ગુજરાત રાજ્યના શિક્ષણવિભાગના પત્ર-ક્રમાંક
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MATHEMATICS

Standard 12

(Semester IV)

PLEDGE

India is my country.
All Indians are my brothers and sisters.
I love my country and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall respect my parents, teachers and all my elders and treat everyone with courtesy.
I pledge my devotion to my country and its people.
My happiness lies in their well-being and prosperity.

રાજ્ય સરકારની વિનામૂલ્યે યોજના હેઠળનું પુસ્તક

Gujarat State Board of School Textbooks
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PREFACE

The Gujarat State Secondary and Higher Secondary Education Board have prepared new syllabi in accordance with the new national syllabi prepared by the N.C.E.R.T. These syllabi are sanctioned by the Government of Gujarat.

It is a pleasure for the Gujarat State Board of School Textbooks, to place this textbook of **Mathematics** before the students for **Standard 12 (Semester IV)** prepared according to the new syllabus.

The manuscript has been fully reviewed by experts and teachers teaching at this level. Following the suggestions given by teachers and experts, we have made necessary changes in the manuscript before publishing the textbook.

The Board has taken special care to make this textbook interesting, useful and free from errors. However, we welcome suggestions, to enhance the quality of the textbook.

Dr. Bharat Pandit
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Date : 05-08-2015

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FUNDAMENTAL DUTIES

It shall be the duty of every citizen of India :

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongist all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers and wide life, and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievement.

(k) to provide opportunities for education by the parent or the guardian, to his child, or a ward between the age of 6-14 years as the case may be.

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About This Textbook...

We are very pleased to present before you the textbook for Mathematics of semester IV for standard XII following the new syllabus prepared by Gujarat Secondary and Higher Secondary Board on the basis of NCERT syllabus, in extension of Mathematics textbooks of semester I and semester II for standard XI and semester III of standard XII.

This textbook has been prepared originally in English as in the case of textbooks of semester I and II for standard XI and semester III of standard XII. The manuscript has been thoroughly examined by learned teachers from schools and colleges through a workshop organized in the month of June. The suggestions and proper amendments had been accepted and the revised manuscript has been translated in Gujarati. The Gujarati version was also examined by teachers from schools and colleges and the necessary amendments were made. The English manuscript and the translated version in Gujarati were examined by language experts and the corrections were made. This way the final draft of the manuscript was prepared.

A second review had been carried out in the end of July by subject experts from universities and technological colleges. They were retired mathematics professors of eminence. Their recommendations were accepted and amendments were made.

In chapter 1, we apply differentiation to various problems in mathematics like coordinate geometry, approximation; maximum and minimum values of a function and rates of change of one variable with respect to another, especially with respect to time which will consequently help to study applications of differentiation to science. Chapter 2 continues the study of integration which has began in semester III. Here, since the study continues, the prerequisite is knowledge of indefinite integration studied in semester III. Some examples can be studied by techniques of any of the methods from these two chapters. Chapter 3 introduces definite integration. Theorems and examples freely make use of indefinite integration techniques. Chapter 4 is about an application of integration. It shows how to calculate certain areas bounded by some known curves. Chapter 5 is further application of integration to solve differential equations. Here, only some simple techniques are studied. Chapter 6 is the study of algebra of vectors useful in three dimensional geometry. The concept of Vectors was introduced in semester II. These concepts are revised. Abstract approach to vectors and geometrical significance are studied. Chapter 7 deals with applications of vectors to three dimensional geometry of lines and planes.

In between, some explanations are given in boxes. They are meant to explain further the concept introduced earlier or to add some comments on them. They are for more understanding only.

Attractive four-colour title, four-colour printing and figures make this textbook visually rich and adds more to its utility value. Plenty of illustrations and exercises are integrated to explain various concepts and variety of problems. They will help the students in achieving good marks in semester examination as well as competitive examinations.

We thank all who have helped to prepare this textbook. We hope that all students, teachers and parents would like this textbook. Positive suggestions to enhance the quality of this textbook are welcome.

– Authors

1 Applications of Derivatives

Life is good only for two things - discovering mathematics and teaching mathematics.
– Siméon Poisson

Each problem that I solved became a rule, which served afterwards to solve other problems.
– René Des Cartes

1.1 Introduction

We have defined the derivative of a function and studied several methods to find the derivative of a function.

In the introductory article in std. XI, semester II, we had introduced the notion of a derivative using the slope of a tangent to a curve intuitively. Now we will study this application and several other applications of a derivative such as rate of change of a quantity *w.r.t.* another quantity, finding approximate values of a function at some value in its domain, equations of tangents and normals to a curve at a point and the orthogonality of curves, increasing and decreasing functions and maximum and minimum values of a function. These mathematical concepts are used to apply differentiation to find optimum values in Physics, Economics, Social Science, Biology, Chemistry etc. **Des Cartes** and **Newton** explained creation, the shape and colour of rainbows using these ideas. Geophysicists use differential calculus when studying the structure of the earth's crust while searching for oil.

1.2 Rates of Change

Let $s = f(t)$ be the equation of rectilinear motion of a particle, where s represents displacement at time t (i.e. directed distance from origin). If the displacements at time t_1 and t_2 are respectively s_1 and s_2, its average velocity during time interval $t_2 - t_1$ is given by the ratio $\frac{s_2 - s_1}{t_2 - t_1}$. Let $\Delta s = s_2 - s_1$, $\Delta t = t_2 - t_1$ and average velocity $= \frac{\Delta s}{\Delta t}$.

As $t_2 \to t_1$, we get instantaneous velocity v of the particle at time t_1.

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Thus rate of change of displacement $s = f(t)$ *w.r.t.* time t is the instantaneous velocity of the particle at time t.

Similarly for any function $y = f(x)$, $\frac{dy}{dx}$ is the rate of change of $y = f(x)$ *w.r.t.* x.

For another example if volume $V = f(r)$, r radius, $\frac{dV}{dr}$ is the rate of change of volume of a sphere *w.r.t.* radius.

For a 'smooth' continuous curve $y = f(x)$, let $P(x, f(x))$ and $Q(x + h, f(x + h))$ be two points on the curve. (Fig. 1.1)

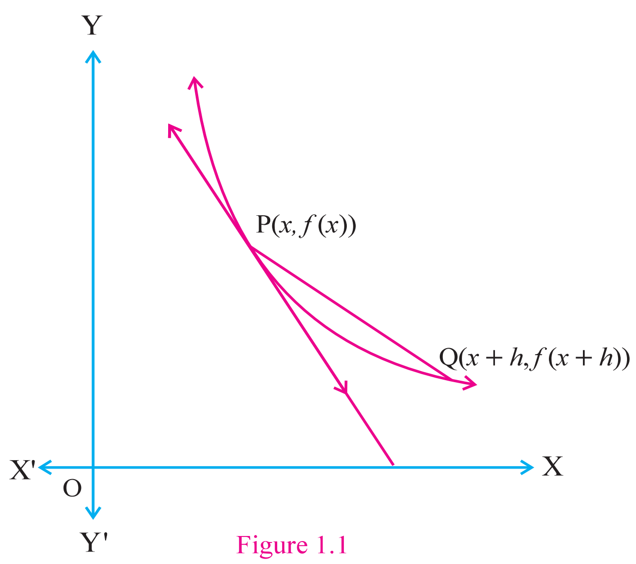


Figure 1.1

Slope of the secant $\overleftrightarrow{PQ} = \dfrac{f(x+h) - f(x)}{x+h-x}$

$$= \frac{f(x+h) - f(x)}{h}$$

As $h \to 0$, Q $\to$ P, P remaining on the curve. Since the curve is 'smooth and continuous',

slope of tangent at P $= \lim\limits_{Q \to P}$ (slope of $\overleftrightarrow{PQ}$)

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f'(x)$$

$\therefore$ The slope of the tangent at $P(x, f(x))$ to the curve $y = f(x)$ is $f'(x)$.

In practice, we encounter many problems in which the rate *w.r.t.* time is required. In these circumstances x, y etc. are functions of time t.

So by Chain rule $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$ will be useful to calculate such rates.

Example 1 : Find the rate of change of volume of a sphere *w.r.t.* radius. Find this rate when $r = 3$ *cm.*

Solution : For a sphere, $V = \frac{4}{3}\pi r^3$, where V is the volume and r is the radius of the sphere.

$\therefore \dfrac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

$\therefore \left(\dfrac{dV}{dr}\right)_{r=3} = 4\pi \times 9 = 36\pi \ cm^3/cm$

$\therefore$ The rate of change of volume of a sphere, *w.r.t.* radius when the radius is 3, is $36\pi \ cm^3/cm$.

Example 2 : The rate of change of volume of a sphere *w.r.t.* time is $16\pi \ cm^3/sec$. Find the rate of change of its surface area *w.r.t.* time at the moment when the radius is 2 *cm*.

Solution : Volume of a sphere, $V = \frac{4}{3}\pi r^3$, where r is the radius

Volume changes *w.r.t.* time. So r and V are functions of time t.

$\therefore \dfrac{dV}{dt} = \dfrac{dV}{dr} \cdot \dfrac{dr}{dt} = \frac{4}{3}\pi \times 3r^2 \dfrac{dr}{dt}$

$$= 4\pi r^2 \frac{dr}{dt}$$

$\therefore 16\pi = 4\pi r^2 \dfrac{dr}{dt}$ $\qquad \left(\dfrac{dV}{dt} = 16\pi \ cm^3/sec\right)$

$\therefore \dfrac{dr}{dt} = \dfrac{4}{r^2} \ cm/sec$

Now surface area of a sphere, $S = 4\pi r^2$

$\therefore \dfrac{dS}{dt} = \dfrac{dS}{dr} \cdot \dfrac{dr}{dt}$

$$= 8\pi r \frac{dr}{dt}$$

$$= 8\pi r \cdot \frac{4}{r^2}$$

$= \dfrac{32\pi}{r} = 16\pi \ cm^2/sec$ $\qquad (r = 2)$

$\therefore \left(\dfrac{dS}{dt}\right)_{r=2} = \dfrac{32\pi}{2} = 16\pi \ cm^2/sec$

$\therefore$ The rate of change of surface area of the sphere is $16\pi \ cm^2/sec$, when $r = 2$ *cm*.

Example 3 : A stone is dropped into a quiet lake and circular ripples are formed. Circular wave fronts move at the speed of radius increasing at the rate of 5 *cm/sec*. How fast is the area increasing when the radius is 10 *cm* ?

