, ..., a_{n-1}$ (n in number) give n identical permutations on a circle. Hence the **number of circular permutations is $\frac{n!}{n} = \frac{n(n-1)!}{n} = (n - 1)!$**

Example 24 : In how many ways can seven members of a managing committee take their positions on a round table ? If the chairman sits on a particular reserved chair, how many arrangements are possible ?

Solution : In circular permutation, 7 persons can be seated in $(7 - 1)! = 6! = 720$ ways.

According to the condition given six persons can be seated in $6! = 720$ ways. (linear) Chairman has a fixed position.

$\therefore$ Number of arrangements with chairman in a fixed chair is 720.

7.3 Combination

How many subsets with two elements from A = $\{a, b, c, d\}$ are there ? 6, namely $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$ and $\{c, d\}$. Here $\{a, b\} = \{b, a\}$ and order of elements in a set is not important. We say that number of selections of 2 elements from set A with 4 elements is denoted by $\binom{4}{2}$ or ${}_4C_2$ or ${}^4C_2$ or C(4, 2) is 6.

How many ordered pairs will be there with distinct elements ? (a, b), (b, a), (a, c), (a, d), (c, a), (d, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c) i.e. 12 in all.

Each subset of two elements will give $2! = 2$ ordered pairs and hence there will $\binom{4}{2} \times 2! = 12$ ordered pairs. But this is the same as arranging 4 elements of set A taking 2 at a time i.e. ${}_4P_2$

$\therefore \quad {}_4P_2 = \binom{4}{2} \times 2!$

Similarly from digits 2, 4, 6, 8, we can select three digits in $\binom{4}{3} = 4$ ways.

Now, how many numbers of three digits can be formed using 2, 4, 6, 8 ?

Each triplet 2, 4, 6 will give 6 numbers as follows :

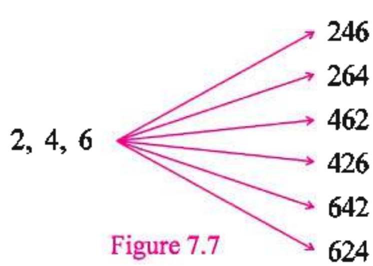


Figure 7.7

Thus there will be $\binom{4}{3} \times 6 = 4 \times 6 = 24$ three digit numbers that can be formed using 2, 4, 6, 8. But this is also the number of arrangements of 4 digits in 3 places.

$\therefore \quad {}_4P_3 = \binom{4}{3} \times 6$

or $\binom{4}{3} = \frac{{}_4P_3}{3!}$ $\qquad (6 = 3!)$

Definition : The number of ways of selecting r things out of n different things is called r combination number of n things and is denoted by $\binom{n}{r}$ or ${}_nC_r$ or ${}^nC_r$ or $C(n, r)$.

Theorem 5 : $\binom{n}{r} = \frac{{}_nP_r}{r!}$ $\qquad 0 < r \le n$

Proof : Let n different objects be given and r objects be selected. This gives $\binom{n}{r}$ selections. Arrange these r objects in r places linearly. This is possible in ${}_rP_r = r!$ ways. Thus, each of $\binom{n}{r}$ selections gives $r!$ arrangements. Hence total number of arrangements is $\binom{n}{r} r!$. But this is the number of arrangements of n distinct objects in r places and it is ${}_nP_r$.

$\therefore \quad {}_nP_r = \binom{n}{r} \times r!$

$\therefore \quad \binom{n}{r} = \frac{{}_nP_r}{r!}$

Definition : $\binom{n}{0} = 1$

We define $\binom{n}{0} = 1$. We justify. Selection of zero objects is same as rejecting all objects. This is possible in one way only.

Formulae For $\binom{n}{r}$:

$$\binom{n}{r} = \frac{{}_nP_r}{r!} \qquad 0 < r \le n$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \qquad 0 < r \le n$$

$$= \frac{n!}{(n-r)!\,r!} \qquad 0 < r \le n$$

But also $\binom{n}{0} = 1 = \frac{n!}{(n-0)!\,0!}$

$$\therefore \quad \binom{n}{r} = \frac{n!}{(n-r)!\,r!} \qquad 0 \le r \le n$$

Theorem 6 : $\binom{n}{r} = \binom{n}{n-r}$ $\quad 0 \le r \le n$

Proof : $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$= \dfrac{n!}{(n-r)!\,r!}$

$= \dfrac{n!}{(n-r)!\,[n-(n-r)]!}$

$= \binom{n}{n-r}$

Theorem 7 : $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$

Proof : $\binom{n}{r} + \binom{n}{r-1} = \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)![n-(r-1)]!}$

$= \dfrac{n!}{r!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)!}$

$= \dfrac{n!}{r(r-1)!(n-r)!} + \dfrac{n!}{(r-1)!(n-r+1)(n-r)!}$

$= \dfrac{n!}{(r-1)!(n-r)!}\left[\dfrac{1}{r} + \dfrac{1}{n-r+1}\right]$

$= \dfrac{n!}{(r-1)!(n-r)!}\left(\dfrac{n-r+1+r}{r(n-r+1)}\right)$

$= \dfrac{n!(n+1)}{r(r-1)!(n-r+1)(n-r)!}$

$= \dfrac{(n+1)!}{r!(n-r+1)!}$

$= \binom{n+1}{r}$

Alternative Proofs :

Proof for Theorem 6 : If we select r things out of n things, we reject remaining $(n-r)$ things.

$\therefore$ The number of selections is same as the number of rejections.

$\therefore \binom{n}{r} = \binom{n}{n-r}$

For example from A = {1, 2, 3, 4, 5}, 2 elements subsets are selected.

Select	Reject	Select	Reject
{1, 2}	{3, 4, 5}	{2, 4}	{1, 3, 5}
{1, 3}	{2, 4, 5}	{2, 5}	{1, 3, 4}
{1, 4}	{2, 3, 5}	{3, 4}	{1, 2, 5}
{1, 5}	{2, 3, 4}	{3, 5}	{1, 2, 4}
{2, 3}	{1, 4, 5}	{4, 5}	{1, 2, 3}

$\therefore$ Every two element subset corresponds to unique three element subset and vice-versa.

$$\binom{5}{2} = \binom{5}{3} = 10$$

Proof for Theorem 7 : Let A = $\{x_1, x_2, x_3, ..., x_{n+1}\}$

Number of r element subsets of A is $\binom{n+1}{r}$.

x_1 may or may not be a member of these subsets.

If x_1 is a member of r element subset, remaining $(r - 1)$ elements can be chosen from remaining n elements in $\binom{n}{r-1}$ ways.

If x_1 is not a member of r element subset, all the r elements should be selected from remaining n elements in $\binom{n}{r}$ ways,

$$\therefore \quad \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

> **Note** (1) $\binom{n}{r}$ increases as r increases and is maximum if $r = \frac{n}{2}$ for n even and $r = \frac{n-1}{2}$ and $\frac{n+1}{2}$ for n odd.
>
> Then $\binom{n}{r}$ decreases with the same values in the same order because $\binom{n}{r} = \binom{n}{n-r}$.
>
> (2) $\binom{n}{r} = k$, $k \in N \cup \{0\}$ has at most two solutions :
>
> For example $\binom{4}{r} = 6$ has only one solution $r = 2$.
>
> $\binom{4}{r} = 5$ has no solution.
>
> $\binom{4}{r} = 4$ has two solutions $r = 1, 3$.

Example 25 : Prove $\binom{2n}{r}$ is maximum for $r = n$.

Solution : $\binom{2n}{r+1} > \binom{2n}{r} \Leftrightarrow \frac{(2n)!}{(r+1)!(2n-r-1)!} > \frac{(2n)!}{r!(2n-r)!}$

$$\Leftrightarrow \frac{(2n-r)!}{(2n-r-1)!} > \frac{(r+1)!}{r!}$$

$$\Leftrightarrow 2n - r > r + 1$$

$$\Leftrightarrow r < n - \tfrac{1}{2}$$

$$\Leftrightarrow r \le n - 1$$

Thus, $\binom{2n}{1} < \binom{2n}{2} < ... < \binom{2n}{n-1} < \binom{2n}{n}$ and $\binom{2n}{n}$ is largest.

$\binom{2n}{n+1} = \binom{2n}{n-1} > \binom{2n}{n+2} = \binom{2n}{n-2}$...

Example 26 : Solve for n : $\binom{n}{5} = \binom{n}{13}$. Then find $\binom{n}{2}$.

Solution : We know $\binom{n}{r} = \binom{n}{n-r}$

Here let $r = 5$, $n - r = 13$

$\therefore \quad n - 5 = 13$

$\therefore \quad n = 18$

$\binom{n}{2} = \binom{18}{2} = \frac{18 \times 17}{2} = 153$

Example 27 : Solve : $\binom{2n}{3} = 11\binom{n}{3}$ and find $\binom{n}{2}$.

Solution :

$$\frac{2n(2n-1)(2n-2)}{3!} = \frac{11\,n(n-1)(n-2)}{3!} \qquad \left(\binom{n}{r} = \frac{n(n-1)(n-2)....(n-r+1)}{r!}\right)$$

$\therefore \quad 4(2n - 1) = 11(n - 2)$

$\therefore \quad 8n - 4 = 11n - 22$

$\therefore \quad 3n = 18$

$\therefore \quad n = 6$

Also, $\binom{n}{2} = \binom{6}{2} = \frac{6 \cdot 5}{2} = 15$

Example 28 : If $\binom{n}{r-1} = 36$, $\binom{n}{r} = 84$, $\binom{n}{r+1} = 126$, find n and r.

Solution : We have $\binom{n}{r-1} = \frac{n!}{(r-1)!(n-r+1)!} = 36$ **(i)**

$\binom{n}{r} = \frac{n!}{r!(n-r)!} = 84$ **(ii)**

$\binom{n}{r+1} = \frac{n!}{(r+1)!(n-r-1)!} = 126$ **(iii)**

Dividing (ii) by (i) and (iii) by (ii),

$$\frac{n!}{r!(n-r)!} \; \frac{(r-1)!(n-r+1)!}{n!} = \frac{84}{36} \qquad \textbf{(iv)}$$

and $$\frac{n!}{(r+1)!(n-r-1)!} \; \frac{r!(n-r)!}{n!} = \frac{126}{84} \qquad \textbf{(v)}$$

$\therefore$ (iv) gives $\frac{(r-1)!(n-r+1)(n-r)!}{r(r-1)!(n-r)!} = \frac{84}{36}$

$\therefore \quad \dfrac{n-r+1}{r} = \dfrac{7}{3}$

$\therefore \quad 3n - 3r + 3 = 7r$

$\therefore \quad 10r - 3n = 3$ **(vi)**

(v) reduces to $\dfrac{n-r}{r+1} = \dfrac{3}{2}$ or $2n - 2r = 3r + 3$

$\therefore \quad 5r - 2n = -3$ **(vii)**

Solving (vi) and (vii) we get $n = 9$, $r = 3$

Example 29 : Solve (1) $\binom{8}{r} = 28$ (2) $\binom{12}{r} = \binom{12}{r+2}$

Solution : $\binom{8}{r} = 28$

$r = 4$ gives maximum value $\binom{8}{4} = \dfrac{8 \times 7 \times 6 \times 5}{24} = 70$

$\therefore \quad \binom{8}{0} = \binom{8}{8} = 1 \neq 28$

$\therefore \quad \binom{8}{1} = \binom{8}{7} = 8 \neq 28$

$\therefore \quad \binom{8}{2} = \binom{8}{6} = \dfrac{8 \times 7}{2} = 28$

Since there are at most two solutions of $\binom{8}{r} = 28$, $r = 2$ or 6

(2) $\binom{12}{r} = \binom{12}{r+2}$

$r \neq r + 2$

Here $n = 12$, $r + 2 = n - r = 12 - r$

$\therefore \quad 2r = 10$

$\therefore \quad r = 5$

Verification : $\binom{12}{5} = \binom{12}{7}$

Example 30 : Solve $\binom{2n}{3} \div \binom{n}{2} = 12$

Solution : $\dfrac{2n(2n-1)(2n-2)}{3!} \times \dfrac{2!}{n(n-1)} = 12$

$\therefore \quad \dfrac{2(2n-1) \cdot 2}{3} = 12$

$\therefore \quad 2n - 1 = 9$

$\therefore \quad n = 5$

Example 31 : Find n and r. $\binom{n+1}{r+1} : \binom{n}{r} : \binom{n-1}{r-1} = 11 : 6 : 3.$

Solution : $\binom{n+1}{r+1} : \binom{n}{r} = \frac{11}{6}$ and $\binom{n}{r} : \binom{n-1}{r-1} = \frac{6}{3}$

$\therefore \quad \frac{(n+1)!}{(r+1)!(n-r)!} \; \frac{r!(n-r)!}{n!} = \frac{11}{6}$ and $\frac{n!}{r!(n-r)!} \; \frac{(r-1)!(n-r)!}{(n-1)!} = \frac{6}{3}$

$\therefore \quad \frac{n+1}{r+1} = \frac{11}{6}$ and $\frac{n}{r} = 2$

$\therefore \quad 6n + 6 = 11r + 11$ and $n = 2r$

$\therefore \quad 12r + 6 = 11r + 11$

$\therefore \quad r = 5$ and $n = 2r = 10$

Example 32 : Prove that the product of n consecutive integers is divisible by $n!$.

Solution : Let the n consecutive intergers be $r + 1, r + 2, ..., r + n$

Product $p = (r + 1)(r + 2)...(r + n)$

$$= \frac{1 \cdot 2 \cdot 3.... r(r+1).....(r+n)}{1 \cdot 2 \cdot 3.... r}$$

$$= \frac{(n+r)!}{r!} = \frac{(n+r)!}{r!\,n!} \times n! = \binom{n+r}{r} n! \text{ is divisible by } n!.$$

Example 33 : Prove $\binom{2n}{n} = \frac{2^n [1 \cdot 3 \cdot 5 \cdot 7....(2n-1)]}{n!}$

Solution : $\binom{2n}{n} = \frac{(2n)!}{n!\,n!}$

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8.... (2n)}{n!\,n!}$$

$$= \frac{[1 \cdot 3 \cdot 5 \cdot 7.... (2n-1)]\,[2 \cdot 4 \cdot 6 \cdot 8.... 2n]}{n!\,n!}$$

$$= \frac{[1 \cdot 3 \cdot 5 \cdot 7.... (2n-1)]\,2^n\,[1 \cdot 2 \cdot 3 \cdot 4.... n]}{n!\,n!}$$

$$= \frac{2^n\,[1 \cdot 3 \cdot 5 \cdot 7.... (2n-1)]}{n!}$$

Example 34 : If ${}_nP_r = {}_nP_{r+1}$ and $\binom{n}{r} = \binom{n}{r-1}$, find n, r.

Solution : $\frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$ and $\frac{n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r+1)!}$

$\therefore \quad (n - r)! = (n - r - 1)!$ and $\frac{r!}{(r-1)!} = \frac{(n-r+1)!}{(n-r)!}$

$\therefore$ $(n - r)(n - r - 1)! = (n - r - 1)!$ and $r = n - r + 1$

$\therefore$ $n - r = 1$ and $r = n - r + 1$

$\therefore$ $r = (n - r) + 1 = 1 + 1 = 2$ and $n = r + 1 = 3$

Example 35 : Prove $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} = \binom{n+2}{r}$

Solution : L.H.S. $= \binom{n}{r} + \binom{n}{r-1} + \binom{n}{r-1} + \binom{n}{r-2}$

$$= \binom{n+1}{r} + \binom{n+1}{r-1}$$

$$= \binom{n+2}{r} = \text{R.H.S.}$$

Example 36 : If $\binom{n-1}{4}, \binom{n-1}{5}, \binom{n-1}{6}$ are in A.P., find n.

Solution : $\binom{n-1}{4} + \binom{n-1}{6} = 2\binom{n-1}{5}$ **(in A.P.)**

$\therefore$ $\binom{n-1}{4} + \binom{n-1}{5} + \binom{n-1}{5} + \binom{n-1}{6} = 4\binom{n-1}{5}$

$\therefore$ $\binom{n}{5} + \binom{n}{6} = 4\binom{n-1}{5}$

$\therefore$ $\binom{n+1}{6} = 4\binom{n-1}{5}$

$\therefore$ $\dfrac{(n+1)n(n-1)(n-2)(n-3)(n-4)}{720} = \dfrac{4(n-1)(n-2)(n-3)(n-4)(n-5)}{120}$

$\therefore$ $n^2 + n = 24(n - 5)$

$\therefore$ $n^2 - 23n + 120 = 0$

$\therefore$ $n = 15$ or 8

EXERCISE 7.4

1. Find : (1) $\binom{8}{2}$ (2) $\binom{5}{3}$ (3) $\binom{10}{4}$
2. If $\binom{n}{8} = \binom{n}{6}$, find n.
3. Solve (1) $\binom{15}{r+3} = \binom{15}{r-2}$ (2) $\binom{16}{r+5} = \binom{16}{r-5}$
4. If ${}_nP_r = 1680$ and $\binom{n}{r} = 70$ find n and r.
5. If $\binom{n-1}{r} : \binom{n}{r} : \binom{n+1}{r} = 6 : 9 : 13$, find n and r.

6. Prove $\binom{n}{r} \times \binom{r}{p} = \binom{n}{p} \times \binom{n-p}{r-p}$.

7. If $\binom{10}{2} + \binom{13}{6} + \binom{12}{5} + \binom{11}{4} + \binom{10}{3} = \binom{14}{r}$, find r.

8. Prove $n\binom{n-1}{r-1} = (n - r + 1)\binom{n}{r-1}$

*

Practical Problems on Permutations and Combinations

Example 37 : 7 points are given in a plane, no three of which are collinear. How many line segments can be drawn using them ?

Solution : Any two points will determine a segment and $\overline{AB} = \overline{BA}$. So order of selection is immaterial. Hence $\binom{7}{2} = \frac{7 \cdot 6}{2} = 21$ line segments can be drawn using them.

Example 38 : 8 points of which 3 are collinear lie in a plane. How many triangles can be formed by them ? How many lines will pass through any two distinct points from them ? How many line segments are formed ?

Solution : 8 non-collinear points can determine $\binom{8}{3}$ triangles like ΔADE. We are considering $\binom{3}{3} = 1$ triangle formed by A, B and C. But A, B, C are collinear.

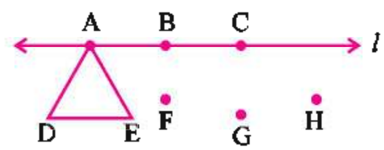


Figure 7.8

Number of triangles is $\binom{8}{3} - 1 = \frac{8 \cdot 7 \cdot 6}{6} - 1 = 55$

Two points determine a line. Hence $\binom{8}{2} = 28$ lines should be formed.

But $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{CA}$ are not three lines but it is one line only as A, B, C are collinear.

Hence from 28 lines considered, we should remove three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{CA}$ counted as three lines and add one line l containing A, B, C.

$\therefore$ 28 − 3 + 1 (line l) = 26 lines are formed.

All $\binom{8}{2} = \frac{8 \cdot 7}{2} = 28$ line segments are different. $\overline{AB} \neq \overline{BC}$ etc.

$\therefore$ 28 line segments are formed.

Example 39 : In how many ways can a committee consisting of 3 boys and 2 girls to celebrate Swarnim Gujarat be selected from 5 boys and 4 girls ? In how many of them a certain boy Kiran will always be selected ? In how many of them a certain girl Reshma will be selected ?

Solution : Selection of 3 boys out of 5 boys and 2 girls out of 4, according to the fundamental principle of counting, will take place in $\binom{5}{3} \times \binom{4}{2}$ ways.

Now, $\binom{5}{3}\binom{4}{2} = \frac{5 \cdot 4 \cdot 3}{3!} \times \frac{4 \cdot 3}{2!}$

$= 10 \times 6 = 60$

$\therefore$ We can select the committee in 60 ways. **(i)**

To select Kiran we have to select 2 boys out of 4 boys and 2 girls also out of 4 girls.

$\therefore$ Number of selections $= \binom{4}{2} \times \binom{4}{2} = \frac{4 \cdot 3}{2!} \times \frac{4 \cdot 3}{2!} = 6 \times 6 = 36$ **(ii)**

36 committees will have Kiran as a member.

To select Reshma, we have to select 3 boys out of 5 boys and 1 girl out of 3 girls.

$\therefore$ Number of selections $= \binom{5}{3} \times \binom{3}{1}$

$= 10 \times 3 = 30$ **(iii)**

30 committees will have Reshma as a member.

Example 40 : 3 cards are chosen from a pack of 52 cards. (1) In how many ways can this be done ? (2) In how many ways can you select three cards so that all of them are face cards ? (3) In how many of the selections, all cards are of the same colour ? (4) In how many of them all cards are of the same suit ?

Solution : (1) We can select three cards in $\binom{52}{3} = \frac{52 \cdot 51 \cdot 50}{3!}$

$= \frac{52 \cdot 51 \cdot 50}{6}$

$= 22100$

$\therefore$ Three cards can be selected in 22100 ways.

(2) There are 12 face cards. Hence we can have $\binom{12}{3}$ selections.

$\binom{12}{3} = \frac{12 \cdot 11 \cdot 10}{3!} = 220$

$\therefore$ Three face cards can be selected in 220 ways.

(3) There are 26 cards of red colour (heart and diamond) and 26 cards of black colour (club and spade).

$\therefore$ $\binom{26}{3} + \binom{26}{3}$ is number of ways to select three cards of black or red colour.

$\binom{26}{3} = \frac{26 \cdot 25 \cdot 24}{6} = 2600$

$\therefore$ There 5200 ways to select cards of the same colour.

(4) There are four suits. One suit can be chosen from them in $\binom{4}{1}$ ways and 3 cards from the selected suit in $\binom{13}{3}$ ways.

$\therefore$ Numbers of selections $= \binom{13}{3} \times \binom{4}{1} = \frac{13 \cdot 12 \cdot 11}{6} \times 4 = 1144$

Example 41 : A reception committee consisting of 6 students for the annual function of a school is to be formed from 8 boys and 5 girls. In how many ways can we do it if the committee is to contain (1) exactly 4 girls (2) at most 2 girls (3) at least 3 girls ?

Solution : (1) Selection of 6 students will involve 4 girls out of 5 girls and 2 boys out of 8 boys.

$\therefore$ Numbers of selections $= \binom{5}{4} \times \binom{8}{2}$

$= \binom{5}{1} \times \binom{8}{2}$

$= \frac{5 \cdot 8 \cdot 7}{2!} = 140$

(2) At most 2 girls means 2 or less number of girls. Following alternatives are possible.

Boys	Girls
4	2
5	1
6	0

$\therefore$ The number of selections $= \binom{8}{4} \times \binom{5}{2} + \binom{8}{5} \times \binom{5}{1} + \binom{8}{6} \times \binom{5}{0}$

$= \frac{8 \cdot 7 \cdot 6 \cdot 5}{24} \times \frac{5 \cdot 4}{2} + \frac{8 \cdot 7 \cdot 6}{6} \times 5 + \frac{8 \cdot 7}{2} \times 1$

$\left[\binom{8}{5} = \binom{8}{3} \text{ and } \binom{8}{6} = \binom{8}{2}\right]$

$= 700 + 280 + 28 = 1008$

(3) At least 3 girls means 3 or more girls :

Boys	Girls
3	3
2	4
1	5

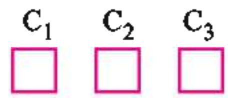
$\therefore$ The number of selections $= \binom{8}{3} \times \binom{5}{3} + \binom{8}{2} \times \binom{5}{4} + \binom{8}{1} \times \binom{5}{5}$

$= \frac{8 \cdot 7 \cdot 6}{6} \times \frac{5 \cdot 4 \cdot 3}{6} + \frac{8 \cdot 7}{2} \times 5 + 8$

$\left[\binom{5}{4} = \binom{5}{1} = 5\right]$

$= 560 + 140 + 8 = 708$

Example 42 : Three couples go to see a movie. In how many ways can they occupy a seat if spouses occupy adjacent seats ? In how many ways can ladies be seated together ?



Solution : Couples C_1, C_2, C_3 can be arranged $3! = 6$ ways. In an arrangement of a couple husband and wife can be arranged in $2! \times 2! \times 2! = 8$ ways.

$\therefore$ There are 48 ways to make the required arrangement. **(i)**

Arrange three men and one group of ladies together i.e. four units in ${}_4P_4 = 4! = 24$ ways.

In each arrangement ladies can be permuted in $3! = 6$ ways.

$\therefore$ Total number of arrangements is $24 \times 6 = 144$ **(ii)**

EXERCISE 7.5

1. How many triangles can be formed from nine points in plane, four of which are collinear ? How many lines can be formed using them ?

2. In how many ways can a committee consisting of 4 male and 4 female members be formed from 8 male and 6 female members of a city club ? How many committees of eight members can be formed with (1) at least 3 female members (2) exactly 2 female members (3) at most 2 female members.

3. Four cards are selected from a pack of 52 cards. (1) In how many ways can they be selected so that they are all from different suits ? (2) In how many ways can they be selected so that they are face cards ? (3) In how many ways can they be selected so that they are all of the same colour ?

4. There are two white, three red and four green marbles. Three marbles are drawn. In how many ways can this be done so that at least one red marble is selected ?

5. 15 students are divided into three groups with equal number. In how many ways can this be done ?

6. In an award function 12 celebrities occupy place around two round tables with accommodation for 8 and 4 persons. In how many ways can this take place ?

7. In how many ways can n persons be arranged in a queue so that certain 2 persons are not adjacent to each other ?

8. What is the number of diagonals of a convex polygon with n sides ?

9. A convex polygon has 44 diagonals. How many sides does it have ?

10. In a convex polygon of n sides, how many triangles are formed joining vertices ? How many of them will have one side in common with the polygon ? How many of them will have two sides in common with the polygon ? How many of them will have no side in common with the polygon ?

*

Miscellaneous Problems

Example 43 : Assuming that the highest power of a prime p occuring in $n!$ is $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots$ find the highest power of 5 occuring in 25!

Solution : The highest power of 5 in 25! is $\left[\frac{25}{5}\right] + \left[\frac{25}{25}\right] = 5 + 1 = 6$

Example 44 : How many zeroes will occur at the end of 52! ?

Solution : The highest power of 5 in 52! is $\left[\frac{52}{5}\right] + \left[\frac{52}{25}\right] = 10 + 2 = 12$

The highest power of 2 in 52! $= \left[\frac{52}{2}\right] + \left[\frac{52}{4}\right] + \left[\frac{52}{8}\right] + \left[\frac{52}{16}\right] + \left[\frac{52}{32}\right]$

$= 26 + 13 + 6 + 3 + 1 = 49$

$\therefore$ The highest power of 10 in 52! = 12

$\therefore$ 52! has 12 zeroes at the end.

Example 45 : If a set A has 3 elements and B has 5 elements how many functions $f : A \to B$ are there ? How many of them will have range as B ?

Solution : Let $A = \{x_1, x_2, x_3\}$, $B = \{y_1, y_2, y_3, y_4, y_5\}$

For a function $f : A \to B$ ordered pairs (x_i, y_j) are to be formed with no x_i from A repeating in a pair and all x_i have to be used once.

Thus $f = \{(x_1, y_p), (x_2, y_q), (x_3, y_r)\}$ is a typical function $f : A \to B$. Thus selection of y's can be done in $5 \times 5 \times 5 = 125$ ways.

Hence number of functions $f : A \to B$ is 125. Each function will have range consisting of at most three elements. Hence no function will have range as B.

Example 46 : How many 5 card combinations of 52 cards are there containing at least one ace ?

Solution : We can select,

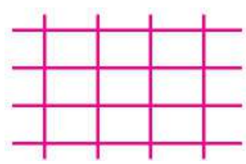
Ace (4)	Other cards (48)
1	4
2	3
3	2
4	1

$\therefore$ Number of selections is $\binom{4}{1} \times \binom{48}{4} + \binom{4}{2} \times \binom{48}{3} + \binom{4}{3} \times \binom{48}{2} + \binom{4}{4} \times \binom{48}{1}$

$= 8{,}86{,}656$

Example 47 : If m vertical bars meet n horizontal bars, how many rectangles will be formed ?

Solution : A rectangle is formed when two horizontal and two vertical lines are selected.



$\therefore$ Number of rectangles formed $= \binom{m}{2} \times \binom{n}{2}$

$$= \frac{m(m-1)}{2!} \times \frac{n(n-1)}{2!} = \frac{mn(m-1)(n-1)}{4}$$

Example 48 : In a plane, we have 25 lines out of which 15 are concurrent in A and 5 are concurrent in B. No two lines are parallel and out of other lines, no three of them are concurrent. Find the number of the points of intersection.

Solution : 25 lines can intersect in $\binom{25}{2}$ points. But instead of $\binom{15}{2}$ points of intersection, we get only one point A and instead of $\binom{5}{2}$ points of intersection we get only one point B.

$\therefore$ Number of points of intersection $= \binom{25}{2} - \binom{15}{2} - \binom{5}{2} + 2$ (A and B)

$= 300 - 105 - 10 + 2 = 187$

Example 49 : A student taking an examination has to answer 20 questions from sections A and B containing 12 questions each. It is necessary to answer at least 8 questions from each section. In how many ways can he take the examination ?

Solution : Several options are available.

Questions from section A	Questions from section B
8	12
9	11
10	10
11	9
12	8
(Out of 12)	(Out of 12)

$\therefore$ Number of ways to take examination is

$$\binom{12}{8}\binom{12}{12} + \binom{12}{9}\binom{12}{11} + \binom{12}{10}\binom{12}{10} + \binom{12}{11}\binom{12}{9} + \binom{12}{12}\binom{12}{8}$$

$$= 2\binom{12}{4} + 2\binom{12}{3}\binom{12}{1} + \binom{12}{2}\binom{12}{2} \qquad \left(\binom{n}{r} = \binom{n}{n-r}\right)$$

$$= \frac{2 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{24} + \frac{2 \cdot 12 \cdot 11 \cdot 10}{6} \times 12 + \left(\frac{12 \cdot 11}{2}\right)^2$$

$$= 990 + 5280 + 4356 = 10626$$

Example 50 : How many triangles can be formed by 12 points 7 of which lie on one line and 5 on another parallel line ?

Solution : Obviously number of triangles is

$$= \binom{7}{2}\binom{5}{1} + \binom{5}{2}\binom{7}{1}$$

$$= 21 \times 5 + 10 \times 7$$

$$= 105 + 70 = 175$$

EXERCISE 7

1. A group of socially active friends has 8 boys and 5 girls as members. In how many ways can a task be assigned to a team of 5 members containing
   (1) at least one boy and at least one girl. (2) no boy
   (3) at least three girls (4) at most two boys
2. Find the numbers of integers greater than 7000 obtained by using 2, 5, 6, 8, 9, no digit being repeated.
3. A convex polygon has 54 diagonals. Find the number of vertices.
4. Out of 8 teachers $T_1, T_2, T_3, ..., T_8$, five are to be selected for training. T_1 is the junior most and he has to go for training. T_8 is retiring next year and does not require any training. In how many ways should the squad be selected ? If only one teacher goes for training every week in how many ways can their order be arranged ? The training is to last for five weeks.
5. Find the number of selections of r things out of n distinct things such that none of particular things are selected at a time.
6. Out of five vowels a, e, i, o, u words with four letters are to be formed. Find number of words in which any one vowel is repeated at least three times.
7. In a test of multiple choice questions, there are four choices for each question and there are 5 such questions. In how many ways can a student fail to get complete correct solution ?
8. There are two parallel lines l_1 and l_2 in plane.
   l_1 contains m different points $A_1, A_2, ..., A_m$.
   l_2 contains n different points $B_1, B_2, ..., B_n$.
   How many triangles are possible with these vertices ?
9. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

   (1) If $\binom{n}{r} = \binom{n}{r+2}$, then r = ... (n is even) ☐
   (a) n (b) $n - 1$ (c) 0 (d) $\frac{n-2}{2}$

   (2) If $\binom{n}{8} = \binom{n}{12}$, then n = ... ☐
   (a) 20 (b) 10 (c) 15 (d) not possible

   (3) If $\binom{n}{r} = \frac{{}^nP_r}{k}$, then k = ... ☐
   (a) $r!$ (b) $(n - r)!$ (c) $(n - r)!\, r!$ (d) $(r - 1)!$

   (4) If $\binom{n}{r} = \frac{n!}{k}$, then k = ... ☐
   (a) $r!$ (b) $(n - r)!$ (c) $(n - r)!\, r!$ (d) $[r(n - 1)]!$

(5) The number of possible outcomes when a coin is tossed 8 times is ......

(a) 32 (b) 64 (c) 128 (d) 256

(6) The sum of digits of unit place of all four digit numbers formed using 3, 4, 5, 6 without repetition is ......

(a) 24 (b) 108 (c) 72 (d) 96

(7) A four digit number is formed using 0, 1, 2, 3, 4. It is divisible by 20. The number of such numbers is....

(a) 12 (b) 6 (c) 24 (d) 96

(8) In a get-to-gether function, everybody handshakes with everybody else. The total number of handshakes is 105. The number of persons in the hall is ....

(a) 12 (b) 11 (c) 15 (d) 14

(9) The total number of 9 digit numbers with all the digits different is....

(a) 9! (b) 10! (c) 10! − 8! (d) 9(9!)

(10) If there are eight points in a plane out of which three are collinear. The number of lines formed by them is.....

(a) 28 (b) 26 (c) 56 (d) 55

(11) If there are 12 points in a plane out of which 6 each lie on two parallel lines, the number of triangles formed by them is .....

(a) 120 (b) 180 (c) 60 (d) 40

(12) If $\binom{100}{r}$ is maximum, r = ...

(a) 100 (b) 0 (c) 50 (d) 51

(13) The number of rectangles formed when 10 horizontal bars intersect 8 vertical bars is ......

(a) 1880 (b) 800 (c) 80 (d) 1260

(14) The number of ways in which seven + signs and four − signs can be arranged so that − signs do not occupy adjacent places is ......