Solution : Area of a circle, A = πr^2, where r is the radius.

$$\therefore \quad \frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$= 2\pi r \frac{dr}{dt}$$

Now $r = 10$ *cm* and $\frac{dr}{dt} = 5$ *cm/sec*

$$\therefore \quad \frac{dA}{dt} = 2\pi \times 10 \times 5 = 100\pi \ cm^2/sec.$$

$\therefore$ The area enclosed by the waves increases at the rate of 100π cm^2/sec.

We say as x increases, y increases if and only if $\frac{dy}{dx} > 0$. We say as x increases, y decreases if and only if $\frac{dy}{dx} < 0$. Later on in this chapter, we will study the concept of an increasing (decreasing) function. If $\frac{dy}{dx} > 0$, then y is an increasing function of x and if $\frac{dy}{dx} < 0$, then y is a decreasing function of x.

Example 4 : Air is being pumped into a spherical balloon so that its volume increases at the rate 80 cm^3/sec. How fast is the radius of the balloon increasing when the diameter is 32 *cm* ?

Solution : Volume of a sphere, V = $\frac{4}{3}\pi r^3$, where r is its radius.

$$\therefore \quad \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = \frac{4}{3}\pi(3r^2) \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now $\frac{dV}{dt} = 80$ cm^3/sec, $r = \frac{32}{2} = 16$ *cm*

$$\therefore \quad 80 = 4\pi \cdot 256 \frac{dr}{dt}$$

$$\therefore \quad \frac{dr}{dt} = \frac{5}{64\pi} \ cm/sec$$

$\therefore$ The radius increases at the rate of $\frac{5}{64\pi}$ *cm/sec*

Example 5 : A ladder 5 *m* long is leaning against a wall. The bottom of the ladder is pulled away along the floor away from the wall at the rate 3 *cm/sec*. How fast is its height on the wall decreasing when the foot of the ladder is 4 *m* away from the wall ?

Solution : Let l be the length of the ladder. A is the end-point of the ladder on the wall. C is the point where the ladder touches the ground. $\overline{AB}$ is a part of the wall.

From the figure 1.2, $x^2 + y^2 = l^2$.

$$\therefore \quad 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\therefore \quad x \frac{dx}{dt} + y \frac{dy}{dt} = 0$$

Now $l = 5$ *m*, $y = 4$ *m*

$$\therefore \quad x = \sqrt{l^2 - y^2}$$

$$= \sqrt{25 - 16}$$

$$= 3 \ m$$

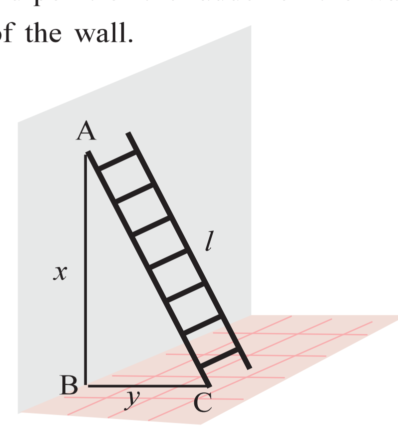


Figure 1.2

$\frac{dy}{dt} = 3 \ cm/sec$ $\left(\frac{dy}{dt} > 0 \text{ as } y \text{ is increasing when } t \text{ increases}\right)$

$\therefore \quad \frac{dy}{dt} = 0.03 \ m/sec$

$\therefore \quad 3\frac{dx}{dt} + 4(0.03) = 0$

$\therefore \quad \frac{dx}{dt} = -0.04$ $\left(\frac{dx}{dt} < 0 \text{ as } x \text{ is decreasing when } t \text{ is increasing}\right)$

$\therefore$ The height of the ladder on the wall is decreasing at the rate of 4 *cm/sec*.

Example 6 : Find the point on the curve $y = x^3 + 7$, where the non-zero rate of change of y *w.r.t.* time is 3 times the rate of change of x *w.r.t.* time.

Solution : We have $y = x^3 + 7$.

Given $\frac{dy}{dt} = 3\frac{dx}{dt}$ **(i)**

Now $\frac{dy}{dt} = 3x^2\frac{dx}{dt}$ **(ii)**

$\therefore$ From (i) and (ii) $3\frac{dx}{dt} = 3x^2\frac{dx}{dt}$

$\therefore \quad x^2 = 1$ $\left(\frac{dx}{dt} \neq 0\right)$

$\therefore \quad x = 1$ or -1

$\therefore \quad y = 8$ or 6

$\therefore$ The required points on $y = x^3 + 7$ where the non-zero rate of change of y *w.r.t.* t is 3 times rate of change of x *w.r.t.* t are $(1, 8)$ and $(-1, 6)$.

Example 7 : On a national highway, a car is driven East at a speed of 60 *km/hr* and a staff bus is driven South at a speed of 50 *km/hr*. Both are headed for the intersection of the roads. The car is 600 *m* away and the bus is 800 *m* away from the intersection. Find the rate at which the car and the bus are approaching each other.

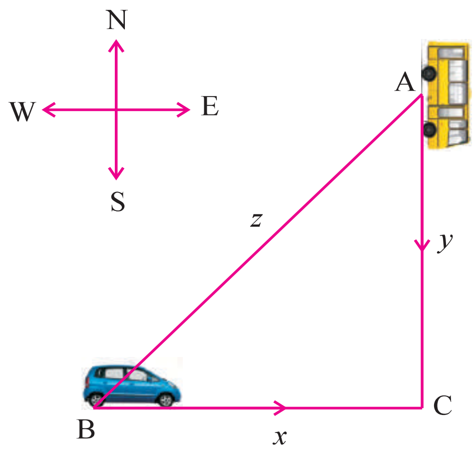


Figure 1.3

Solution : C is the intersection of the roads. B represents the position of the car and A represents the position of the bus at a time. Let $BC = x$, $AC = y$ at a moment. The distance between the car and the bus is $AB = z$.

From figure 1.3, $x^2 + y^2 = z^2$.

$\frac{dx}{dt} = -60 \ km/hr$, $\frac{dy}{dt} = -50 \ km/hr$, negative as x and y are decreasing functions of time.

$x = 0.6 \ km$ and $y = 0.8 \ km$

$\therefore \quad z = \sqrt{(0.6)^2 + (0.8)^2} = 1 \ km$

Now, $x^2 + y^2 = z^2$

$$\therefore \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\therefore \quad \frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{1}(0.6(-60) + 0.8(-50))$$

$$= -76 \ km/hr$$

$\therefore$ The bus and the car are approaching each other at the rate of 76 *km/hr*

Example 8 : The total cost in rupees associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 10x + 10000$. Find the marginal cost, when 20 units are produced.

[Note : Marginal cost means the rate of change of total cost *w.r.t.* the output *x*.]

Solution : We have $C(x) = 0.005x^3 - 0.02x^2 + 10x + 10000$

$$\therefore \quad \text{Marginal cost MC} = \frac{dC}{dx} = (0.005)3x^2 - (0.02)2x + 10$$

$$\therefore \quad \left(\frac{dC}{dx}\right)_{x=20} = (0.005)1200 - (0.02)40 + 10$$

$$= 6 - 0.8 + 10$$

$$= 15.2$$

$\therefore$ The required marginal cost is ₹ 15.2.

Example 9 : The total revenue in rupees received from the sale of x units is given by $R(x) = 10x^2 + 20x + 1500$. Find the marginal revenue when $x = 5$.

[Note : Marginal revenue means the rate of change of total revenue *w.r.t.* the number of units sold.]

Solution : We have $R(x) = 10x^2 + 20x + 1500$

$$\therefore \quad \frac{dR}{dx} = 20x + 20$$

$$\therefore \quad \left(\frac{dR}{dx}\right)_{x=5} = 100 + 20 = 120$$

$\therefore$ The marginal revenue is ₹ 120.

Example 10 : The volume of a cube is increasing at the rate of 12 cm^3/sec. Find the rate at which the surface area is increasing, when the length of the edge of the cube is 10 *cm*.

Solution : Volume of a cube, $V = x^3$, where x is the length of an edge.

$$\therefore \quad \frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$$

$$= 3x^2\frac{dx}{dt}$$

But $\frac{dV}{dt} = 12 \ cm^3/sec$

$$\therefore \quad 12 = 3x^2\frac{dx}{dt}$$

$$\therefore \quad \frac{dx}{dt} = \frac{4}{x^2}$$

Now surface area of the cube, S = $6x^2$

$$\therefore \quad \frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt}$$

$$= 12x \frac{dx}{dt}$$

$$= 12x \times \frac{4}{x^2}$$

$$= \frac{48}{x}$$

$$\therefore \quad \left(\frac{dS}{dt}\right)_{x = 10} = \frac{48}{10}$$

$$\therefore \quad \frac{dS}{dt} = 4.8 \ cm^2/sec$$

$\therefore$ The rate of increase of surface area is 4.8 cm^2/sec.

Example 11 : A water tank is in the shape of an inverted cone. The radius of the base is 4 *m* and the height is 6 *m*. The tank is being emptied for cleaning at the rate of 2 m^3/min. Find the rate at which the water level will be decreasing, when the water is 3 *m* deep.

Solution : Let the height of the water level at any instant be *h* and the radius of water cone be *r*.

Using similarity of triangles, $\frac{OA}{BC} = \frac{OD}{BD}$

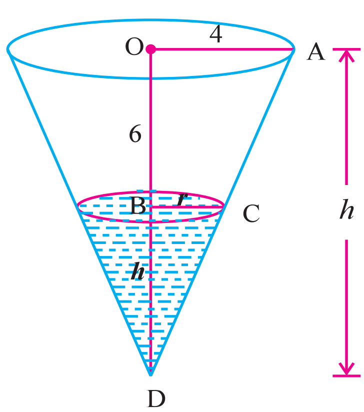


Figure 1.4

$$\therefore \quad \frac{4}{r} = \frac{6}{h}$$

$$\therefore \quad \frac{r}{h} = \frac{2}{3}$$

$$\therefore \quad r = \frac{2h}{3}$$

Now the volume of water at any time *t* is,

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{4h^2}{9}\right) h$$

$$= \frac{4\pi h^3}{27}$$

$$\therefore \quad \frac{dV}{dt} = \frac{4\pi}{27}\left(3h^2 \frac{dh}{dt}\right)$$

$$\therefore \quad \frac{dV}{dt} = \frac{4\pi h^2}{9}\frac{dh}{dt}$$

$$\therefore \quad \frac{dh}{dt} = \frac{9}{4\pi h^2}\frac{dV}{dt}$$

Now $\frac{dV}{dt} = -2 \ m^3/min$ **(Volume is decreasing)**

$$\therefore \quad \frac{dh}{dt} = \frac{9}{4\pi h^2}(-2)$$

$$\therefore \quad \left(\frac{dh}{dt}\right)_{h = 3} = \frac{-9}{2\pi(9)}$$

$$= -\frac{1}{2\pi}$$

$\therefore$ The height is decreasing at the rate $\frac{1}{2\pi}$ *m/min*.

Exercise 1.1

1. The surface area of a cube increases at the rate of 12 cm^2/sec. Find the rate at which its volume increases, when its edge has length 5 cm.
2. Find the rate of change of volume of a cone *w.r.t.* its radius, when the height is kept constant.
3. Find the rate of change of lateral surface area of a cone *w.r.t.* to its radius, when the height is kept constant.
4. The volume of a sphere increases at the rate 8 cm^3/sec. Find the rate of increase of its surface area, when the radius is 4 cm.
5. The volume of a closed hemisphere increases at the rate of 4 cm^3/sec. Find the rate of increase of its surface area, when the radius is 4 cm.
6. A cylinder is heated so that its radius remains twice of its height at any moment. Find the rate of increase of its volume, when the radius is 3 cm and the radius increases at the rate 2 cm/sec. Find the rate of increase of its total surface area also in this case.
7. A stone is dropped into a quiet lake and ripples move in circles with radius increasing at a speed 4 cm/sec. At the time when the radius of a circular wave is 10 cm, find the rate at which the area enclosed by the waves increases.
8. A rectangular plate is expanding. Its length x is increasing at the rate 1 cm/sec and its width y is decreasing at the rate 0.5 cm/sec. At the moment when $x = 4$ and $y = 3$, find the rate of change of (1) its area (2) its perimeter (3) its diagonal.
9. A ladder 7.5 m long leans against a wall. The ladder slides along the floor away from the wall at the rate of 3 cm/sec. How fast is the height of the ladder on the wall decreasing, when the foot of the wall is 6 m away from the wall ?
10. A concrete mixture is pouring on ground at the rate of 8 cm^3/sec to form a cone in such a way that the height of the cone is always $\frac{1}{4}th$ of the radius at the time. Find the rate of increase of the height, when the radius is 8 cm.
11. The total cost in rupees associated with the production of x units is given by
    $C(x) = 0.005x^3 - 0.004x^2 + 20x + 1000$. Find the marginal cost when $x = 10$.
12. The total revenue in rupees received from the sale of x units of a product is given by
    $R(x) = 20x^2 + 15x + 50$. Find the marginal revenue when $x = 15$.
13. A man 2 m tall walks away at a rate of 4 m/min from source of light 6 m high from the ground. How fast is the length of his shadow changing ?
14. Area of a triangle is increasing at a rate of 4 cm^2/sec and its altitude is increasing at a rate of 2 cm/sec. At what rate is the length of the base of the triangle changing, when its altitude is 20 cm and area is 30 cm^2 ?
15. Two sides of a triangle have lengths 4 m and 5 m. The measure of the angle between them is increasing at a rate of 0.05 rad/sec. Find the rate at which the area of the triangle increases, when the angle between the sides (fixed) has measure $\frac{\pi}{3}$.

16. Two sides of a triangle have lengths 10 *m* and 15 *m*. The angle between them has the measure increasing at a rate of 0.01 *rad/sec*. How fast is the third side increasing when the angle between sides having lengths 10 *m* and 15 *m* (fixed) has measure $\frac{\pi}{3}$?

17. The radius of a spherical balloon increases at the rate of 0.3 *cm/sec*. Find the rate of increase of its surface area, when the radius is 5 *cm*.

18. If $y = 3x - x^3$ and x increases at the rate of 3 units per second, how fast is the slope of the curve changing when $x = 2$?

19. A particle moves on the curve $y = x^3$. Find the points on the curve at which the y-coordinate changes *w.r.t.* time thrice as fast as x-coordinate.

20. Find the points on the parabola $y^2 = 4x$ for which the rate of change of abscissa and ordinate is same.

*

1.3 Increasing and Decreasing Functions

We have seen in the third semester that $f(x) = a^x$, $a \in R^+$, $x \in R$ is an increasing function of x for $a > 1$ i.e. as x increases, the value of $f(x)$ also increases. This was observed looking at the graph of $f(x) = a^x$. But this is not always possible or even convenient for all functions. Let us find a criterion for this.

Consider $f(x) = 2x + 3$, $x \in R$. Here obviously,

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2$$
$$\Rightarrow 2x_1 + 3 < 2x_2 + 3$$
$$\Rightarrow f(x_1) < f(x_2), \quad \forall x_1, x_2 \in R$$

Thus f is 'increasing' on R. We have observed *sine* is increasing in $\left(0, \frac{\pi}{2}\right)$.

Consider $f(x) = x^2$, $x \in R$ (Fig. 1.5)

In the first quadrant $f(x) = x^2$ increases with x and as x proceeds towards right of Y-axis, y-coordinate increases. But on the left of Y-axis, as x increases, y decreases.

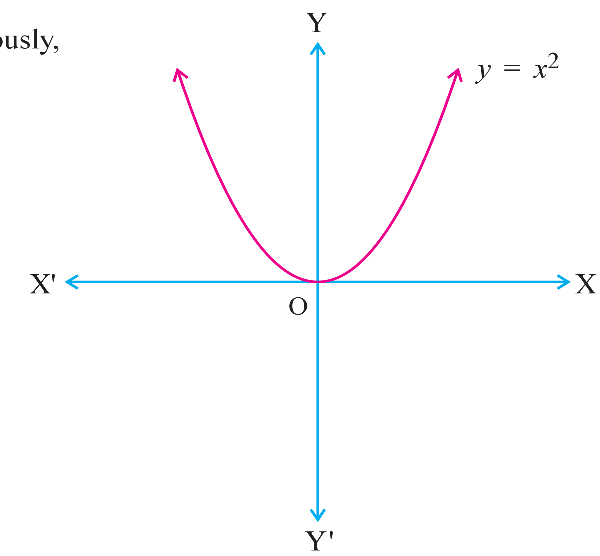


Figure 1.5

Now let us formally define this concept.

Definition : Let (a, b) be a subset of the domain of a function. We say,

(1) ***f* is increasing on (*a, b*) (denoted by *f* ↑) if**

$$x_1 < x_2 \Rightarrow f(x_1) \le f(x_2), \forall x_1, x_2 \in (a, b)$$

(2) ***f* is strictly increasing on (*a, b*) if** $x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \forall x_1, x_2 \in (a, b)$

(3) ***f* is decreasing on (*a, b*) (denoted by *f* ↓) if** $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2), \forall x_1, x_2 \in (a, b)$

(4) ***f* is strictly decreasing on (*a, b*) if** $x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \forall x_1, x_2 \in (a, b)$

We say f is increasing (or decreasing or strictly increasing or strictly decreasing) on R or a subset of R which is a subset of its domain D, if f is increasing in every open interval (or decreasing or strictly increasing or strictly decreasing) which is a subset of R or of D as the case may be.

Consider following graphs :

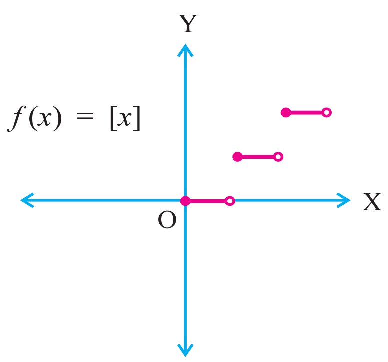


Figure 1.6

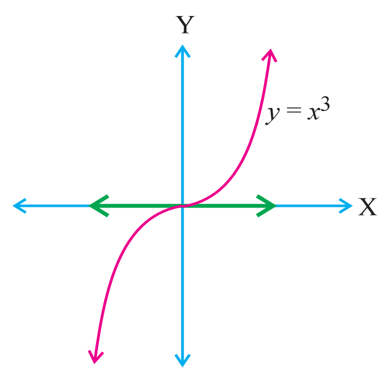


Figure 1.7

Figure 1.6 is the graph of the increasing function $f(x) = [x]$ in $[0, 1)$, $[1, 2)$... It is increasing on R.

Note : See that increasing actually means non-decreasing.

Figure 1.7 represents the graph of a strictly increasing function.

Figure 1.8 is the graph of $f(x) = \begin{cases} 2 - x & 0 \le x \le 1 \\ 1 & 1 < x < 2 \\ 3 - x & x \ge 2 \end{cases}$

Here f is decreasing for $x \ge 0$.

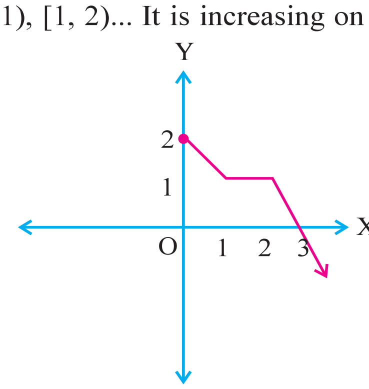


Figure 1.8

Note : f is constant, so f is non-increasing and non-decreasing for $1 < x < 2$.

$f(x) = x^2$, $x < 0$ represents the graph of a decreasing function. (Fig. 1.9)

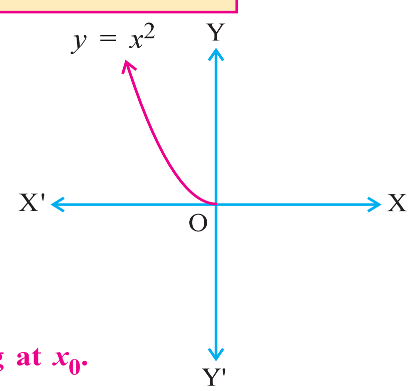


Figure 1.9

A function increasing or decreasing at a point :

Let f be defined on a domain containing an open interval I. Let $x_0 \in$ I and let some h, $h > 0$ be so small that $(x_0 - h, x_0 + h) \subset$ I.

If f is increasing in $(x_0 - h, x_0 + h)$, we say f is increasing at x_0.

If f is decreasing in $(x_0 - h, x_0 + h)$, we say f is decreasing at x_0.

If f is strictly increasing in $(x_0 - h, x_0 + h)$, we say f is strictly increasing at x_0.

If f is strictly decreasing in $(x_0 - h, x_0 + h)$, we say f is strictly decreasing at x_0.

If f is increasing for all $x_0 \in$ I (decreasing, strictly decreasing or strictly increasing), then we say f is increasing (decreasing, strictly decreasing or strictly increasing) on I.

Now we will find some criteria to determine the nature of a function whether increasing or decreasing.