(a) ${}_8P_4$ (b) $\binom{7}{4}$ (c) $\binom{8}{4}$ (d) None of these

(15) 7 persons can sit around a round dining table for dinner in ...... ways.

(a) 720 (b) 80 (c) 60 (d) 5040

(16) If $\binom{44}{r-2} = \binom{44}{r+2}$, then r = ...

(a) 33 (b) 11 (c) 22 (d) 44

(17) The number of diagonals of a decagon is ......

(a) 35 (b) 45 (c) 55 (d) 30

(18) If $\binom{20}{r} = \binom{20}{r+2}$, then $\binom{r}{2}$ = ...

(a) 11 (b) 9 (c) 45 (d) 36

(19) If $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{x}$, then x = ...

(a) $n - r$ (b) $r + 1$ (c) n (d) $n - r + 1$

(20) If $\binom{a^2+a}{3} = \binom{a^2+a}{9}$, then a = ...

(a) 3 (b) 9 (c) 12 (d) 6

(21) $\binom{10}{1} + \binom{10}{2} + \binom{11}{3} + \binom{12}{4} + \binom{13}{5}$ = ...

(a) $\binom{14}{6}$ (b) $\binom{13}{7}$ (c) $\binom{13}{6}$ (d) $\binom{14}{5}$

(22) If $\binom{77}{r}$ is maximum, r = ...

(a) 35 (b) 38.5 (c) 39 (d) 40

(23) $\binom{33}{10}$ ...... $\binom{33}{8}$.

(a) $>$ (b) $<$ (c) $=$ (d) $\geq$

(24) $\binom{n}{r}$ increases as r increases from 0 to ....

(a) n (b) $n - 1$ (c) $\frac{n}{2}$ (d) $\left[\frac{n}{2}\right]$

(25) If $\binom{18}{10} = \binom{18}{k}$, k = ... $(n > 10)$

(a) n (b) 8 (c) 0 (d) $n + 1$

Summary

1. Fundamental principal of counting
2. Linear permutation and formulae
3. Permutation with repetitions
4. Permutation with identical objects
5. Circular permutation
6. Combination and its formulae and theorems
7. Practical problems related to permutation and combination.

Chapter 8

LINEAR INEQUALITIES

8.1 Introduction

We have solved linear equations in one variable and simultaneous linear equations in two variables in lower standards. We have also transformed some statement problems in such equations and solved them. In practical situation, it is not always possible to translate a problem into an equation. They may involve $<$, $>$, $\leq$ or $\geq$ signs of inequality. For example, temperature in the month of May this year in Ahmedabad had ranged between 28° C to 44° C. If x is the temperature recorded on a particular day

$$28 < x < 44 \text{ i.e. } 28 < x \text{ and } x < 44.$$

The salary of government employees had a rise between ₹ 800 to ₹ 30,000. So if x is the rise in the salary, $800 < x < 30000$. Rucha can solve atleast 20 problems in an hour. So if x is the number of problems solved by Rucha in an hour, $x \geq 20$. Dev can buy at most 10 pencils from his pocket money. So if x is the number of pencils Dev can buy, $x \leq 10$. These mathematical expressions like $x \leq 10$ or $x \geq 20$ or $28 < x < 44$ are called **linear inequalities in one variable**. $20x + 31y \leq 500$ is an example of a **linear inequality in two variables**.

Inequalities are useful in mathematics, science, optimisation problems in statistics etc.

8.2 Inequalities

How do inequalities arise in day-to-day practice ? Whenever we compare two quantities, they are more likely to be unequal than equal. If x is the number of minutes required by you to answer annual question paper completely, $0 \leq x \leq 180$. Mumbai is 500 km from Ahmedabad. If Vadodara is at a distance of x km from Ahmedabad, then $x < 500$ and if y is the distance of Pune from Ahmedabad, $y > 500$.

Consider following problems.

(1) Devi goes to buy some full-scape books with ₹ 500 in her purse. If one dozen such note-books can be purchased at the rate of ₹ 60 per dozen and x is the number (in dozen) of note-books she buys, then $60x \leq 500$. If y is the number of note-books purchased, $5y \leq 500$. (Cost of one note-book = $\frac{60}{12}$ = 5)

(2) Dev goes to a shopping mall to buy some shirts and pants. Each shirt costs ₹ 200 and a pant costs ₹ 500. He buys x shirts and y pants. He has ₹ 5000 with him. Then $200x + 500y$ is the amount he has to pay. So $200x + 500y \le 5000$.

In the above two problems x and y are non-negative integers. But consider following situation.

(3) We wish to collect the data of solutions of linear equations $ax + b = 0$ lying in the interval $[3, 5]$, $a, b \in R$. This will be the set of all real numbers x satisfying $3 \le x \le 5$.

We know for any two real numbers a, b; $a < b$ or $a = b$ or $a > b$ **(Law of Trichotomy)**. If $a = b$, we say a, b are equal. If **$a \ne b$, then $a < b$ or $a > b$ and both $a < b$ and $a > b$ are called strict inequalities.** Sometimes we come across inequalities like $a \le b$ or $a \ge b$ (as seen above).

$a \le b$ means $a < b$ or $a = b$. (read : a less than or equal to b)

$a \ge b$ means $a > b$ or $a = b$. (read : a greater than or equal to b)

These are called slack inequalities.

We remember following postulates for inequalities. For $a, b, c \in R$.

(1) If $a > 0$ and $b > 0$, then $a + b > 0$ and $ab > 0$.

(2) $a < 0 \Leftrightarrow -a > 0$

We will also use following properties :

(1) $a > b \Leftrightarrow a - b > 0$

(2) $a > b$ and $b > c \Rightarrow a > c$ (Can you prove ?) **(Transitivity)**

(3) $a > b \Rightarrow a + c > b + c$

(4) $a > b, c > 0 \Rightarrow ac > bc$ and

$a > b, c < 0 \Rightarrow ac < bc$

(5) $a > b, c > d \Rightarrow a + c > b + d$ (Prove !)

We will have following type of linear inequalities in one variable.

(1) $ax + b < c$ (2) $ax + b > c$ (3) $ax + b \le c$ (4) $ax + b \ge c$

or you may restructure them as

(1) $ax + b < 0$ (2) $ax + b > 0$ (3) $ax + b \le 0$ (4) $ax + b \ge 0$

(consider $b - c$ as b)

8.3 Solution Set of a Linear Inequalitity in One Variable

Consider the inequalitity in problem (1) in 8.2, $60x \le 500$

If $x = 2$, $120 \le 500$ is a true statement.

For $x = 5$, $300 \le 500$ is also a true statement.

But if $x = 10$, we get $600 \le 500$ which is a false statement.

The values of variable x which give a true statement from an inequality after substitution are called solutions of the inequality. All such solutions constitute the solution set of the inequality.

Definition : The set of values of the variable which give a true statement from an inequality after substituting the value is called the solution set of the inequality.

Example 1 : Solve : $20x + 9 < 300$ for (1) $x \in N$ (2) $x \in Z$ (3) $x \in R$.

Solution : $20x + 9 < 300 \quad \Leftrightarrow 20x < 291$

$$\Leftrightarrow x < \frac{291}{20} = 14.55$$

(1) If x is a natural number $x = 1, 2, 3, 4,...,14$.

The solution set is $\{1, 2, 3, 4,...,14\} = \{x \mid x \in N, x \le 14\}$

(2) If $x \in Z$ we can have $x = 0, -1, -2,...$ etc.

$\therefore$ The solution set is $\{..., -3, -2, -1, 0, 1, 2,..., 14\} = \{x \mid x \in Z, x \le 14\}$

(3) In R, the solution set is $\{x \mid x < 14.55, x \in R\}$

Example 2 : Solve : $2x - 3 > 5$ where (1) $x \in N$ (2) $x \in R$.

Solution : $2x - 3 > 5 \Leftrightarrow 2x > 8$

$$\Leftrightarrow x > 4$$

(1) The solution set is $\{5, 6, 7, 8,...\} = \{x \mid x \ge 5, x \in N\}$

(2) The solution set in R is $(4, \infty) = \{x \mid x \in R, x > 4\}$

Example 3 : Solve : $5x < 7$, where (1) $x \in N$ (2) $x \in R$

Solution : $5x < 7 \Leftrightarrow x < \frac{7}{5} = 1.4$

(1) The solution set in N is $\{1\}$.

(2) The solution set in R is $\left\{x \mid x \in R, x < \frac{7}{5}\right\}$

$$= \left(-\infty, \frac{7}{5}\right)$$

Example 4 : Solve : $-2x \ge 10$, (1) $x \in N$ (2) $x \in Z$ (3) $x \in R$

Solution : $-2x \ge 10 \Leftrightarrow x \le -5$ (see that we divide by -2 which is $-$ve)

(1) If $x \in N$, the solution set is $\emptyset$

(2) If $x \in Z$, the solution set is $\{..., -8, -7, -6, -5\}$

(3) If $x \in R$, the solution set is $(-\infty, -5]$

Example 5 : Solve : $2(x - 1) + 1 > 3 - (-1 - 2x)$, $x \in R$

Solution : $2(x - 1) + 1 > 3 - (-1 - 2x) \Leftrightarrow 2x - 2 + 1 > 3 + 1 + 2x$

$$\Leftrightarrow 2x - 2x > 5$$

$$\Leftrightarrow 0 > 5$$

This is always false.

$\therefore$ The solution set is $\emptyset$.

☛ Note If $>$ is replaced by $<$ in the above problem, we get $0 < 5$ which is true for all real x. Hence the solution set would be R.

Example 6 : Solve : $3(2x - 1) + 5 \leq \frac{1}{2}(x + 15)$, $x \in$ R.

Solution : $3(2x - 1) + 5 \leq \frac{1}{2}(x + 15) \iff 6(2x - 1) + 10 \leq x + 15$

$$\iff 12x - 6 + 10 \leq x + 15$$

$$\iff 11x \leq 15 + 6 - 10 = 11$$

$$\iff x \leq 1$$

$\therefore$ The solution set is $(-\infty, 1]$.

Example 7 : Solve : $\frac{3-2x}{5} < \frac{x}{3} + 2$.

Solution : We have $\frac{3-2x}{5} < \frac{x}{3} + 2 \iff \frac{3-2x}{5} - \frac{x}{3} < 2$

$$\iff \frac{3(3-2x)-5x}{15} < 2$$

$$\iff 9 - 6x - 5x < 30 \qquad \textbf{(15 > 0)}$$

$$\iff 9 - 11x < 30$$

$$\iff -11x < 21$$

$$\iff x > \frac{-21}{11} \qquad \textbf{(–11 < 0)}$$

Hence the solution set is $\left(\frac{-21}{11}, \infty\right)$

Example 8 : Solve : $\frac{2x+3}{x-2} < \frac{1}{2}$, $x \in$ R.

Solution : $\frac{2x+3}{x-2} < \frac{1}{2} \iff \frac{2x+3}{x-2} - \frac{1}{2} < 0$

$$\iff \frac{2(2x+3)-(x-2)}{2(x-2)} < 0$$

$$\iff \frac{3x+8}{2(x-2)} < 0$$

$\iff (3x + 8 > 0$ and $2x - 4 < 0)$ or $(3x + 8 < 0$ and $2x - 4 > 0)$

$\iff (x > -\frac{8}{3}$ and $x < 2)$ or $(x < -\frac{8}{3}$ and $x > 2)$

$\iff x \in \left(\frac{-8}{3}, 2\right)$ as it is impossible that $x < -\frac{8}{3}$ and $x > 2$

$\therefore$ The solution set is $\left(\frac{-8}{3}, 2\right)$.

Example 9 : Solve : $\frac{x}{x-3} > 1, x \in R$.

Solution : $\frac{x}{x-3} > 1 \Leftrightarrow \frac{x}{x-3} - 1 > 0$

$$\Leftrightarrow \frac{x-(x-3)}{x-3} > 0$$

$$\Leftrightarrow \frac{3}{x-3} > 0$$

$$\Leftrightarrow x - 3 > 0 \text{ as } 3 > 0$$

$$\Leftrightarrow x > 3$$

$\therefore$ The solution set is $(3, \infty)$.

EXERCISE 8.1

Find the solution set of following inequalities as required :

1. $x + 2 < -8$, where (1) $x \in N$ (2) $x \in Z$ (3) $x \in R$
2. $4x \geq 16$, where (1) $x \in N$ (2) $x \in Z$ (3) $x \in R$
3. $-5x \leq -20$, where (1) $x \in N$ (2) $x \in Z$ (3) $x \in R$
4. $-6x \leq 18$, where (1) $x \in N$ (2) $x \in Z$ (3) $x \in R$
5. $5x - 17 > 8, x \in R$
6. $2x - 8 < 10, x \in R$
7. $3x - 22 \geq 5, x \in R$
8. $4x - 17 \leq -1, x \in R$
9. $\frac{x+1}{2} > 6(x + 2), x \in R$
10. $\frac{x+7}{3} - 5 < \frac{2x+1}{9} - 3, x \in R$
11. (1) $\frac{4}{x-3} < 1, x \in R$ (2) $\frac{3x-2}{2x-7} > 0, x \in R$
12. (1) $\frac{x-2}{x} > \frac{1}{3}, x \in R$ (2) $\frac{1}{x-2} \leq 5, x \in R$ (3) $\frac{x-2}{x+5} > 3, x \in R$
13. $\frac{x-1}{x} + 2 > 5, x \in R$
14. $2(x - 1) + 3(x - 2) \leq 5(x + 1), x \in R$
15. $3(x - 1) + 2(x - 2) \leq 5(x + 2), x \in R$

*

8.4 Representation of Solution of Linear Inequality in One Variable on the Number Line

Consider the solution of $5x < 20$ in N, which is {1, 2, 3}. We represent it on the number line as in figure 8.1 :

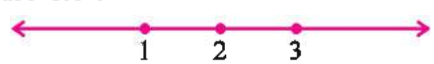


Figure 8.1

If we consider its solution in R, $(-\infty, 4)$, it would be represented as in figure 8.2 :

Figure 8.2

The solution of $2x \geq 4$ namely $[2, \infty)$ will be represented as in figure 8.3

Figure 8.3

In an interval, if an end point is included in the solution, we put a dark circle. around it and show the interval by a thick line. If an end point is excluded from the solution, we put a hollow circle o around it. $-\infty$ and ∞ are always just symbols and not a part of a solution. In case of a finite solution set, all of its members are represented with a dark circle around them.

Example 10 : Solve : $\frac{1}{x-4} < 0$ and represent it on the number line.

Solution : $\frac{a}{b} < 0$ and $a > 0 \Rightarrow b < 0$

$$\frac{1}{x-4} < 0 \iff x - 4 < 0$$

$$\iff x < 4$$

$\therefore$ The solution set is $(-\infty, 4)$.

It is shown on the number line in figure 8.4 :

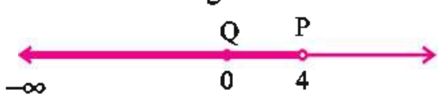


Figure 8.4

It is represented by $\overrightarrow{PQ} - \{P\}$.

Example 11 : Solve : $\frac{x+3}{x+5} > 1$, $x \in$ R and represent it on the number line.

Solution : $\frac{x+3}{x+5} > 1 \iff \frac{x+3}{x+5} - 1 > 0$

$$\iff \frac{x+3-x-5}{x+5} > 0$$

$$\iff \frac{-2}{x+5} > 0$$

$$\iff \frac{2}{x+5} < 0$$

$$\iff x + 5 < 0$$

$$\iff x < -5$$

∴ The solution set is $(-\infty, -5)$.

The solution is shown on number line as in figure 8.5 :

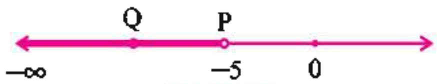


Figure 8.5

This is also represented by $\overrightarrow{PQ} - \{P\}$.

Example 12 : Represent graphically the solution set of $3x \le 18$, where (1) $x \in N$ (2) $x \in R$.

Solution : (1) $3x \le 18 \Leftrightarrow x \le 6$

$\Leftrightarrow x = 1, 2, 3, 4, 5, 6$

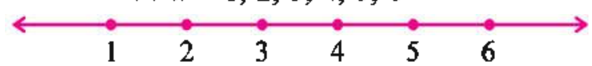


Figure 8.6

The above figure 8.6 represents the solution set.

(2) $x \le 6 \Leftrightarrow x \in (-\infty, 6]$

Hence the solution is represented as in figure 8.7 :

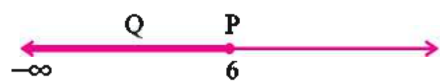


Figure 8.7

It can be named $\overrightarrow{PQ}$.

Example 13 : Represent the solution set of $5x \ge -10$ graphically.

(1) $x \in N$ (2) $x \in Z$ (3) $x \in R$.

Solution : (1) $5x \ge -10 \Leftrightarrow x \ge -2$. The solution set in N is N itself.

Figure 8.8

(2) $5x \ge -10 \Leftrightarrow x \ge -2$

Hence the solution set in Z is $\{-2, -1, 0, 1, 2, ...\}$

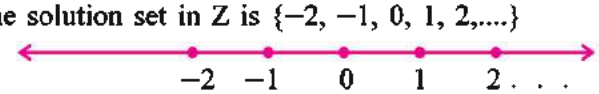


Figure 8.9

(3) The solution set is $[-2, \infty)$ in R.

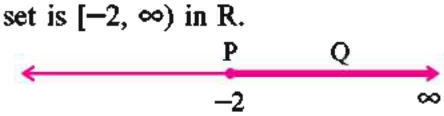


Figure 8.10

$\overrightarrow{PQ}$ shows the solution set in figure 8.10.

Now onwards we will consider solution set in R unless mentioned otherwise.

Example 14 : Solve and represent on the number line : $5x - 3 > 3x - 5$.

Solution : $5x - 3 > 3x - 5 \Leftrightarrow 5x - 3x > 3 - 5$

$\Leftrightarrow 2x > -2$

$\Leftrightarrow x > -1$

Hence solution set is $(-1, \infty)$.

Its representation is given in figure 8.11.

Figure 8.11

It can be named $\overrightarrow{PQ} - \{P\}$.

Example 15 : Solve and show the solution on the number line : $\frac{2x}{3} \geq \frac{5x-2}{5} - \frac{7x-5}{2}$.

Solution : Here the L.C.M. of denominator is 30. Multiply by 30 on both the sides.

$\frac{2x}{3} \geq \frac{5x-2}{5} - \frac{7x-5}{2} \Leftrightarrow 20x \geq 6(5x - 2) - 15(7x - 5)$

$\Leftrightarrow 20x \geq 30x - 12 - 105x + 75$

$\Leftrightarrow 105x + 20x - 30x \geq 75 - 12$

$\Leftrightarrow 95x \geq 63$

$\Leftrightarrow x \geq \frac{63}{95}$

$\therefore$ Hence the solution set is $\left[\frac{63}{95}, \infty\right)$ and is represented in figure 8.12.

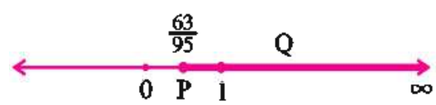


Figure 8.12

$\overrightarrow{PQ}$ represents the solution set.

Example 16 : Show the solution set of following inequality on the number line :

$\frac{2x-1}{3} + 5 < \frac{3x-1}{2} - 2$.

Solution : $2(2x - 1) + 30 < 3(3x - 1) - 12$

$\Leftrightarrow 4x + 28 < 9x - 15$

$\Leftrightarrow 5x > 43$

$\Leftrightarrow x > \frac{43}{5} = 8.6$

The solution set is $(8.6, \infty)$

It is shown as in figure 8.13 on the number line

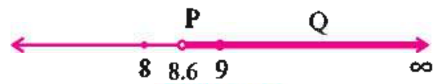


Figure 8.13

It can be named $\overrightarrow{PQ} - \{P\}$.

EXERCISE 8.2

Solve the following inequalities and represent the solution set on the number line ($x \in R$).

1. $5x - 7 > 7x - 5$

2. $3x + 5 < 5x + 3$

3. $\frac{x}{3} + 5 \geq \frac{x}{2} + 7$

4. $\frac{3x}{2} + 15 \leq \frac{2x}{3} + 6$

5. $\frac{x-1}{2} + 5 \geq \frac{2x-1}{3} + 15$

6. $\frac{2x+3}{5} + \frac{7x+1}{3} \geq \frac{3x-1}{2}$

7. $\frac{4x+1}{9} > \frac{9x+1}{4} - 2$

8. $\frac{x}{3} \geq \frac{2x-1}{3} + \frac{5x-3}{7}$

9. $\frac{x}{x-2} < 0$

10. $\frac{1}{x-1} \geq 0$

11. $\frac{x-2}{x} > 1$

12. $\frac{x+3}{x} \leq 1$

*

Simultaneous Linear Inequalities in One Variable

Surendra must score 36 or more marks to get credit in a 100 marks examination. So if he passes the examination and obtains x marks, then $x \geq 36$ and $x \leq 100$.

Figures 8.14 and 8.15 represent solutions of $x \geq 36$ and $x \leq 100$ respectively.

$\overrightarrow{AB}$ is the solution of $x \geq 36$.

Figure 8.14

$\overrightarrow{BA}$ is the solution of $x \leq 100$.

Figure 8.15

To get the solution of the pair of inequalities $36 \leq x \leq 100$, we get $\overline{AB}$ as the solution set of simultaneous inequalities.

Figure 8.16

Example 17 : Solve $4x \geq 8$ and $5x < 20$ and represent the solution set on number line.

Solution : $4x \geq 8 \Leftrightarrow x \geq 2$

Hence the solution set is $[2, \infty)$.

Figure 8.17

It is represented by $\overrightarrow{AB}$ in figure 8.17.

$5x < 20 \Leftrightarrow x < 4$

Hence the solution set is $(-\infty, 4)$.

Figure 8.18

It is represented by $\overrightarrow{BA} - \{B\}$ in figure 8.18.

Hence the solution set of the system of inequalities is $[2, 4)$.

Figure 8.19

$\overline{AB} - \{B\}$ represents the solution set in figure 8.19.

Example 18 : Show the solution set on the number line : $x \geq 3$ and $-3x \leq 6$

Solution : $-3x \leq 6 \Leftrightarrow x \geq -2$

Thus $x \geq -2$ and $x \geq 3$

$\therefore$ $x \geq 3$ will satisfy both inequalities. Therefore solution set is $[3, \infty)$

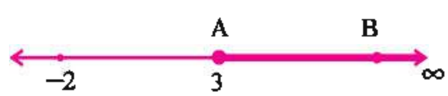


Figure 8.20

The solution set is $\overrightarrow{AB}$ in figure 8.20.

Example 19 : Solve : $2x > 4$ and $2x < 4$, $x \in$ R and represent graphically.

Solution : $2x > 4 \Leftrightarrow x > 2$

$\Leftrightarrow$ The solution set is $(2, \infty)$ **(i)**

$2x < 4 \Leftrightarrow x < 2$

$\Leftrightarrow$ The solution set is $(-\infty, 2)$ **(ii)**

The solution sets of (i) and (ii) are shown in figure 8.21 (i) and (ii) respectively.

The intersection is $\emptyset$.

Hence the solution set is $\emptyset$.

(No thick line or point. Only number line)

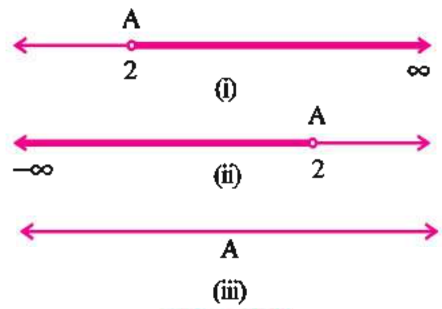
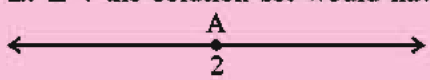


Figure 8.21

> **☛ Note** (1) Had we $2x \geq 4$ or $2x \leq 4$ (only one of them) the solution set would still have been $\emptyset$.
>
> (2) Had we $2x \geq 4$ and $2x \leq 4$ the solution set would have been $\{2\}$.
>
> 
>
>
> Figure 8.22

Example 20 : Solve : $x \geq 17$ and $x \leq 15$, $x \in R$.

Solution : Solutions are represented by $[17, \infty)$ and $(-\infty, 15]$ in figure 8.23.

Figure 8.23

$\overrightarrow{BQ}$ and $\overrightarrow{AP}$ represent solutions. There is no common point.

Hence solution set is $\emptyset$.

(Obviously no number can be less than or equal to 15 and greater than or equal to 17.)

EXERCISE 8.3

Solve following system of inequalities and represent them on number line :

1.	$x \geq 3$	$x \leq 5$	$x \in R$
2.	$x > 3$	$x < 8$	$x \in R$
3.	$x \geq 4$	$x < 6$	$x \in R$
4.	$x > 4$	$x \leq 6$	$x \in R$
5.	$x \geq 3$	$x \leq 2$	$x \in R$
6.	$-2x \geq 4$	$3x \leq -6$	$x \in R$
7.	$-2x \geq -10$	$2x \geq 4$	$x \in R$
8.	$3x - 1 \geq 5$	$x + 2 \leq -1$	$x \in R$

9. $2x - 7 \leq 11$ $3x + 4 < -5$ $x \in$ R

10. $x - 3 < 5$ $3x + 5 > 2$ $x \in$ R

*

8.5 Linear Inequalities in Two Variables

Standard form of a linear expression in two variables is $ax + by + c$. The equation $ax + by + c = 0$ is called a linear equation in x and y, where $a, b, c \in$ R and $a^2 + b^2 \neq 0$. ($a^2 + b^2 \neq 0$ means at least one of a or b is non-zero.)

Its graph in plane or R^2 is a straight line. All points on the line satisfy $ax + by + c = 0$ and conversely any point (x, y) satisfying $ax + by + c = 0$ is on the straight line graph of $ax + by + c = 0$. A solution of $ax + by + c = 0$ is the set of ordered pairs (x, y), where $y = \frac{-ax - c}{b}$ if $b \neq 0$.

If $b = 0$, then $a \neq 0$. Then the solution set is $\left\{\left(\frac{-c}{a}, y\right)\right\}$, where $y \in$ R is arbitrary.

$ax + by + c < 0$, $ax + by + c \leq 0$, $ax + by + c > 0$, $ax + by + c \geq 0$ are linear inequalities corresponding to the expression $ax + by + c$. We have to study how to solve them graphically.

For example consider the inequality $x + y - 2 > 0$.

Taking $x = 1$ and $y = 2$, we get $1 + 2 - 2 > 0$, a true statement.

Thus (1, 2) and similarly (2, 1), (2, 3), (3, 2), (5, 7) etc. are solutions of $x + y - 2 > 0$. But (1, 1) is not a solution of $x + y - 2 > 0$ as it gives $1 + 1 - 2 > 0$ i.e. $0 > 0$. Similarly (−1, −2), (1, −5), (−4, 1) are not solutions of $x + y - 2 > 0$.

An ordered pair (x, y) satisfying $ax + by + c > 0$ (or $ax + by + c < 0$ or $ax + by + c \leq 0$ or $ax + by + c \geq 0$) is said to be a solution of the inequality $ax + by + c > 0$ (or $ax + by + c < 0$ or $ax + by + c \leq 0$ or $ax + by + c \geq 0$) and the set of all ordered pairs (x, y) satisfying $ax + by + c > 0$ constitute the solution set of the inequality. All such points give solution region of $ax + by + c > 0$.

Graphical Solution : A straight line gives rise to a partition of plane in three disjoint subsets.

(1) Points on the line.

(2) All points lying on either side of the line giving two half-planes.

All points on the line satisfy the equation of the line. Now we will visualise that for all points (x', y') lying in any one half plane on either side of the line $ax + by + c = 0$, $ax' + by' + c$ has same sign. (i.e. for all (x', y'), $ax' + by' + c$ is positive or negative.)

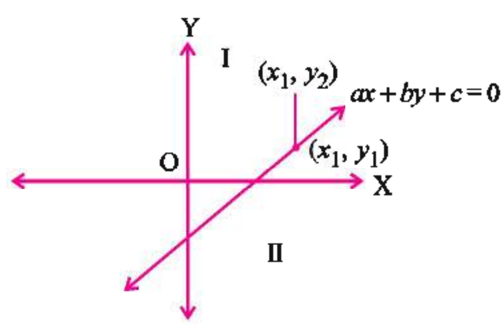


Figure 8.24

Let $b > 0$: Let (x_1, y_1) be any point on the line $ax + by + c = 0$.

$\therefore \quad ax_1 + by_1 + c = 0$

Consider point (x_1, y_2) in half-plane I. Obviously $y_2 > y_1$.

$\therefore \quad by_2 > by_1$

$\therefore \quad ax_1 + by_2 + c > ax_1 + by_1 + c = 0$

$\therefore \quad ax_1 + by_2 + c > 0$

This can be verified for any point in the half-plane I.

Conversely let $ax_1 + by_2 + c > 0$. Let (x_1, y_1) be on the line.

$\therefore \quad ax_1 + by_1 + c = 0$

$\therefore \quad ax_1 + by_2 + c > ax_1 + by_1 + c$ **$(ax_1 + by_1 + c = 0)$**

$\therefore \quad by_2 > by_1$

$\therefore \quad y_2 > y_1$ **$(b > 0)$**

Thus all points (x', y') in half-plane I satisfy $ax' + by' + c > 0$ and vice-versa.

It can be seen that all points (x, y) in half-plane II satisfy $ax + by + c < 0$ and vice-versa.

Therefore the line $ax + by + c = 0$ divides R^2 in three parts.

(1) Points (x, y) on the line for which $ax + by + c = 0$.

(2) Points (x, y) in one half-plane satisfying $ax + by + c > 0$.

(3) Points (x, y) in the other half-plane satisfying $ax + by + c < 0$.

Since for any point (x, y) in one definite half-plane the expression $ax + by + c$ has same sign, we may consider $(0, 0)$ as testing point. If the line passes through $(0, 0)$, we may have to consider another point like $(1, 0)$, $(0, 1)$ etc. **Similar discussion applies if $b < 0$.**

If $b = 0$, then $a \neq 0$. Then the expression is $ax + c$.

If $a > 0$, then $ax + c > 0 \Leftrightarrow x > \frac{-c}{a}$

$ax + c < 0 \Leftrightarrow x < \frac{-c}{a}$

If $a < 0$, then $ax + c > 0 \Leftrightarrow x < \frac{-c}{a}$

$ax + c < 0 \Leftrightarrow x > \frac{-c}{a}$

We draw vertical line $x = \frac{-c}{a}$.

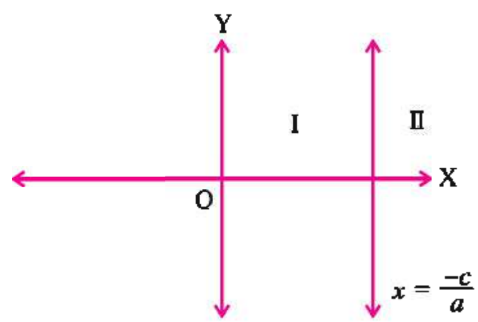


Figure 8.25

Half-plane I is on the left of line $x = \frac{-c}{a}$ where $x < \frac{-c}{a}$ and half-plane II is on the right of line $x = \frac{-c}{a}$ where $x > \frac{-c}{a}$.

A Convention :

(1) For solution of $ax + by + c \geq 0$ (or ≤ 0) points on the lines $ax + by + c = 0$ are also included in the solution set and we draw a dark line $ax + by + c = 0$.

(2) For solution of $ax + by + c > 0$ (or < 0) points on $ax + by + c = 0$ are not included in the solution set and we draw a dotted line or a broken line $ax + by + c = 0$.

Example 21 : Solve graphically $x \geq 0$.

Solution : $x = 0$ is the straight line Y-axis. Consider (1, 0). $x = 1$ satisfies $x \geq 0$.

Hence the coloured region I is the solution set of $x \geq 0$. It includes Y-axis. (see that the line passes through origin. So we considered (1, 0)). (figure 8.26)

Also we know that for all points (x, y) on the right of Y-axis and on Y-axis, $x \geq 0$.

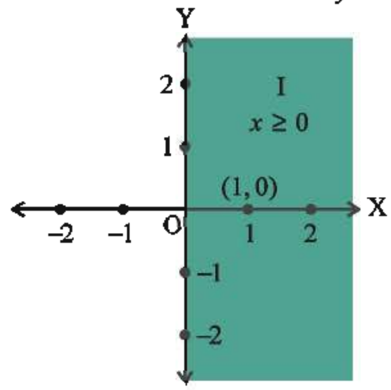


Figure 8.26

Example 22 : Solve graphically $y \leq 0$.

Solution : The equation $y = 0$ represents X-axis. Here also the line passes through (0, 0). Let us consider (0, –1). $y = -1$ satisfies $y \leq 0$.

Thus the solution set is the region below X-axis and includes X-axis. (The coloured region in figure 8.27)

Also we know that all points (x, y) below X-axis and on X-axis have $y \leq 0$.

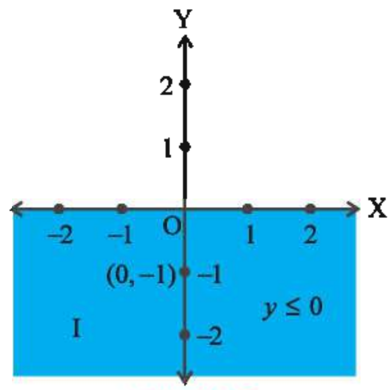


Figure 8.27

Example 23 : Solve graphically $x > 2$.

Solution : $x = 2$ is a vertical line.

For solution of $x > 2$, we draw dotted line $x = 2$.

(0, 0) does not satisfy $x > 2$. Hence (0, 0) is not in the solution set. Thus the solution set is the region on the right of $x = 2$ excluding points on the line $x = 2$. (The coloured region in figure 8.28)

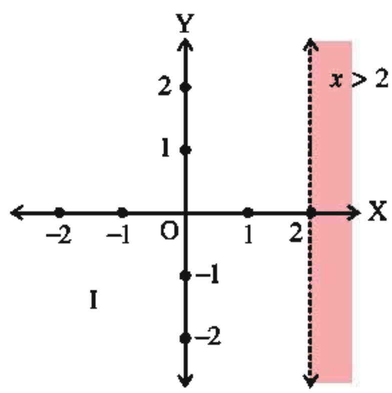


Figure 8.28

Example 24 : Solve graphically $y < 1$.

Solution : $y = 1$ represents horizontal line. The inequality is $y < 1$. Hence dotted line $y = 1$ is drawn. (0, 0) satisfies $y < 1$ since substituting $y = 0$ in the inequality we get $0 < 1$ which is true. Thus (0, 0) is in the solution set. Thus solution set is the region below the line $y = 1$ (The coloured region in figure 8.29)

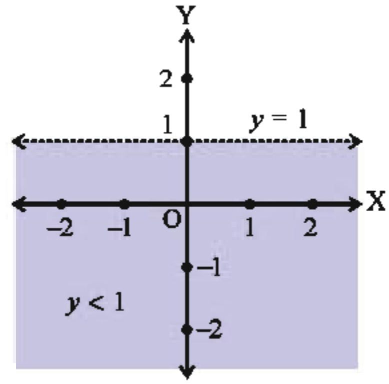


Figure 8.29

Example 25 : Solve graphically $x + y - 4 \geq 0$.

Solution : $x + y - 4 = 0$ is the line passing through (1, 3) and (3, 1). (You can choose any two points on the line to draw the line.) (0, 0) does not satisfy $x + y - 4 \geq 0$ since it gives $0 + 0 - 4 \geq 0$ i.e. $-4 \geq 0$.

Hence the solution set does not contain (0, 0).

Thus the solution set is the region above the line $x + y - 4 = 0$ and contains points on the line also. (The coloured region in figure 8.30)

(We select the region in which (0, 0) does not lie.)

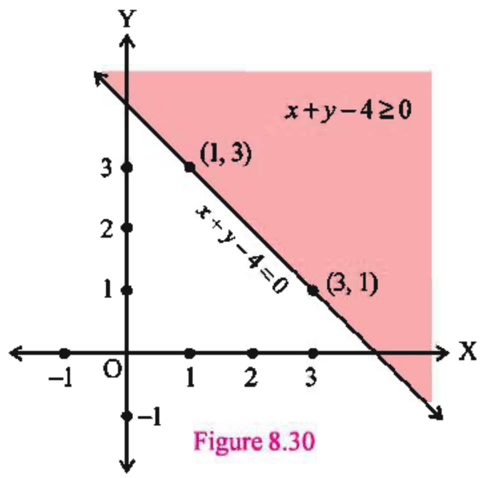


Figure 8.30

Example 26 : Solve graphically (1) $x - 2y - 2 < 0$ (2) $x - y \geq 0$ (3) $3x - 2y \geq x - y + 5$.

Solution :

(1) Consider $x - 2y - 2 < 0$. Let us draw the line $x - 2y - 2 = 0$.

Substituting $x = 0$ and $y = 0$ respectively in the equation we get (0, −1) and (2, 0) on the line. The line is the dotted line as shown. Substituting $x = 0$ and $y = 0$ in $x - 2y - 2 < 0$, we get $-2 < 0$ which is true. Hence (0, 0) lies in the solution region. Thus the solution region is as shown in figure 8.31 as the coloured and it excludes points on the line $x - 2y - 2 = 0$.

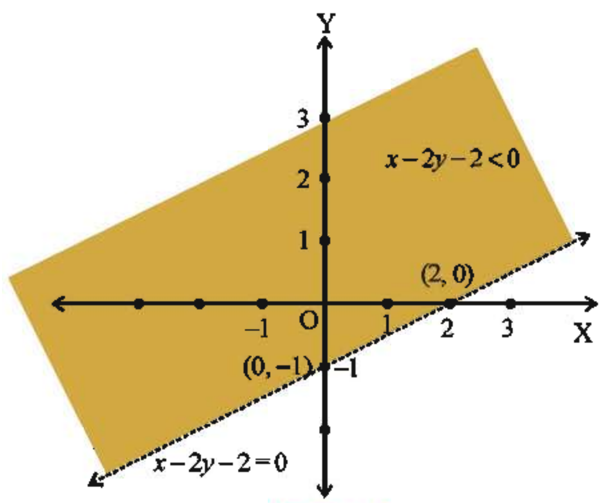


Figure 8.31

(2) $x - y \geq 0$

Let us draw the line $x - y = 0$. It passes through origin and obviously also through (1, 1). Since (0, 0) lies on the line, we consider another point, say (2, 1). Substituting $x = 2$, $y = 1$ in $x - y$ we see that (2, 1) satisfies $2 - 1 \geq 0$. (In fact $2 - 1 > 0$).

Thus the required region is as shown in figure 8.32 as the coloured region and it contains points on the line $x - y = 0$

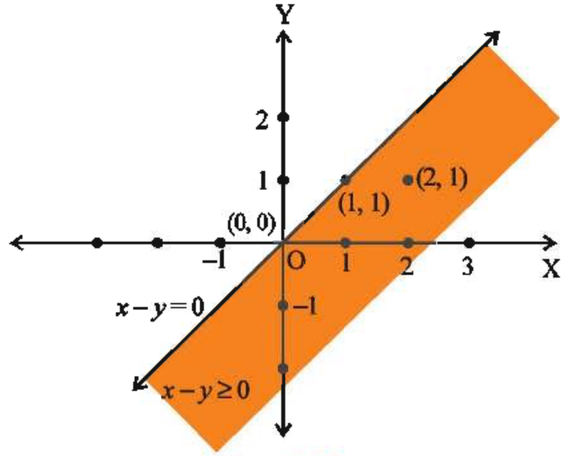


Figure 8.32

(3) $3x - 2y \geq x - y + 5$

$\therefore \quad 2x - y - 5 \geq 0$

Let us draw $2x - y - 5 = 0$. Obviously it passes through $(0, -5)$ and $\left(\frac{5}{2}, 0\right)$.

Substituting $x = 0$, $y = 0$ in $2x - y - 5$ we see that (0, 0) gives $0 - 0 - 5 \geq 0$ which is not true.

The required region does not contain (0, 0) and is as shown in figure 8.33 as the coloured region. It includes points on the line.

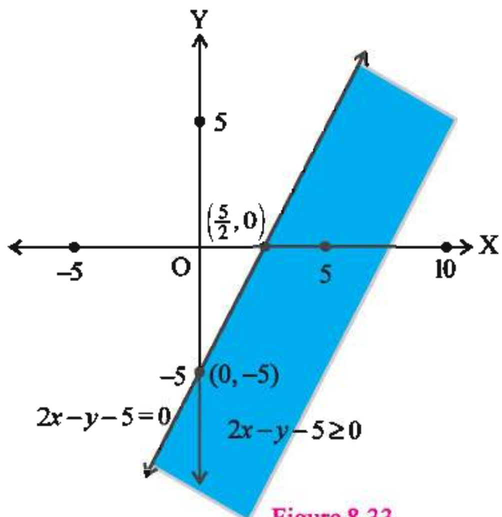


Figure 8.33

EXERCISE 8.4

Solve following inequalities graphically.

1. $x < -1$ $\quad x \in R$ (In R^2)

2. $x \geq 3$ $\quad x \in R$ (In R^2)

3. $y \leq -2$ $\quad y \in R$ (In R^2)

4. $y > -5$ $\quad y \in R$ (In R^2)

5. $x + y - 5 > 0$ $\quad x, y \in R$

6. $x + y \geq 0$ $\quad x, y \in R$

7. $2x + y - 3 < 0$ $\quad x, y \in R$

8. $x + y > 1$ $\quad x, y \in R$

9. $2x - y + 7 > 3x + 2y - 9$ $\quad x, y \in R$

10. $2x + y - 3 > x + 2y + 5$ $\quad x, y \in R$

11. $3x - y \geq 0$ $\quad x, y \in R$

12. $x - 2y \leq 0$ $\quad x, y \in R$

*

8.6 System of Linear Inequalities in Two Variables

Sometimes we have to solve two or more inequalities such as $x + y - 5 \geq 0$, $x \geq 0$, $y \geq 0$. In such a case we find the solution set of each inequality and intersection set of their solution regions is the required solution set.

Let us understand this by some examples.

Example 27 : Solve system of inequalities $x \geq 0$, $y \geq 0$.

Solution : The solution region of $x \geq 0$ would be, as we know, the region to the right of Y-axis including Y-axis. Similarly the solution of $y \geq 0$ is the region above X-axis including X-axis.

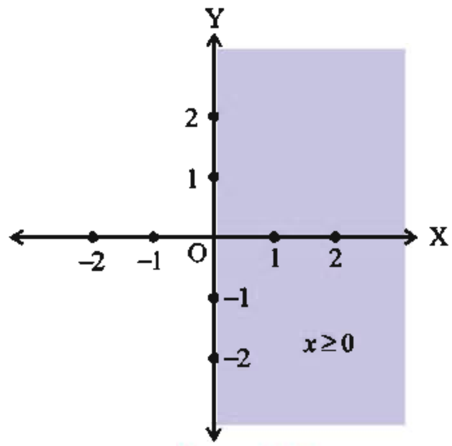


Figure 8.34

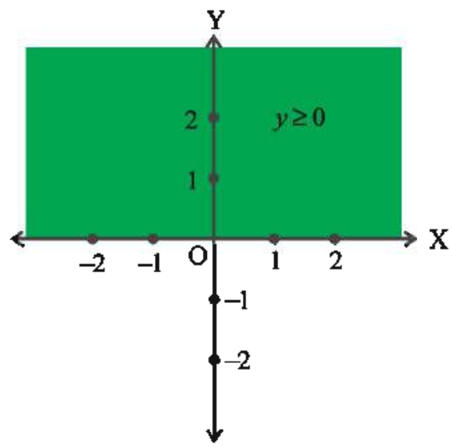


Figure 8.35

The intersection of these two regions would be as in figure 8.36 represented as the coloured region.

It is the union of first quadrant and $\overrightarrow{OX}$ and $\overrightarrow{OY}$.

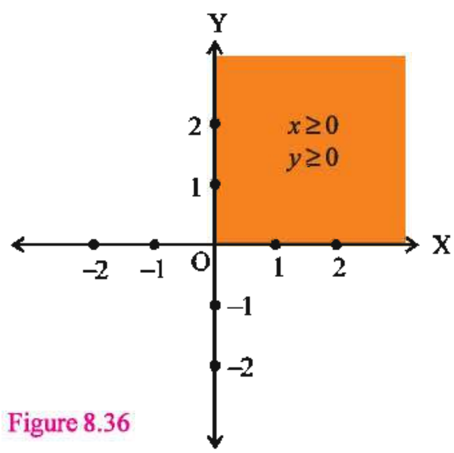


Figure 8.36

Note In many practical problems constraints are $x \geq 0$ or $x > 0$ or $y \geq 0$ or $y > 0$.)

Example 28 : Solve graphically $x - y + 2 \geq 0$ and $2x + y - 5 \leq 0$.

Solution : To draw $x - y + 2 = 0$, we take points (0, 2) and (–2, 0) on the line.

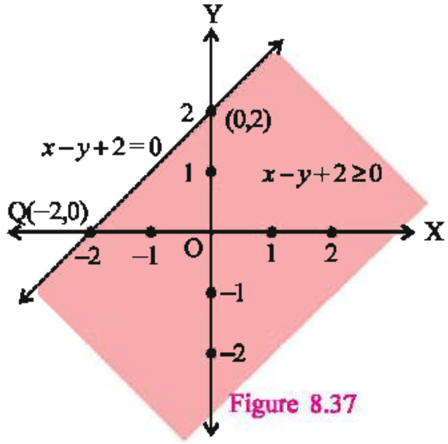


Figure 8.37

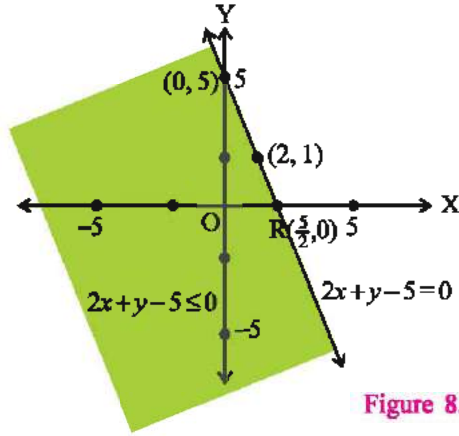


Figure 8.38

Taking $x = 0$, $y = 0$ in $x - y + 2 \geq 0$, we get $0 - 0 + 2 \geq 0$ which is true.

$\therefore$ (0, 0) is in the solution set of $x - y + 2 \geq 0$.

Similarly (2, 1), $(\frac{5}{2}, 0)$ lie on $2x + y - 5 = 0$.

If we put $x = 0$, $y = 0$ in $2x + y - 5 \leq 0$ we get $0 + 0 - 5 \leq 0$ which is true.

$\therefore$ (0, 0) lies in the solution set of $2x + y - 5 \leq 0$.

Now consider both the lines on the same pair of axes. The solution region shown also contains points on both the rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

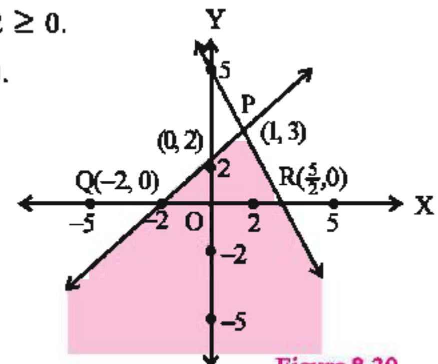


Figure 8.39

The coloured region in figure 8.39 is the solution region of the system of inequalities.

Example 29 : Solve inequalities $x > 0$, $y > 1$ and $x + y < 2$ graphically.

Solution : The solution region of $x > 0$ is shown by points to the right of Y-axis not including points on Y-axis in figure 8.40.

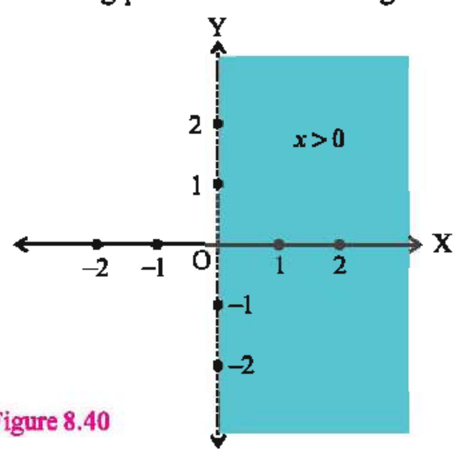


Figure 8.40

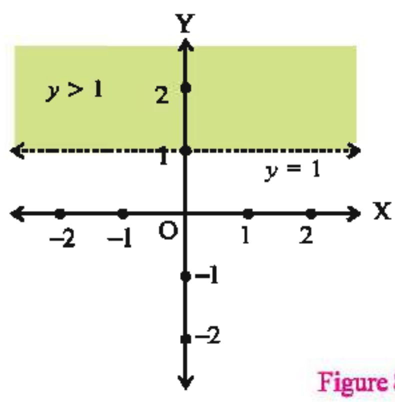


Figure 8.41

Points in the half-plane above $y = 1$ represent the solution set of $y > 1$.

Straight line $x + y = 2$ passes through (2, 0) and (0, 2). Substituting $x = 0$, $y = 0$ in $x + y < 2$, we get $0 + 0 < 2$ which is true. Hence (0, 0) is in the solution set of $x + y < 2$.

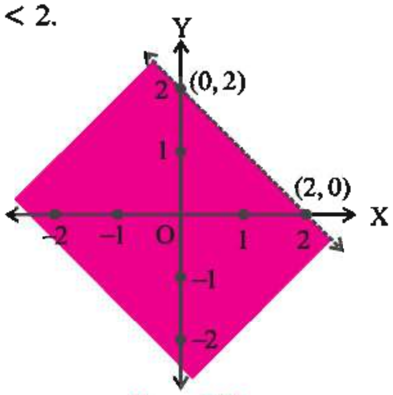


Figure 8.42

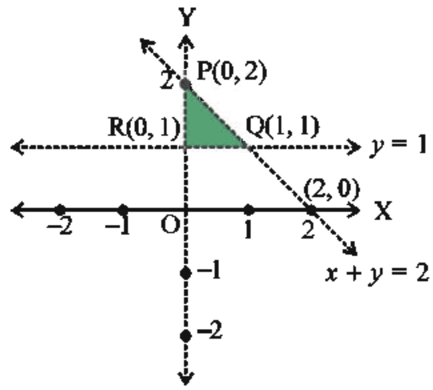


Figure 8.43

Interior of ΔPQR represents the solution set of the system of inequalities in figure 8.43.

Example 30 : Solve graphically $x \geq 0$, $y \geq 0$, $5x + 3y - 15 \leq 0$, $4x + 5y - 20 \leq 0$.

Solution : Now solution set of $x \geq 0$, $y \geq 0$ is shown in figure 8.44 as the coloured region.

The straight line $5x + 3y - 15 = 0$ passes through (3, 0) and (0, 5). (0, 0) lies in the solution set of $5x + 3y - 15 \leq 0$. The solution set is as shown in figure 8.45 as the coloured region.

The line $4x + 5y - 20 = 0$ passes through (5, 0) and (0, 4). Here also (0, 0) lies in the solution set of $4x + 5y - 20 \leq 0$.

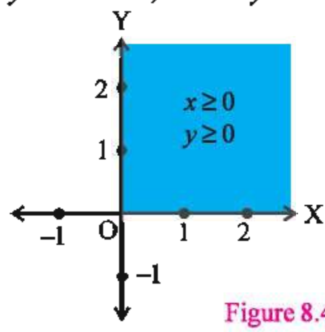


Figure 8.44

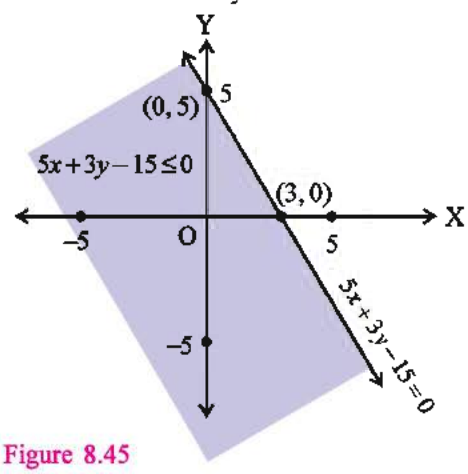


Figure 8.45

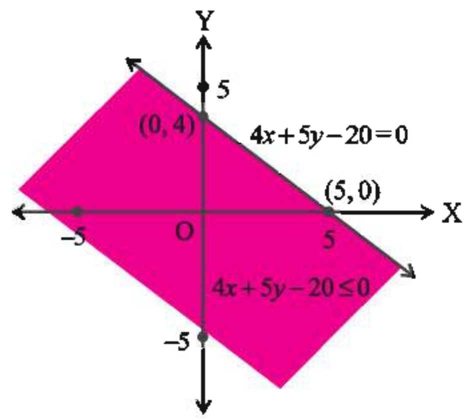


Figure 8.46

The solution set of $4x + 5y - 20 \leq 0$ is shown in figure 8.46 by the coloured region.

Thus the solution set of the system of equations is the closed region shown by quadrilateral OABC as shown in the figure 8.47 by the coloured region.

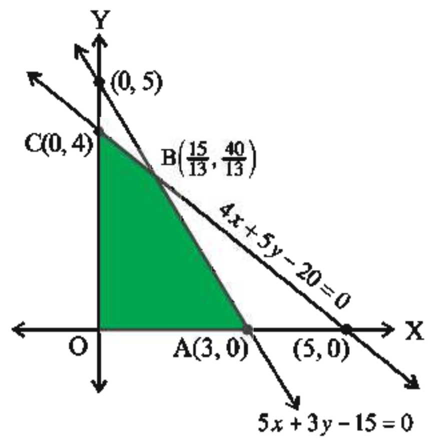


Figure 8.47

Example 31 : Solve the following system of inequations.

(1) $x - 2y \geq 0, 2x - y \leq -4, x \geq 0, y \geq 0$

Solution : (1) Since $x \geq 0$, $y \geq 0$. The solution region will contain points only in the first quadrant and on semi-axes for $x \geq 0$ or $y \geq 0$

$x - 2y = 0$ is the straight line passing through origin. It passes through (2, 1). (Check it !) Taking $x = 3$, $y = 1$ in $x - 2y$, $3 - 2 \geq 0$. Hence (3, 1) lies in the region specified by $x - 2y \geq 0$.

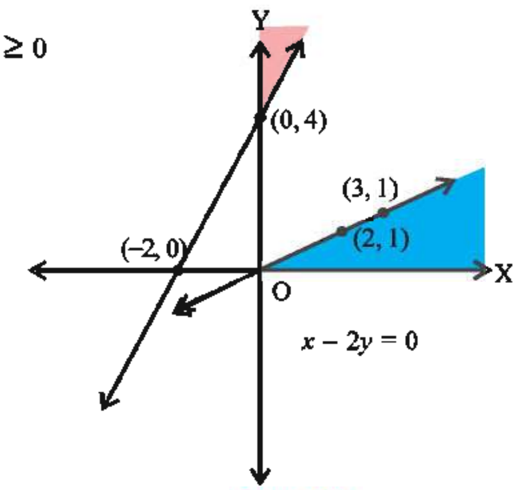


Figure 8.48

Thus the region for which $x - 2y \geq 0$ is as shown in figure 8.48. It containts points below the line $x - 2y = 0$ in the first quadrant and on $\overrightarrow{OX}$.

Line $2x - y = -4$ passes through points (–2, 0) and (0, 4). (–3, 0) lies in the solution region of $2x - y \leq -4$ as $-6 - 0 \leq -4$.

Thus there is no point satisfying $2x - y \leq -4$, $x - 2y \geq 0$ and $x \geq 0$, $y \geq 0$.

(**Note :** Obviously if $x \geq 2y$, then $2x \geq 4y$.

$\therefore \quad 3y \leq 2x - y \leq -4 \Rightarrow y < 0$.

But as per condition $y \geq 0$.)

Hence the solution region is null set.

EXERCISE 8.5

Solve following system of inequalities graphically. ($x \in R$, $y \in R$)

1. $y \geq 0, y \leq 4, x < 5$
2. $x \geq 0, y \geq 0, x \leq 3, y \leq 2$
3. $x > 0, y > 0, x \leq 3, y \leq 2$
4. $x > 0, y > 0, x + 2y < 12, x + y \geq 2$
5. $2x + y \leq 12, x + 2y \leq 7, x \geq 0, y \geq 0$
6. $x \geq 0, y \geq 0, x - y \geq 0$
7. $x \geq 0, y \geq 0, x + y \leq 6, 3x + 4y \leq 12$
8. $y > 0, x > 2y, x + y > 4, x + y < 6$
9. $x < 1, y < 0, x \geq -3, x + y \geq 0$
10. $3x + y > 0, 3x + y < 3$
11. Write down the system of inequalities whose solution set is given by the coloured regions in figure 8.49..

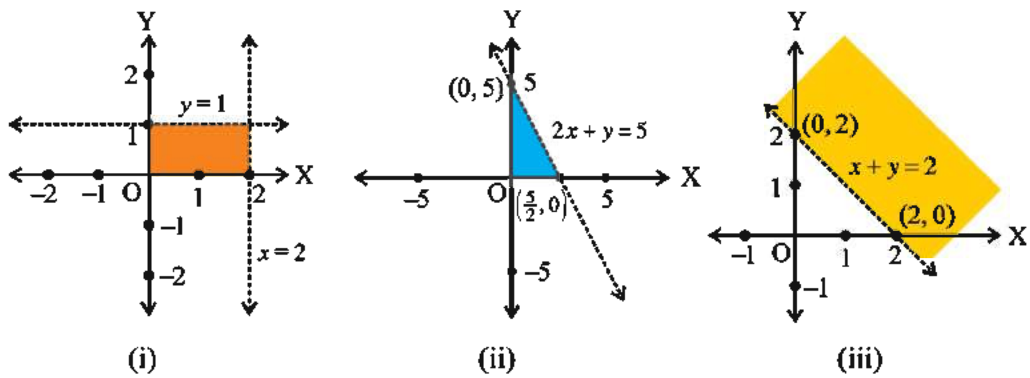


Figure 8.49

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Some Miscellaneous Examples and Problems

Now we will consider some more problems on what we have learnt. We remember following inequalities.

(1) $|x| < a \Leftrightarrow -a < x < a \qquad x \in R, a \in R^+$

(2) $|x| \leq a \Leftrightarrow -a \leq x \leq a \qquad x \in R, a \in R^+$

(3) $|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a \qquad x \in R, a \in R^+$

(4) $|x| > a \Leftrightarrow x < -a$ or $x > a \qquad x \in R, a \in R^+$

These can be easily proved.

For example let us prove (1),

Let $|x| < a$.

If $x \geq 0$, $x < a$ and if $x < 0$, $-x < a$ implying $x > -a$

$\therefore \quad -a < x < a$

Conversely, let $-a < x < a$

If $x > 0$, $| x | = x < a$

If $x < 0$, $| x | = -x$ and $-a < x$ gives $a > -x$ or $a > | x |$ i.e. $| x | < a$

$\therefore \quad -a < x < a \Leftrightarrow | x | < a$

(3) is negation of (1) and hence $-a < x$ and $x < a$ gives $x \leq -a$ or $x \geq a$ on taking negation. Similarly (2) and (4) can be proved.

To show the solution region in case of a system of inequalities in two variable, we can use different colours for solution of individual inequalities or shade the regions differently.

For example to solve $-2 < y < 2$, we can follow the following procedure.

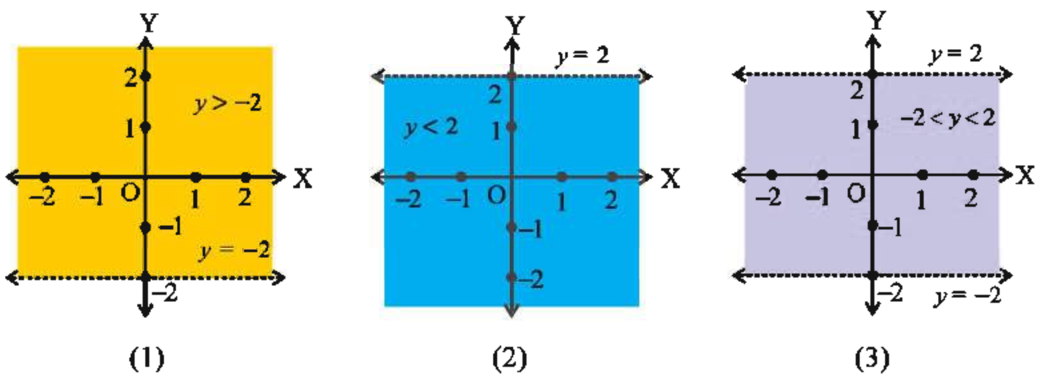


Figure 8.50

Also see that $x < 2$ can be represented on the real line as

Figure 8.51

and in plane as the coloured region as in figure 8.52.

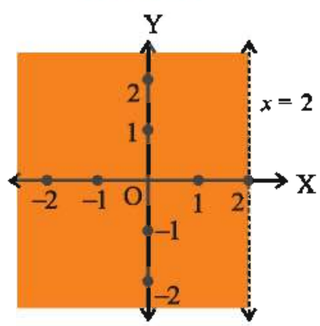


Figure 8.52

Example 32 : Obtain the solution set of the following inequalities in a plane :

(1) $|y| \geq 1$ (2) $|x| \leq 2$ (3) $|x - y| \leq 2$ (4) $|x - y| \leq 0$.

Solution : (1) If $y \geq 0$ then $y \geq 1$ and if $y < 0$, then $-y \geq 1$ or $y \leq -1$.

Thus the solution region is union of two regions shown in figure 8.53.

(or using the relations (1) to (4) we get $y \geq 1$ or $y \leq -1$)

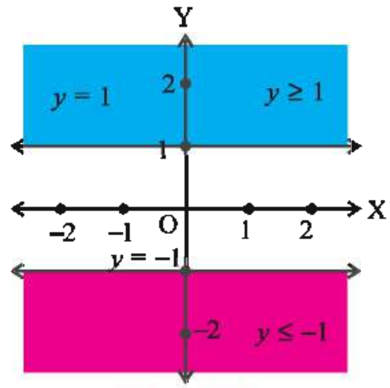


Figure 8.53

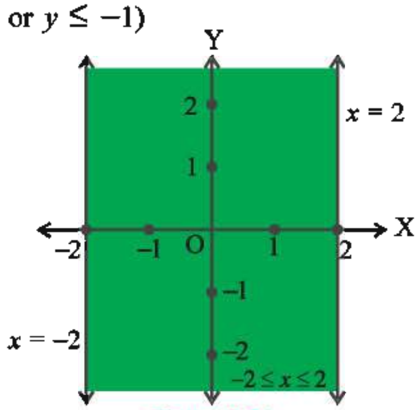


Figure 8.54

(2) $|x| \leq 2 \Leftrightarrow -2 \leq x \leq 2$

The coloured region in figure 8.54 represents the solution region of $|x| \leq 2$.

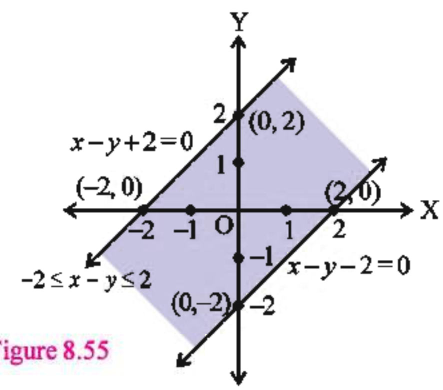


Figure 8.55

(3) $|x - y| \leq 2$

$-2 \leq x - y \leq 2$.

$(|x| \leq a \Leftrightarrow -a \leq x \leq a)$

$\therefore$ We have inequalities;

$x - y + 2 \geq 0$ and $x - y - 2 \leq 0$

Consider the lines $x - y - 2 = 0$ and $x - y + 2 = 0$.

For $x - y + 2 \geq 0$, (0, 0) gives $2 \geq 0$. Hence (0, 0) lies in the solution set of $x - y + 2 \geq 0$. Similarly $x - y - 2 \leq 0$ also has the solution region containing (0, 0).

Hence the solution set is as shown in figure 8.55 by the coloured region.

(4) $|x - y| \leq 0$

$|x - y| < 0$ is impossible.

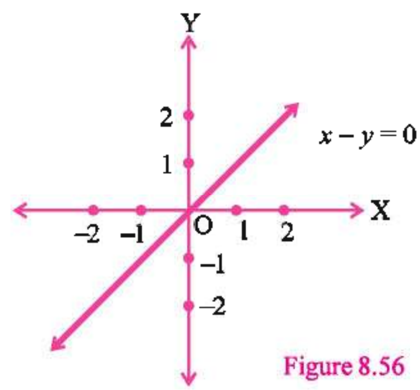


Figure 8.56

Hence $| x - y | = 0$

$\therefore \quad x - y = 0$

The solution region is line $x = y$.

Example 33 : Solve : (1) $-5 < 3x - 8 < 28$ (2) $-20 < -5(x - 3) < 40$. $x \in R$

Solution : (1) The inequality means $-5 < 3x - 8$ and $3x - 8 < 28$.

$\therefore \quad -5 < 3x - 8 \Leftrightarrow 8 - 5 < 3x \Leftrightarrow 3 < 3x \Leftrightarrow x > 1$

Also $3x - 8 < 28 \Leftrightarrow 3x < 36 \Leftrightarrow x < 12$

Thus $-5 < 3x - 8 < 28 \Leftrightarrow 1 < x < 12$

Hence the solution set is $(1, 12)$

(2) $-20 < -5(x - 3)$ and $-5(x - 3) < 40$

$\Leftrightarrow \quad -4 < -x + 3$ and $-x + 3 < 8$

$\Leftrightarrow \quad x < 4 + 3$ and $x > 3 - 8$

$\Leftrightarrow \quad -5 < x < 7$

Hence the solution set is $(-5, 7)$.

Example 34 : Solve and show the solution set on the number line : $(x \in R)$

(1) $2(x - 1) < x + 5$ (2) $5(2x - 7) - 3(2x + 3) < 0$, $2x + 19 < 6x + 47$.

Solution : (1) $2(x - 1) < x + 5 \Leftrightarrow 2x - 2 < x + 5$

$\Leftrightarrow 2x - x < 5 + 2$

$\Leftrightarrow x < 7$

Hence the solution set is $(-\infty, 7)$.

Figure 8.57

(2) $5(2x - 7) - 3(2x + 3) < 0 \Leftrightarrow 10x - 35 - 6x - 9 < 0$

$\Leftrightarrow 4x - 44 < 0$

$\Leftrightarrow x < 11$

The solution set is $(-\infty, 11)$

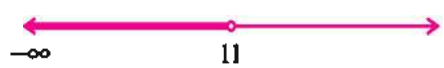


Figure 8.58

$2x + 19 < 6x + 47 \Leftrightarrow -4x < 28$

$\Leftrightarrow x > -7$

The solution set is $(-7, \infty)$.

Figure 8.59

Combining the two solutions, the required solution set is $(-7, 11)$. (figure 8.60)

Figure 8.60

Example 35 : Solve and represent on a number line.

(1) $|4 - x| + 2 < 5$ (2) $\left|x + \frac{7}{3}\right| > \frac{2}{3}$

Solution : (1) $|4 - x| + 2 < 5 \Leftrightarrow |4 - x| < 3$

$\Leftrightarrow -3 < x - 4 < 3$

$\therefore$ $1 < x < 7$

$\therefore$ The solution set is $(1, 7)$. (figure 8.61)

Figure 8.61

(2) $\left|x + \frac{7}{3}\right| > \frac{2}{3}$

$\Leftrightarrow$ $x + \frac{7}{3} > \frac{2}{3}$ or $x + \frac{7}{3} < -\frac{2}{3}$

$\Leftrightarrow$ $x > \frac{-5}{3}$ or $x < -\frac{9}{3} = -3$

Figure 8.62

Hence the solution set is $\left(\frac{-5}{3}, \infty\right) \cup (-\infty, -3)$. (figure 8.62)

Example 36 : Solve and represent graphically.