Theorem 1.1 : If f is continuous on $[a, b]$ and differentiable in (a, b), then

(1) f is increasing on (a, b) if $f'(x) \geq 0 \quad \forall x \in (a, b)$

(2) f is decreasing on (a, b) if $f'(x) \leq 0 \quad \forall x \in (a, b)$

(3) f is strictly increasing on (a, b) if $f'(x) > 0 \quad \forall x \in (a, b)$

(4) f is strictly decreasing on (a, b) if $f'(x) < 0 \quad \forall x \in (a, b)$

(5) f is constant on (a, b) if $f'(x) = 0 \quad \forall x \in (a, b)$

Proof : Let $x_1 \in (a, b)$, $x_2 \in (a, b)$ and $x_1 < x_2$. Since f is continuous on $[a, b]$ and differentiable in (a, b), there exists $c \in (x_1, x_2) \subset (a, b)$ so that $f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$.

(Mean value theorem)

(1) If $f'(x) \geq 0$, $\forall x \in (a, b)$, $f'(c) \geq 0$ as $c \in (x_1, x_2) \subset (a, b)$.

Since $x_1 < x_2$, $x_2 - x_1 > 0$

$\therefore \quad f'(c)(x_2 - x_1) \geq 0$

$\therefore \quad f(x_2) - f(x_1) \geq 0$

$\therefore \quad f(x_1) \leq f(x_2)$

$\therefore \quad x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2), \quad \forall x_1, x_2 \in (a, b)$

$\therefore$ f is increasing on (a, b).

(2) If $f'(x) \leq 0$, $\forall x \in (a, b)$, $f'(c) \leq 0$.

$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2), \quad \forall x_1, x_2 \in (a, b)$,

$\therefore$ f is decreasing on (a, b).

(3) If $f'(x) > 0$, $\forall x \in (a, b)$, $f'(c) > 0$.

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \quad \forall x_1, x_2 \in (a, b)$,

$\therefore$ f is strictly increasing on (a, b).

(4) If $f'(x) < 0$, $\forall x \in (a, b)$, $f'(c) < 0$.

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2), \quad \forall x_1, x_2 \in (a, b)$,

$\therefore$ f is strictly decreasing on (a, b).

(5) If $f'(x) = 0$, $\forall x \in (a, b)$, $f'(c) = 0$.

$f(x_2) - f(x_1) = 0, \quad \forall x_1, x_2 \in (a, b)$

$\therefore \quad f(x_2) = f(x_1) \qquad \forall x_1, x_2 \in (a, b)$

$\therefore$ f is a constant function on (a, b).

Note : Do you remember how arbitrary constant was introduced in indefinite integration ?

In view of the remark preceding the theorem, f is increasing or decreasing on $[a, b]$ also according as $f'(x) \geq 0$ or $f'(x) \leq 0$ respectively in (a, b).

Similar remarks apply for strictly increasing and strictly decreasing functions.

Example 12 : Prove that *sine* function is strictly increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution : $\frac{d}{dx} \sin x = \cos x$

$\cos x > 0$, if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore$ *sine* function is strictly increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example 13 : Prove that $f(x) = \left(\frac{1}{2}\right)^x$ is strictly decreasing on R.

Solution : $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$

$\therefore$ $f'(x) = -2^{-x} \log 2 < 0$ as $\log_e 2 > 0$ and $2^{-x} > 0$.

$\therefore$ f is strictly decreasing on any interval $(a, b) \subset$ R.

$\therefore$ $f(x) = \left(\frac{1}{2}\right)^x$ is strictly decreasing on R.

Example 14 : Prove that $f(x) = \tan x$, $x \in \mathrm{R} - \left\{(2k-1)\frac{\pi}{2} \;\middle|\; k \in \mathrm{Z}\right\}$ is strictly increasing in every quadrant.

Solution : $f(x) = \tan x$

$\therefore$ $f'(x) = sex^2 x > 0 \quad \forall x \in \mathrm{R} - \left\{(2k-1)\frac{\pi}{2} \;\middle|\; k \in \mathrm{Z}\right\}$.

$\therefore$ $f(x) = \tan x$ is strictly increasing in all intervals like $\left(0, \frac{\pi}{2}\right)$, $\left(\frac{\pi}{2}, \pi\right)$,... etc.

$\therefore$ $f(x) = \tan x$ is strictly increasing in all quadrants.

Example 15 : Prove that $f:$ R $\to$ R, $f(x) = ax + b$ is strictly increasing for $a > 0$ and strictly decreasing for $a < 0$.

Solution : $f(x) = ax + b$

$\therefore$ $f'(x) = a$

$\therefore$ If $a > 0$, $f'(x) > 0$ and so f is strictly increasing on R.

$\therefore$ If $a < 0$, $f'(x) < 0$ and so f is strictly decreasing on R.

As an example $f(x) = 5x + 7$ is strictly $\uparrow$ and $f(x) = -2x + 3$ is strictly $\downarrow$.

Example 16 : Prove that $f(x) = x^3$, $x \in$ R is increasing on R.

Solution : $f'(x) = 3x^2 \geq 0$

$\therefore$ f is $\uparrow$ on any $(a, b) \subset$ R

$\therefore$ f is $\uparrow$ on R.

Example 17 : Prove that $f:$ R $\to$ R, $f(x) = x^3 + 3x^2 + 5x$ is strictly increasing on R.

Solution : $f(x) = x^3 + 3x^2 + 5x$

$$\therefore \quad f'(x) = 3x^2 + 6x + 5$$
$$= 3x^2 + 6x + 3 + 2$$
$$= 3(x+1)^2 + 2 > 0, \quad \forall x \in \mathrm{R}$$

$\therefore$ f is strictly increasing on R.

Example 18 : Find the intervals in which $f: R \to R$, $f(x) = x^2 - 6x + 15$ is strictly increasing or strictly decreasing.

Solution : $f(x) = x^2 - 6x + 15$

$\therefore \quad f'(x) = 2x - 6$

If $x < 3$, $2x < 6$ and $f'(x) < 0$.

$\therefore$ f is strictly decreasing for $x \in (-\infty, 3)$.

If $x > 3$, $2x > 6$ and $f'(x) > 0$.

$\therefore$ f is strictly increasing for $x \in (3, \infty)$.

Example 19 : Determine in which intervals the function $f: R \to R$, $f(x) = x^3 - 6x^2 - 36x + 2$ is increasing and where it is decreasing.

Solution : $f(x) = x^3 - 6x^2 - 36x + 2$

$$\therefore \quad f'(x) = 3x^2 - 12x - 36$$
$$= 3(x^2 - 4x - 12)$$
$$= 3(x - 6)(x + 2)$$

(1) If $x < -2$, then $x < 6$

$\therefore$ $x + 2 < 0$, $x - 6 < 0$

$\therefore$ $f'(x) = 3(x - 6)(x + 2) > 0$

$\therefore$ f is $\uparrow$ in $(-\infty, -2)$. **(Infact strictly $\uparrow$)**

(2) If $-2 < x < 6$, then $x + 2 > 0$, $x - 6 < 0$

$\therefore$ $f'(x) = 3(x - 6)(x + 2) < 0$

$\therefore$ f is $\downarrow$ in $(-2, 6)$.

(3) If $x > 6$, then $x + 2 > 0$, $x - 6 > 0$

$\therefore$ $f'(x) > 0$

$\therefore$ f is $\uparrow$ in $(6, \infty)$.

Example 20 : Determine where $f(x) = tan^{-1}(sinx + cosx)$, $x \in (0, \pi)$ is increasing and in which interval it is decreasing.

Solution : $f(x) = tan^{-1}(sinx + cosx)$

$$\therefore \quad f'(x) = \frac{1}{1 + (sinx + cosx)^2} \times (cosx - sinx)$$
$$= \frac{cosx - sinx}{1 + (sinx + cosx)^2}$$

(1) If $x \in \left(0, \frac{\pi}{4}\right)$, then $cosx > sinx$

$\left(cosx \in \left(\frac{1}{\sqrt{2}}, 1\right) \text{ and } sinx \in \left(0, \frac{1}{\sqrt{2}}\right)\right)$

Also $1 + (sinx + cosx)^2 > 0$

$\therefore$ $f'(x) > 0$ for $x \in \left(0, \frac{\pi}{4}\right)$

$\therefore$ f is increasing in $\left(0, \frac{\pi}{4}\right)$.

(2) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, $\cos x < \sin x$. Thus, $\cos x - \sin x < 0$ and if $x \in \left(\frac{\pi}{2}, \pi\right)$, $\cos x < 0$, $\sin x > 0$

$\therefore$ $\cos x - \sin x < 0$. For $x = \frac{\pi}{2}$, $\cos x - \sin x = 0 - 1 = -1 < 0$

$\therefore$ If $x \in \left(\frac{\pi}{4}, \pi\right)$, $f'(x) < 0$

$\therefore$ f is decreasing in $\left(\frac{\pi}{4}, \pi\right)$.

Example 21 : Prove that $f(x) = x^{100} + \sin x - 1$ is increasing for $x \in (0, \pi)$.

Solution : $f(x) = x^{100} + \sin x - 1$

$\therefore$ $f'(x) = 100x^{99} + \cos x$

For $x \in \left(0, \frac{\pi}{2}\right)$, $x^{99} > 0$, $\cos x > 0$. So $f'(x) > 0$.

For $x = \frac{\pi}{2}$, $x^{99} > 0$, $\cos x = 0$. So $f'(x) > 0$.

If $x \in \left(\frac{\pi}{2}, \pi\right)$, $x^{99} > 1$ and $-1 < \cos x < 0$.

$\therefore$ $f'(x) > 0$.

$\therefore$ f is (strictly) increasing in $(0, \pi)$.

Example 22 : Prove $f(x) = \log \sin x$ is increasing in $\left(0, \frac{\pi}{2}\right)$.

Solution : $f(x) = \log \sin x$

$\therefore$ $f'(x) = \frac{1}{\sin x} \times \cos x = \cot x > 0$ in $\left(0, \frac{\pi}{2}\right)$.

$\therefore$ f is increasing in $\left(0, \frac{\pi}{2}\right)$.

Example 23 : Determine intervals in which $f(x) = \frac{x}{\log x}$, $x > 1$ is increasing and where it is decreasing.

Solution : $f(x) = \frac{x}{\log x}$

$$\therefore \quad f'(x) = \frac{\log x - x \cdot \frac{1}{x}}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}$$

(1) $x < e$, then $\log x < \log e = 1$

$\therefore$ $\log x - 1 < 0$. Also $(\log x)^2 > 0$

$\therefore$ $f'(x) < 0$.

$\therefore$ f is $\downarrow$ in $(1, e)$.

(2) If $x > e$, then $\log x > 1$. So $\log x - 1 > 0$ and $(\log x)^2 > 0$

$\therefore$ $f'(x) > 0$.

$\therefore$ f is $\uparrow$ in (e, ∞).

Example 24 : Prove $f(x) = \frac{tanx}{x}$ is increasing on $\left(0, \frac{\pi}{2}\right)$.

Solution : $f(x) = \frac{tanx}{x} = \frac{sinx}{xcosx}$

$$\therefore \quad f'(x) = \frac{xcosx \cdot cosx - sinx\,(cosx - xsinx)}{(xcosx)^2}$$

$$= \frac{x(cos^2x + sin^2x) - sinx\,cosx}{(xcosx)^2}$$

$$= \frac{x - sinx\,cosx}{(xcosx)^2}$$

Now, $0 < x < \frac{\pi}{2}$. So $0 < sin\,x < x$, $0 < cos\,x < 1$

$\therefore \quad 0 < sinx\ cosx < x$

$\therefore \quad x - sinx\ cosx > 0$. Also $(xcos\,x)^2 > 0$

$\therefore \quad f'(x) > 0$

$\therefore \quad f$ is $\uparrow$ in $\left(0, \frac{\pi}{2}\right)$.

Exercise 1.2

1. Prove that $cot : R - \{k\pi \mid k \in Z\} \to R$ is decreasing in all quadrants.
2. Prove that *cosine* is a decreasing function in $(0, \pi)$.
3. Prove that *sec* is an increasing function in $\left(0, \frac{\pi}{2}\right)$.
4. Prove that *cosec* is an increasing function in $\left(\frac{\pi}{2}, \pi\right)$.
5. Prove that $f(x) = a^x$ is $\uparrow$, if $a > 1$.
6. Prove that $f(x) = \log_e x$ is $\uparrow$, $x \in R^+$.
7. **Determine the intervals in which f is increasing and the intervals in which f is decreasing :**

(1) $f : R \to R$, $\quad f(x) = 3x + 7$

(2) $f : R \to R$, $\quad f(x) = 8 - 5x$

(3) $f : R \to R$, $\quad f(x) = x^2 - 2x + 5$

(4) $f : R \to R$, $\quad f(x) = 9 + 3x - x^2$

(5) $f : R \to R$, $\quad f(x) = x^3 + 3x + 10$

(6) $f : R \to R$, $\quad f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

(7) $f : (0, \pi) \to R$, $\quad f(x) = sin\,x + cos\,x$

(8) $f : R \to R$, $\quad f(x) = -2x^3 - 9x^2 - 12x + 1$

(9) $f : R \to R$, $\quad f(x) = (x + 1)^3 (x - 3)^3$

(10) $f : \left(0, \frac{\pi}{2}\right) \to R$, $\quad f(x) = \log\ cosx$

(11) $f : \left(\frac{\pi}{2}, \pi\right) \to R, \quad f(x) = \log |\cos x|$

(12) $f : R - \{0\} \to R, \quad f(x) = e^{\frac{1}{x}}$

8. Prove that if I is an open interval and $I \cap [-1, 1] = \emptyset$, then $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

9. Prove that $f(x) = x^3 - 3x^2 + 3x + 100$ is increasing on R.

10. Prove that $f(x) = x^{100} + \sin x - 1$ is increasing on (0, 1).

11. Find intervals in which $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is increasing and intervals in which it is decreasing.

12. Find in which intervals, $f : R \to R$, $f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is decreasing and intervals in which it is increasing.

13. Prove $f(x) = x^x$, $x \in R^+$ is increasing if $x > \frac{1}{e}$ and decreasing if $0 < x < \frac{1}{e}$.

14. Decide the intervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or intervals in which it is decreasing. $x \in \left(0, \frac{\pi}{2}\right)$.

15. Find the value of a for which the function $f(x) = ax^3 - 3(a + 2)x^2 + 9(a + 2)x - 1$ is decreasing for all $x \in$ R.

16. Find the values of a for which $f(x) = ax^3 - 9ax^2 + 9x + 25$ is increasing on R.

17. Prove that $f(x) = (x - 1)e^x + 1$ is increasing for all $x > 0$.

18. Prove that $f(x) = x^2 - x \sin x$ is increasing on $\left(0, \frac{\pi}{2}\right)$.

19. Prove $f : R \to R$, $f(x) = x^2$ is increasing for $x \in R^+$ and decreasing for $x < 0$ without using derivative test and using the definition only.

20. Prove $f : R \to R$, $f(x) = 2^x + 2^{-x}$ is increasing for $x \in (0, \infty)$ and decreasing for $x \in (-\infty, 0)$.

21. Determine intervals in which following functions are strictly increasing or strictly decreasing :

(1) $f : R \to R, \quad f(x) = x^3 - 6x^2 - 36x + 2$

(2) $f : R \to R, \quad f(x) = x^4 - 4x$

(3) $f : R \to R, \quad f(x) = (x - 1)(x - 2)^2$

(4) $f : R \to R, \quad f(x) = 2x^3 - 12x^2 + 18x + 15$

(5) $f : R^+ \to R, \quad f(x) = x\sqrt{x + 1}$

(6) $f : R^+ \to R, \quad f(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$

(7) $f : (0, \pi) \to R, \quad f(x) = 2x + \cot x$

(8) $f : R \to R, \quad f(x) = 2\cos x + \sin^2 x$

(9) $f : R \to R, \quad f(x) = \log(1 + x^2)$

(10) $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^6 + 192x + 10$

(11) $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = xe^x$

(12) $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2e^x$

(13) $f: \mathbb{R}^+ \to \mathbb{R}^+, \quad f(x) = \frac{\log x}{\sqrt{x}}$

(14) $f: \mathbb{R}^+ \to \mathbb{R}^+, \quad f(x) = x \log x$

*

1.4 Applications to Geometry

(1) Tangents and Normals : We know that if $y = f(x)$ is a differentiable function in (a, b), $f'(x_0)$ is the slope of the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$, $x_0 \in (a, b)$.

So a tangent to a curve $y = f(x)$ at $(x_0, f(x_0))$ is the line passing through (x_0, y_0) and having slope $f'(x_0)$, where $y_0 = f(x_0)$. If a tangent at (x_0, y_0) is vertical, it does not have a slope.

The equation of tangent at (x_0, y_0) to the curve $y = f(x)$ is $y - y_0 = f'(x_0)(x - x_0)$, where the tangent is not vertical. If the tangent is a vertical line through (x_0, y_0), its equation is $x = x_0$.

Note : A tangent may intersect the curve again. The tangents $y = 1$ or $y = -1$ intersect the graph of $y = \sin x$, $x \in \mathbb{R}$ in infinitely many points. (Touch)

A normal to a curve $y = f(x)$ at (x_0, y_0) is a line perpendicular to the tangent at that point and passing through (x_0, y_0). If the tangent is not horizontal, $f'(x_0) \neq 0$. Then the slope of the normal at (x_0, y_0) is $-\frac{1}{f'(x_0)}$, since slopes m_1, m_2 of perpendicular lines satisfy $m_1 m_2 = -1$.

$\therefore$ The equation of the normal at (x_0, y_0) is $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0) \quad (f'(x_0) \neq 0)$

If $f'(x_0) = 0$, the equation of the normal at (x_0, y_0) is $x = x_0$. If the tangent at (x_0, y_0) is vertical, the equation of the normal at (x_0, y_0) is $y = y_0$.

Example 25 : Find the slope of the tangent and the normal to $y = x^3 - 2x + 4$ at $(1, 3)$.

Solution : The equation of the curve is $y = x^3 - 2x + 4$.

$$\frac{dy}{dx} = 3x^2 - 2$$

$$\therefore \left(\frac{dy}{dx}\right)_{x = 1} = 1$$

$\therefore$ The slope of the tangent to $y = x^3 - 2x + 4$ at $(1, 3)$ is 1.

Since a normal at a point is perpendicular to the tangent at the point , its slope at $(1, 3)$ is -1.

$(m_1 m_2 = -1)$

Example 26 : Find the equation of the tangent and the normal to the circle $x^2 + y^2 = a^2$ at (x_1, y_1).

Solution : The equation of the circle is $x^2 + y^2 = a^2$.

$\therefore \quad 2x + 2y\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$, if $y \neq 0$

$\therefore$ The equation of the tangent at (x_1, y_1) is,

$y - y_1 = -\dfrac{x_1}{y_1}(x - x_1)$ **$(y_1 \neq 0)$**

$\therefore \quad yy_1 - y_1^2 = -xx_1 + x_1^2$

$\therefore \quad xx_1 + yy_1 = x_1^2 + y_1^2$

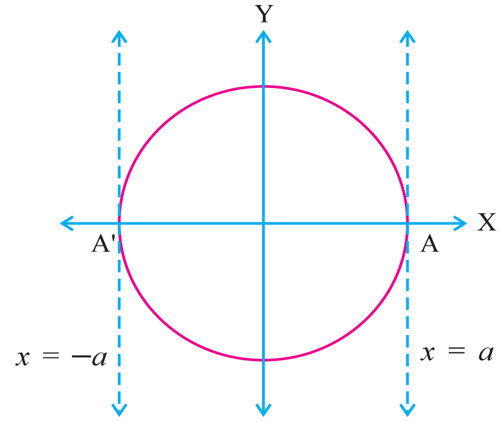


Figure 1.10

But $x_1^2 + y_1^2 = a^2$ as (x_1, y_1) lies on the circle $x^2 + y^2 = a^2$.

$\therefore$ $xx_1 + yy_1 = a^2$ is the equation of tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$. **$(y_1 \neq 0)$**

Corresponding to $y_1 = 0$, A$(a, 0)$, A'$(-a, 0)$ are two points on the circle.

$\therefore$ The tangents at A and A' are vertical and have equations $x = a$ and $x = -a$ respectively.

Taking $(x_1, y_1) = (a, 0)$ or $(-a, 0)$ repectively in the equation $xx_1 + yy_1 = a^2$ also, we get

$xa + 0 = a^2$ i.e. $xa = a^2$ or $-xa = a^2$

$\therefore$ $x = a$ and $x = -a$ are tangents at A and A'. **$(a \neq 0)$**

$\therefore$ At all points (x_1, y_1) on $x^2 + y^2 = a^2$ the equation of tangent to $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

A normal to $x^2 + y^2 = a^2$ is perpendicular to $xx_1 + yy_1 = a^2$ and passes through (x_1, y_1).

$\therefore$ Its equation is $xy_1 - yx_1 = x_1y_1 - y_1x_1 = 0$.

> A line perpendicular to $ax + by + c = 0$ and passing through (x_1, y_1) has equation
> $bx - ay = bx_1 - ay_1$.

$\therefore$ The equation of the normal to $x^2 + y^2 = a^2$ at (x_1, y_1) is $xy_1 - yx_1 = 0$ and it passes through the centre $(0, 0)$ of the circle.