(1) $\frac{|x-1|}{x-1} \le 0$ (2) $\frac{|x+3| - x}{x} < 3$ (3) $|x - 1| + |x - 2| + |x - 3| < 6$

Solution : (1) $\frac{|x-1|}{x-1} \le 0$

Obviously $x \ne 1$ and hence $\frac{|x-1|}{x-1} \ne 0$

If $x > 1$, $\frac{|x-1|}{x-1} = \frac{x-1}{x-1} = 1$ is neither less than 0 nor equal to zero.

If $x < 1$, $\frac{|x-1|}{x-1} = \frac{-(x-1)}{x-1} = -1 \le 0$ is true.

Hence the solution set is $(-\infty, 1)$. (figure 8.63)

Figure 8.63

(2) $\frac{|x+3| - x}{x} < 3 \Leftrightarrow \frac{|x+3|}{x} - 1 < 3$

$\Leftrightarrow \frac{|x+3|}{x} < 4$

If $x < 0$, then $\frac{|x+3|}{x} < 0 < 4$ is true.

Hence all negative real numbers are in the solution set.

Also, $x \neq 0$. So, let us think about $x > 0$.

$\therefore \quad x + 3 > 0$

$\therefore \quad |x + 3| = x + 3$

$\therefore \quad \frac{|x+3|}{x} < 4 \Leftrightarrow \frac{x+3}{x} < 4$

$\Leftrightarrow x + 3 < 4x$ $\qquad$ **($x > 0$)**

$\Leftrightarrow 3x > 3$

$\Leftrightarrow x > 1$

$\therefore$ Hence the solution set is $(-\infty, 0) \cup (1, \infty)$. (figure 8.64)

Figure 8.64

$(\overrightarrow{AB} - \{A\}) \cup (\overrightarrow{PQ} - \{P\})$ is the required solution set.

(3) $|x - 1| + |x - 2| + |x - 3| < 6$

Here we consider several cases.

(a) Let $x \leq 1$

$\therefore \quad x - 1 \leq 0, x - 2 < 0, x - 3 < 0$

$\therefore$ We have $1 - x + 2 - x + 3 - x < 6$

$\therefore \quad 3x > 0$ i.e. $x > 0$

Thus the solution region in this case contains $(0, 1]$.

(b) Let $1 < x \leq 2$.

Then $x - 1 > 0, x - 2 \leq 0, x - 3 < 0$

$\therefore \quad x - 1 + 2 - x + 3 - x < 6 \Leftrightarrow -x < 2$ i.e. $x > -2$ which is true as $x > 1$.

Then the inequality is true for $x \in (1, 2]$.

(c) Let $2 < x \leq 3$.

Then $x - 1 > 0, x - 2 > 0, x - 3 \leq 0$.

$\therefore$ We have $x - 1 + x - 2 + 3 - x < 6$

$\therefore \quad x < 6$ and since $2 < x \leq 3$, $x < 6$ is true.

$\therefore$ The solution set also contains $(2, 3]$.

(d) If $x > 3$ obviously $x - 1 + x - 2 + x - 3 < 6 \Leftrightarrow x < 4$

Thus solution set contains $(3, 4)$.

(e) If $x < 0$, let $x = -y$, $y > 0$.

Then $|x - 1| + |x - 2| + |x - 3| = |-y - 1| + |-y - 2| + |-y - 3|$

$= |y + 1| + |y + 2| + |y + 3|$

$= y + 1 + y + 2 + y + 3 < 6 \quad (y > 0)$

$\Rightarrow y < 0$ which is false.

Hence summing up, the solution set is (0, 4).

Figure 8.65

Example 37 : A container contains 1120 litres of a 40 % solution of acid. How many litres of acid have to be added so that the resulting mixture contains more than 40 % but less than 50 % of water ?

Solution : Here acid content is 40 %. Therefore water content is 60 %. Let x litres of acid be added. Water content in $(1120 + x)$ litres is more than $(1120 + x)\frac{40}{100}$ and less than $(1120 + x)\frac{50}{100}$. This gives following inequality.

$$(1120 + x)\frac{40}{100} < 1120 \times \frac{60}{100} < (1120 + x)\frac{50}{100}$$

Consider $(1120 + x)\frac{40}{100} < 1120 \times \frac{60}{100}$

$\therefore \quad (1120 + x)2 < 1120 \times 3$

$\therefore \quad 2x < 1120$

$\therefore \quad x < 560$

Also, $1120 \times \frac{60}{100} < (1120 + x)\frac{50}{100}$

$\therefore \quad 1120 \times 6 < 1120 \times 5 + 5x$

$\therefore \quad 1120 < 5x$

$\therefore \quad 224 < x$

$\therefore \quad$ Thus $224 < x < 560$.

In other words the quantity of acid added should be more than 224 litres but less than 560 litres.

Example 38 : A solution is to be kept between 77° F and 95° F. The conversion formula from Celsius to Faherenheit is $F = \frac{9}{5}C + 32$. What is the range of temperature in degree celsius ?

Solution : We have $77 < x < 95$, where x is the temperature of the solution in degree Fahrenheit.

$77 < \frac{9}{5}y + 32 < 95$ where y is the temperature of the solution in degree Celcius.

$\therefore$ $(77 - 32)\frac{5}{9} < y < (95 - 32)\frac{5}{9}$

$\therefore$ $25 < y < 35$

$\therefore$ The solution must be kept between 25° C and 35° C.

Example 39 : IQ of a person is given by $IQ = \frac{MA}{CA} \times 100$

where MA is the mental age of the person, CA is her chronological age.

If $75 < IQ < 125$, find the mental age, provided the chronological age is 16. (CA = 16)

Solution : We have $75 < \frac{MA}{CA} \times 100 < 125$

$\Leftrightarrow$ $\frac{75}{100} \times CA < MA < \frac{125}{100} \times CA$

$\Leftrightarrow$ $\frac{3}{4} \times CA < MA < \frac{5}{4} \times CA$

$\Leftrightarrow$ $\frac{3}{4} \times 16 < MA < \frac{5}{4} \times 16$ **(CA = 16)**

$\Leftrightarrow$ $12 < MA < 20$

Thus the range of the mental age is (12, 20).

Example 40 : The longest side of a triangle is twice its shortest side. The other side is 3 *cm* longer than the shortest side and perimeter of the triangle is at least 51 *cm*. Find the minimum length of the side which is neither shortest nor longest.

Solution : Let x be the shortest side. Then the sides are $2x$, $x + 3$, x.

$\therefore$ Since perimeter is at least 51,

we have $2x + x + 3 + x \geq 51 \Leftrightarrow 4x \geq 48$

$\Leftrightarrow x \geq 12$

$\Leftrightarrow x + 3 \geq 15$

Hence the required side is at least 15 *cm* long.

EXERCISE 8

Solve following inequalities : (1 to 5) $x \in R$

1. $|x + 1| + |x - 1| > 2$ **2.** $\frac{|x-2|-2}{|x-1|-1} \leq 0$ **3.** $\frac{1}{|x|-5} \leq \frac{1}{3}$

4. $|x - 1| < 5$ and $|x| \leq 2$ **5.** $\frac{2}{x+3} \leq 5 \leq \frac{4}{x+3}$ $(x > 0)$

6. The water acidity in a pool is considered normal if three daily measurements have average between 8.2 and 8.5. What should be the third measurement if two of them on a day are 8.25 and 8.40 ?

7. Cost function and revenue functions in a small scale unit are $C = 500 + \frac{5}{2}x$ and $R = 3x$ respectively, where x is the number of manufactured units. What is the number of units to be manufactured for (1) break even (2) profit ?

8. Find pairs of consecutive odd integers each larger than 30 and such that their sum is less than 75.

9. Answer following questions in short :

(1) What is the solution set of $\frac{x^2}{x-5} < 0$? (2) Solve $\left| x + \frac{1}{x} \right| \geq 2$.

(3) Solve $(x^4 - 2x^2 + 1)(x - 2) > 0$ (4) Solve $| x - 3 | = x - 3$

(5) Solve $\left| \frac{1}{x} - 3 \right| > 5$ (6) Solve in integers $\frac{x+2}{x^2+1} < \frac{1}{2}$

(7) Solve $| x - 2 | \geq | x - 4 |$ (8) Solve $x + \frac{1}{x} < -2$

10. Fill in the blanks :

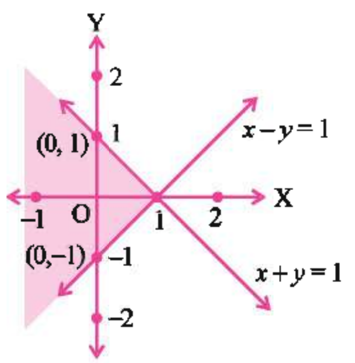
(1) If $| x - 2 | > 1$, then $x \in$ ...... (2) If $| x | \leq 0$, then $x =$ ......

(3) If $\frac{1}{x-4} < 0$, then x ...... 4 (4) If $| x - 2 | < 3$, then 5 ...... x ...... -1.

(5) Solution set of $\frac{x^2}{x^2+1} < 0$ is .... $(x \in R)$

11. Find the linear inequalities whose solution set is given as the coloured region in the following graphs.

(1)



(2)

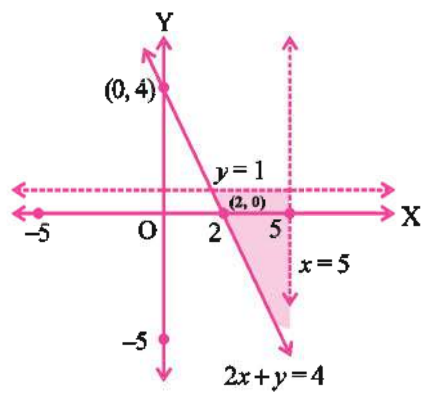


Figure 8.66

12. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) If $| x - 2 | \geq 8$, then... ☐

(a) $x \in (-6, 10)$ (b) $x \in (-\infty, -6) \cup (10, \infty)$

(c) $x \in (-\infty, -6] \cup (10, \infty)$ (d) $x \in (-\infty, -6] \cup [10, \infty)$

(2) If $| x + 2 | \leq 9$, then... ☐

(a) $x \in (-11, 7)$ (b) $x \in [-11, 7]$

(c) $x \in (-\infty, -11] \cup [7, \infty)$ (d) $x \in (-\infty, -11] \cup (7, \infty)$

(3) The inequality representing the coloured region in figure 8.67 is ......

(a) $|x| < 2$

(b) $|x| \leq 2$

(c) $|x| \geq 2$

(d) $-2 < x \leq 2$

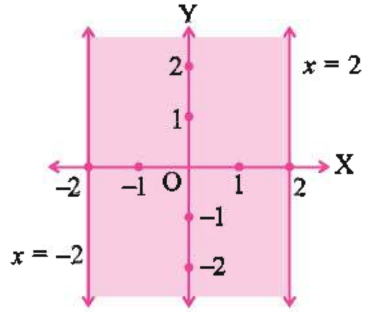


Figure 8.67

(4) The inequality to represent following on the number line is...

Figure 8.68

(a) $x \geq 2$ (b) $x \in (-\infty, 2)$ (c) $x > 2$ (d) $x \leq 2$

(5) The inequality to represent the coloured region in figure 8.69 is...

(a) $x \geq 0$ (b) $y \geq 0$

(c) $x > 0$ (d) $x \leq 0$

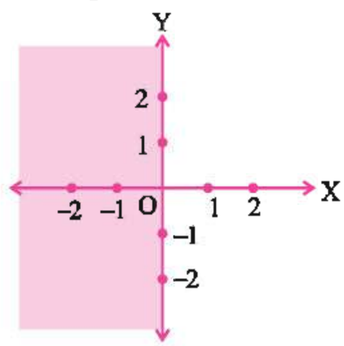


Figure 8.69

(6) The solution set of $x < 5$ and $x \geq 2$ is...

(a) $(2, 5)$ (b) $[2, 5)$ (c) $(2, 5]$ (d) $[2, 5]$

(7) The coloured region in figure 8.70 is the solution set of ......

(a) $x \geq 0, y \geq 0$

(b) $x \leq 0, y \geq 0$

(c) $x > 0, y > 0$

(d) $x \geq 0, y \leq 0$

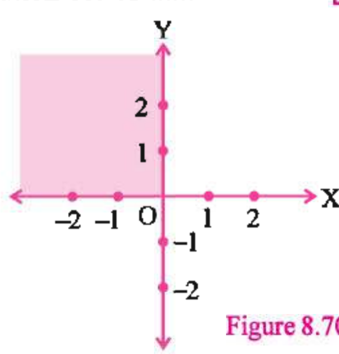


Figure 8.70

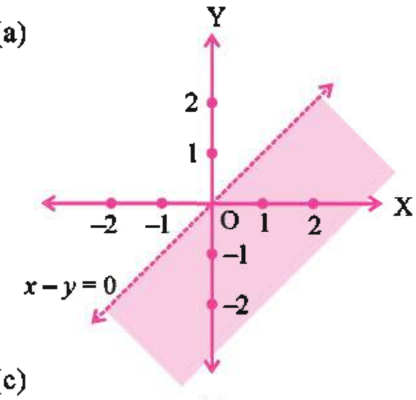
(8) If $\dfrac{x^2 - 1}{x^2 + 1} \geq 0$, then $x \in$ ......

(a) $x \in (-\infty, -1] \cup [1, \infty)$ (b) $x \in [-1, 1]$

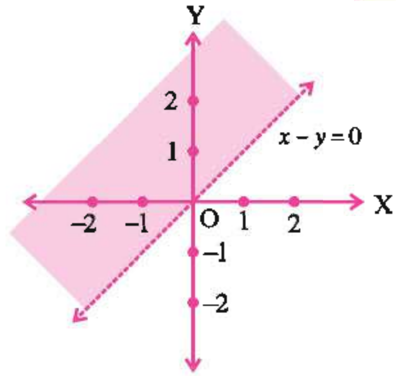
(c) $x \in \{-1, 1\}$ (d) $x \in (-\infty, -1) \cup [1, \infty)$

(9) The solution set of $x - y \geq 0$ is shown graphically by

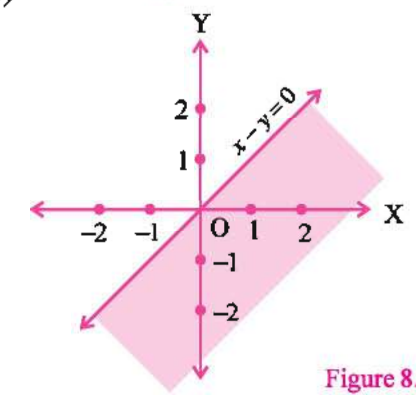
(a)



(b)



(c)



(d)

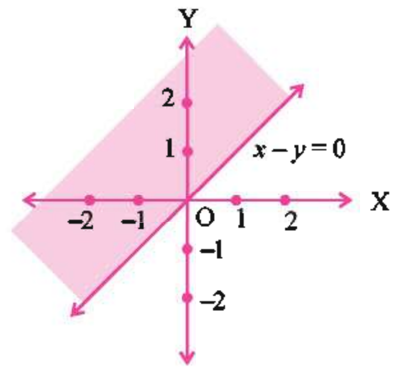


Figure 8.71

(10) The inequalities represented by the coloured region in figure 8.72 are...

(a) $x \geq 1$

(b) $y < 2$

(c) $x \geq 1$ and $y < 2$

(d) $x \leq 1$ and $y \geq 2$

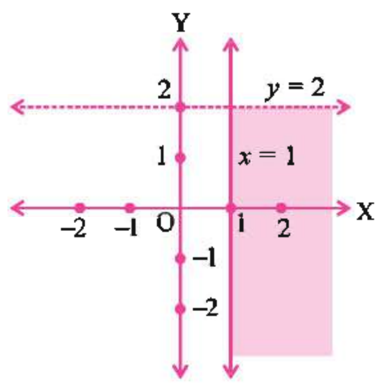


Figure 8.72

(11) The solution set of $| x - 1 | \leq -1$ is...

(a) $(0, 2)$ (b) $[0, 2]$ (c) $(-\infty, -1] \cup [1, \infty)$ (d) $\emptyset$

(12) The solution set of $| x - 1 | + | x - 2 | < 3$ is...

(a) $(0, 3)$ (b) $(1, 2)$ (c) $(0, 2)$ (d) $(2, 3)$

(13) The solution set of $|x| + | x - 2 | < 2$ is shown on the number line by...

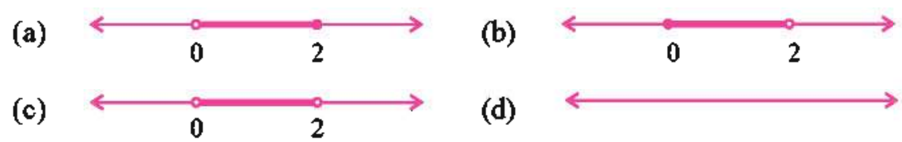


Figure 8.73

(14) The solution set of $| x - 1 | + | x + 1 | < 2$ is...

(a) $(-1, 1)$ (b) $[-1, 1]$ (c) $\emptyset$ (d) $\{-1, 1\}$

(15) The solution set of $x^2 \leq 4$ is...

(a) $[-2, 2]$ (b) $(-2, 2)$

(c) $(-\infty, -2] \cup [2, \infty)$ (d) $\emptyset$

Summary

1. Inequalities
2. A linear inequality in one variable and representation of its solution on the number line.
3. System of linear inequalities in one variable.
4. Linear inequalities in two variables and their graphs
5. System of linear inequalities in two variables
6. Inequalities related to modulus, problems and miscellaneous problems.

Chapter 9

DISPERSION

9.1 Introduction

We know that the statistics deals with data collection for specific purposes. We have learnt how to classify the numerical data using discrete or continuous variables in standard 8, 9 and 10. We are familiar with the words like class, class length, class boundary points, frequency, mid-value etc. We have also studied the method of finding mean, median and mode. These values are the measures of central tendency. A measure of central tendency gives us a rough idea about frequency distribution where the observations are centered. However when two given data have the same mean, the observations in two cases need not be distributed about mean in the same manner. Two data may have the same mean but very different spread. For example if Ritu gets 2 marks in the first examination and 98 marks in another examination, then the mean of Ritu's score is 50 marks. On the other hand, if Riya got 48 marks in the first and 52 marks in the second examination, the mean of her score is also 50 marks. But from the data we can observe that Ritu's score is not consistent. There is a gap of 96 marks between scores of two examinations. While Riya's score is consistent, because there is a gap of only 4 marks. For a consistent data, observations should be spread in neighbourhood of mean. They should be clustered around mean. This is not the case with Ritu's marks. Thus, we cannot have a correct picture of data just from its mean. Some other statistical measure is necessary to have correct and reliable comparison of two data. We shall study such measures in this chapter. They are called measures of dispersion of data. We begin with an illustration to understand this. Runs scored by two batsmen in their last five matches are as follows :

Match Number	Batsman A	Batsman B
1	55	40
2	65	55
3	65	65
4	65	65
5	75	100
Total	**325**	**325**
Mean $\bar{x}$ =	65	65
Median M =	65	65
Mode Z =	65	65

Here the mean, median, mode for both the data are the same i.e. 65 runs. Yet looking at the runs obtained by them in 5 matches, we can observe that A is more consistent in his performance than B. His runs are close to mean i.e. 65 and the difference between his minimum and maximum score is 20. On the other hand there is a variation in the performance of B in various matches.

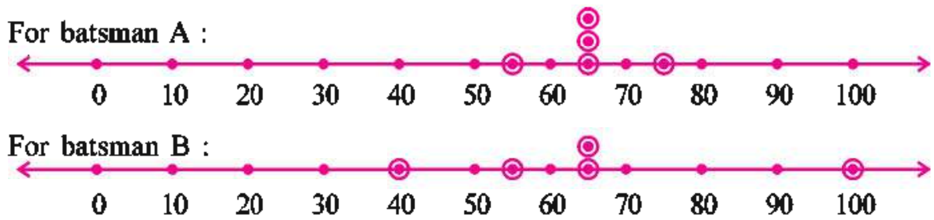


Figure 9.1

In one match he scores 40 and in another match he scores 100, a difference of 60 runs. So we can judge how far they are from mean 65. So it is not proper to judge the performance of two cricketers on the basis of their average only. In order to get better understanding of the data, it is necessary to know how far the observations are scattered from the average. This scattering or the spread of the observations is called **dispersion**. Now we will discuss some measures of dispersion here.

9.2 Measures of Dispersion

As discussed above a measure of dispersion is a number showing how scattered the observations are from their mean. If the difference between the observations and their mean is less, then the dispersion is small and the data can be regarded as consistent or stable. If the difference between the observations and their mean is more, then the dispersion will be large and the data cannot be regarded as consistent or stable. In such cases the mean cannot be considered as a good representative of observations. Following are commonly used measures of dispersion.

(1) Range (2) Average Deviation (3) Standard Deviation

9.3 Range

The range is the difference between two extreme observations of the given data. i.e. the difference between the maximum and the minimum observations of the given data.

Range of a data = Maximum value of observation − Minimum value of observation

Range of scores of batsman A = $75 - 55 = 20$

Range of scores of batsman B = $100 - 40 = 60$

The smaller range of the data indicates consistency (stability) of the variable. As the range of batsman A is less, A is considered to be more consistent or more stable than B. Range is the simplest measure of dispersion. It gives us a rough idea about the variability or scatteredness. It is based upon two extreme observations. So it does not measure the dispersion of the data from a central value. Range of observation 2 and 102 is 100. Range of 101 observations 2, 3, 4,..., 102 is also 100. These data obviously are of very different nature. For this purpose we need some other measure of dispersion which measures the deviation of the observed values from the central value. Among such measures of dispersion are average deviation and standard deviation.

9.4 Average Deviation (Mean Deviation)

In the definition of range, the use of deviations of observations of the data from the mean is not considered. Deviations are called **deviations of the observations from the mean**. These deviations can be negative, zero or positive and their sum can also be zero or positive or negative. So we consider the absolute values of the deviations of observations of the data from the mean.

Average Deviation : The average of the positive differences of observations of data from their mean is called the average deviation of the data. It is denoted by the symbol $\delta\bar{x}$. (read : delta x bar)

The average deviation of the data defined above is called the average (mean) deviation from the mean. If the differences from the median instead of the mean are taken, we get the average (mean) deviation from the median δm.

We shall now discuss the differences from the mean and the method of computing average deviation about the mean for ungrouped and grouped data for discrete variable and their formulae.

(1) Ungrouped Data :

Let $x_1, x_2, x_3, ..., x_n$ be n observations of given ungrouped data. These n values can also be written as x_i, $(1 \leq i \leq n)$. If $\bar{x}$ denotes the mean of the data, then $\bar{x} = \frac{\sum x_i}{n}$.

Average deviation from the mean $\delta\bar{x} = \frac{\sum |x_i - \bar{x}|}{n}$, where $|x_i - \bar{x}|$ is the positive difference between the ith observation and the mean $\bar{x}$.

Example 1 : The weights in kg of 10 persons are given below. Find the average deviation about the mean of this data.

37, 70, 48, 50, 32, 56, 63, 46, 54, 44

Solution : The mean $\bar{x} = \frac{\sum x_i}{n} = \frac{37 + 70 + 48 + 50 + 32 + 56 + 63 + 46 + 54 + 44}{10}$

$$= \frac{500}{10} = 50 \text{ kg}$$

x_i	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$
37	–13	13
70	20	20
48	–2	2
50	0	0
32	–18	18
56	6	6
63	13	13
46	–4	4
54	4	4
44	–6	6
$\Sigma x_i = 500$	–	$\Sigma \lvert x_i - \bar{x} \rvert = 86$

$$\delta\bar{x} = \frac{\sum \lvert x_i - \bar{x} \rvert}{n}$$

$$= \frac{86}{10}$$

$$= 8.6 \text{ kg}$$

Example 2 : Preyaa and Karishma obtained the following marks out of 50 in ten weekly tests. Find whose performance is more consistent using average deviation about mean.

Preyaa's marks	30	35	40	42	38	25	31	36	40	41
Karishma's marks	25	48	40	38	42	37	44	28	46	40

Solution : Mean of Preyaa's marks :

$$\bar{x} = \frac{\sum x_i}{n} = \frac{30+35+40+42+38+25+31+36+40+41}{10}$$

$$= \frac{358}{10}$$

$$= 35.8 \text{ marks}$$

Mean of Karishma's marks :

$$\bar{y} = \frac{\sum y_i}{n} = \frac{25+48+40+38+42+37+44+28+46+40}{10}$$

$$= \frac{388}{10}$$

$$= 38.8 \text{ marks}$$

x_i	$\lvert x_i - \bar{x} \rvert$
30	5.8
35	0.8
40	4.2
42	6.2
38	2.2
25	10.8
31	4.8
36	0.2
40	4.2
41	5.2
$\Sigma x_i = 358$	$\Sigma \lvert x_i - \bar{x} \rvert = 44.4$

y_i	$\lvert y_i - \bar{y} \rvert$
25	13.8
48	9.2
40	1.2
38	0.8
42	3.2
37	1.8
44	5.2
28	10.8
46	7.2
40	1.2
$\Sigma y_i = 388$	$\Sigma \lvert y_i - \bar{y} \rvert = 54.4$

$$\delta\bar{x} = \frac{\sum |x_i - \bar{x}|}{n} = \frac{44.4}{10}$$

$$\delta\bar{x} = 4.44 \text{ marks}$$

$$\delta\bar{y} = \frac{\sum |y_i - \bar{y}|}{n} = \frac{54.4}{10}$$

$$\delta\bar{y} = 5.44 \text{ marks}$$

The average deviation of Preyaa's marks is less than that of Karishma's marks. Hence Preyaa is more consistent in study.

(2) Grouped Data :

(i) Discrete Frequency Distribution : Let $x_1, x_2, x_3, ..., x_k$ be the values assumed by the discrete variable x of the discrete frequency distribution with corresponding frequencies $f_1, f_2, f_3, ..., f_k$. The average deviation about the mean of the discrete frequency distribution is defined by the following formula $\delta\bar{x} = \frac{\sum f_i |x_i - \bar{x}|}{n}$.

where, $n = \Sigma f_i$ = sum of all frequencies. $i = 1, 2, 3, ..., k$

$\bar{x} = \frac{\sum f_i x_i}{n}$ = mean of the frequency distribution.

$|x_i - \bar{x}|$ = absolute difference between the observation x_i and the mean $\bar{x}$.

Example 3 : Find the average deviation about the mean for the following frequency distribution.

x_i	3	9	17	23	27
f_i	8	10	12	9	5

Solution :

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
3	8	24	−12	12	96
9	10	90	−6	6	60
17	12	204	2	2	24
23	9	207	8	8	72
27	5	135	12	12	60
	$n = 44$	$\Sigma f_i x_i = 660$			$\Sigma f_i \lvert x_i - \bar{x} \rvert = 312$

$$\bar{x} = \frac{\sum f_i x_i}{n} = \frac{660}{44} = 15$$

$$\delta\bar{x} = \frac{\sum f_i \lvert x_i - \bar{x} \rvert}{n} = \frac{312}{44} = 7.09$$

(ii) Continuous Frequency Distribution : Let the mid-values of k classes of the continuous frequency distribution be $x_1, x_2, ..., x_k$ with corresponding frequencies as $f_1, f_2, ..., f_k$. Then the average deviation about the mean of the continuous frequency distribution is defined by the formula

$$\delta\bar{x} = \frac{\sum f_i \lvert x_i - \bar{x} \rvert}{n}$$

where, $n = \Sigma f_i$ = sum of all frequencies, $\bar{x} = \dfrac{\sum f_i x_i}{n}$

Example 4 : In a language test of 60 marks, the frequency distribution of marks secured by 50 students is given below. Find the average deviation about the mean of the frequency distribution.

Class	0-10	10-20	20-30	30-40	40-50	50-60
Frequency	6	8	14	16	4	2

Solution :

Class	x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
0-10	5	6	30	−22	22	132
10-20	15	8	120	−12	12	96
20-30	25	14	350	−2	2	28
30-40	35	16	560	8	8	128
40-50	45	4	180	18	18	72
50-60	55	2	110	28	28	56
		$n = 50$	$\Sigma f_i x_i = 1350$			$\Sigma f_i \lvert x_i - \bar{x} \rvert = 512$

Mean $\bar{x} = \frac{\sum f_i x_i}{n} = \frac{1350}{50} = 27.$ So, $\bar{x} = 27$

Mean deviation $\delta\bar{x} = \frac{\sum f_i |x_i - \bar{x}|}{n} = \frac{512}{50} = 10.24.$ So, $\delta\bar{x} = 10.24$

Shortcut method for calculating mean deviation about mean :

(i) For Discrete Frequency Distribution :

If the values x_i of the variable are relatively large, an assumed mean A is to considered to simplify calculations. Any one of the mid-value or value of a variable or a suitable value is taken as assumed mean A and differences between x_i and A is obtained $d_i = x_i - A$. Then each d_i is multiplied by f_i and sum $\Sigma f_i d_i$ is obtained. Then the formula for the mean by shortcut method is given by $\bar{x} = A + \frac{\sum f_i d_i}{n}$.

Example 5 : Find the average deviation about mean of the following frequency distribution by shortcut method.

x_i	12	13	14	15	17	20	25
f_i	10	18	20	25	15	10	2

Solution :

x_i	f_i	$d_i = x_i - A$ A = 15	$f_i d_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
12	10	−3	−30	3.14	31.40
13	18	−2	−36	2.14	38.52
14	20	−1	−20	1.14	22.80
15 = A	25	0	0	0.14	3.50
17	15	2	30	1.86	27.90
20	10	5	50	4.86	48.60
25	2	10	20	9.86	19.72
	100		$\Sigma fidi = 14$		$\Sigma f_i \lvert x_i - \bar{x} \rvert = 192.44$

Mean $\bar{x} = A + \frac{\sum f_i d_i}{n}.$

$= 15 + \frac{14}{100}$

$= 15 + 0.14$

$\bar{x} = 15.14$

$\delta\bar{x} = \frac{\sum f_i |x_i - \bar{x}|}{n}$

$= \frac{192.44}{100}$

$\delta\bar{x} = 1.92$

(ii) For Continuous Frequency Distribution :

If the class-length of all classes of the continuous frequency distribution is the same, then the formula for calculating mean $\bar{x}$ can be written in a suitable form so as to make the computation of the mean simpler.

Suppose the length of the classes are the same and equal to c. Suppose x_i is the mid-value of the ith class for $i = 1, 2, 3, \ldots, k$. Any one of the mid-values is taken as the assumed mean A and the difference between x_i and A is divided by class-length c to obtain deviation $d_i = \frac{x_i - A}{c}$ for the ith class. The deviation d_i is multiplied by the frequency f_i of the ith class and sum $\Sigma f_i d_i$ is obtained. We can compute the mean by short-cut method by the formula :

$$\bar{x} = A + \frac{\Sigma f_i d_i}{n} \times c, \quad d_i = \frac{x_i - A}{c}$$

Example 6 : Find the average deviation about mean for the following data using shortcut method :

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of students	6	8	14	16	4	2

Solution :

Marks	f_i	Mid-value x_i	$d_i = \frac{x_i - 25}{10}$	$f_i d_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
0-10	6	5	–2	–12	22	132
10-20	8	15	–1	–8	12	96
20-30	14	25 = A	0	0	2	28
30-40	16	35	1	16	8	128
40-50	4	45	2	8	18	72
50-60	2	55	3	6	28	56
	$n = 50$			$\Sigma f_i d_i = 10$		$\Sigma f_i \lvert x_i - \bar{x} \rvert = 512$

$$\bar{x} = A + \frac{\Sigma f_i d_i}{n} \times c$$
$$= 25 + \frac{10}{50} \times 10$$
$$= 25 + 2 = 27$$

Mean deviation about mean $\delta \bar{x} = \frac{\Sigma f_i \lvert x_i - \bar{x} \rvert}{n}$

$$= \frac{512}{50}$$
$$= 10.24$$

9.5 Median

To find average deviation about median, we have to find the median first. Median is that value of the variable of the given data for which the number of observations with values less than it and greater than this value are equal. It is clear from this statement that the observations of the data are to be arranged in an increasing order of their magnitude. A value which divides these ordered observations into two equal parts is called the median of the data. The median is denoted by M. In class IX we have studied about the method for finding the median of ungrouped data. Suppose x_1, x_2, $x_3, \ldots, x_n$ are n observations of an ungrouped data. First arrange the observations in increasing order of their magnitude. The middle-most value of the variable x is called the median of the ungrouped data. If n is odd, then the median is the value of $\frac{n+1}{2}$th observation. If n is even, then the median is

$$M = \frac{\text{Value of } \left(\frac{n}{2}\right)\text{th observation} + \text{value of } \left(\frac{n}{2}+1\right)\text{th observation}}{2}$$

For example, (1) The marks obtained by seven students in an examination of 100 marks are 38, 48, 50, 87, 60, 49 and 70. First we arrange these observations in an increasing order of their magnitude.

38, 48, 49, 50, 60, 70, 87

Now, $n = 7$, which is an odd integer.

$$M = \text{Value of } \frac{n+1}{2}\text{th observation}$$

$$= \text{Value of } \left(\frac{7+1}{2}\right)\text{th observation}$$

$$= \text{Value of 4th observation}$$

$$= 50$$

$\therefore$ The median = 50 marks

(2) The daily pocket-expenses of ten students are ₹ 20, 25, 17, 18, 8, 15, 22, 10, 9, 14.

We arrange these observations in an increasing order of their magnitude.