$\therefore$ A radius (i.e. line containing radius) is always a normal to the circle.

Example 27 : Find the equation of the tangent and the normal to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at $x = a\cos^3\theta$, $y = a\sin^3\theta$. $\theta \in \left[0, \dfrac{\pi}{2}\right)$. **$(a > 0)$**

Solution : See that $x^{\frac{2}{3}} + y^{\frac{2}{3}} = (a\cos^3\theta)^{\frac{2}{3}} + (a\sin^3\theta)^{\frac{2}{3}}$

$= a^{\frac{2}{3}}(\cos^2\theta + \sin^2\theta)$

$= a^{\frac{2}{3}}$

$\therefore$ $(a\cos^3\theta, a\sin^3\theta)$ lies on $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Now $\dfrac{2}{3}x^{-\frac{1}{3}} + \dfrac{2}{3}y^{-\frac{1}{3}}\dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = \dfrac{-(a\sin^3\theta)^{\frac{1}{3}}}{-(a\cos^3\theta)^{\frac{1}{3}}} = -\tan\theta$

$\therefore$ The equation of the tangent at $(a\cos^3\theta, a\sin^3\theta)$ is $y - a\sin^3\theta = -\dfrac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$

$\therefore \quad y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\sin\theta\cos^3\theta$

$\therefore$ $x\sin\theta + y\cos\theta = a\sin\theta\cos\theta\,(\sin^2\theta + \cos^2\theta)$

$= a\sin\theta\cos\theta$

$\therefore$ The equation of the tangent at $(a\cos^3\theta, a\sin^3\theta)$, $\theta \in \left(0, \frac{\pi}{2}\right)$ is

$$x\sin\theta + y\cos\theta = a\sin\theta\cos\theta$$

$\therefore$ The equation of the normal at $(a\cos^3\theta, a\sin^3\theta)$ is

$$\begin{aligned} x\cos\theta - y\sin\theta &= a\cos^3\theta\cos\theta - a\sin^3\theta\sin\theta \\ &= a(\cos^4\theta - a\sin^4\theta) \\ &= a(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) \\ &= a\cos 2\theta \end{aligned}$$

$\therefore$ The equation of the normal at $(a\cos^3\theta, a\sin^3\theta)$ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

Note : **Remember :** A line perpendicular to $ax + by + c = 0$ has equation $bx - ay = bx_1 - ay_1$, if it passes through (x_1, y_1).

Example 28 : Find the equation of the tangent and the normal to $y^2 = 4ax$ at $(at^2, 2at)$

Solution : The equation of the curve is $y^2 = 4ax$.

$\therefore$ $2y\frac{dy}{dx} = 4a$

$\therefore$ $2(2at)\frac{dy}{dx} = 4a$

$\therefore$ $\frac{dy}{dx} = \frac{1}{t}$, if $t \neq 0$

$\therefore$ The equation of the tangent at $(at^2, 2at)$ is

$y - 2at = \frac{1}{t}(x - at^2)$ **$(t \neq 0)$**

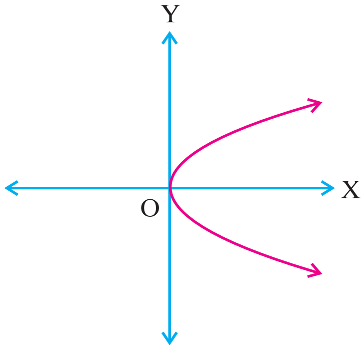


Figure 1.11

$\therefore$ $ty - 2at^2 = x - at^2$

$\therefore$ $x - ty + at^2 = 0$ is the equation of the tangent to

$y^2 = 4ax$ at $(at^2, 2at)$ where $t \neq 0$

$\therefore$ The equation of normal at $(at^2, 2at)$ is $tx + y = t(at^2) + 2at$.

$\therefore$ $tx + y - 2at - at^3 = 0$ is the equation of the normal to $y^2 = 4ax$ at $(at^2, 2at)$. **$(t \neq 0)$**

Now if $t = 0$, the corresponding point on parabola is $(0, 0)$. The tangent at $(0, 0)$ is vertical and its equation is $x = 0$. Normal at $t = 0$ is perpendicular to $x = 0$ and passes through $(0, 0)$.

Hence its equation is $y = 0$.

Note : See that these equations can also be obtained from general equations by putting $t = 0$.

Example 29 : Find the equation of the tangent to $y = \sqrt{3x-2}$ parallel to $4x - 2y + 5 = 0$.

Solution : The slope of the line $4x - 2y + 5 = 0$ is $m = -\frac{a}{b} = -\frac{4}{-2} = 2$.

∴ The slope of tangent to $y = \sqrt{3x-2}$ must be 2 as parallel lines have same slopes.

∴ $\frac{dy}{dx} = 2$

∴ Since $y = \sqrt{3x-2}$ is the equation of the curve,

∴ $\frac{dy}{dx} = \frac{1 \cdot 3}{2\sqrt{3x-2}} = 2$

∴ $9 = 16(3x - 2)$

Let (x_0, y_0) be the point of contact.

Then $x_0 = \frac{1}{3}\left(\frac{9}{16}+2\right) = \frac{41}{48}$, $y_0 = \sqrt{3 \times \frac{41}{48} - 2}$

$= \sqrt{\frac{41}{16} - 2} = \frac{3}{4}$

∴ The equation of tangent at $\left(\frac{41}{48}, \frac{3}{4}\right)$ is $y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$ **($m = 2$)**

∴ $24y - 18 = 48x - 41$

∴ $48x - 24y = 23$ is the equation of the tangent to $y = \sqrt{3x-2}$ parallel to $4x - 2y + 5 = 0$.

[Verify that $48x - 24y = 23$ is parallel to $4x - 2y + 5 = 0$ and is not coincident with $4x - 2y + 5 = 0$.]

Example 30 : Find the points on $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to X-axis.

Solution : The equation of the curve is $x^2 + y^2 - 2x - 3 = 0$

∴ $2x + 2y\frac{dy}{dx} - 2 = 0$ **(i)**

The tangent is parallel to X-axis. So its slope is zero.

∴ $\frac{dy}{dx} = 0$

∴ $2x - 2 = 0$ **(using (i))**

∴ $x = 1$

Now, $x^2 + y^2 - 2x - 3 = 0$

∴ $1 + y^2 - 2 - 3 = 0$ **($x = 1$)**

∴ $y^2 = 4$

∴ $y = \pm 2$

∴ The tangents at (1, 2) and (1, −2) to the circle are $y = \pm 2$ and they are parallel to X-axis.

Example 31 : Find the point or points on $y = x^3 - 11x + 5$ at which the equation of the tangent is $y = x - 11$.

Solution : The equation is $y = x^3 - 11x + 5$.

∴ $\frac{dy}{dx} = 3x^2 - 11$ **(i)**

The slope of $y = x - 11$ is 1.

∴ The slope of the tangent is 1.

∴ $\frac{dy}{dx} = 1$

$\therefore \quad 3x^2 - 11 = 1$ **(using (i))**

$\therefore \quad 3x^2 = 12$

$\therefore \quad x^2 = 4$

$\therefore \quad x = \pm 2$

$\therefore$ If $x = 2$, $y = x^3 - 11x + 5 = -9$. If $x = -2$, $y = x^3 - 11x + 5 = 19$

$\therefore$ Point of contact may be $(2, -9)$ or $(-2, 19)$.

At $(2, -9)$, the equation of the tangent is $y + 9 = 1(x - 2)$ **(slope = 1)**

$\therefore \quad y = x - 11.$

$\therefore$ But the tangent at $(-2, 19)$ cannot have equation $y = x - 11$ as $(-2, 19)$ does not lie on $y = x - 11$.

$\therefore$ The tangent at $(2, -9)$ has equation $y = x - 11$.

Example 32 : Show that tangents to $y = 7x^3 + 11$ at $x = 2$ and at $x = -2$ are parallel.

Solution : The equation of the curve is $y = 7x^3 + 11$.

$\therefore \quad \frac{dy}{dx} = 21x^2 = 84$ at $x = \pm 2$

If $x = 2$, $y = 7x^3 + 11 = 67$. If $x = -2$, $y = -45$.

$\therefore$ The equations of tangents at $(2, 67)$ and $(-2, -45)$ are respectively $y - 67 = 84(x - 2)$ and $y + 45 = 84(x + 2)$. **(m = 84)**

$\therefore$ $84x - y = 101$ and $84x - y + 123 = 0$ are equations of the tangents at $(2, 67)$ and $(-2, -45)$ respectively.

They are having same slopes and are distinct lines.

$\therefore$ They are parallel.

Example 33 : Find the equation of the normal to $x^2 = 4y$ passing through $(1, 2)$.

Solution : The equation of the curve is $x^2 = 4y$

$\therefore \quad 2x = 4\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{x}{2}$

$\therefore$ The slope of the normal at (x_0, y_0) is $-\frac{2}{x_0}$ **($x_0 \neq 0$)**

$\therefore$ The equation of the normal at (x_0, y_0) is $y - y_0 = -\frac{2}{x_0}(x - x_0)$ **(i)**

If it passes through $(1, 2)$, $2 - y_0 = -\frac{2}{x_0}(1 - x_0)$

$\therefore \quad x_0\left(2 - \frac{x_0^2}{4}\right) = -2 + 2x_0$ **($x_0^2 = 4y_0$)**

$\therefore \quad 8x_0 - x_0^3 = -8 + 8x_0$

$\therefore \quad x_0^3 = 8$

$\therefore \quad x_0 = 2, y_0 = \frac{x_0^2}{4} = 1$

$\therefore$ The equation of the normal at $(2, 1)$ is $y - 1 = -\frac{2}{2}(x - 2) = -x + 2$ **(using (i))**

$\therefore$ $x + y = 3$ is the equation of the normal to $x^2 = 4y$ passing through $(1, 2)$.

Note : **(1)** If $x_0 = 0$, then $y_0 = 0$. Normal at (x_0, y_0) is $x = 0$. It does not pass through (1, 2)

(2) Here the normal passes **through (1, 2)** and is not at **(1, 2)**. It is proved to be a normal at **(2, 1)**. **(1, 2) does not lie on $x^2 = 4y$.**

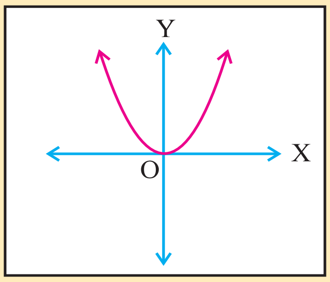


Figure 1.12

Example 34 : Prove that the sum of the intercepts (if they exist) on axes by any tangent to $\sqrt{x} + \sqrt{y} = \sqrt{c}$ is constant. $(c > 0)$. $(x \neq 0, y \neq 0)$

Solution : The equation of the curve is $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$\therefore \quad \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$ **$(x \neq 0)$**

$\therefore$ The equation of the tangent at (x_1, y_1) is $y - y_1 = -\sqrt{\frac{y_1}{x_1}} (x - x_1)$

$\therefore \quad \frac{y}{\sqrt{y_1}} - \frac{y_1}{\sqrt{y_1}} = -\frac{x}{\sqrt{x_1}} + \frac{x_1}{\sqrt{x_1}}$ **$(x_1 \neq 0, y_1 \neq 0)$**

$\therefore \quad \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1} = \sqrt{c}$ **((x_1, y_1) lies on $\sqrt{x} + \sqrt{y} = \sqrt{c}$)**

$\therefore$ It intersects axes at $(\sqrt{x_1}\sqrt{c}, 0)$, $(0, \sqrt{y_1}\sqrt{c})$.

$\therefore$ The sum of the intercepts on axes is $\sqrt{x_1}\sqrt{c} + \sqrt{y_1}\sqrt{c} = \sqrt{c}(\sqrt{x_1} + \sqrt{y_1})$

$= \sqrt{c}\sqrt{c}$

$= c$

$\therefore$ The sum of the intercepts of any tangent to $\sqrt{x} + \sqrt{y} = \sqrt{c}$ on axes is constant.

Note : If $x_1 = 0$ or $y_1 = 0$, the points on the curve are $(0, c)$ or $(c, 0)$. The tangents at these points are respectively $x = 0$ and $y = 0$ and do not have both the intercepts.

Example 35 : Prove that any normal to $x = a\cos\theta + a\theta \sin\theta$, $y = a\sin\theta - a\theta \cos\theta$ is at a constant distance from origin. $\theta \neq \frac{k\pi}{2}$, $k \in Z$

Solution : Since $x = a\cos\theta + a\theta \sin\theta$ and $y = a\sin\theta - a\theta \cos\theta$

$\frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta \cos\theta = a\theta \cos\theta$

$\frac{dy}{d\theta} = a\cos\theta - a \cos\theta + a\theta \sin\theta = a\theta \sin\theta$

$\therefore \quad \frac{dy}{dx} = \frac{\sin\theta}{\cos\theta}$ **$(\cos\theta \neq 0)$**

$\therefore$ The slope of the normal at θ-point is $-\frac{\cos\theta}{\sin\theta}$. **$(\sin\theta \neq 0)$**

$\therefore$ The equation of the normal at θ-point is $(y - a\sin\theta + a\theta\cos\theta) = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\theta\sin\theta)$

$\therefore$ $y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$

$\therefore$ $x\cos\theta + y\sin\theta = a(\cos^2\theta + \sin^2\theta) = a$

$\therefore$ $x\cos\theta + y\sin\theta = a$ is the equation of the normal at θ-point. $\left(\theta \neq \frac{k\pi}{2}\right)$

If its distance from origin is p, then $p = \frac{|c|}{\sqrt{a^2 + b^2}}$

$= \frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = |a|$ which is a constant.

[What happens if $\theta = \frac{k\pi}{2}$?]

(2) Angle between two curves :

The measure of the angle between two curves is defined to be the measure of the angle between the tangents to them at their point of intersection.

A result : Let $y = f(x)$ and $y = g(x)$, $x \in (a, b)$, be equations of two curves and $f(x)$ and $g(x)$ are differentiable in (a, b). If they intersect at (x_0, y_0), $x_0 \in (a, b)$. The measure α of the angle between them is given by

$$\tan\alpha = \left|\frac{f'(x_0) - g'(x_0)}{1 + f'(x_0)\, g'(x_0)}\right|$$

Explanation : We know if m_1 and m_2 are slopes of two lines, the measure α of the angle between them is given by

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Also the slopes of tangents at (x_0, y_0) are $f'(x_0)$ and $g'(x_0)$.

So $m_1 = f'(x_0)$ and $m_2 = g'(x_0)$. Hence the result.

If $f'(x_0)\, g'(x_0) = -1$, $\alpha = \frac{\pi}{2}$ and we say the curves intersect orthogonally.

If $f'(x_0) = g'(x_0)$, the curves touch each other at (x_0, y_0).

Example 36 : Prove that $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ intersect orthogonally at every point of intersection.

Solution : Let us first find the points of intersection.

$x^2 - y^2 = 5, \quad 4x^2 + 9y^2 = 72$ **(i)**

$4x^2 - 4y^2 = 20$ using $x^2 - y^2 = 5$. **(ii)**

Solving (i) and (ii), $13y^2 = 52$

$\therefore$ $y^2 = 4$. So $y = \pm 2$

$\therefore$ $x^2 - 4 = 5$ **$(x^2 - y^2 = 5)$**

$\therefore$ $x^2 = 9$. So, $x = \pm 3$

$\therefore$ The points of intersection are (3, 2), (3, −2), (−3, −2), (−3, 2).

For the first curve $2x - 2y\frac{dy}{dx} = 0$

The slope of the tangent to $x^2 - y^2 = 5$ denoted by m_1 is given by $m_1 = \frac{x}{y}$.

For the second curve $8x + 18y\frac{dy}{dx} = 0$.

$\therefore$ The slope of the tangent to $4x^2 + 9y^2 = 72$ at (x, y) denoted by m_2 is given by $m_2 = -\frac{4x}{9y}$

$\therefore$ $m_1 m_2 = -\frac{4x^2}{9y^2} = -\frac{36}{36} = -1$

$\therefore$ At all the points of intersection the curves (hyperbola and ellipse) intersect orthogonally.

Example 37 : Prove that $y = ax^3$, $x^2 + 3y^2 = b^2$ are orthogonal.

Solution : The slope of the tangent to $y = ax^3$ at (x, y) is denoted by m_1. So $m_1 = \frac{dy}{dx} = 3ax^2$.

$x^2 + 3y^2 = b^2$ implies $2x + 6y\frac{dy}{dx} = 0$

$\therefore$ The slope of the tangent to $x^2 + 3y^2 = b^2$ at (x, y) is denoted by m_2. So $m_2 = \frac{dy}{dx} = -\frac{x}{3y}$.

$\therefore$ $m_1 m_2 = (3ax^2)\left(-\frac{x}{3y}\right) = -\frac{ax^3}{y} = -1$ as at the point of intersection $y = ax^3$

$\therefore$ The curves intersect at right angles.

[The curves do intersect as substituting $y = ax^3$ in $x^2 + 3y^2 = b^2$, we get $x^2 + 3a^2b^6 = b^2$. This equation has a solution.]

Example 38 : Find the measure of the angle between $x^2 + y^2 - 4x - 1 = 0$ and $x^2 + y^2 - 2y - 9 = 0$.

Solution : The equations of curves are $x^2 + y^2 - 4x - 1 = 0$, $x^2 + y^2 - 2y - 9 = 0$.

$\therefore$ At the point of intersection, $x^2 + y^2 = 4x + 1 = 2y + 9$.

$\therefore$ $4x - 2y = 8$

$\therefore$ $2x - y = 4$

$\therefore$ $y = 2x - 4$

$\therefore$ Substituting $y = 2x - 4$ in $x^2 + y^2 - 4x - 1 = 0$, $x^2 + (2x - 4)^2 - 4x - 1 = 0$

$\therefore$ $5x^2 - 20x + 15 = 0$

$\therefore$ $x^2 - 4x + 3 = 0$

$\therefore$ $x = 3$ or 1. So correspondingly $y = 2x - 4 = 2$ or -2

$\therefore$ The points of intersection of the circles are $(3, 2)$ and $(1, -2)$.

Now for $x^2 + y^2 - 4x - 1 = 0$, $2x + 2y\frac{dy}{dx} - 4 = 0$ **(i)**

and for $x^2 + y^2 - 2y - 9 = 0$, $2x + 2y\frac{dy}{dx} - 2\frac{dy}{dx} = 0$. **(ii)**

(1) At (3, 2) : $6 + 4\frac{dy}{dx} - 4 = 0$, $6 + 4\frac{dy}{dx} - 2\frac{dy}{dx} = 0$ **(Using (i) and (ii))**

$\therefore$ For $x^2 + y^2 - 4x - 1 = 0$ slope of tangent $m_1 = -\frac{1}{2}$.

For $x^2 + y^2 - 2y - 9 = 0$ slope of tangent $m_2 = -3$.

$\therefore$ $tan\alpha = \left|\frac{-\frac{1}{2} + 3}{1 + \frac{3}{2}}\right| = 1$ $\left(tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|\right)$

$\therefore$ $\alpha = \frac{\pi}{4}$

(2) At (1, −2) : $2 - 4\frac{dy}{dx} - 4 = 0$, $2 - 4\frac{dy}{dx} - 2\frac{dy}{dx} = 0$ **(Using (i) and (ii))**

$\therefore$ As before $m_1 = -\frac{1}{2}$, $m_2 = \frac{1}{3}$

$\therefore$ $tan\alpha = \left|\frac{-\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{6}}\right| = 1$

$\therefore \quad \alpha = \frac{\pi}{4}$

$\therefore$ The circles intersect at both the points at an angle having measure $\frac{\pi}{4}$.

Example 39 : Where does the normal to $x^2 - xy + y^2 = 3$ at $(-1, 1)$ intersect the curve again ?

Solution : $x^2 - xy + y^2 = 3$ is the equation of the curve.

$\therefore \quad 2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$

$\therefore$ At $(-1, 1)$, $-2 - \left(-\frac{dy}{dx} + 1\right) + 2\frac{dy}{dx} = 0$

$\therefore \quad 3\frac{dy}{dx} = 3$

$\therefore$ The slope of the tangent at $(-1, 1)$ is $\frac{dy}{dx} = 1$.

So the slope of the normal at $(-1, 1)$ is -1.

$\therefore$ The equation of the normal at $(-1, 1)$ is $y - 1 = -1(x + 1)$

$\therefore \quad x + y = 0$ is the equation of the normal at $(-1, 1)$.

To find the points of intersection, let us solve.

$x + y = 0$ and $x^2 - xy + y^2 = 3$

Substitution $y = -x$ in $x^2 - xy + y^2 = 3$,

$\therefore \quad 3x^2 = 3$

$\therefore \quad x = \pm 1$

Since $x = -y$, the point of intersection is $(1, -1)$ as $x \neq -1$.

The normal drawn at $(-1, 1)$ intersects the curve at $(1, -1)$.

[$(-1, 1)$ is the point at which normal is drawn. So it is the foot of the normal. Hence $x \neq -1$.]