We have, 8, 9, 10, 14, 15, 17, 18, 20, 22, 25

Now, $n = 10$, which is an even integer.

$$M = \frac{\text{Value of } \left(\frac{n}{2}\right)\text{th observation} + \text{value of } \left(\frac{n}{2}+1\right)\text{th observation}}{2}$$

$$= \frac{\text{Value of 5th observation} + \text{value of 6th observation}}{2}$$

$$= \frac{15+17}{2} = 16$$

$\therefore$ The median = ₹ 16

Median of grouped data :

We know that data can be grouped into two ways **(i) Discrete frequency distribution (ii) Continuous frequency distribution.**

(i) Discrete frequency distribution : Let $x_1, x_2, ..., x_k$ be the values of discrete variable with frequencies $f_1, f_2, ..., f_k$ respectively in a discrete frequency distribution. To find the median, the cumulative frequencies are obtained from the frequency distribution. Then, we identify the observation whose cumulative frequency is equal or just greater than $\frac{n+1}{2}$th observation, where $n = \Sigma f_i$. This observation is the required median.

For a given n, the value of $\frac{n+1}{2}$th observation against the column of cumulative frequency for discrete frequency distribution is the median.

For example let us obtain the median for the following frequency distribution :

x_i	0	1	2	3	4
f_i	4	1	6	11	3

x_i	f_i	cf
0	4	4
1	1	5
2	6	11
3	11	22
4	3	25

$n = \Sigma f_i = 25$

M = Value of $\frac{n+1}{2}$th observation

= Value of $\left(\frac{26}{2}\right)$th observation

= Value of 13th observation

We find that the cumulative frequency just greater than 13 is 22 and the value of x corresponding to 22 is 3.

$\therefore$ The median = 3

(ii) Median for continuous frequency distribution : Continuous grouped distribution is in the form of classes. In frequency distribution, values of continuous variable are in the increasing order in the form of classes. To find the median of a grouped frequency distribution, first cumulative frequencies are obtained and we let $n = \Sigma f_i$. Then we find the cumulative frequency just greater than $\frac{n}{2}$ and determine the corresponding class. This class is known as the median class. After finding the median class we compute the median using following formula,

$$\mathbf{M = L + \frac{\left(\frac{n}{2} - F\right)}{f} \times c}$$

L = Lower boundary point of the median class.

n = Σf_i = number of observations.

F = cumulative frequency of the class preceding the class in which the median lies.

f = frequency of the median class.

c = length of the class interval of the class in which the median lies.

For example : Calculate the median from the following distribution.

Class	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	4	12	14	16	20	16	10	8

Solution :

Class	f_i	cf
20-30	4	4
30-40	12	16
40-50	14	30
50-60	16	46
60-70	20	66
70-80	16	82
80-90	10	92
90-100	8	100

$n = \Sigma f_i = 100$

Now, cumulative frequency just greater than $\frac{n}{2} = 50$ is 66 and the corresponding class is 60-70. So, 60-70 is the median class.

$L = 60, f = 20, F = 46, c = 10$

$$M = L + \frac{\left(\frac{n}{2} - F\right)}{f} \times c$$

$$= 60 + \left(\frac{50 - 46}{20}\right) \times 10$$

$$= 60 + \frac{4}{20} \times 10$$

$$= 60 + 2$$

$$M = 62$$

Average Deviation from the Median :

We shall now discuss the method of computing the average deviation about the median for the ungrouped and grouped data and their formulae.

(i) Ungrouped Data :

Suppose that $x_1, x_2, \ldots, x_n$ are observations of an ungrouped data and M is their median where, **$M = \frac{n+1}{2}$ th observation, if n is odd. If n is even,**

$$M = \frac{\text{Value of } \left(\frac{n}{2}\right)\text{th observation} + \text{value of } \left(\frac{n}{2}+1\right)\text{th observation}}{2}$$

The difference $x_i - M$ between ith observation x_i and the median M is obtained and then $| x_i - M |$ is the absolute value of the deviation of the observation x_i from the median M. The average of the absolute values is called the average deviation about the median. Thus, we define the average deviation about the median M as,

$$\delta M = \frac{\sum | x_i - M |}{n}$$

Example 7 : Find the average deviation about the median for the following data :

37, 70, 48, 50, 32, 56, 63, 46, 54, 44

Solution : Here the number of observations is 10 which is even. Arranging the data into ascending order, we have, 32, 37, 44, 46, 48, 50, 54, 56, 63, 70.

$$\text{Now, The median M} = \frac{\text{5th observation} + \text{6th observation}}{2}$$

$$= \frac{48 + 50}{2}$$

$$M = 49$$

x_i	$x_i - M$	$\| x_i - M \|$
37	−12	12
70	21	21
48	−1	1
50	1	1
32	−17	17
56	7	7
63	14	14
46	−3	3
54	5	5
44	−5	5
		$\Sigma\| x_i - M \| = 86$

$$\delta M = \frac{\sum | x_i - M |}{n} = \frac{86}{10} = 8.6$$

(2) Grouped Data :

(i) Discrete Frequency Distribution : To find average deviation about the median, we have to find the median of the given discrete frequency distribution. To do this, observations are arranged in increasing order of their magnitude. In order to determine the median, the cumulative frequencies are obtained using the frequencies of the given frequency distribution. Then, we identify the observation whose cumulative frequency is equal or just greater than $\frac{n+1}{2}$ where $n = \Sigma f_i$ = sum of frequencies. This observation is the required median. After finding the median, we obtain the mean of the absolute values of the deviations from median. Thus

$$\delta M = \frac{\sum f_i |x_i - M|}{n}$$

Example 8 : Find the average deviation about the median for the following data :

x_i	2	5	6	8	10	12
f_i	2	8	10	7	8	5

Solution :

x_i	f_i	cf	$\|x_i - M\|$	$f_i\|x_i - M\|$
2	2	2	6	12
5	8	10	3	24
6	10	20	2	20
8	7	27	0	0
10	8	35	2	16
12	5	40	4	20
	$n = 40$			$\Sigma f_i\|x_i - M\| = 92$

The median M = The value of $\left(\frac{n+1}{2}\right)$th observation

= The value of $\left(\frac{40+1}{2}\right)$th observation

= The value of (20.5)th observation

The cf just greater than 20 is 27 and it occurs against observation 8.

$\therefore$ M = 8

$$\delta M = \frac{\sum f_i |x_i - M|}{n}$$

$$= \frac{92}{40}$$

$\therefore$ $\delta M = 2.3$

(ii) Continuous Frequency Distribution : To find the average deviation about the median, we have to find the median first from given continuous frequency distribution. After finding the median, the absolute value of the deviation of mid-value x_i of each class from the median i.e. $|x_i - M|$ is obtained. Then,

$$\delta M = \frac{\sum f_i |x_i - M|}{n}, \text{ where } n = \Sigma f_i$$

Example 9 : Calculate the average deviation about the median for the following data :

Class	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Frequency	2	3	8	14	8	3	2

Solution :

Class	Frequency f_i	Mid-value x_i	cf	$\lvert x_i - M \rvert$	$f_i \lvert x_i - M \rvert$
10-20	2	15	2	30	60
20-30	3	25	5	20	60
30-40	8	35	13	10	80
40-50	14	45	27	0	0
50-60	8	55	35	10	80
60-70	3	65	38	20	60
70-80	2	75	40	30	60
	$n = 40$				$\Sigma f_i \lvert x_i - M \rvert = 400$

The class containing $\left(\frac{n}{2}\right)$th observation

= The class containing 20th observation

For the class 30-40, $cf = 13$ and for the class 40-50, $cf = 27$

∴ The median class is 40-50.

∴ $L = 40,\ F = 13,\ \frac{n}{2} = \frac{40}{2} = 20,\ f = 14,\ c = 10$

$$M = L + \left(\frac{\frac{n}{2} - F}{f}\right) \times c$$

$$= 40 + \frac{20 - 13}{14} \times 10$$

$$= 40 + 5$$

$$M = 45$$

$$\delta M = \frac{\sum f_i \lvert x_i - M \rvert}{n} = \frac{400}{40}$$

$$\delta M = 10$$

EXERCISE 9.1

1. Find the average deviation about the mean for the following observations :
   18, 20, 28, 15, 17, 22, 25, 29, 32, 34

2. The figures of the monthly wages of ten workers are given below. Find the average deviation about the mean of the data.

1150, 1140, 1230, 1200, 1100, 1300, 1190, 1180, 1160, 1150.

3. Find the average deviation about the median for the following data :

3, 9, 5, 3, 12, 10, 18, 4, 7, 19, 21.

4. Find the average deviation about the median of the daily wages of ten workers in ₹ from the following data :

36, 72, 46, 42, 60, 45, 53, 46, 51, 49

5. Find the average deviation about the mean for the following frequency distribution :

x_i	8	16	24	32	40	48	56	64
f_i	5	13	12	8	6	10	9	3

6. Find the average deviation about the mean for the following frequency distribution :

x_i	10	30	50	70	90
f_i	4	24	28	16	8

7. Find the average deviation about the median for the following frequency distribution :

x_i	3	6	9	12	13	15	21	22
f_i	3	4	5	2	4	5	4	3

8. Find the average deviation about the median for the following frequency distribution :

x_i	15	21	27	30	35
f_i	3	5	6	7	8

9. The score of a batsman in ten innings is 48, 80, 58, 44, 52, 65, 73, 56, 64, 54. Find the average deviation from the median.

10. The length (in *cm*) of 10 plants are given below :

42, 52.3, 55.2, 72.9, 52.8, 79, 32.5, 15.2, 27.9, 30.2

Find the average deviation from the median and also from the mean.

11. Find the mean (average) deviation from the mean from the following frequency distribution.

Marks	0–10	10–20	20–30	30–40	40–50	50–60	60–70
Number of students	4	6	10	20	10	6	4

12. Find the mean deviation about the mean from the following frequency distribution.

Income per day	0–100	100–200	200–300	300–400	400–500	500–600	600–700	700–800
Number of person	4	8	9	10	7	5	4	3

13. Calculate the mean (average) deviation about the median for the following data :

Class	0–10	10–20	20–30	30–40	40–50	50–60
Frequency	6	7	15	16	4	2

14. Calculate the average deviation about the median for the age distribution of 100 persons given below :

Age	16–20	21–25	26–30	31–35	36–40	41–45	46–50	51–55
Number	5	6	12	14	26	12	16	9

(**Hint :** Convert the given data into classes using class boundary points)

*

9.6 Standard Deviation

We have seen that the definition of average deviation is based on absolute values of the deviations of observations of the data from the mean. Absolute values of these deviations are all non-negative. The absolute values were taken to give meaning to the average deviation otherwise the deviations may cancel among themselves. Instead of taking the absolute value of deviation of each observation from the mean, the square of the deviation is taken and the sum of the squares of these deviations is divided by the total number of observations. We obtain an important measure of dispersion. This measure of dispersion is known as variance and is denoted by symbol s^2 and its positive square root is called standard deviation and is denoted by s. In 1893 the famous statistician Karl Pearson gave the definition of standard deviation as a measure of dispersion. Among all the measures of dispersion, the standard deviation is most important and widely used as a measure of dispersion. Its definition is as given below.

Standard Deviation : The positive square root of the number, obtained by dividing the sum of the squares of deviations from mean of the observations of given data, by the number of observations is called the standard deviation of the data. Thus, if $\overline{x}$ is the mean of n observations $x_1, x_2, ..., x_n$, then

$$s = \sqrt{\frac{\Sigma(x_i - \overline{x})^2}{n}}$$

Calculation of Standard Deviation for Ungrouped Data :

Direct Method : If $x_1, x_2, ..., x_n$ are n observations of ungrouped data and $\overline{x}$ is the mean, then we obtain deviation $x_i - \overline{x}$ for each x_i from the mean. Square of all the deviations are taken and their sum $\Sigma(x_i - \overline{x})^2$ is obtained. Dividing the sum by the total number of observations n we get variance s^2.

So, $s^2 = \dfrac{\Sigma(x_i - \overline{x})^2}{n}$.

Taking the positive square root of s^2 we get standard deviation :

Standard deviation $s = \sqrt{s^2} = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}}$

Let us understand this by an example.

Example 10 : The runs scored by a cricket player in six innings are 60, 45, 25, 40, 70 and 30. Find the mean of the runs scored and the standard deviation.

Solution : First we find $\bar{x}$.

Mean $\bar{x} = \frac{\sum x_i}{n} = \frac{270}{6} = 45$. Thus, $\bar{x} = 45$.

Now, values of $x_i - \bar{x}$ are calculated.

Then the square $(x_i - \bar{x})^2$ for each x_i and the sum $\Sigma(x_i - \bar{x})^2$ is calculated. Then s^2 and s are calculated.

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
60	15	225
45	0	0
25	–20	400
40	–5	25
70	25	625
30	–15	225
$\Sigma x_i = 270$		$\Sigma(x_i - \bar{x})^2 = 1500$

$\therefore$ Variance $s^2 = \frac{\sum(x_i - \bar{x})^2}{n}$

$= \frac{1500}{6} = 250$

and standard deviation

$s = \sqrt{250}$

$= 15.81$

Example 11 : The marks obtained by 9 students in a test of 100 marks in mathematics are given as, 69, 67, 66, 69, 64, 63, 68, 65, 72. Find the standard deviation of the data.

x_i	$(x_i - \bar{x})$	$(x_i - \bar{x})^2$
69	2	4
67	0	0
66	–1	1
69	2	4
64	–3	9
63	–4	16
68	1	1
65	–2	4
72	5	25
$\Sigma x_i = 603$		$\Sigma(x_i - \bar{x})^2 = 64$

Mean $\bar{x} = \frac{\sum x_i}{n} = \frac{603}{9}$

$\bar{x} = 67$ marks

Variance $s^2 = \frac{\sum(x_i - \bar{x})^2}{n}$

$= \frac{64}{9} = 7.11$

Standard deviation

$s = \sqrt{\frac{64}{9}}$

$= \sqrt{7.11}$

$= 2.67$ marks

Example 12 : Find the standard deviation from the following observations.

10, 17, 15, 21, 19, 23, 19, 25, 30, 26.

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
10	–10.5	110.25
17	–3.5	12.25
15	–5.5	30.25
21	0.5	0.25
19	–1.5	2.25
23	2.5	6.25
19	–1.5	2.25
25	4.5	20.25
30	9.5	90.25
26	5.5	30.25
$\Sigma x_i = 205$		$\Sigma(x_i - \bar{x})^2 = 304.50$

Mean $\bar{x} = \frac{\sum x_i}{n} = \frac{205}{10}$

$\bar{x} = 20.5$

Variance $s^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$

$= \frac{304.50}{10} = 30.45$

Standard deviation $= \sqrt{s^2}$

$s = \sqrt{30.45}$

$= 5.518$

In example 11 above, $\bar{x}$ is 67 and this made the calculation of standard deviation easy. But in example 12, $\bar{x}$ is 20.5 which is a not an integer and so the calculations are bit lengthy and numerically large. We have the following alternative formula for standard deviation.

$$\text{Standard deviation, } s = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2} = \sqrt{\frac{\sum x_i^2}{n} - (\bar{x})^2}$$

Let us prove the formula.

By definition,

$$s = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$$

$$= \sqrt{\frac{1}{n}\Sigma(x_i^2 - 2x_i\,\bar{x} + \bar{x}^2)}$$

$$= \sqrt{\frac{1}{n}\sum x_i^2 - 2\bar{x}\frac{\sum x_i}{n} + \frac{\bar{x}^2}{n}(1+1+\ldots+1\ n \text{ times})}$$

$$= \sqrt{\frac{1}{n}\sum x_i^2 - 2\bar{x}^2 + \bar{x}^2}$$

$$= \sqrt{\frac{1}{n}\sum x_i^2 - (\bar{x})^2}$$

$$\therefore \quad s = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2} \qquad (1)$$

While using this formula we need the sum Σx_i as well as Σx_i^2 of the observations. Let us solve the examples 11 and 12 again using this formula.

Example 13 : The marks obtained by 9 students in a test of 100 marks in mathematics are as follows 69, 67, 66, 69 64, 63, 68, 65, 72. Find the standard deviation for the data.

Solution :

x_i	x_i^2
69	4761
67	4489
66	4356
69	4761
64	4096
63	3969
68	4624
65	4225
72	5184
$\Sigma x_i = 603$	$\Sigma x_i^2 = 40,465$

$$s = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$= \sqrt{\frac{40465}{9} - \left(\frac{603}{9}\right)^2}$$

$$= \sqrt{4496.11 - 4489}$$

$$= \sqrt{7.11}$$

$$\therefore \quad s = 2.67$$

Example 14 : Find the standard deviation from the following observations.

10, 17, 15, 21, 19, 23, 19, 25, 30, 26

Solution :

x_i	x_i^2
10	100
17	289
15	225
21	441
19	361
23	529
19	361
25	625
30	900
26	676
$\Sigma x_i = 205$	$\Sigma x_i^2 = 4507$

$$s = \sqrt{\frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n}\right)^2}$$

$$= \sqrt{\frac{4507}{10} - \left(\frac{205}{10}\right)^2}$$

$$= \sqrt{450.7 - 420.25}$$

$$= \sqrt{30.45}$$

$$\therefore \quad s = 5.518$$

We can see that even in alternative formula for s, finding s by this formula is tedious as x_i^2 may be relatively large. The first method is also time consuming if the mean is not an integer. In that case we take deviation from an appropriately chosen number A, called assumed mean. Let us find the formula for s for this shortcut method.

Let $x_1, x_2, x_3, ..., x_n$ be n observations. Let $\bar{x}$ be their mean. Let d_i be the deviation of x_i from A.

If $d_i = x_i - A$, then $x_i = d_i + A$ (i)

$\therefore \quad \Sigma d_i = \Sigma(x_i - A)$

$\therefore \quad \Sigma d_i = \Sigma x_i - (A + A + ... \, n \text{ times})$

$\therefore \quad \Sigma d_i = \Sigma x_i - A \cdot n$

$$\therefore \quad \frac{\sum d_i}{n} = \frac{\sum x_i}{n} - A$$

$$\therefore \quad \bar{d} = \bar{x} - A, \text{ where } \bar{d} = \frac{\sum d_i}{n}$$

Then $\bar{x} = \bar{d} + A$ (ii)

From (i) and (ii)

$$x_i - \bar{x} = d_i + A - \bar{d} - A$$

$$\therefore \quad x_i - \bar{x} = d_i - \bar{d}$$

$$\therefore \quad s = \sqrt{\frac{\Sigma(x_i - \bar{x})^2}{n}}$$

$$= \sqrt{\frac{\Sigma(d_i - \bar{d})^2}{n}}$$

$$= \sqrt{\frac{1}{n}\Sigma(d_i^2 - 2d_i\bar{d} + \bar{d}^2}$$

$$= \sqrt{\frac{1}{n}(\Sigma d_i^2 - 2\bar{d}\,\Sigma d_i + \bar{d}^2\,\Sigma 1)}$$

$$= \sqrt{\frac{1}{n}(\Sigma d_i^2 - 2\bar{d}\,\Sigma d_i + \bar{d}^2 \cdot n)}$$

$$= \sqrt{\frac{1}{n}\Sigma d_i^2 - 2\bar{d}\frac{\Sigma d_i}{n} + \bar{d}^2}$$

$$= \sqrt{\frac{1}{n}\Sigma d_i^2 - 2\bar{d}^2 + \bar{d}^2} \qquad \left(\frac{\Sigma d_i}{n} = \bar{d}\right)$$

$$= \sqrt{\frac{1}{n}\Sigma d_i^2 - \bar{d}^2}$$

$$\therefore \quad s = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} \qquad (2)$$

Now, let us again revisit the example 13 and 14.

☛ **Note** From formulae (1) and (2) for s, it follows that s remains same, if a constant is subtracted from all observations.

Example 15 : The marks obtained by 9 students in a test of 100 marks in mathematics are as follows : 69, 67, 66, 69, 64, 63, 68, 65, 72.

Find the standard deviation for the data.

Solution : Let A = 67

x_i	$d_i = x_i - A$	d_i^2
69	2	4
67	0	0
66	−1	1
69	2	4
64	−3	9
63	−4	16
68	1	1
65	−2	4
72	5	25
	$\Sigma d_i = 0$	$\Sigma d_i^2 = 64$

$$s = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

$$= \sqrt{\frac{64}{9} - \left(\frac{0}{9}\right)^2}$$

$$= \frac{8}{3}$$

$$\therefore \quad s = 2.67$$

Example 16 : Find the standard deviation from the following observations, 10, 17, 15, 21, 19, 23, 19, 25, 30, 26.

Solution : Let A = 20.

x_i	$d_i = x_i - A$	d_i^2
10	−10	100
17	−3	9
15	−5	25
21	1	1
19	−1	1
23	3	9
19	−1	1
25	5	25
30	10	100
26	6	36
	$\Sigma d_i = 5$	$\Sigma d_i^2 = 307$

$$s = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

$$= \sqrt{\left(\frac{307}{10}\right) - \left(\frac{5}{10}\right)^2}$$

$$= \sqrt{30.7 - 0.25}$$

$$= \sqrt{30.45}$$

$$\therefore \quad s = 5.518$$

Example 17 : The marks obtained by 10 students in a test are as follows : 65, 58, 68, 44, 48, 45, 60, 62, 60, 50.

Find the standard deviation of the data.

Solution : Here the short-cut method of assumed mean is more appropriate as observations are large. We take the assumed mean A = 55.

x_i	$d_i = x_i - A$	d_i^2
65	10	100
58	3	9
68	13	169
44	–11	121
48	–7	49
45	–10	100
60	5	25
62	7	49
60	5	25
50	–5	25
	$\Sigma d_i = 10$	$\Sigma d_i^2 = 672$

$$s = \sqrt{\frac{\sum d_i^2}{n} - \left(\frac{\sum d_i}{n}\right)^2}$$

$$= \sqrt{\left(\frac{672}{10}\right) - \left(\frac{10}{10}\right)^2}$$

$$= \sqrt{67.2 - 1}$$

$$= \sqrt{66.2}$$

$$\therefore \quad s = 8.136$$

Computation of Standard Deviation for Grouped Data :

(i) For Discrete Frequency Distribution :

Direct Method : Suppose the values of variable x of the discrete frequency distribution are $x_1, x_2, ..., x_k$ and their corresponding frequencies are $f_1, f_2, ..., f_k$ respectively. Then for the given frequency distribution,

Standard deviation $s = \sqrt{\dfrac{\sum f_i (x_i - \bar{x})^2}{n}}$ **(i)**

where, f_i = The frquency of x_i, $n = \Sigma f_i$

(1) First find the mean $\bar{x} = \dfrac{\sum f_i x_i}{n}$

(2) Find the deviation of x_i from the mean $\bar{x}$ as $x_i - \bar{x}$

(3) Calculate $(x_i - \bar{x})^2$

(4) Find $f_i (x_i - \bar{x})^2$ for each i and then $\Sigma f_i (x_i - \bar{x})^2$.

(5) Finally find s using the formula.

$$s = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{n}}$$

$$s^2 = \frac{\sum f_i x_i^2}{n} - \bar{x}^2$$

We can prove following formula as before.

$$s = \sqrt{\frac{\sum f_i x_i^2}{n} - (\bar{x})^2} \quad \text{(ii)}$$

If the values of variable x or frequencies f are relatively large the calculation of standard deviation using the above two formulae become quite tedious and time consuming. In such a case, we take deviations of the values of variable x from an arbitary number A, say assumed mean. If $d_i = x_i - A$ and $n = \Sigma f_i$, $i = 1, 2, ..., k$, then the above formula reduces to,

$$s = \sqrt{\frac{\sum f_i d_i^2}{n} - \left(\frac{\sum f_i d_i}{n}\right)^2}.$$

This can also be proved as earlier formulae.

Example 18 : Find the standard deviation for the following frequency distribution by direct method and by shortcut method.

x_i	6	7	8	9	10	11	12
f_i	3	6	9	13	8	5	4

Solution :

Direct Method :

x_i	f_i	$f_i x_i$	$x_i - \overline{x}$	$(x_i - \overline{x})^2$	$f_i (x_i - \overline{x})^2$
6	3	18	–3	9	27
7	6	42	–2	4	24
8	9	72	–1	1	9
9	13	117	0	0	0
10	8	80	1	1	8
11	5	55	2	4	20
12	4	48	3	9	36
	$n = 48$	$\Sigma f_i x_i = 432$			$\Sigma f_i (x_i - \overline{x})^2 = 124$

$$\overline{x} = \frac{\sum f_i x_i}{n} = \frac{432}{48} = 9$$

$$s = \sqrt{\frac{\sum f_i (x_i - \overline{x})^2}{n}}$$

$$= \sqrt{\frac{124}{48}} = \sqrt{2.58} = 1.6$$

$\therefore \quad s = 1.6$

Let us calculate the standard deviation by alternative formula given by,

$$s = \sqrt{\frac{\sum f_i x_i^2}{n} - (\overline{x})^2}$$

x_i	f_i	x_i^2	$f_i x_i^2$
6	3	36	108
7	6	49	294
8	9	64	576
9	13	81	1053
10	8	100	800
11	5	121	605
12	4	144	576
	$n = 48$		$\Sigma f_i x_i^2 = 4012$

$$s = \sqrt{\frac{\sum f_i x_i^2}{n} - (\bar{x})^2}$$

$$= \sqrt{\frac{4012}{48} - (9)^2}$$

$$= \sqrt{83.58 - 81}$$

$$= \sqrt{2.58} = 1.6$$

$\therefore \quad s = 1.6$

Shortcut Method : Let us take A = 10.

x_i	f_i	$d_i = x_i - A$	d_i^2	$f_i d_i$	$f_i d_i^2$
6	3	–4	16	–12	48
7	6	–3	9	–18	54
8	9	–2	4	–18	36
9	13	–1	1	–13	13
A = 10	8	0	0	0	0
11	5	1	1	5	5
12	4	2	4	8	16
	$n = 48$			$\Sigma f_i d_i = -48$	$\Sigma f_i d_i^2 = 172$

$$s = \sqrt{\frac{\sum f_i d_i^2}{n} - \left(\frac{\sum f_i d_i}{n}\right)^2}$$

$$= \sqrt{\frac{172}{48} - \left(\frac{-48}{48}\right)^2}$$

$$= \sqrt{3.58 - 1}$$

$$= \sqrt{2.58} = 1.6$$

$\therefore \quad s = 1.6$

Example 19 : Calculate the standard deviation from the data given below :

Size of Item	3.5	4.5	5.5	6.5	7.5	8.5	9.5
Frequency	3	7	22	60	85	32	8

Solution : Let the assumed mean A = 6.5

x_i	f_i	$d_i = x_i - A$	d_i^2	$f_i d_i$	$f_i d_i^2$
3.5	3	–3	9	–9	27
4.5	7	–2	4	–14	28
5.5	22	–1	1	–22	22
6.5 = A	60	0	0	0	0
7.5	85	1	1	85	85
8.5	32	2	4	64	128
9.5	8	3	9	24	72
	$n = 217$			$\Sigma f_i d_i = 128$	$\Sigma f_i d_i^2 = 362$

$$s = \sqrt{\frac{\sum f_i d_i^2}{n} - \left(\frac{\sum f_i d_i}{n}\right)^2}$$

$$= \sqrt{\frac{362}{217} - \left(\frac{128}{217}\right)^2}$$

$$= \sqrt{1.668 - 0.347}$$

$$= \sqrt{1.321}$$

$$\therefore \quad s = 1.149$$

(ii) Calculation of Standard Deviation for Continuous Frequency Distribution :

Direct Method : Suppose $x_1, x_2, \ldots, x_k$ are the mid-values of n classes of a continuous frequency distribution and $f_1, f_2, \ldots, f_k$ are the respective frequencies of classes. Then the formula for computing the standard deviation of continuous frequency distribution is given by,

$$s = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{n}} = \sqrt{\frac{\sum f_i x_i^2}{n} - \bar{x}^2}$$

where x_i = mid-value of the ith class

f_i = frequency of the ith class

$\bar{x} = \dfrac{\sum f_i x_i}{n}$ = mean

$x_i - \bar{x}$ = deviation of mid-value x_i from the mean $\bar{x}$.

Shortcut Method : In this method for finding the standard deviation, when the continuous frequency distribution is given, we take any real number as assumed mean

A. But to make the calculation simple we take the central value of the class of maximum frequency or of the class close to the middle of the distribution as A. Let us derive the formula for s, when the assumed mean is A and class length is c.

Let $d_i = \dfrac{x_i - A}{c}$

$\therefore \quad x_i = cd_i + A \qquad (1)$

$\Sigma f_i x_i = \Sigma f_i(cd_i + A)$

$= c\,\Sigma f_i d_i + A\Sigma f_i$

$\Sigma f_i x_i = c\,\Sigma f_i d_i + A \cdot n \qquad (\Sigma f_i = n)$

Now, $\dfrac{\Sigma f_i x_i}{n} = c\,\dfrac{\Sigma f_i d_i}{n} + A$

Let $\dfrac{\Sigma f_i d_i}{n} = \bar{d}$

then $\bar{x} = c\bar{d} + A \qquad (2)$

From (1) and (2)

$x_i - \bar{x} = c\,(d_i - \bar{d})$

Now, $s^2 = \dfrac{\Sigma f_i (x_i - \bar{x})^2}{n}$

$= \dfrac{1}{n}\Sigma f_i\,(c\,(d_i - \bar{d}))^2$

$= \dfrac{\Sigma f_i (d_i - \bar{d})^2 \times c^2}{n}$

$s^2 = \left[\dfrac{\Sigma f_i d_i^2}{n} - \left(\dfrac{\Sigma f_i d_i}{n}\right)^2\right] \times c^2$ as before

$$s = \sqrt{\frac{\Sigma f_i d_i^2}{n} - \left(\frac{\Sigma f_i d_i}{n}\right)^2} \times c$$

where, c = class length, $d_i = \dfrac{x_i - A}{c}$, $\Sigma f_i = n$.

Let us understand the method of finding the standard deviation for continuous frequency distribution from the following example.

Example 20 : The age distribution of the workers of a factory is as shown below. Find the standard deviation of the age of the workers by shortcut method.

Age (in years)	15–20	20–25	25–30	30–35	35–40	40–45	45–50
Number of workers	5	12	10	8	2	2	1

Solution :

Mid-value of the class 25-30 is 27.5. Let A = 27.5 and c = 5.

Class	x_i	f_i	$d_i = \frac{x_i - A}{c}$	$f_i d_i$	$f_i d_i^2$
15–20	17.5	5	–2	–10	20
20–25	22.5	12	–1	–12	12
25–30	27.5 = A	10	0	0	0
30–35	32.5	8	1	8	8
35–40	37.5	2	2	4	8
40–45	42.5	2	3	6	18
45–50	47.5	1	4	4	16
		n = 40		$\Sigma f_i d_i = 0$	$\Sigma f_i d_i^2 = 82$

$$s = \sqrt{\frac{\sum f_i d_i^2}{n} - \left(\frac{\sum f_i d_i}{n}\right)^2} \times c$$

$$= \sqrt{\frac{82}{40} - \left(\frac{0}{40}\right)^2} \times 5 = \sqrt{2.05} \times 5$$

$\therefore \quad s$ = 7.16 years

Example 21 : Find the mean and the standard deviation for the following data :

Wages in ₹	0-15	15-30	30-45	45-60	60-75	75-90	90-105	105-120
Number of workers	12	18	35	42	50	45	20	8

Solution :

Class	x_i	f_i	$d_i = \frac{x_i - A}{c}$	$f_i d_i$	$f_i d_i^2$
0-15	7.5	12	–4	–48	192
15-30	22.5	18	–3	–54	162
30-45	37.5	35	–2	–70	140
45-60	52.5	42	–1	–42	42
60-75	A = 67.5	50	0	0	0
75-90	82.5	45	1	45	45
90-105	97.5	20	2	40	80
105-120	112.5	8	3	24	72
		n = 230		$\Sigma f_i d_i = -105$	$\Sigma f_i d_i^2 = 733$

Here we take A = 67.5, which is the mid-value of the class 60-75. The class length c = 15.

$$\bar{x} = A + \frac{\sum f_i d_i}{n} \times c$$

$$= 67.5 + \frac{-105}{230} \times 15 = 67.5 - 6.85 = 60.65$$

$$s = \sqrt{\frac{\sum f_i d_i^2}{n} - \left(\frac{\sum f_i d_i}{n}\right)^2} \times c$$

$$= \sqrt{\frac{733}{230} - \left(\frac{-105}{230}\right)^2} \times 15$$

$$= \sqrt{3.18 - 0.208} \times 15$$

$$= \sqrt{2.972} \times 15$$

$\therefore$ s = 25.86

EXERCISE 9.2

1. Find the standard deviation for the following data :
   (1) 6, 7, 10, 12, 13, 4, 8, 12
   (2) 38, 70, 48, 34, 42, 55, 63, 46, 54, 44
   (3) 20, 24, 23, 26, 19, 25, 26, 18, 20, 21, 16, 27
2. Find the standard deviation for the following data :

(1)

x_i	10	11	12	13	14	15	16	17	18
f_i	3	10	20	17	22	16	13	9	5

(2)

x_i	6	10	14	18	24	28	30
f_i	2	4	7	12	8	4	3

3. Find the mean and the standard deviation for the following data :

(1)

x_i	2	4	6	8	10	12	14	16
f_i	4	4	5	15	8	5	4	5

(2)

x_i	4	8	12	16	20	24
f_i	6	14	4	8	11	7

4. Find the mean and the standard deviation for the following data by short-cut method :

x_i	60	61	62	63	64	65	66	67	68
f_i	2	1	12	29	25	12	10	4	5

5. Find the mean, the variance and the standard deviation for the following frequency distribution :

Class	0–10	10–20	20–30	30–40	40–50
Frequency	5	8	15	16	6

6. Find the mean and the standard deviation for the following by short-cut method :

(1)

Class	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency	3	7	12	15	8	3	2

(2)

Height (cms)	70–75	75–80	80–85	85–90	90–95	95–100	100–105	105–110	110–115
Number of students	3	4	7	7	15	9	6	6	3

(3)

Class	0–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40	40–45
Frequency	20	24	32	28	20	11	26	15	24

7. The following table gives the distribution of income of 100 families. Calculate the standard deviation.

Income (₹)	0–1000	1000–2000	2000–3000	3000–4000	4000–5000	5000–6000
Number of families	18	26	30	12	10	4

8. The measurements of the diameters (in mm) of the heads of 107 screws are given below. Find the standard deviation.

Diameter (mm)	33–35	36–38	39–41	42–44	45–47
Number of screws	17	19	23	21	27

*

9.7 Co-efficient of Variation

In this chapter we have seen average deviation and standard deviation as the measures of dispersion. The measures of dispersion are generally used to compare the dispersion of two or more groups of data. More appropriate comparison of the dispersion of two or more groups of data can be made by the co-efficient of variation suggested by Karl Pearson. The co-efficient of variation is the relative measure of dispersion depending upon standard deviation. It is defined as the quotient obtained when the standard deviation is divided by the mean and it is denoted by C.V.

Thus, C.V. $= \frac{s}{\bar{x}}$

The co-efficient of variation is usually expressed in percentage.

Hence, C.V. = $\frac{s}{\bar{x}} \times 100$

Here s and $\bar{x}$ are the standard deviation and the mean of the data respectively. It is clear that the co-efficient of variation depends on the standard deviation as well as on the mean. Smaller the variation in observations and larger the mean, smaller will be the co-efficient of variation. Hence, it is a better measure of comparison of data. A sequence of observations whose co-efficient of variation is small is said to have less dispersion or is said to be more stable. Such a sequence is also said to be consistent from the point of view of variability. A sequence of observations whose co-efficient of variation is large is said to have more dispersion or is said to be less stable from the variation point of view.

Comparison of Two Frequency Distributions with the Same Mean :

Let $\bar{x}_1$ and s_1 be the mean and the standard deviation of the first frequency distribution and $\bar{x}_2$ and s_2 be the mean and the standard deviation of the second frequency distribution respectively.

If $\bar{x}_1 = \bar{x}_2 = \bar{x}$

Then C.V. (1st distribution) = $\frac{s_1}{\bar{x}_1} \times 100 = \frac{s_1}{\bar{x}} \times 100$ **(1)**

C.V. (2nd distribution) = $\frac{s_2}{\bar{x}_2} \times 100 = \frac{s_2}{\bar{x}} \times 100$ **(2)**

It is clear from (1) and (2) that two C.V. can be compared on the basis of values of s_1 and s_2 only.

Example 22 : On analysis of monthly wages paid to the workers of two firms A and B belonging to the same industry, we get following :

	A	**B**
Number of workers	1000	1200
Average monthly wages	₹ 3000	₹ 3000
Variance of distribution of wages	81	100

In which firm A or B is the variation in the individual wages greater ?

Solution : The variance of distribution of wages in the firm A, $s^2 = 81$. Therefore, standard deviation of the wages in the firm A is 9.

Also, the variance of distribution of wages in the firm B is 100, therefore, standard deviation of wages in the firm B is 10. Since the average monthly wages that is the mean in both the firms is same namely $\bar{x} = 3000$, so the firm with the greater standard deviation will have more variation. Thus the firm B has greater variation in the individual wages.

Example 23 : An analysis of monthly wages paid to workers in two plants A and B of a factory, gives the following results :

	A	B
No. of workers	650	550
Average monthly wages	₹ 5250	₹ 5250
Variance of distribution of wages	121	100

(1) Which plant A or B pays large amount as monthly wages ?

(2) Which plant A or B, shows greater variation in individual wages ?

Solution :

(1) Plant A : Number of workers $n_1 = 650$

Mean of monthly wages $\bar{x}$ = ₹ 5250

$$\bar{x} = \frac{\sum x_i}{n_1}$$

$$\therefore \quad 5250 = \frac{\sum x_i}{650}$$

$$\therefore \quad \Sigma x_i = 5250 \times 650$$

Total monthly wages = ₹ 34,12,500

Plant B : Number of workers $n_2 = 550$

Mean of monthly wages $\bar{y}$ = ₹ 5250

$$\bar{y} = \frac{\sum y_i}{n_2}$$

$$\therefore \quad 5250 = \frac{\sum y_i}{550}$$

$$\therefore \quad \Sigma y_i = 5250 \times 550$$

Total monthly wages = ₹ 28,87,500

Clearly plant A pays larger amount as monthly wages.

(2) Since plants A and B have the same mean, the plant with greater variance will have more variation in individual wages.

Hence, plant A has greater variation in individual wages.

Example 24 : The runs scored by two cricket players A and B in three test matches are as given below :

Player A	60	45	5	105	45	25
Player B	100	25	35	25	70	45

Which of them is a better player ?

Solution : We shall find the co-efficient of variation for players A and B.

Player A :

x_i	x_i^2
60	3600
45	2025
5	25
105	11025
45	2025
25	625
$\Sigma x_i = 285$	$\Sigma x_i^2 = 19325$

$\bar{x} = \frac{\sum x_i}{n} = \frac{285}{6}$

$\bar{x} = 47.5$

$s = \sqrt{\frac{\sum x_i^2}{n} - (\bar{x})^2}$

$= \sqrt{\frac{19325}{6} - (47.5)^2}$

$= \sqrt{3220.83 - 2256.25}$

$= \sqrt{964.58}$

$= 31.058$

For A; C.V. $= \frac{s}{\bar{x}} \times 100$

$= \frac{31.058}{47.5} \times 100$

$= 65.385\ \%$

Player B :

y_i	y_i^2
100	10000
25	625
35	1225
25	625
70	4900
45	2025
$\Sigma y_i = 300$	$\Sigma y_i^2 = 19400$

$\bar{y} = \frac{\sum y_i}{n} = \frac{300}{6}$

$\bar{y} = 50$

$s = \sqrt{\frac{\sum y_i^2}{n} - (\bar{y})^2}$

$= \sqrt{\frac{19400}{6} - (50)^2}$

$= \sqrt{3233.33 - 2500}$

$= \sqrt{733.33}$

$= 27.08$

For B; C.V. $= \frac{s}{\bar{y}} \times 100$

$= \frac{27.08}{50} \times 100$

$= 54.16\ \%$

$\therefore$ C.V. of B < C.V. of A

$\therefore$ The performance of B could be rated better from consistency point of view.

EXERCISE 9.3

1. Ten students took 80, 89, 69, 74, 91, 96, 71, 86, 75 and 81 seconds to run 400 metre race. Find the co-efficient of variation of this data.

2. The runs scored by two cricket players A and B in 5 innings are as shown below. Decide which player performed better.

A	39	56	47	43	50
B	34	75	63	38	40

3. The means and standard deviations of heights and weights of 50 students of a class are as follows :

	Weight	**Height**
Mean	63.2 kg	63.2 inch
Standard deviation	5.6 kg	11.3 inch

Which shows more variation, heights or weights ?

4. The following values are calculated with respect of heights and weights of the students of a section of class XII.

	Height	**Weight**
Mean	165 *cm*	52.50 kg
Variance	132.25 cm^2	23.04 kg^2

Can we say that the weights show greater variation than the heights ?

5. Price fluctuations of two shares A and B are given in the following table. Which type of share has more variation in its price ?

A	318	319	316	323	320	324	322	325	322	321
B	152	132	134	132	145	142	146	130	146	141

6. The sum of values and the sum of squares of values corresponding to length x (in *cm*) and weight y (in *gm*) of 50 plant products are given below :

$\Sigma x_i = 212$, $\Sigma x_i^2 = 902.8$, $\Sigma y_i = 261$, $\Sigma y_i^2 = 1457.6$.

Which has more variation, length or weight ?

7. The mean and standard deviation of marks obtained by 50 students of a class in three subjects, Mathematics, Physics and Chemistry are given below.

Subject	**Mathematics**	**Physics**	**Chemistry**
Mean	40	30	40
Standard Deviation	10	15	22

Which of the three subjects shows the highest variation in marks and which shows the lowest variation ?

8. From the data given below state which group has more variations G_1 or G_2 ?

Marks	10-20	20-30	30-40	40-50	50-60	60-70	70-80
G_1	9	17	32	33	40	10	9
G_2	10	20	30	25	43	15	7

*

Miscellaneous Examples

Example 25 : If the mean and the standard deviation of $x_1, x_2, x_3, ..., x_n$ are $\bar{x}$ and s, then find the mean and standard deviation of $ax_1 + b, ax_2 + b, ..., ax_n + b$.

Solution : Let observations be x_i and $y_i = ax_i + b$, where $i = 1, 2, 3, ..., n$

$$y_i = ax_i + b$$

$$\therefore \quad \Sigma y_i = a\Sigma x_i + (b + b + ... n \text{ times})$$

$$= a\Sigma x_i + bn$$

$$\therefore \quad \frac{\sum y_i}{n} = a\frac{\sum x_i}{n} + b$$

$$\therefore \quad \bar{y} = a\bar{x} + b$$

$\therefore$ The mean of $ax_1 + b, ax_2 + b, ..., ax_n + b$ is $a\bar{x} + b$.

Now, let the standard deviation of $ax_1 + b, ax_2 + b, ..., ax_n + b$ be s'.

Now $y_i - \bar{y} = (ax_i + b) - (a\bar{x} + b)$

$$= a(x_i - \bar{x})$$

$$s' = \sqrt{\frac{\Sigma(y_i - \bar{y})^2}{n}}$$

$$= \sqrt{\frac{\Sigma(a^2)(x_i - \bar{x})^2}{n}}$$

$$= |a| \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}$$

$$s' = |a|\, s$$

$\therefore$ The standard deviation of $ax_1 + b, ax_2 + b, ..., ax_n + b$ is $|a|\, s$ and variance is a^2s^2.

Example 26 : The mean and the standard deviation of 20 observations are 10 and 2 respectively. On checking, it was found that one observation with value 8 is introduced by error. (1) Find the new mean and standard deviation by removing the incorrect observation (2) Find the new mean and standard deviation if the incorrect observation is replaced by an observation with value 12.

Solution : We have $n = 20$, $\bar{x} = 10$ and $s = 2$

$$\therefore \quad \bar{x} = \frac{\sum x_i}{n}$$

$$\Sigma x_i = n \cdot \bar{x} = 20 \times 10 = 200 \qquad (1)$$

$$s^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2$$

$4 = \frac{\Sigma x_i^2}{20} - 100$

$\Sigma x_i^2 = 104 \times 20 = 2080$ (2)

$\therefore \quad \Sigma x_i = 200$ and $\Sigma x_i^2 = 2080$

(1) If 8 is removed from the data, then 19 observations are remain.

Now corrected $\Sigma x_i = 200 - 8 = 192$

and corrected $\Sigma x_i^2 = 2080 - 8^2 = 2080 - 64 = 2016$

Corrected mean $= \frac{192}{19} = 10.105$

$$\text{Corrected variance} = \frac{\text{corrected } \Sigma x_i^2}{19} - (\text{corrected mean})^2$$

$$= \frac{2016}{19} - (10.105)^2$$

$$= 106.105 - 102.11 = 3.994$$

Corrected standard deviation $= \sqrt{3.994} = 1.998$

(2) When the incorrect observaiton 8 is replaced by 12.

Corrected $\Sigma x_i = 200 - 8 + 12 = 204$

Corrected $\Sigma x_i^2 = 2080 - 8^2 + 12^2 = 2160$

Now,

Corrected mean $= \frac{204}{20} = 10.2$

$$\text{Corrected variance} = \frac{\text{corrected } \Sigma x_i^2}{20} - (\text{corrected mean})^2$$

$$= \frac{2160}{20} - (10.2)^2$$

$$= 108 - 104.04 = 3.96$$

Corrected standard deviation $= \sqrt{3.96} = 1.99$

EXERCISE 9

1. The mean of five observations is 4.4 and their variance is 8.24. Three of the given observations are 1, 2 and 6. Find the remaining two observations.
2. The mean and the variance of 8 observations are 9 and 9.25 respectively. Six of the given observations are 6, 7, 10, 12, 12 and 13. Find the remaining two observations.

3. The variance of n observations $x_1, x_2, x_3,..., x_n$ is s^2. If the observations are changed to $x_1 + a, x_2 + a,..., x_n + a$, where $a \in$ R show that variance remains unchanged.

4. The mean and the standard deviation of 6 observations are 8 and 4 respectively. If each of the observation is multiplied by 3, find the mean and the standard deviation of the new observations.

5. For a group of 200 candidates the mean and the standard deviation were found to be 40 and 15 respectively. Later on it was found that the score 43 was misread as 34. Find the correct mean and the correct standard deviation.

6. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) In an experiment with 15 observations, the following results were available : $\Sigma x_i^2 = 2830$, $\Sigma x_i = 170$. One observation 20 was found to be wrong and was replaced by correct value 30. Thus the correct variance is.... ☐

(A) 188.66 (B) 177.33 (C) 8.33 (D) 78.00

(2) The median of a set of 9 distinct observations is 20.5. If each of the last 4 observations of the set is increased by 2, then the median of the new set ☐

(A) is decreased by 2

(B) is two times the original median

(C) remains the same as that of the original set

(D) is increased by 2

(3) Suppose a population A has 100 observations 101, 102,..., 200 and another pupulation B has 100 observations 151, 152,..., 250. If V_A and V_B represent the variances of the two populations respectively, then $\frac{V_A}{V_B}$ is... ☐

(A) 1 (B) $\frac{9}{4}$ (C) $\frac{4}{9}$ (D) $\frac{2}{3}$

(4) The mean and the S.D. of the marks of 200 candidates were found to be 40 and 15 respectively. Later, it was discovered that a score 40 was read as 50. The correct mean and the correct S.D. are respectively... ☐

(A) 14.98, 39.95 (B) 39.95, 14.98 (C) 39.95, 224.5 (D) 224.5, 39.95

(5) If S.D. of a variate x is 4 and $y = \frac{3x+7}{4}$, then S.D. of y is... ☐

(A) 4 (B) 3.5 (C) 3 (D) 2.5

(6) The variance of the data 3, 4, 5, 8 is... ☐

(A) 4.5 (B) 3.5 (C) 5.5 (D) 6.5

(7) The mean deviation of the data 3, 10, 4, 10, 7, 10, 5 from the mean is ... ☐

(A) 2 (B) 2.57 (C) 3 (D) 3.75

(8) The marks obtained by 9 students in a mathematics test are :

50, 69, 20, 33, 39, 53, 65, 40, 59.

The mean deviation from the median is... ☐

(A) 9 (B) 10.5 (C) 12.67 (D) 14.76

(9) The standard deviation of the data 6, 9, 5, 12, 13, 8, 10 is... ☐

(A) $\sqrt{\frac{52}{7}}$ (B) $\frac{52}{7}$ (C) $\sqrt{6}$ (D) 6

(10) The mean of 100 observations is 50 and their S.D. is 5. The sum of squares of all the observations is... ☐

(A) 50,000 (B) 2,50,000 (C) 2,52,500 (D) 2,55,000

(11) Let a, b, c, d, e be the observations with mean $\bar{x}$ and S.D. s. The standard deviation of the observations $a + m, b + m, c + m, d + m, e + m$ is... ☐

(A) s (B) ms (C) $s + m$ (D) $\frac{s}{m}$

(12) Let a, b, c, d, e be the observations with mean $\bar{x}$ and S.D. s. The standard deviation of the observations ma, mb, mc, md, me is... ☐

(A) $m + s$ (B) $\frac{s}{m}$ (C) s (D) ms

(13) Consider the numbers 1, 2, 3, 4,..., 10. If 1 is added to each number, the variance of the numbers so obtained is... ☐

(A) 6.5 (B) 2.87 (C) 8.25 (D) 3.57

(14) If the S.D. of $x_1, x_2, \ldots, x_n$ is 3.5, then the S.D. of $-2x_1 - 3, -2x_2 - 3, \ldots, -2x_n - 3$ is... ☐

(A) -7 (B) 10 (C) 7 (D) -10

(15) Consider the first 10 positive integers. If we multiply each number by -1 and then add 1 to each number, the variance of the numbers so obtained is... ☐

(A) 8.25 (B) 6.5 (C) 3.87 (D) 2.87

(16) In a series of $2n$ observations, half of them are equal a and remaining half are equal $-a$. If the standard deviation of the observations is 2, then $|a|$ equals. ☐

(A) 2 (B) $\sqrt{2}$ (C) $\frac{1}{n}$ (D) $\frac{\sqrt{2}}{n}$

Summary

1. Range, Average (mean) deviation, variance, standard deviation are measures of dispersion.
2. Range = Maximum value – Minimum value.
3. Average deviation about the mean for ungrouped data.

$$\delta\bar{x} = \frac{\sum |x_i - \bar{x}|}{n}, \text{ where } \bar{x} \text{ is the mean}$$

4. Average deviation about the median for ungrouped data.

$$\delta M = \frac{\sum |x_i - M|}{n}, \text{ where M is the median}$$

5. Average deviation about the mean for grouped data.

$$\delta\bar{x} = \frac{\sum f_i |x_i - \bar{x}|}{n}, \text{ where } n = \Sigma f_i$$

6. Average deviation about the median for grouped data.

$$\delta M = \frac{\sum f_i |x_i - M|}{n}, \text{ where } n = \Sigma f_i$$

7. Variance and standard deviation for ungrouped data

$$s^2 = \frac{\Sigma (x_i - \bar{x})^2}{n}, \qquad s = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{n}}$$

$$s^2 = \frac{\Sigma x_i^2}{n} - (\bar{x})^2, \qquad s = \sqrt{\frac{\Sigma x_i^2}{n} - (\bar{x})^2}, \text{ where } \bar{x} = \frac{\Sigma x_i}{n}$$

8. Short-cut method to find variance and standard deviation for ungrouped data.

$$s^2 = \frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2 \qquad s = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2}$$

where $d_i = x_i - A$, A is the assumed mean.

9. Variance and standard deviation of discrete frequency distribution

$$s^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{n} \qquad s = \sqrt{\frac{\Sigma f_i (x_i - \bar{x})^2}{n}}$$

$$s^2 = \frac{\Sigma f_i x_i^2}{n} - (\bar{x})^2 \qquad s = \sqrt{\frac{\Sigma f_i x_i^2}{n} - (\bar{x})^2}$$

where $d_i = x_i - A$, A is the assumed mean.

10. Shortcut method to find variance and standard deviation of discrete frequency distribution

$$s^2 = \frac{\Sigma f_i d_i^2}{n} - \left(\frac{\Sigma f_i d_i}{n}\right)^2 \qquad s = \sqrt{\frac{\Sigma f_i d_i^2}{n} - \left(\frac{\Sigma f_i d_i}{n}\right)^2}$$

where $d_i = x_i - A$, A is the assumed mean.

11. Variance and standard deviation of a continuous frequency distribution.

$$s^2 = \frac{\sum f_i (x_i - \bar{x})^2}{n} \qquad s = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{n}}$$

$$s^2 = \frac{\sum f_i x_i^2}{n} - \left(\frac{\sum f_i x_i}{n}\right)^2 \qquad s = \sqrt{\frac{\sum f_i x_i^2}{n} - (\bar{x})^2}$$

12. Shortcut method to find standard deviation of a continuous frequency distribution.

$$s = c\sqrt{\frac{\sum f_i (d_i - \bar{d})^2}{n}}, \quad d_i = \frac{x_i - A}{c}, \quad \bar{d} = \frac{\sum f_i d_i}{n}$$

$$= c\sqrt{\frac{\sum f_i d_i^2}{n} - (\bar{d})^2}$$

13. Coefficient of variation

$$\text{C.V.} = \frac{s}{\bar{x}}$$

Historical Note

The word 'Statistics' is derived from the Latin word 'status' which means a political state. This suggests that statistics is as old as human civilisation. In the year 3050 B.C., perhaps the first census was held in Egypt. In India also, about 2000 years ago, we had an efficient system of collecting administrative statistics, particularly, during the regime of Chandra Gupta Maurya (324-300 B.C.). The system of collecting data related to births and deaths is mentioned in Kautilya's *Arthashastra* (around 300 B.C.) A detailed account of administrative surveys conducted during Akbar's regime is given in *Ain-I-Akbari* written by Abul Fazl.

Captain John Graunt of London (1620-1674) is known as father of vital statistics due to his studies on statistics of births and deaths. Jacob Bernoulli (1654-1705) stated the Law of Large numbers in his book "Ars Conjectandi", published in 1713.

The theoretical development of statistics came during the mid seventeenth century and continued after that with the introduction of theory of games and chance (i.e., probability). Francis Galton (1822-1921), an Englishman, pioneered the use of statistical methods, in the field of Biometry. Karl Pearson (1857-1936) contributed a lot to the development of statistical studies with his discovery of *Chi square test* and foundation of *statistical laboratory* in England (1911). Sir Ronald A. Fisher (1890-1962), known as the Father of modern statistics, applied it to various diversified fields such as Genetics, Biometry, Education, Agriculture, etc.

Chapter 10

PROBABILITY

10.1 Introduction

The words probable, possible chance etc. are quite familiar to us. We use these words when we are not sure of the result of certain events. These words convey the sense of uncertainty of occurrence of events.

Probability is the word we use calculating the degree of the certainty of events in ideal conditions. An experiment means an operation which can produce some well defined outcomes. There are mainly two approaches in defining in the theory of the probability of the events of a random experiment (1) Classical approach (2) Axiomatic approach. The classical approach is given by **Blaise Pascal** and the axiomatic approach is given by a Russian mathematician **A. Kolmogorov** in 1937. We will discuss these two approaches in this chapter.

10.2 Random Experiment

Definition : An experiment, in which we know all the possible outcomes in advance but which of them will occur is known only after the experiment is performed, is called a random experiment.

A coin has two sides, one side we know as head denoted by H and the other side as tail denoted by T. In advance we can not predict whether head will occur or tail will occur when the coin is tossed. So in the experiment of tossing of a coin, we know the possible outcomes but one cannot foretell the outcomes before tossing.

Consider another example. The selection of a card from a pack of 52 cards. Nobody can predict which card will be selected. But all possible outcomes of selection are known before the selection.

There are six faces of a die. They are marked with integers 1, 2, 3, 4, 5, 6. When the die is tossed any one of these integers can appear on the upper face as an outcome of the experiment. We can't say which of the digit will occur in advance. So these type of experiments are known as random experiments.

10.3 Sample Space

Sample Space : The set of all possible outcomes of a random experiment is called a sample space.

The sample space is denoted by U. The possible outcomes in the experiment of tossing a coin are H and T. So, the sample space associated with the experiment of tossing a coin is U = {H, T} Similarly, the possible outcomes in the experiment of tossing a die are 1, 2, 3, 4, 5, 6. Hence the sample space is U = {1, 2, 3, 4, 5, 6}

A sample space can be (1) A finite sample space or (2) An infinite sample space.

If a sample space is in one-one correspondence with a finite set, $\{x \in N \mid 1 \le x \le n, n \in N\}$, then it is called a finite sample space.

A sample space which is not finite is called an infinite sample space. Each element of the sample space is called a sample point. Let $a_1, a_2, a_3, ..., a_n$ be all the outcomes of a random experiment. Then $a_1, a_2, ... a_n$ are called sample points and sample space $U = \{a_1, a_2, a_3, ..., a_n\}$.

Example 1 : Find the sample space associated with the experiment of tossing of two coins simultaneously.

Solution : As we know the two possible outcomes of tossing a balanced coin are head H or tail T.

If we get head (H) on first coin and head on second coin, the result is denoted by (H, H). Similarly if we get H on first coin and T on second coin, the result is denoted by (H, T). Similarly considering other results, the sample space is

U = {H, T} × {H, T},

U = {(H, H), (H, T), (T, H), (T, T)}.

Here sample points are ordered pairs formed by H and T. For simplicity we will write HT for (H, T) etc.

∴ U = {HH, HT, TH, TT}

Example 2 : Write the sample space associated with the experiment of tossing of two dice simultaneously.

Solution : Consider the experiment of tossing a die. There are six possible outcomes of this experiment namely 1, 2, 3, 4, 5, 6.

Thus, the sample space associated with the random experiment of throwing a die is S = {1, 2, 3, 4, 5, 6}.

Here two dice are tossed once. So the sample space is

$$U = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

$$= \{(1, 1), (1, 2), ..., (1, 6), (2, 1), (2, 2), ..., (2, 6), (3, 1), (3, 2), ... (3, 6), ..., (6, 1), (6, 2), ..., (6, 6)\}$$

Here (x, y) denotes the result x on the first die and result y on the second die.

$$\therefore \quad U = \{(x, y) \mid x = 1, 2, 3, 4, 5, 6; y = 1, 2, 3, 4, 5, 6\}$$

Example 3 : From a group of 2 boys and 3 girls, two children are selected at random. Write the sample space of this experiment.

Solution : Let us denote the two boys as B_1, B_2 and 3 girls as G_1, G_2, G_3. Out of the five children, two children are selected.

So the sample space is

$U = \{B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3, G_2G_3\}$

The following tree diagram can help us to understand about outcomes of the sample space.

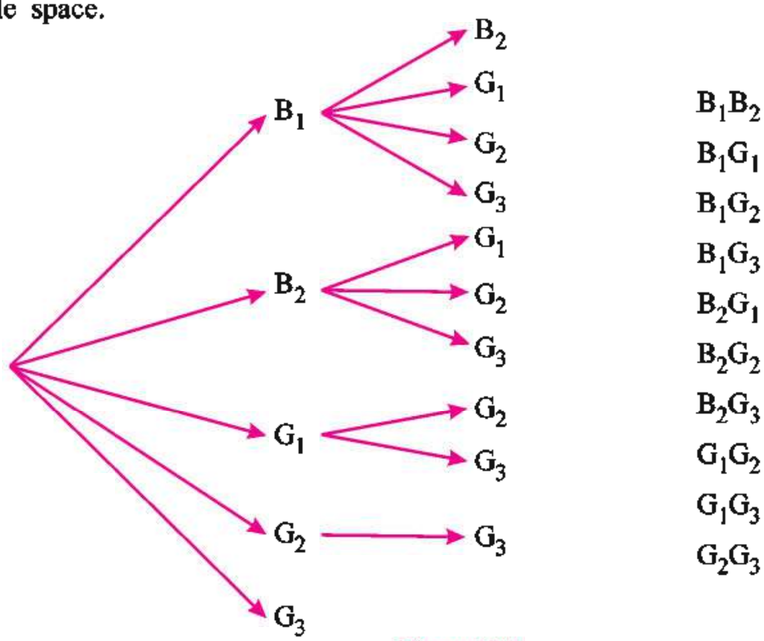


Figure 10.1

> **☛ Note** Here there is selection. Order is not important. Hence B_1B_2 is same as B_2B_1.

Example 4 : Consider an experiment in which a coin is tossed repeatedly until a head comes up for the first time. Describe the sample space.

Solution : In this experiment you may get H on the first trial or T on the first trial and H on the second trial or T on the first two trials and H on the third trial and we continue the experiment till we get H. So sample space U = {H, TH, TTH, TTTH, TTTTH, ...}

Here TH means T on the first trial and H on the second trial. This experiment is continued till we get H and hence tails precede H till we get H on *n*th trial.

> **☛ Note** This is an example of an infinite sample space. We shall be studying problems with finite sample spaces only.

EXERCISE 10.1

1. Let a coin be tossed. If it shows head we draw a ball from a box containing 3 identical red and 4 identical green balls and if it shows a tail, we throw a die. What is the sample space of this experiment ?

2. A box contains 2 identical red and 3 identical white balls. Two balls are drawn at random simultaneously. Write the sample space for this experiment.

3. Describe the sample space for the indicated random experiments.

1. A coin is tossed three times.
2. A coin and die are tossed simultaneously.

4. An experiment consists of tossing a coin and then tossing it second time if a head occurs. If a tail occurs on the first toss, then a die is tossed once. Find the sample space for this experiment.

5. An experiment consists of rolling a die and then tossing a coin once if the number on the die is odd. If the number on the die is even the coin is tossed twice. Write the sample space for the experiment.

6. A box contains 3 identical red balls, 2 identical white balls and 1 black ball. The experiment consists of drawing one ball from the box and then putting it back into the box and again drawing a ball. Write the sample space associated with this experiment.

7. Four cards are labelled with A, B, C and D. We select any two cards at random without replacement. Describe the sample space for the experiment.

8. Three distinct balls are to be distributed in two cells. Write the sample space associated with the experiment.

9. A balanced coin is tossed thrice. If three tails are obtained, a balance die is tossed. Otherwise the experiment is terminated. Write down the elements of the sample space.

10. One die is marked with letters a,b,c,d,e,f and other is marked with integers 1,2,3,4,5,6 on its six faces. Write the sample space associated with the random experiment of tossing two dice simultaneously.

*

10.4 Event

Event : A subset of a sample space is called an event.

We have studied the sample space associated with a random experiment. Consider the sample space as a universal set. Subsets of U are called events.

The set of all subsets of the sample space U is the power set of U. It is denoted by P(U). It is clear that the elements of P(U) are events.

U = {HH, HT, TH, TT} is the sample space of the experiment associated with tossing of two coins once. A = {TT} is a subset of U. So A is the event of getting T on both the coins.

B = {HT, TH, TT} is the event of getting at least one T.

Now we shall study different events with the help of set operations.

Elementary Event : (Simple Event) Let U be a finite sample space. U = {x_1, x_2,..., x_n}. The singleton {x_i}, (i = 1, 2, 3,..., n) of U are called elementary events. An elementary event is also known as a simple event.

For example in the experiment of tossing of two coins once, sample space is U = {HH, HT, TH, TT}. Here {HH}, {HT}, {TH} and {TT} are the elementary events corresponding to this sample space.

Impossible event : The subset $\emptyset$ of the sample space U is called the impossible event.

Sure event or certain Event : The subset U of the sample space U is called the certain event.

Compound event : If an event has more than one element (sample point), it is called a compound event. It is also known as decomposable event.

For example in the experiment of tossing of two coins once,

U = {HH, TH, HT, TT}

The subsets of U, A = {TH, HT} and B = {TH, HT, TT} are compound events.

Complementary Event : A $\in$ P(U). The set consisting of all elements of the sample space U other than the elements of A, is called the complementary event of A.

The complementary event of A is denoted by A'. It is also called 'not A'.

In set notation **A' = {x | $x \in$ U, $x \notin$ A}**

For example, consider the experiment of tossing two coins. The sample space U = {HH, HT, TH, TT}. Let A = {TT} be the event where only tail appears.

The complementary event of A (not A) is A' = {HH, HT, TH}

It is clear that $\emptyset$' = U and U' = $\emptyset$.

The Event 'A or B' (Union of events) : A, B $\in$ P(U). The set consisting of all the elements of the sample space U which are in A or in B is called the union of events A and B. It is denoted by A $\cup$ B.

In the set notation A $\cup$ B = {x | $x \in$ U, $x \in$ A or $x \in$ B}

Intersection of Events 'A and B' : A, B $\in$ P(U). The set consisting of all the elements of the sample space U which are in A as well as in B is called the intersection of events A and B. It is denoted by A $\cap$ B.

In the set notation A $\cap$ B = {x | $x \in$ U, $x \in$ A and $x \in$ B}

Mutually Exclusive Events : A, B $\in$ P(U). Let U be the sample space associated with a random experiment and let A and B be two events. Then A and B are called mutually exclusive events if A $\cap$ B = $\emptyset$.

Elementary events associated with a random experiment are mutually exclusive.

Consider the random experiment of throwing a die. A and B are the events given by

A = getting an even number, B = getting an odd number

A = {2, 4, 6} and B = {1, 3, 5}

Clearly A $\cap$ B = $\emptyset$

Exhaustive Events : For events A, B $\in$ P(U), if A $\cup$ B = U, then A and B are called exhaustive events.

Mutually Exclusive and Exhaustive Events : If for events A and B, A $\cap$ B = $\emptyset$ and A $\cup$ B = U, then A and B are called mutually exclusive and exhaustive events.

For any event A, A $\cap$ A' = $\emptyset$, A $\cup$ A' = U. A and A' are mutually exclusive and exhaustive events.

As we have seen in previous example, U = {1, 2, 3, 4, 5, 6}, A = {2, 4, 6} and B = {1, 3, 5} are events. Here A $\cap$ B = $\emptyset$ and A $\cup$ B = U. So events, A and B are mutually exclusive and exhaustive.

Difference Event : Let A, B $\in$ P(U) be events. The set consisting of all the elements of U which are in A but not in B is called difference event of A and B. It is denoted by A–B.

Similarly we can define the event B – A.

A – B and B–A are represented in the set notation as

$$A - B = \{x \mid x \in U, x \in A \text{ and } x \notin B\} \text{ and}$$

$$B - A = \{x \mid x \in U, x \in B \text{ and } x \notin A\}$$

We can also see A – B = A $\cap$ B' for the two events A and B.

We have defined union and intersection of two events. Similarly let $A_1, A_2, ..., A_n$ $(n \geq 2)$ be events.

The union of events $A_1, A_2, ..., A_n$ denoted by $A_1 \cup A_2 \cup ... \cup A_n$ or $\bigcup_{i=1}^{n} A_i$.

and the intersection of events $A_1, A_2, ..., A_n$ denoted by $A_1 \cap A_2 \cap ... \cap A_n$ or $\bigcap_{i=1}^{n} A_i$ are defined as follows :

$$\bigcup_{i=1}^{n} A_i = \{x \mid x \in U, x \in A_i, \text{ for at least one } i = 1, 2, 3, ..., n\}$$

$$\bigcap_{i=1}^{n} A_i = \{x \mid x \in U, x \in A_i, \text{ for all } i = 1, 2, 3, ..., n\}$$

10.5 Mutually Exclusive Events and Exhaustive Events

If $A_i \cap A_j = \emptyset$ for $i = 1, 2,..., n$ and $j = 1, 2,..., n, (i \neq j)$

Then $A_1, A_2,..., A_n$ are called mutually exclusive events.

Further if $\bigcup_{i=1}^{n} A_i = U$ also, then $A_1, A_2,..., A_n$ are called mutually exclusive and exhaustive events.

Partition of a sample space : If $A_1, A_2,..., A_n$ are mutually exclusive and exhaustive events for the sample space U, then $\{A_1, A_2,..., A_n\}$ is called a partition of U.

10.6 Elementary Events

Let $U = \{x_1, x_2,..., x_n\}$ be a finite sample space. Singleton $\{x_i\}$ of U for $i = 1, 2,..., n$ are called elementary events. Description of different events in words and set notation are given in the following table.

No.	Description of an Event in words	Set theoretic notation of event
1.	A is an event	$A \subset U$
2.	Event A does not occur	A'
3.	Event B surely occurs when A occurs	$A \subset B$
4.	Impossible event	$\emptyset$
5.	Certain event	U
6.	Events A and B are mutually exclusive	$A \cap B = \emptyset$
7.	Events A and B are exhaustive	$A \cup B = U$
8.	Of the two events A and B, only B occurs	$B - A$ or $B \cap A'$
9.	Of the two events A and B only one occurs	$(A - B) \cup (B - A)$ $A \Delta B$
10.	Events A and B occur together	$A \cap B$
11.	At least one of the events A and B occurs	$A \cup B$
12.	Of the events A, B, C only A occurs	$A - (B \cup C)$ or $A \cap B' \cap C'$
13.	Of the events A, B, C only A and B occur	$A \cap B \cap C'$

Example 5 : One fair coin is tossed twice and the outcomes noted. Give the sample space of the experiment and hence, give the elements of the following events :

(1) Event A : getting T twice, (2) Event B : getting H exactly once

(3) Event C : getting T at most once.

Solution : We know that the sample space associated with the experiment is,

U = {HH, HT, TH, TT}

Event A : getting T twice, A = {TT}

Event B : getting H exactly once, B = {HT, TH}

Event C : getting T at most once, C = {HH, HT, TH}

Example 6 : An experiment has sample space U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} and A = {1, 3, 4}, B = {4, 5, 7, 8}, C = {5, 6, 7} are three events. Write down the elements of the following events :

(1) $A \cup B'$ (2) $A \cap B'$ (3) $A' \cap B'$ (4) $A \cap (B \cap C)'$ (5) $(B \cap C) \cup A$

Solution : Here U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A = {1, 3, 4}, B = {4, 5, 7, 8}, C = {5, 6, 7}

(1) A = {1, 3, 4}, B' = {1, 2, 3, 6, 9, 10}

$\therefore$ $A \cup B'$ = {1, 2, 3, 4, 6, 9, 10}

(2) Now $A \cap B'$ = {1, 3, 4} $\cap$ {1, 2, 3, 6, 9, 10} = {1, 3}

$\therefore$ $A \cap B'$ = {1, 3}

(3) A' = {2, 5, 6, 7, 8, 9, 10} and B' = {1, 2, 3, 6, 9, 10}

$\therefore$ $A' \cap B'$ = {2, 6, 9, 10}

(4) $A \cap (B \cap C)'$

$B \cap C$ = {4, 5, 7, 8,} $\cap$ {5, 6, 7} = {5, 7}

$(B \cap C)'$ = {1, 2, 3, 4, 6, 8, 9, 10}

$A \cap (B \cap C)'$ = {1, 3, 4} $\cap$ {1, 2, 3, 4, 6, 8, 9, 10} = {1, 3, 4}

(5) $(B \cap C) \cup A$ = {5, 7} $\cup$ {1, 3, 4} = {1, 3, 4, 5, 7}

Example 7 : Describe the sample space associated with the experiment of selecting a child at random from three families each with a boy and a girl. Also write the elements of the following events :

(1) There is at most one boy in the selection.

(2) The selection consists of only girls

(3) The selection has exactly two girls.

Solution : A child in a family could either be a boy or a girl. If we denote a boy by letter '*b*' and a girl by letter '*g*', then the sample space associated with this experiment is given by

$$U = \{b, g\} \times \{b, g\} \times \{b, g\}$$
$$= \{bbb, bbg, bgb, gbb, ggb, gbg, bgg, ggg\}$$

(1) Suppose A denotes the event that there is at most one boy in the selection.
A = {*ggg, ggb, gbg, bgg*}

(2) Suppose B denotes the event that the selection has only girls
B = {*ggg*}

(3) Suppose C denotes the event that the selection has exactly two girls.
C = {*ggb, gbg, bgg*}

EXERCISE 10.2

1. A box contains one ball each of red, black, yellow and white colour. One ball is drawn out of the box at random and its colour is noted and it is put back in the box. Then another ball is drawn out, its colour noted and it is put back in the box. List the elements of the sample space of this experiment and hence, list the elements of the following events.
   (1) A : two balls of the same colour are drawn.
   (2) B : exactly one ball of white colour is drawn.
   (3) C : at least one white ball is drawn.
   (4) D : two balls of different colours are drawn.
   Write the elements of $A \cap B$, $B \cup C$, $A \cup D$, $A \cap D$. What can be said about events B and C ? About A and D ?
2. A coin is tossed three times. Give the elements of the following events :
   (1) Event A : Getting at least two heads
   (2) Event B : Getting exactly two tails
   (3) Event C : Getting at most one tail
   (4) Event D : Getting at least one tail
   Find $A \cap B$, $C \cap D'$, $A \cup C$, $B \cap C$, $A' \cup C'$
3. U is the sample space consisting of positive integers from 1 to 30. If A_i denotes the event consisting of those elements of U which are divisible by i, write down the elements of A_2, A_3, A_4, A_5, Check the validity of the following statements.
   (1) Events A_2 and A_3 are mutually exclusive.
   (2) Event A_4 is a subset of event A_2.
   (3) A_3, A_4 and A_5 are not exhaustive.
4. A box contains 2 white, 1 red and 2 green identical balls. Two balls are drawn at random without replacement. Write the sample space associated with experiment. Write the elements of the following events.
   Event A : Both the balls selected are white.
   Event B : At least one ball selected is white.
   Event C : Both the balls selected are of different colours.

5. An integer from 1 to 50 is selected at random. Write the elements of the following events.

   A : Randomly selected integer is a multiple of 2.

   B : Randomly selected integer is a multiple of 10

   C : Randomly selected integer is a multiple of 4.

6. Two fair dice are rolled simultaneously. Write down the sample space of this experiment. Give the elements of the following events :

   A : The sum of the numbers on two dice is divisible by 4.

   B : The sum of the numbers on two dice is divisible by 3.

   C : The sum of the numbers on two dice is less than 7.

   D : Numbers on both the dice are even integers.

7. There are three identical balls, marked with a, b, c in a box. One ball is picked up from the box at random. The letter on it is noted and the ball is put back in the box. Then another ball is picked up from the box and the letter on it is noted. Write the sample space of the experiment. Write down the elements of the following events :

   Event A : Ball marked a is selected exactly once.

   Event B : Balls selected have same letters marked.

   Event C : Ball with mark c is selected at least once.

8. From a group of 3 boys and 2 girls, two children are selected at random. Describe the following events :

   E : Both the children selected are girls.

   F : Selected group consists of one boy and one girl.

   G : At least one boy is selected.

9. A die is thrown. The number that appears on the top face is observed. Write the elements of the following events.

   A : A number less than 7 appears on the top face.

   B : A multiple of 3 appears on the top face.

   C : A number greater than 4 appears on the top face.

   D : A number less than 2 appears on the top face.

   Find $A \cap C$, $B \cup C$, $D' \cup C'$

*

10.7 Set Function

Let U be a finite sample space. The set of all subsets of U is the power set of U and it is denoted by P(U). Elements of P(U) are called events. Now we shall denote P(U) by S.

Let S be the power set of the sample space U and R be the set of real numbers. A function $T : S \to R$ is called a real valued set function.

10.8 Additive Set Function

Let $T : S \rightarrow R$ be a set function.

If $T(A_1 \cup A_2) = T(A_1) + T(A_2)$, whenever $A_1, A_2 \in S$ and $A_1 \cap A_2 = \emptyset$, then T is called an additive set function on S.

Example 8 : Let $U = \{a, b\}$ be a sample space. $T : S \rightarrow R$ is a set function. $T(A)$ = Number of elements in A, $A \in S$. Find the range of T. Determine whether the function is an additive set function or not.

Solution : Here given sample space $U = \{a, b\}$

$\therefore$ $S = P(U) = \{\emptyset, \{a\}, \{b\}, U\}$

Now let $A_1 = \emptyset, A_2 = \{a\}, A_3 = \{b\}, A_4 = U$

$T(A_1) = 0, T(A_2) = T(A_3) = 1, T(A_4) = 2$

$\therefore$ Range of $T = \{0, 1, 2\}$

$A_1 \cap A_2 = \emptyset \qquad A_1 \cup A_2 = \emptyset \cup \{a\} = \{a\} = A_2$

$T(A_1 \cup A_2) = T(A_2) = 1$

$T(A_1) + T(A_2) = 0 + 1 = 1$

$\therefore$ $T(A_1 \cup A_2) = T(A_1) + T(A_2)$

$A_2 \cup A_3 = A_4 \qquad A_2 \cap A_3 = \emptyset$

$T(A_2 \cup A_3) = T(A_4) = 2$

$T(A_2) + T(A_3) = 1 + 1 = 2$

$\therefore$ $T(A_2 \cup A_3) = T(A_2) + T(A_3)$

$A_1 \cap A_3 = \emptyset \qquad A_1 \cup A_3 = A_3$

$T(A_1 \cup A_3) = T(A_3) = 1$

$T(A_1) + T(A_3) = 0 + 1 = 1$

$\therefore$ $T(A_1 \cup A_3) = T(A_1) + T(A_3)$

$A_1 \cap A_4 = \emptyset \qquad A_1 \cup A_4 = U$

$\therefore$ $T(A_1 \cup A_4) = T(U) = 2$

$\therefore$ $T(A_1) + T(A_4) = 0 + 2 = 2$

$\therefore$ $T(A_1 \cup A_4) = T(A_1) + T(A_4)$

Hence, T is an additive set function.

10.9 Axiomatic Definition of Probability

In 1933, the Russian mathematician A. N. Kolmogorov was the first mathematician who gave the definition of probability.

Definition : Let U be a finite sample space and S be its power set. Suppose that a set function P : S → R satisfies following axioms.

Axiom 1 : For every A ∈ S, P(A) ≥ 0

Axiom 2 : P(U) = 1

Axiom 3 : $\forall A_1 \in S, A_2 \in S$, if $A_1 \cap A_2 = \emptyset$,

then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

Then P is called a probability function. For event A ∈ S, P(A) is called the probability of the event A. The triplet (U, S, P) is called probability space.

From the axiomatic definition of probability we can deduce following results :

1. According to axiom 1, probability of any event is non-negative real number.
2. According to axiom 2, the probability of certain event U is 1 i.e. P(U) = 1.
3. The probability function P is an additive set function.
4. If $A_1, A_2, ..., A_n \in S$ are mutually exclusive ($n \geq 2$),

   $P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n)$

Example 9 : A set function P is defined on the power set of the sample space U = {a, b, c} as follows.

Event A	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, c\}$	U
P(A)	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{3}{4}$	$\frac{7}{12}$	1

Determine whether P is a probability function on S.

Solution : Here U = {a, b, c}

The power set S of the given sample space U is,

$S = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, U\}$

From the given table it is clear that $\forall A \in S$, $P(A) \geq 0$. Hence axiom 1 is satisfied.

Now P(U) = 1. Hence axiom 2 is also satisfied. Let $A_1 = \{a\}$ and $A_2 = \{b\}$. Then $A_1 \cap A_2 = \emptyset$. So A_1 and A_2 are mutually exclusive events.

$A_1 \cup A_2 = \{a, b\}$

$P(A_1 \cup A_2) = P(\{a, b\}) = \frac{1}{6}$

$P(A_1) + P(A_2) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

$\therefore \quad P(A_1 \cup A_2) \neq P(A_1) + P(A_2)$

Thus P does not satisfy axiom 3. So P is not a probability function.

Example 10 : For mutually exclusive and exhaustive events A, B, C, a set function P is defined as follows :

(1) $P(A) = 0.40, P(B) = 0.36, P(C) = 0.24$

(2) $P(A) = \frac{3}{5}, P(B) = \frac{1}{5}, P(C) = \frac{2}{5}$

(3) $P(A) = 0.31, P(B) = 0.36, P(C) = -0.23$

In each of the above determine if P is a probability function or not.

Solution : (1) $P(A) > 0, P(B) > 0, P(C) > 0$.

A, B, C are mutually exclusive and exhaustive events. **(Given)**

$\therefore \quad A \cup B \cup C = U$

$\therefore \quad P(A \cup B \cup C) = P(U)$

also $P(A) + P(B) + P(C) = 0.40 + 0.36 + 0.24 = 1.00$

$\therefore \quad P(U) = 1.00$

So P satisfies all the axioms

$\therefore$ P is a probability function

(2) According to the axiom 3, we must have,

$P(A) + P(B) + P(C) = 1$

but according to the allocation

$P(A) + P(B) + P(C) = \frac{3}{5} + \frac{1}{5} + \frac{2}{5} = \frac{6}{5} = 1.2 > 1$

$\therefore$ P is not a probability function.

(3) $\forall A, P(A) \geq 0$ is necessary as per the first axiom. Probability of all the events should be non-negative. But $P(C) = -0.23 < 0$. So given allocation of probabilities is not possible. Hence P is not a probability function.

10.8 Theorems on Probability

Using the definition of probability we shall prove some useful theorems on probability.

We shall assume that every event is a subset of a finite sample space. S is the power set of U. Suppose P is a probability function.

Theorem 1 : For the impossible event $\emptyset$, $P(\emptyset) = 0$

Proof : For the events $\emptyset$ and U,

$\emptyset \cap U = \emptyset$ and $\emptyset \cup U = U$

Hence by axiom 3,

$P(\emptyset \cup U) = P(\emptyset) + P(U)$

$\therefore \quad P(U) = P(\emptyset) + P(U)$

$\therefore \quad 1 = P(\emptyset) + 1$ **(axiom 2)**

$\therefore \quad P(\emptyset) = 0$

Theorem 2 : For every event A,

$P(A') = 1 - P(A)$

Proof : For events A and A'

$A \cap A' = \emptyset$ and $A \cup A' = U$

$P(A \cup A') = P(U)$

$\therefore$ $P(A) + P(A') = P(U)$ **(axiom 3)**

$\therefore$ $P(A) + P(A') = 1$ **(axiom 2)**

$\therefore$ $P(A') = 1 - P(A)$

Theorem 3 : If A and B are events such that $A \subset B$, then,

(1) $P(B - A) = P(B) - P(A)$

(2) $P(A) \leq P(B)$

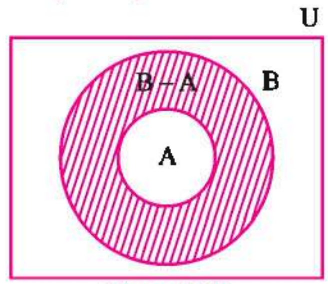


Figure 10.2

Proof : $A \subset B$ is given.

From figure 10.2 is clear that,

$A \cap (B - A) = \emptyset$ and $A \cup (B - A) = B$

So $P(A \cup (B - A)) = P(B)$

$\therefore$ $P(A) + P(B - A) = P(B)$ **(axiom 3)**

$\therefore$ $P(B - A) = P(B) - P(A)$

Now using the axiom 1

$P(B - A) \geq 0$

$\therefore$ $P(B) - P(A) \geq 0$

$\therefore$ $P(B) \geq P(A)$

$\therefore$ $P(A) \leq P(B)$

Corollary 1 : For event A, $0 \leq P(A) \leq 1$

Proof : We know $P(A) \geq 0$ by axiom 1. **(i)**

Also $A \subset U$

$\therefore$ $P(A) \leq P(U)$ **(Theorem 3)**

$\therefore$ $P(A) \leq 1$ **(axiom 2) (ii)**

From (i) and (ii) we can say that $0 \leq P(A) \leq 1$

Corollary 2 : For any events A, B,

$P(A \cap B') = P(A) - P(A \cap B)$

Proof : For any events A, B, using Venn-diagram 10.3, it is clear that

$A \cap B' = A - (A \cap B)$

$\therefore$ $P(A \cap B') = P(A - (A \cap B))$

$= P(A) - P(A \cap B)$ **$((A \cap B) \subset A)$**

Theorem 4 : For events A and B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

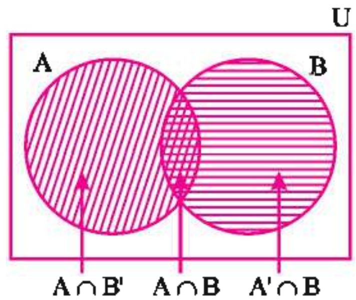


Figure 10.3

Proof : For events B and $A \cap B'$

$B \cap (A \cap B') = \emptyset$ and

$B \cup (A \cap B') = A \cup B$

So,

$\therefore \quad P(B \cup (A \cap B')) = P(A \cup B)$

$\therefore \quad P(B) + P(A \cap B') = P(A \cup B)$ **(axiom 3)**

$\therefore \quad P(B) + P(A) - P(A \cap B) = P(A \cup B)$ **(corollary 2)**

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

10.9 Classical Definition of Probability

Laplace and Bernoulli had given the classical definition of probability. In the definition they have considered that certain basic events have equal probabilities. This definition is useful in understanding the probability distribution among the events of a sample space.

Equiprobable Events : Let $U = \{x_1, x_2,..., x_n\}$ be a finite sample space. If $P(\{x_1\}) = P(\{x_2\}) = \cdots = P(\{x_n\})$ then the elementary events $\{x_1\}, \{x_2\},..., \{x_n\}$ are called equiprobable events.

Now $\bigcup_{i=1}^{n}\{x_i\} = U$ and $\{x_i\} \cap \{x_j\} = \emptyset$; $i \neq j$, $i = 1, 2, 3,..., n$; $j = 1, 2, 3,..., n$

$P(\{x_1\} \cup \{x_2\} \cup ... \cup \{x_n\}) = P(U)$

$P(\{x_1\}) + P(\{x_2\}) + ... + P(\{x_n\}) = P(U)$ **(axiom 3)**

$\therefore \quad P(\{x_1\}) + P(\{x_2\}) + ... + P(\{x_n\}) = 1$ **(axiom 2)**

As $\{x_1\}, \{x_2\},..., \{x_n\}$ are equiprobable, so $n\,P(\{x_i\}) = 1$, $i = 1, 2,..., n$

$\therefore \quad P(\{x_i\}) = \frac{1}{n}$ for each i, $1 \le i \le n$

Now Let $U = \{x_1, x_2,..., x_n\}$ be a finite space. Suppose the elementary events are equiprobable. Suppose A is non-empty event of U with r elements. Suppose that the elements of A are $x_1, x_2,..., x_r$

$\therefore \quad A = \{x_1\} \cup \{x_2\} \cup ... \cup \{x_r\}$

So according to axiom 3,

$P(A) = P(\{x_1\}) + P(\{x_2\}) + ... + P(\{x_r\})$

$$P(A) = \frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n} \qquad (r \text{ times})$$

$\therefore \quad P(A) = \frac{r}{n}$

The result is true for $A = \emptyset$ also.

Definition : If a finite sample space associated with a random experiment has n equally likely (equiprobable) outcomes (elements) and of these r $(0 \le r \le n)$ outcomes are favourable for the occurrence of an event A, then probability of A, P(A) is given by $P(A) = \frac{r}{n}$.

Example 11 : If probability of an event A is $\frac{9}{10}$, What is the probability of the event 'not A'.

Solution : Here $P(A) = \frac{9}{10}$. So $P(A') = 1 - P(A)$

$$= 1 - \frac{9}{10}$$

$$= \frac{1}{10}$$

Example 12 : A and B are given events. $P(A) = 0.38$, $P(B) = 0.52$ and $P(A \cap B) = 0.18$.

Determine the probability of the following events :

(1) $A \cup B$ (2) $A - B$ (3) $B - A$ (4) B'

Solution : (1) $P(A) = 0.38$, $P(B) = 0.52$ and $P(A \cap B) = 0.18$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.38 + 0.52 - 0.18 = 0.72$$

(2) Now $P(A - B) = P(A) - P(A \cap B)$

$\therefore \quad P(A - B) = 0.38 - 0.18 = 0.20$

(3) Now $P(B - A) = P(B) - P(A \cap B)$

$$= 0.52 - 0.18 = 0.34$$

(4) $P(B') = 1 - P(B)$

$$= 1 - 0.52 = 0.48$$

Example 13 : A fair coin is tossed twice. Find the probability of (1) getting H exactly once. (2) getting T at least once.

Solution : The sample space associated with the experiment is

U = {HH, HT, TH, TT}

$\therefore$ Number of elements in U is 4. So $n = 4$

Suppose A denotes the event of getting H exactly once.

$\therefore$ A = {HT, TH}. So n(A) = 2

According to the classical definition of probability,

$\therefore$ $P(A) = \frac{r}{n} = \frac{2}{4} = \frac{1}{2}$

In the same way for the event B of getting T at least once,

B = {HT, TH, TT}. So $r = 3$.

$P(B) = \frac{r}{n} = \frac{3}{4}$

Example 14 : One card is drawn at random from a well shuffled pack of 52 cards. Calculate the probability that the card will be.

(1) an ace (2) a card of black colour

(3) a diamond (4) not an ace

Solution : One card is drawn at random from a well shuffled pack of cards. So the possible number of outcomes is 52.

$\therefore$ $n = 52$

(1) Number of aces in a pack of cards is 4. So $r = 4$.

$\therefore$ P (card of an ace) = $\frac{r}{n} = \frac{4}{52} = \frac{1}{13}$

(2) A card of black colour is a club card or a spade card. Number of cards of black colour is 26

$\therefore$ $r = 26$

$\therefore$ P (card of black colour) = $\frac{r}{n} = \frac{26}{52} = \frac{1}{2}$

(3) Number of diamond cards is 13. So $r = 13$.

$\therefore$ P (a diamond card) = $\frac{13}{52} = \frac{1}{4}$

(4) P (Not an ace card) = 1 − P (an ace card)

$= 1 - \frac{1}{13} = \frac{12}{13}$

Example 15 : A box contains 5 red, 6 white and 2 black balls. The balls are identical in all respect other than colour.

(1) One ball is drawn at random from the box. Find the probability that the selected ball is black.

(2) Two balls are drawn at random from the box. Find the probability that one ball is white and one is red.

Solution : Total number of balls in the box is 13. One ball is drawn at random from the box.

We can select one ball in $\binom{13}{1}$ ways. So $n = 13$.

There are two black balls. Hence the number of favourable elements for the event is 2.

$\therefore \quad r = \binom{2}{1} = 2$

$\therefore$ The probability that selected ball being black is

$$\frac{r}{n} = \frac{2}{13}$$

(2) Two balls are to be drawn from the box simultaneously. So $n = \binom{13}{2}$. There are 5 red and 6 white balls in the box. The number of ways of selection is $\binom{5}{1}\binom{6}{1}$.

$\therefore \quad r = \binom{5}{1}\binom{6}{1}$

$\therefore$ The probability of this event is,

$$\frac{r}{n} = \frac{\binom{5}{1}\binom{6}{1}}{\binom{13}{2}} = \frac{30}{78} = \frac{5}{13}$$

EXERCISE 10

1. A and B are two events such that $P(A \cup B) = 0.89$, $P(A) = 0.54$, $P(B) = 0.59$. Find (1) $P(A \cap B)$ (2) $P(A' \cap B')$ (3) $P(A \cap B')$ (4) $P(B \cap A')$
2. A box contains 15 white, 10 blue and 5 black marbles. From the box 5 marbles are drawn at random. What is the probability that,
   (1) all the five marbles drawn are black.
   (2) Either blue or white marble is drawn.
   (3) at least one marble drawn is black.
3. If A, B, C are three events associated with a random experiment, prove that,
   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$
4. Write the sample space associated with the experiment of tossing of a coin thrice. Find the probability of the following events.
   (1) At least one head appears.
   (2) Exactly two tails occur.
   (3) At most one head occurs.

5. Two unbiased dice are rolled simultaneously. Find the probability of the following events :

   (1) A = Getting a double (same number on two dice)

   (2) B = The sum of numbers on two dice is divisible by 3.

   (3) C = The product of numbers on two dice is divisible by 2

   (4) D = The sum of numbers on two dice is greater than 10.

6. There are 100 lottery tickets on which numbers from 1 to 100 have been printed. Find the probability that the number on a randomly drawn ticket is a multiple of 5 or 7.

7. There are 400 screws in a box out of which 50 are defective. If a screw is randomly selected from the box, find the probability that it is non-defective. If two screws are randomly selected, find the probability of both being defective.

8. From a well shuffled pack of 52 cards 13 cards are drawn at random. Find the probability that 6 out of these 13 cards are face cards.

9. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

   (1) A die is rolled. The probability of getting a prime number is... ☐

   (A) $\frac{1}{3}$ (B) $\frac{5}{6}$ (C) $\frac{2}{3}$ (D) $\frac{2}{5}$

   (2) Two cards are drawn at random from a pack of 52 cards. The probability that both the cards are kings is ...... ☐

   (A) $\frac{1}{221}$ (B) $\frac{5}{221}$ (C) $\frac{4}{13}$ (D) $\frac{1}{21}$

   (3) A coin is tossed successively three times. The probability of getting exactly one head or two heads is ...... ☐

   (A) $\frac{1}{8}$ (B) $\frac{3}{4}$ (C) $\frac{3}{8}$ (D) $\frac{5}{8}$

   (4) A box contains 7 red and 5 blue balls. Two balls are drawn at random. The probability that one is red and other is blue is... ☐

   (A) $\frac{31}{66}$ (B) $\frac{32}{66}$ (C) $\frac{35}{66}$ (D) $\frac{23}{66}$

   (5) A letter from the English alphabet is chosen at random. The probability that the letter chosen is vowel is... ☐

   (A) $\frac{21}{26}$ (B) $\frac{5}{26}$ (C) $\frac{3}{26}$ (D) $\frac{1}{13}$

(6) If A and B are mutually exclusive events in a sample space, such that $P(B) = 0.4$ and $P(A) = 0.5$, then $P(A' \cap B')$...

(A) 0.9 (B) 0.1 (C) 0.2 (D) 0.23

(7) Each number is equiprobable. Probability that a number selected at random from the numbers 1 to 25 is a prime number is...

(A) $\frac{9}{25}$ (B) $\frac{16}{25}$ (C) $\frac{7}{25}$ (D) $\frac{18}{25}$

(8) 8 boys and 2 girls are to be seated in a row. The probability of an arrangement in which two girls are not seated side by side is...

(A) $\frac{1}{5}$ (B) $\frac{4}{5}$ (C) $\frac{3}{5}$ (D) $\frac{2}{5}$

(9) A fair die is rolled once. The probability of the event of getting a number on the die being divisible 3 is...

(A) $\frac{1}{6}$ (B) $\frac{2}{3}$ (C) $\frac{1}{3}$ (D) $\frac{5}{6}$

(10) If $P(A) = 0.38$ and A, B are mutually exclusive events, then $P(A \cap B') =$ ......

(A) 0.38 (B) 1 (C) 0.12 (D) 0.62

(11) A, B, C are mutually exclusive and exhaustive events. $P(A) = 0.40$, $P(B) = P(C)$. Then, $P(B) =$ ......

(A) 0.40 (B) 0.60 (C) 0.20 (D) 0.30

(12) Two balanced dice are rolled. The probability that the same number will appear on each of them is...

(A) $\frac{1}{36}$ (B) $\frac{1}{18}$ (C) $\frac{1}{6}$ (D) $\frac{3}{28}$

Summary

1. Random experiment, sample space
2. Event, union event, intersection event, complementary event, exhaustive event, mutually exclusive events, difference event
3. Additive set function, probability set function
4. Axiomatic definition of probability
5. Theorems on probability function.

ANSWERS

(Answers to questions involving some calculations only are given.)

Exercise 1.1

1. Yes **2.** Yes **3.** No **4.** No **5.** Yes **6.** No **7.** Yes **8.** Yes

Exercise 1.2

2. $2 + 2 \neq 5$ **2.** Area of a square is not given by the formula $A = \pi r^2$

3. A cube is not a plane figure or it is not true that a cube is a plane figure

4. Georg Cantor did not develop set theory

5. Amitabh Bachchan is not the brand ambassador of Gujarat tourism.

6. $2 + 2 \neq 2^2$ **7.** For natural numbers $x \geq 3$, $x + x \neq x^2$ **8.** Ice is not hot.

Exercise 1.3

1. (1) $p : 3 + 7 = 5$
$q : 5^2 = 25$ $p \wedge q$: F

(2) $p : 3 + 7 = 10$
$q : 10^2 = 100$ $p \wedge q$: T

(3) p : A triangle has three sides.
q : A triangle has three angles. $p \wedge q$: T

(4) p : A quadrilateral has four sides.
q : A quadrilateral has four angles. $p \wedge q$: T

(5) p : The sum of the measures of angles of a triangle is 180.
q : The sum of the measures of angles of a triangle is 360. $p \vee q$: T

(6) $p : 2 + 2 = 5$
$q : 5 + 2 = 25$ $p \wedge q$: F

(7) p : 1 is a root of $x^2 - 3x + 2 = 0$.
q : 2 is a root of $x^2 - 3x + 2 = 0$. $p \wedge q$: T

(8) $p : 1^3 = 1$
$q : 3^2 = 9$ $p \vee q$: T

(9) p : $x^2 = x$ is satisfied by 1.
q : $x^2 = x$ is satisfied by 0. $p \wedge q$: T

(10) p : 0 is the identity for addition.
q : 1 is the identity for multiplication. $p \wedge q$: T

2. (1) $3 + 7 \neq 5$ or $5^2 \neq 25$ (2) $3 + 7 \neq 10$ or $10^2 \neq 100$

(3) A triangle does not have three sides or does not have three angles.

(4) A quadrilateral does not have four sides or it does not have four angles.

(5) The sum of the measures of angles of a triangle is not 180 and
The sum of the measures of angles of a triangle is not 360.

(6) $2 + 2 \neq 5$ or $5 + 2 \neq 25$ (7) 1 or 2 is not a root of $x^2 - 3x + 2 = 0$.

(8) $1^3 \neq 1$ and $3^2 \neq 9$ (9) $x^2 = x$ is not satisfied by 1 or 0.

(10) 0 is not the identity for addition or 1 is not the identity for multiplication.

3. (1) inclusive (2) exclusive (3) exclusive (4) inclusive (5) exclusive

5. p : 30 is divisible by 2.
q : 30 is divisible by 3. $p \wedge q \wedge r$: T
r : 30 is divisible by 5.
Negation : 30 is not divisible by 2 or by 3 or by 5.

6. p : 1 is a prime.
q : 1 is composite. $p \vee q$: F
Negation : 1 is not a prime and 1 is not composite.

Exercise 1.4

1. (1) Universal quantifier :
Negation : There exists a pair of natural numbers a and b such that $a + b$ is not even integer.

(2) Universal quantifier :
Negation : There exists an income tax payer who does not have a PAN card.

(3) Existential quantifier :
Negation : For all positive integers x, $\sqrt{x} \notin$ R

(4) Existential quantifier :
Negation : For every element x, $x \notin \emptyset$

(5) Universal quantifier :
Negation : There exists some $\theta \in$ R such that $sin^2\theta + cos^2\theta \neq 1$

(6) Universal quantifier :
Negation : There exists an angle which can not be constructed by using a straight edge and compass only.

(7) Universal quantifier :
Negation : There exists a person of age exceeding 18 years who is not a voter.

(8) Existential quantifier :
Negation : There exists a subset of N which does not have a smallest element.

(9) Universal quantifier :
Negation : There exists a number ending in zero which is not divisible by 10.

(10) Existential quantifier :
Negation : Every multiple of 5 ends in 5.

2. (1) p : n is odd.
q : n^2 is odd. , T

(2) p : 2 divides n.
q : 4 divides n. , F

(3) p : 9 divides n.
q : 3 divides n. , T

(4) p : All the angles of a quadrilateral have measure 90.
q : It is a rectangle. , T

(5) p : All the angles of a triangle have same measure.
q : It is equilateral. , T

(6) p : A triangle is isosceles.
q : It is equilateral. , F

(7) p : A triangle is a right angle triangle.
q : The largest side occurs opposite to the right angle. , T

(8) p : A triangle has sides $2uv$, $u^2 - v^2$, $u^2 + v^2$ for $u, v \in Z$ $(u > v)$.
q : It is a right angle triangle. , T

(9) p : A triangle has sides $2mn$, $m^2 - n^2$, $m^2 + n^2$ for all $m, n \in N$ $(m > n)$.
q : It is a right angle triangle. , T

(10) p : A number is divisible by 1001.
q : It is divisible by 7, 11 and 13. , T

3. (1) Quadrilateral ABCD is a rectangle if and only if it is a square. F

(2) ΔABC is isoscles if and only if it is equilateral. F

(3) Quadrilateral ABCD has all angles and all the sides congruent if and only if it is a square. T

(4) Integer n is positive if and only if it is even. F

(5) A real number x is positive if and only if it is a square of another real number. T

Exercise 1.5

1. (1) Converse : If 2 divides n, 30 divides n.
Contrapositive : If 2 does not divide n, 30 does not divide n.

(2) Converse : If 16 divides n, 8 divides n.
Contrapositive : If 16 does not divide n, 8 does not divide n.

(3) Converse : If Sanjay will fail, he does not take the examination.
Contrapositive : If Sanjay will not fail, he takes the examination.

(4) Converse : If the square root of an integer is an integer, it is the square of an integer.
Contrapositive : If square root of an integer is not an integer, it is not the square of an integer.

(5) Converse : If n has three real cube roots, it is the cube of an integer.
Contrapositive : If n does not have three real cube roots, it is not the cube of an integer.

(6) Converse : If two lines in a plane are not parallel, they intersect.
Contrapositive : If two lines in a plane are parallel, they do not intersect.

(7) Converse : If the two sides opposite to two angles of a triangle are not congruent, then the angles opposite to them are not congruent.
Contrapositive : If the two sides opposite to two angles of a triangle are congruent, then the angles opposite to them are congruent.

(8) Converse : If in a plane $l \parallel n$ or $l = n$, then $l \parallel m$ and $m \parallel n$.
Contrapositive : If in a plane $l \nparallel n$ and $l \neq n$, then $l \nparallel m$ or $m \nparallel n$.

(9) Converse : If $a = \pm b$, then $a^2 = b^2$. $(a, b \in R)$
Contrapositive : if $a \neq \pm b$, then $a^2 \neq b^2$.

(10) Converse : If $a = b$, then $a^3 = b^3$. $(a, b \in R)$
Contrapositive : If $a \neq b$, then $a^3 \neq b^3$.

Exercise 1

1. (1) Yes (2) Yes (3) No (4) No (5) Yes (6) Yes (7) No (8) No
(9) Yes (10) Yes

2. (1) Science and mathematics are not useful for development.
(2) One can not opt for engineering or medicine course.
(3) n is a perfect square and last digit of n is 3.
(4) There exists a prime number which is not odd.
(5) There is an odd number which is not prime.
(6) There exists an integer which is not a rational number.
(7) For all even integers n, n is not a prime.
(8) For all real numbers x, $x^2 \neq -1$
(9) There exists $a \in R$ such that $a + 0 \neq a$.
(10) For every $a \in R$, $a \cdot 1 = a$.
(11) There exists a real number x such that $x^2 = x$.
(12) For all $x \in R$, $x^3 \nless x$.

4. (1) Converse : If you have an umbrella, it is raining outside.
Contrapositive : If you do not have an umbrella, it is not raining outside.
(2) Converse : If an integer has at least three factors, it is composite.
Contrapositive : If an integer does not have at least three factors, it is not composite.

(3) Converse : If $n = 1$, n is not a prime or not composite.

Contrapositive : If $n \neq 1$, n is a prime and composite.

(4) Converse : If opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.

Contrapositive : If opposite sides of a quadrilateral are not congruent, the quadrilateral is not a parallelogram.

(5) Converse : If a quadrilateral is a parallelogram, its diagonals are bisecting each other.

Contrapositive : If a quadrilateral is not parallellogram, its diagonals are not bisecting each other.

(6) Converse : If I will go to watch a new movie, it is Friday.

Contrapositive : If I will not go to watch a new movie, it is not Friday.

(7) Converse : If x^2 is positive, x is negative.

Contrapositive : If x^2 is not positive, x is not negative.

(8) Converse : If xy is positive, x and y are negative.

Contrapositive : If xy is not positive, x or y is not negative.

(9) Converse : If a quadrilateral is a square, it is equiangular.

Contrapositive : If a quadrilateral is not a square, it is not equiangular.

(10) Converse : If $p(a) = 0$, $x - a$ is a factor of polynomial $p(x)$.

Contrapositive : If $p(a) \neq 0$, $x - a$ is not a factor of polynomial $p(x)$.

10. (1) b (2) c (3) a (4) b (5) c (6) b (7) a (8) c (9) a (10) b
(11) a (12) a (13) b (14) c (15) d (16) b (17) b

Exercise 2.1

1. (1) $\{1, 2, 3, ..., 9\}$ (2) $\{6\}$ (3) $\{-1, 6\}$ (4) $\{-1, 0, 1\}$
(5) $\{-3, -2, -1, 0, 1, 2, 3\}$

2. $\emptyset$, $\{1\}$, $\{a\}$, $\{b\}$, $\{1, a\}$, $\{1, b\}$, $\{a, b\}$, A

3. (1) $X = \{a\}$, $X = \{c\}$, $X = \{a, b\}$, $X = \{a, c\}$, $X = \{b, c\}$, $X = \{a, b, c\}$
(2) $X = \{a\}$, $X = \{c\}$, $X = \{a, b\}$, $X = \{a, c\}$, $X = \{b, c\}$
(3) $\{a, b\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, A

4. (1), (2), (3)

Exercise 2.2

1. $A \cup B = \{1, 2, 3, 4, 5, 7, 11, 13\}$, $A \cap B = \{2, 3\}$

4. (1) $\{a, b, c, d, e, f\}$ (2) $\{c, d, e\}$ (3) $\{a, b\}$ (4) $\{f\}$ (5) $\{a, b, f\}$

Exercise 2.3

1. $A \times B = \{(1, 4), (1, 7), (2, 4), (2, 7), (3, 4), (3, 7)\}$

$B \times A = \{(4, 1), (4, 2), (4, 3), (7, 1), (7, 2), (7, 3)\}$

3. $\{(1, 2), (1, 5), (1, 8), (1, 11), (2, 2), (2, 5), (2, 8), (2, 11),$
$(3, 2), (3, 5), (3, 8), (3, 11), (4, 2), (4, 5), (4, 8), (4, 11)\}$

Exercise 2.4

1. 125 **2.** Incorrect data **3.** 11, 6 **4.** 43 **5.** 60

Exercise 2

1. (1) $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$ (2) $\beta = \{A, E, I, O, U\}$

(3) $X = \{6, 7, 8, 9, 10\}$ (4) $X = \{-1, 1\}$ (5) $\emptyset$

2. (1) $A = \{x \mid x \in N, x$ is multiple of 5, less than 25.$\}$

(2) $P = \{x \mid x$ is an odd natural number.$\}$

3. (1) $A - B = \{1, 5, 9\}$ (2) $B - A = \{11\}$

(3) $A \cup B = \{1, 3, 5, 7, 9, 11\}$

6. $\emptyset$, $\{1\}$, $\{5\}$, $\{9\}$, $\{1, 5\}$, $\{1, 9\}$, $\{5, 9\}$, A **9.** 40 **10.** 5

11. (1) c (2) b (3) a (4) c (5) a (6) b (7) c (8) c (9) b (10) b
(11) c (12) a (13) d (14) b (15) c (16) b (17) b (18) a (19) c (20) a
(21) a (22) b (23) c (24) c (25) d (26) b (27) a (28) c (29) c (30) b
(31) c

Exercise 3.1

1. Domain = $\{1, 2, 3, 4, 5, 6, 7\}$, Range = $\{1, 2, 3, 4, 5, 6, 7\}$

2. $S = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

3. $S = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

4. $S = \{(5, 3), (6, 4), (7, 5)\}$

Exercise 3.2

1. (1) $\{3, 4, 5, 6, ...\}$ (2) $\{2, 4, 8, 16, 32, ...\}$ (3) $\{5\}$ (4) Z

3. $f(4) = 27$, $f(16) = 275$ **4.** $a = 3$

Exercise 3.3

2. (1) $R_f = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ (2) $R_h = \{0\}$

Exercise 3

1. $fog = \{(4, 4), (5, 5), (6, 6)\}$, $gof = \{(1, 1), (2, 2), (3, 3)\}$

2. (1) $fog(x) = 2x + 1$ $gof(x) = 2x + 2$

$fof(x) = x + 2$ $gog(x) = 4x$

(2) $fog(x) = 9x^2 + 2$ $\quad$ $gof(x) = 3x^2 + 6$

$fof(x) = x^4 + 4x^2 + 6$ $\quad$ $gog(x) = 9x$

(3) $fog(x) = 4x^2 - 6x + 1$ $\quad$ $gof(x) = 2x^2 + 6x - 1$

$fof(x) = x^4 + 6x^3 + 14x^2 + 15x + 5$ $\quad$ $gog(x) = 4x - 9$

(4) $fog(x) = x$ $\quad$ $gof(x) = x$

$fof(x) = x + 2$ $\quad$ $gog(x) = x - 2$

(5) $fog(x) = 18x^2 + 1$ $\quad$ $gof(x) = 6x^2 + 3$

$fof(x) = 8x^4 + 8x^2 + 3$ $\quad$ $gog(x) = 9x$

3. Domain = {1, 2, 3, 4}, Range = {1, 2, 3, 4} **4.** S = {(1, 1), (2, 4), (3, 9), (4, 16)}

6. (1) R (2) $[0, \infty)$ (3) R (4) {1000} (5) $[0, \infty)$

8. $f(9) = 6$, $f(2) = 2 - \sqrt{2}$

9. (1) $fog(x) = (x - 1)^2$ $\quad$ $gof(x) = x^2 - 1$

$fof(x) = x^4$ $\quad$ $gog(x) = x - 2$

(2) $fog(x) = 5x - 5$ $\quad$ $gof(x) = 5x - 25$

$fof(x) = x - 10$ $\quad$ $gog(x) = 25x$

(3) $fog(x) = x^4 + 6x^2 + 6$ $\quad$ $gof(x) = x^4 - 6x^2 + 12$

$fof(x) = x^4 - 6x^2 + 6$ $\quad$ $gog(x) = x^4 + 6x^2 + 12$

10. $fog(x) = x$ $\quad$ $gof(x) = |x|$

$fof(x) = x^4$ $\quad$ $gog(x) = \sqrt[4]{x}$

12. (1) b (2) b (3) c (4) c (5) b (6) b (7) c (8) a (9) a (10) c (11) d

Exercise 4.1

1. (1) $\left\{\frac{k\pi - 1}{2} \mid k \in Z\right\}$ (2) $\left\{(2k + 1)\frac{\pi}{6} - \frac{2}{3} \mid k \in Z\right\}$ (3) $\left\{(4k - 1)\frac{\pi}{2} \mid k \in Z\right\}$

(4) $\left\{2k\pi \mid k \in Z\right\}$ (5) $\left\{(2k + 1)\frac{\pi}{6} \mid k \in Z\right\}$ (6) $\emptyset$

2. (1) $[-2, 8]$ (2) $\{p \mid p \leq 2, p \in R\}$ (3) $[-4, -1]$ (4) $[0, 3]$

(5) $\{p \mid p \geq 0, p \in R\}$ (6) $R - (-5, 1)$

4. $\dfrac{2x(x+1)}{2x^2 + 2x + 1}$ **5.** 65 **6.** $\frac{5}{13}$

Exercise 4.2

1. (1) $\frac{4\pi}{3}$ (2) $\frac{5\pi}{12}$ (3) $\frac{121\pi}{540}$ (4) $\frac{221\pi}{360}$

2. (1) 12° (2) 458° 10' 54" (3) 5° 37' 30" (4) 14° 19' 5" **3.** 19° 5' 27"

4. 10π **5.** 16° 2' 11" **6.** 22 : 13 **7.** 105°, $\frac{7\pi}{12}$ **8.** $\frac{\pi}{4}$ **9.** $\frac{3}{4}$

Exercise 4.3

1. $sin\theta = \frac{3}{5}$, $cos\theta = \frac{-4}{5}$, $tan\theta = \frac{-3}{4}$, $sec\theta = \frac{-5}{4}$, $cot\theta = \frac{-4}{3}$

2. $cos\theta = \frac{-1}{5}$, $tan\theta = 2\sqrt{6}$, $cosec\theta = \frac{-5}{2\sqrt{6}}$, $cot\theta = \frac{1}{2\sqrt{6}}$, $sec\theta = -5$

3. 7 **4.** $\frac{-1}{2}$ **5.** $sec\theta = \frac{p^2+1}{2p}$, $tan\theta = \frac{p^2-1}{2p}$, $sin\theta = \frac{p^2-1}{p^2+1}$ **6.** $\frac{1}{2}$

Exercise 4

14. (1) c (2) d (3) c (4) c (5) b (6) c (7) b (8) b (9) b (10) a
(11) a (12) d (13) a (14) a (15) b (16) b (17) b (18) d (19) b (20) b
(21) d

Exercise 5.1

1. $\frac{7}{2}$ **2.** 6 **3.** 6 **5.** $\frac{1+3\sqrt{3}}{8}$ **6.** $\frac{90-53\sqrt{3}}{6}$

Exercise 5.3

1. (1) $n = 2$, $\alpha = 30^\circ$ (2) $n = 3$, $\alpha = 45^\circ$ (3) $n = 4$, $\alpha = 45^\circ$

2. (1) $n = 2$, $\alpha = 120^\circ$ (2) $n = 2$, $\alpha = -45^\circ$ (3) $n = 4$, $\alpha = -30^\circ$

Exercise 5

4. (1) $n = 3$, $\alpha = -240^\circ$ (2) $n = 5$, $\alpha = -200^\circ$ (3) $n = 1$, $\alpha = -180^\circ$

5. (1) a (2) a (3) b (4) a (5) a (6) a (7) b (8) d (9) a (10) b

Exercise 6.1

1. $\left(\frac{13}{5}, \frac{-9}{5}\right)$ **2.** (2, 15) **3.** $\lambda = 8:5$ **4.** (3, 4), (5, 6) **5.** $\left(\frac{27}{2}, \frac{9}{2}\right)$, $\left(\frac{-3}{2}, \frac{3}{2}\right)$

6. $\frac{-9}{10}$

Exercise 6.2

1. (7, 0) **2.** Maximum : 27, Minimum : 15 **3.** 6

4. $\overleftrightarrow{AB} = \left\{(x, y) \middle| \begin{array}{l} x = -13t+3 \\ y = -2t+2 \end{array}; t \in R\right\}$; $\overrightarrow{AB} = \left\{(x, y) \middle| \begin{array}{l} x = -13t+3 \\ y = -2t+2 \end{array}; t \geq 0\right\}$

$\overline{AB} = \left\{(x, y) \middle| \begin{array}{l} x = -13t+3 \\ y = -2t+2 \end{array}; t \in [0,1]\right\}$;

$\overleftrightarrow{AB} = \left\{(x, y) \middle| \begin{array}{l} x = -13t+3 \\ y = -2t+2 \end{array}; t \in R - [0,1]\right\}$ **6.** $4x + 3y + 1 = 0$

Exercise 6.3

1. $\frac{-1}{2}$, 2 **2.** $\frac{-5}{6}, \frac{-3}{5}, -2$ **3.** $\frac{2}{3}$, 14 **5.** $\frac{\pi}{4}$ **9.** $\frac{-2}{7}$ **11.** $\frac{3}{2}$

13. 1, 2 or $\frac{1}{2}$, 1 or −1, −2 or $\frac{-1}{2}$, −1

Exercise 6.4

1. $x - 5y + 27 = 0$ **2.** $x + y - 4 = 0$ **3.** $x - y + 1 = 0, x + y - 3 = 0$

5. $15x - 10y + 12 = 0$ **7.** $x + \sqrt{3}y - 3 = 0$ **8.** $\sqrt{3}x + y - 4 = 0$

9. $(2 + \sqrt{3})x - y - \sqrt{3} = 0, (2 - \sqrt{3})x - y + \sqrt{3} = 0$

10. $x + y - 4 = 0, x + 9y = 12$

11. $(\sqrt{3} + 1)x + (\sqrt{3} - 1)y = 4, (\sqrt{3} - 1)x - (\sqrt{3} + 1)y = 4$

12. $7x - 4y - 3 = 0, 7x - 4y + 25 = 0, 4x + 7y - 11 = 0$

13. $7x + y - 9 = 0, x - 7y + 13 = 0$

Exercise 6

3. $3x \pm 4y = 36, -3x \pm 4y = 36$ or $4x \pm 3y = 36, -4x \pm 3y = 36$ **6.** (0, 5), $\left(0, \frac{-15}{2}\right)$

7. $x - y + 1 = 0, x - 7y + 19 = 0$ **8.** (1, −2), (−1, 3)

9. $x + y = 5, 3x = 2y$ **10.** $x + 2y = 6$ **12.** $3x + 4y = 12, 4x + 3y = 12$

13. (8, 0), (−2, 0) **14.** $x = -1$ **15.** $x + 2y = 5$ **16.** $2x - y + 3 = 0$

17. $8x - 3y = 0, 23x + 23y - 11 = 0, 23x - 23y + 5 = 0$

18. (1) d (2) a (3) d (4) b (5) d (6) b (7) d (8) a (9) b (10) d (11) b

Exercise 7.1

1. 320 **2.** 2240 **3.** 120, 24, 24 **4.** 136 **5.** 2339766 **6.** 90, 90, 225, 225

7. 1204533

Exercise 7.2

1. (1) 1680 (2) 504 (3) 720 **2.** (1) 720 (2) 20160 (3) 72

4. (1) $r = 13$ (2) $r = 6$ **5.** $n = 6$ **6.** $r = 41$ **8.** $n = 3$ **9.** $n = 5$

10. 64, 24 **11.** 2^n **12.** 4^{10} **13.** 16 **14.** 67600, 58500 **15.** 729

16. $m!\ {}_{m+1}P_n$ **17.** (i) $2(n-1)!, n! - 2(n-1)!$ **18.** 96 **19.** 8 **20.** 2880, 1152

21. 518400 **22.** 30, 360 **23.** 24, 20th **24.** 108 **25.** 24, 24, 24 **26.** 12

Exercise 7.3

1. 3rd **2.** NIGAA, 60th **3.** 127 **4.** 5274 **5.** 133320 **6.** 1440

7. 36 **8.** 1440 **9.** 12, 0 **10.** 144 **11.** 24, 108th **12.** 1260

13. 1663200 **14.** $x = 6$

Exercise 7.4

1. (1) 28 (2) 10 (3) 210 **2.** $n = 14$ **3.** (i) $r = 7$, (ii) $r = 8$
4. $n = 8$, $r = 4$ **5.** $n = 12$, $r = 4$ **7.** $r = 6$ or 8

Exercise 7.5

1. 80, 31 **2.** (1) 1050 (2) 2534 (3) 420 (4) 469 **3.** (1) 28561 (2) 495 (3) 29900
4. 64 **5.** 756756 **6.** 14968800 **7.** $n! - 2(n-1)!$ **8.** $\frac{n(n-3)}{2}$
9. $n = 11$ **10.** (i) $\binom{n}{3}$ (ii) $n(n-4)$ (iii) n (iv) $\frac{n(n-4)(n-5)}{6}$

Exercise 7

1. (1) 1230 (2) 1 (3) 321 (4) 321 **2.** 168 **3.** 12 **4.** (i) 15 (ii) 120
5. $\binom{n}{r} - \binom{n-2}{r-2}$ **6.** 85 **7.** 1023 **8.** $\frac{mn}{2}(m+n-2)$
9. (1) d (2) a (3) a (4) c (5) d (6) b (7) a (8) c (9) d (10) b
(11) b (12) c (13) d (14) c (15) a (16) c (17) a (18) d (19) d (20) a
(21) d (22) c (23) a (24) d (25) b

Exercise 8.1

1. (1) $\emptyset$ (2) $\{..., -12, -11\}$ (3) $(-\infty, -10)$
2. (1) $\{4, 5, 6, ...\}$ (2) $\{4, 5, 6, ...\}$ (3) $[4, \infty)$
3. (1) $\{4, 5, 6, ...\}$ (2) $\{4, 5, 6, ...\}$ (3) $[4, \infty)$
4. (1) N (2) $\{-3, -2, -1, 0, 1, 2, ...\}$ (3) $[-3, \infty)$
5. $(5, \infty)$ **6.** $(-\infty, 9)$ **7.** $[9, \infty)$ **8.** $(-\infty, 4]$ **9.** $\left(-\infty, \frac{-23}{11}\right)$
10. $(-\infty, -2)$ **11.** (1) $(-\infty, 3) \cup (7, \infty)$ (2) $\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{7}{2}, \infty\right)$
12. (1) $(-\infty, 0) \cup (3, \infty)$ (2) $(-\infty, 2) \cup \left[\frac{11}{5}, \infty\right)$ (3) $\left(\frac{-17}{2}, -5\right)$ **13.** $\left(\frac{-1}{2}, 0\right)$
14. R **15.** R

Exercise 8.2

1. $(-\infty, -1)$ **2.** $(1, \infty)$ **3.** $(-\infty, -12]$ **4.** $(-\infty, -10.8]$ **5.** $(-\infty, -61]$
6. $\left[\frac{-43}{37}, \infty\right)$ **7.** $\left(-\infty, \frac{67}{65}\right)$ **8.** $\left(-\infty, \frac{8}{11}\right]$ **9.** $(0, 2)$
10. $(1, \infty)$ **11.** $(-\infty, 0)$ **12.** $(-\infty, 0)$

Exercise 8.3

1. $[3, 5]$ **2.** $(3, 8)$ **3.** $[4, 6)$ **4.** $(4, 6]$ **5.** $\emptyset$ **6.** $(-\infty, -2]$ **7.** $[2, 5]$
8. $\emptyset$ **9.** $(-\infty, -3)$ **10.** $(-1, 8)$

Exercise 8

1. $R - [-1, 1]$ **2.** $(2, 4]$ **3.** $[R - (-8, 8)] \cup (-5, 5)$ **4.** $[-2, 2]$ **5.** $\emptyset$

6. $\{x \in R \mid 7.95 < x < 8.85\}$ **7.** (1) $x = 1000$ (2) $x > 1000$

8. 31 and 33; 33 and 35; 35 and 37

9. (1) $(-\infty, 5)$ (2) $R - \{0\}$ (3) $(2, \infty)$ (4) $[3, \infty)$ (5) $\left(\frac{-1}{2}, \frac{1}{8}\right) - \{0\}$

(6) $\{..., -4, -3, -2\} \cup \{4, 5, 6,...\}$ (7) $[3, \infty)$ (8) $(-\infty, 0) - \{-1\}$

10. (1) $R - [1, 3]$ (2) 0 (3) $x < 4$ (4) $5 > x > -1$ (5) $\emptyset$

11. (1) $x + y \le 1, x - y \le 1$ (2) $x < 5, y < 1, 2x + y \ge 4$

12. (1) d (2) b (3) b (4) c (5) d (6) b (7) b (8) a (9) c (10) c (11) d (12) a (13) d (14) c (15) a

Exercise 9.1

1. 5.6 **2.** 40 **3.** 5.27 **4.** 7 **5.** 14.52 **6.** 16 **7.** 4.97 **8.** 5.1

9. 8.6 **10.** 16.44, 16.44 **11.** 11.33 **12.** 157.92 **13.** 10.16 **14.** 7.35

Exercise 9.2

1. (i) 3.041 (ii) 10.61 (iii) 3.428 **2.** (i) 2.007 (ii) 6.58

3. (i) $\bar{x} = 9, s = 3.88$ (ii) $\bar{x} = 14, s = 6.7$ **4.** $\bar{x} = 64, s = 1.691$

5. $\bar{x} = 27, s^2 = 132.02, s = 11.49$ **6.** (i) $\bar{x} = 62, s = 14.18$ (ii) $\bar{x} = 93, s = 10.27$ (iii) $\bar{x} = 21.5, s = 12.84$ **7.** 1351.88 **8.** 4.22

Exercise 9.3

1. 0.1062 **2.** Performance of A is better. **3.** Heights show more variability.

4. Yes **5.** Share B has more variation in its price. **6.** Weights show more variability.

7. Chemistry shows highest variability and mathematics shows lowest variability.

8. Group G_2 has more variation.

Exercise 9

1. 4, 9 **2.** 4, 8 **4.** 24, 12 **5.** $\bar{x} = 40.045, s = 14.995$

6. (1) d (2) c (3) a (4) b (5) c (6) b (7) b (8) c (9) a (10) c (11) a (12) d (13) c (14) c (15) a (16) a

Exercise 10.1

1. $U = \{HR_1, HR_2, HR_3, HG_1, HG_2, HG_3, HG_4, T1, T2, T3, T4, T5, T6\}$

2. $U = \{R_1R_2, R_1W_1, R_1W_2, R_1W_3, R_2W_1, R_2W_2, R_2W_3, W_1W_2, W_1W_3, W_2W_3\}$

3. (1) U = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(2) U = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

4. U = {HH, HT, T1, T2, T3, T4, T5, T6}

5. U = {1H, 1T, 3H, 3T, 5H, 5T, 2HH, 2HT, 2TH, 2TT, 4HH, 4HT, 4TH, 4TT, 6HH, 6HT, 6TH, 6TT}

6. U = {R_1R_1, R_1R_2, R_1R_3, R_1W_1, R_1W_2, R_1B
R_2R_1, R_2R_2, R_2R_3, R_2W_1, R_2W_2, R_2B
R_3R_1, R_3R_2, R_3R_3, R_3W_1, R_3W_2, R_3B
W_1R_1, W_1R_2, W_1R_3, W_1W_1, W_1W_2, W_1B
W_2R_1, W_2R_2, W_2R_3, W_2W_1, W_2W_2, W_2B
BR_1, BR_2, BR_3, BW_1, BW_2, BB}

7. U = {AB, AC, AD, BC, BD, CD}

8. U = {(–, *abc*), (*abc*, –), (*ab*, *c*), (*ac*, *b*), (*bc*, *a*), (*a*, *bc*), (*b*, *ac*), (*c*, *ab*)}

9. U = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT1, TTT2, TTT3, TTT4, TTT5, TTT6}

10. $U = \left\{(x, y) \;\middle|\; \begin{matrix} x = a, b, c, d, e, f \\ y = 1, 2, 3, 4, 5, 6 \end{matrix}\right\}$

Exercise 10.2

1. U = {RR, RB, RY, RW, BR, BB, BY, BW, YR, YB, YY, YW, WR, WB, WY, WW}
A = {RR, BB, YY, WW}
B = {RW, BW, YW, WR, WB, WY}
C = {RW, BW, YW, WR, WB, WY, WW}
D = {RB, RY, RW, BR, BY, BW, YR, YB, YW, WR, WB, WY}
$A \cap B = \emptyset$, $B \cup C = C$, $A \cup D = U$, $A \cap D = \emptyset$, $B \subset C$
A and D are mutually exclusive events.

2. A = {HHH, HHT, HTH, THH}
B = {TTH, THT, TTH}
C = {HHH, HHT, HTH, HTT}
D = {THH, HTH, HHT, HTT, THT, TTH, TTT}
$A \cap B = \emptyset$, $C \cap D' = \{HHH\}$
$A \cup C = A = C$, $B \cap C = \emptyset$, $A' \cup C'$ = {HTT, THT, TTH, TTT}

3. $A_2 = \{2, 4, 6, ..., 30\}$, $A_3 = \{3, 6, 9, ..., 30\}$,
$A_4 = \{4, 8, 12, 16, ..., 28\}$, $A_5 = \{5, 10, 15, ..., 30\}$
(1) F (2) T (3) T

4. U = $\{W_1W_2, W_1R, W_1G_1, W_1G_2, W_2R, W_2G_1, W_2G_2, G_1G_2, RG_1, RG_2\}$

A = $\{W_1W_2\}$

B = $\{W_1W_2, W_1R, W_1G_1, W_1G_2, W_2R, W_2G_1, W_2G_2\}$

C = $\{W_1R, W_1G_1, W_1G_2, W_2G_1, W_2G_2, W_2R, RG_1, RG_2\}$

5. A = {2, 4, 6,..., 50}, B = {10, 20, 30, 40, 50}, C = {4, 8, 12,..., 48}

6. U = $\left\{(x, y) \;\middle|\; \begin{matrix} x = 1, 2, 3, 4, 5, 6 \\ y = 1, 2, 3, 4, 5, 6 \end{matrix}\right\}$

A = {(1, 3), (2, 2), (2, 6), (3, 1), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)}

B = {(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)}

C = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (5, 1)}

D = {(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)}

7. U = {*aa, ab, ac, ba, bb, bc, ca, cb, cc*} A = {*ab, ac, ba, ca*}

B = {*aa, bb, cc*} C = {*ac, bc, ca, cb, cc*}

8. U = $\{B_1B_2, B_1B_3, B_1G_1, B_1G_2, B_2B_3, B_2G_1, B_2G_2, B_3G_1, B_3G_2, G_1G_2\}$

E = $\{G_1G_2\}$

F = $\{B_1G_1, B_1G_2, B_2G_1, B_2G_2, B_3G_1, B_3G_2\}$

G = $\{B_1B_2, B_1B_3, B_2B_3, B_1G_1, B_1G_2, B_2G_1, B_2G_2, B_3G_1, B_3G_2\}$

9. A = {1, 2, 3, 4, 5, 6}, B = {3, 6}, C {5, 6}, D = {1}

$A \cap C = \{5, 6\}$, $B \cup C = \{3, 5, 6\}$, $D' \cup C' = \{1, 2, 3, 4, 5, 6\}$

Exercise 10

1. (1) 0.24 (2) 0.11 (3) 0.30 (4) 0.35

2. (1) $\frac{1}{142506}$ (2) $\frac{53130}{142506}$ (3) $\frac{89376}{142506}$ **4.** (1) $\frac{7}{8}$ (2) $\frac{3}{8}$ (3) $\frac{1}{2}$

5. $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{3}{4}$, $P(D) = \frac{1}{12}$

6. $\frac{8}{25}$ **7.** $\frac{7}{8}$, $\frac{7}{456}$ **8.** $\dfrac{\binom{12}{6}\binom{40}{7}}{\binom{52}{13}}$

9. (1) a (2) a (3) b (4) c (5) b (6) b (7) a (8) b (9) c (10) a (11) d (12) c

TERMINOLOGY (In Gujarati)

Antecedent	પૂર્વવિધાન
Arrow diagram	કિરણ આકૃતિ
Associative law	જૂથનો નિયમ
Average deviation	સરેરાશ વિચલન
Biconditional statement	દ્વિપ્રેરણ
Cartesian product	કાર્તેઝિય ગુણાકાર
Cartesian product of sets	ગણોનો કાર્તેઝિય ગુણાકાર
Ceiling function	ન્યૂનતમ પૂર્ણાંક વિધેય
Circular permutation	વર્તુળાકાર ગોઠવણી
Closed interval	સંવૃત્ત અંતરાલ
Closure	સંવૃત્તતા
Codomain	સહપ્રદેશ
Coefficient of variation	ચલનાંક
Combination	સંચય
Complementary event	પૂરક ઘટના
Complementation	પૂરકક્રિયા
Composition of functions	વિધેયોનું સંયોજન
Compound event	સંયુક્ત ઘટના
Compound statement	સંયુક્ત વિધાન
Commutative law	ક્રમનો નિયમ
Conjuction	સંયોજન
Consequent	ઉત્તર વિધાન
Constant function	અચળ વિધેય
Contrapositive	સમાનાર્થી પ્રેરણ
Converse	પ્રતીપ (શરતી વિધાન)
Coordinates of the point of division	વિભાજન બિંદુના યામ
Difference event	તફાવત ઘટના
Difference set	તફાવત ગણ
Disjoint sets	અલગગણ

Disjunction	વિયોજન
Dispersion	પ્રસાર
Distance formula	અંતર સૂત્ર
Division formula	વિભાજન સૂત્ર
Division of a line segment	રેખાખંડનું વિભાજન
Domain	પ્રદેશ
Element	ઘટક (સભ્ય)
Elementary event	મૂળભૂત ઘટનાઓ
Empty set	રિક્ત ગણ
End point	અંત્યબિંદુ
Equal functions	સમાન વિધેયો
Equal sets	સમાન ગણો
Equiprobable events	સમસંભાવી ઘટનાઓ
Event	ઘટના
Even function	યુગ્મ વિધેય
Exclusive or	નિવારક વિકલ્પ
Exhaustive events	નિઃશેષ ઘટનાઓ
Existential quantifier	અસ્તિત્વ કારક
External division	બહિર્વિભાજન
Function	વિધેય
Graph	આલેખ
Greatest integer function	મહત્તમ પૂર્ણાંક વિધેય
Grouped data	વર્ગીકૃત માહિતી
Idempotent law	સ્વયંઘાતી નિયમ
Identical	સમસ્વરૂપ
Identity function	તદેવ વિધેય
Image	પ્રતિબિંબ
Inclusive or	સમાવેશ વિકલ્પ
Implication	પ્રેરણ
Impossible event	અશક્ય ઘટના
Improper subsets	અનુચિત ઉપગણો
Inequality	અસમતા
Intersection operation	છેદક્રિયા

Intersection set	છેદગણ
Intercepts	અંતઃખંડ
Interval	અંતરાલ
Internal division	અંતઃવિભાજન
Linear inequality	સુરેખ અસમતા
Linear permutation	સુરેખ ક્રમચય
Listing method / Roster method	યાદીની રીત
Logical connectives	તાર્કિક કારકો
Mathematically acceptable statement	ગાણિતીક સ્વીકાર્ય વિધાન
Mean	મધ્યક
Measures of dispersion	પ્રસારમાનનાં માપ
Median	મધ્યસ્થ
Method of contradiction	અનિષ્ટાપત્તિની રીત
Mid-point of a line segment	રેખાખંડનું મધ્યબિંદુ
Modulus function	માનાંક વિધેય
Mutually exclusive events	પરસ્પર નિવારક ઘટનાઓ
Negation	નિષેધ
Neutral element	તટસ્થ ઘટક
Odd function	અયુગ્મ વિધેય
Open interval	વિવૃત્ત અંતરાલ
Parallel	સમાંતર
Parametric equations	પ્રચલ સમીકરણો
Period	આવર્તમાન
Periodic function	આવર્તી વિધેય
Permutation	ક્રમચય
Polynomial function	બહુપદી વિધેય
Power set	ઘાતગણ
Pre-image	પૂર્વ-પ્રતિબિંબ
Principal period	મુખ્ય આવર્તમાન
Probability	સંભાવના
Proper subset	ઉચિત ઉપગણો
Property method	ગુણધર્મની રીત
Range	વિસ્તાર

Rational function	સંમેય વિધેય
Reflexivity	સ્વવાચકતા
Relation	સંબંધ
Set function	ગણ વિધેય
Shifting of origin	ઊગમબિંદુનું સ્થાનાંતર
Signum function	ચિહ્ન વિધેય
Simple statement	સાદું વિધાન
Singleton	એકાકી ગણ
Slope	ઢાળ
Standard deviation	પ્રમાણિત વિચલન
Statement	વિધાન
Super set	અધિગણ
Symmetric difference set	સંમિત તફાવત ગણ
Symmetry	સંમિતતા
Transitivity	પરંપરિતા
Trigonometric point function	ત્રિકોણમિતીય બિંદુ વિધેય
Triplet	ત્રય
Truth value	સત્યમૂલ્ય
Ungrouped data	અવર્ગીકૃત માહિતી
Union operation	યોગક્રિયા
Union set	યોગગણ
Unit circle	એકમ વર્તુળ
Unit element	એકમ ઘટક
Universal quantifier	વૈશ્વિક કારક
Universal relation	સાર્વત્રિક સંબંધ
Universal set	સાર્વત્રિક ગણ
Variance	વિચરણ
Visual Presentation of a relation	સંબંધનું દૃશ્ય નિરૂપણ
Void relation	ખાલી સંબંધ