Example 40 : Prove that $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a^2 \neq b^2)$ and $xy = c^2$ cannot intersect orthogonally.

Solution : One of the equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \quad \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\therefore$ The slope of the tangent to the curve, $m_1 = \frac{dy}{dx} = \frac{b^2x}{a^2y}$ **(Why $y \neq 0$)**

The other curve has equation $xy = c^2$

$\therefore \quad x\frac{dy}{dx} + y = 0$

$\therefore$ The slope of the tangent to the curve, $m_2 = -\frac{y}{x}$

$\therefore \quad m_1m_2 = \left(\frac{b^2x}{a^2y}\right)\left(-\frac{y}{x}\right) = -\frac{b^2}{a^2} \neq -1$ as $a^2 \neq b^2$.

$\therefore$ The curves (hyperbolas) cannot intersect at right angles.

Note : If $a^2 = b^2$, they intersect orthogonally. Hence rectangular hyperbolas $x^2 - y^2 = a^2$ and $xy = c^2$ intersect orthogonally. That they do intersect can be verified.

Exercise 1.3

1. Find the equation of the tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1).

2. Find the equation of the tangent to $y^2 = 4ax$ at (x_1, y_1).

3. Find the slope of the tangent to $y = x^3 + 5x + 2$ at (2, 20).

4. Find the slope of the normal to $y^2 = 4x$ at (1, 2).

5. Find the equation of the tangent to $y^2 = 16x$, which is parallel to the line $4x - y = 1$.

6. Find the equation of the normal to $y^2 = 8x$ perpendicular to the line $2x - y - 1 = 0$.

7. Prove that the curves $\frac{x^2}{a^2+\lambda_1} + \frac{y^2}{b^2+\lambda_1} = 1$, $\frac{x^2}{a^2+\lambda_2} + \frac{y^2}{b^2+\lambda_2} = 1$ intersect orthogonally, if they intersect. $(\lambda_1 \neq \lambda_2)$

8. Prove that portion of any tangent to $x = acos^3\theta$, $y = asin^3\theta$ intercepted between axes has constant length.

9. Prove that $2x^2 + y^2 = 3$ and $y^2 = x$ intersect at right angles.

10. Prove that circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ are orthogonal.

11. **(1)** Find the equation of the tangent to $y = sinx$ at $\left(\frac{\pi}{2}, 1\right)$.

(2) Where does it intersect the curve again ?

12. Find equation of tangent to $x = cos\theta$, $y = sin\theta$ $\theta \in [0, 2\pi)$ at $\theta = \frac{\pi}{4}$.

13. Find equation of tangent to $y = 4x^3 - 2x^5$ passing through origin.

14. (2, 3) lies on $y^2 = ax^3 + b$. The slope of the tangent at (2, 3) is 4. Find a and b.

15. The slope of the tangent to $xy + ax + by = 2$ at (1, 1) is 2. Find a and b.

16. Find the equation of tangent to $x = a(\theta - sin\theta)$, $y = a(1 - cos\theta)$.

17. Prove that parabola $y^2 = x$ and hyperbola $xy = k$ intersect at right angles, if $8k^2 = 1$.

18. Where does the normal to $y = x - x^2$ at (1, 0) intersect the curve again ?

19. Find a, b if tangent to $y = ax^2 + bx$ at (1, 1) is $y = 3x - 2$.

20. Find the equation of tangent to $x^3 + y^3 = 6xy$ at (3, 3). At which point is the tangent horizontal or vertical ?

21. Prove $xy = c^2$, $c \neq 0$ and $x^2 - y^2 = k^2$, $k \neq 0$ intersect orthogonally. **(Compare : Example 40)**

22. Find the equation of the tangent to given curves at given point :

(1) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $\left(-5, \frac{9}{4}\right)$

(2) $\frac{x^2}{9} + \frac{y^2}{36} = 1$ at $(-1, 4\sqrt{2})$

(3) $y^2 = x^3 (2 - x)$ at (1, 1)

(4) $y^2 = 5x^4 - x^2$ at (1, 2)

(5) $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at (3, 1)

23. Find points on $x^2y^2 + xy = 2$ where tangent has slope -1.

24. Find the measure of the angle between

(1) $y = x^2, y = (x - 2)^2$ **(2)** $x^2 - y^2 = 3, x^2 + y^2 - 4x + 3 = 0$

25. Find the equations of tangents to $y = cos(x + y)$ parallel to $x + 2y = 0$.

26. Find the equations of tangents to $y = \frac{1}{x-1}, x \neq 1$ parallel to the line $x + y + 7 = 0$.

27. Prove that $\frac{x}{a} + \frac{y}{b} = 2$ touches $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ for all $n \in N - \{1\}$, the point of contact being (a, b).

28. X-axis touches $y = ax^3 + bx^2 + cx + 5$ at P$(-2, 0)$ and intersects Y-axis at Q. The slope of the tangent at Q is 3. Find a, b, c.

*

1.5 Approximation and Differentials

Error : We know that $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$, where f is a differentiable function in (a, b) and $x \in (a, b), x + h \in (a, b)$.

$\therefore$ If h is 'very small',

$\frac{f(x+h) - f(x)}{h} = f'(x) + u(h)$ where $u(h)$ is a function of h and as $h \to 0$, $u(h) \to 0$.

$\therefore$ $f(x + h) - f(x) = f'(x)h + u(h)h$.

Let $f(x + h) - f(x) = \Delta f(x)$ and $h = (x + h) - x = \Delta x$.

$\therefore$ $\Delta f(x)$ is a 'small' change in $f(x)$ caused by a 'small' change Δx in x.

$\therefore$ $\boldsymbol{\Delta f(x) = f'(x)\Delta x + u(\Delta x)\Delta x}$

$f'(x)\Delta x$ is called differential of $y = f(x)$ and is denoted by dy. Also $\Delta f(x) = \Delta y$.

$\boldsymbol{\Delta y = dy + u(\Delta x)\Delta x}$

Since $u(\Delta x)\Delta x$ is very small and can be neglected, we say dy is an approximate value of Δy and we write $\Delta y \simeq dy$.

Also $dy = f'(x)\Delta x$ **(i)**

Moreover for the function $y = x$, $f'(x) = 1$.

$dx = 1 \cdot \Delta x$

$\therefore$ For the independent variable x, $\Delta x = dx$.

Thus from (i) $dy = f'(x)\Delta x = f'(x)dx$

$\therefore$ $\boldsymbol{f'(x) = \frac{(dy)}{(dx)}}$

$\therefore$ $\frac{dy}{dx} = \frac{(dy)}{(dx)}$

On L.H.S. we have derivative of $y = f(x)$ and is not a ratio, but on R.H.S. We have a ratio $\frac{(dy)}{(dx)}$ of differential of y and differential of x.

Δy **is also called an error in calculation of** $f(x)$**.**

$\therefore \quad \Delta y \simeq dy = f'(x)\Delta x.$

Moreover $f(x + \Delta x) \simeq f(x) + f'(x)\Delta x$**.**

Geometrical Interpretation of Differential :

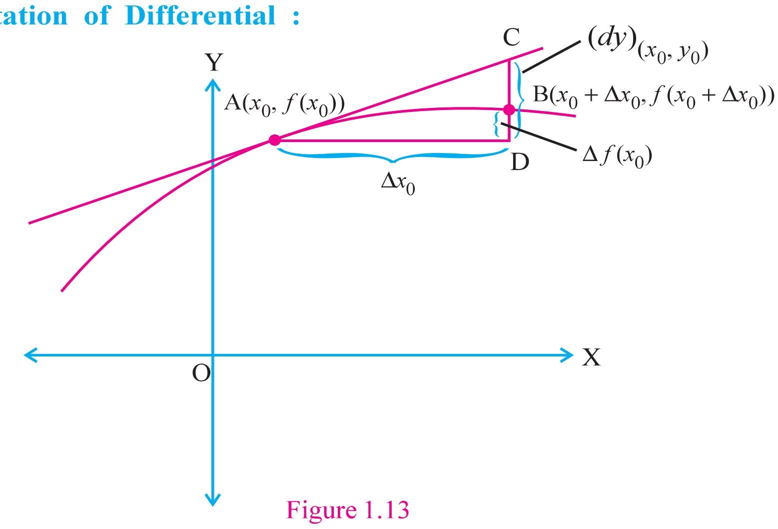


Figure 1.13

Let $A(x_0, f(x_0))$ be a point on the curve $y = f(x)$.

$B(x_0 + \Delta x_0, f(x_0 + \Delta x_0))$, is also on the curve. C is the point on the tangent at A to the curve $y = f(x)$ lying on the vertical line through B.

The equation of the tangent at A is $y - y_0 = f'(x_0)(x - x_0)$ **($f'(x_0)$ is slope of the tangent)**

At C, $x = x_0 + \Delta x_0$

$\therefore$ y-coordinate of C, $y = y_0 + (x_0 + \Delta x_0 - x_0) f'(x_0)$

$= f(x_0) + f'(x_0)\,\Delta x_0$

$= f(x_0) + (dy)_{(x_0, y_0)}$

CD = y-coordinate of C $- f(x_0) = (dy)_{(x_0, y_0)}$

BD = $f(x_0 + \Delta x_0) - f(x_0) = \Delta f(x_0) = \Delta y_0$

$\therefore$ BC $= |\Delta y_0 - (dy)_{(x_0, y_0)}|$

As B moves nearer and nearer to A on the curve, BC $\to 0$. Hence $dy \simeq \Delta y$.

Thus $f(x_0 + \Delta x_0) \simeq f(x_0) + f'(x_0)\,\Delta x_0$ is called the approximate value of $f(x)$ for $x = x_0 + \Delta x_0$ obtained by linear approximation using tangent to $y = f(x)$.

Example 41 : Obtain approximate value of $\sqrt{101}$ and $\sqrt{99}$ using differentiation.

Solution : Let $f(x) = \sqrt{x}$, $x \in R^+$

Let $x = 100$ and $x + \Delta x = 101$ **(We know $\sqrt{100}$.)**

$\therefore \quad \Delta x = 1. \quad (\Delta x = x + \Delta x - x = 101 - 100)$

$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{100}} = \frac{1}{20} = 0.05$

Now $f(x + \Delta x) \simeq f(x) + f'(x)\,\Delta x$

$\therefore \quad f(101) \simeq f(100) + f'(100)\,\Delta x$

$= \sqrt{100} + (0.05)(1) = 10.05$

$\therefore$ An approximate value of $\sqrt{101}$ is 10.05.

For $\sqrt{99}$, let $x = 100$, $x + \Delta x = 99$, $\Delta x = -1$ **($\Delta x = 99 - 100 = -1$)**

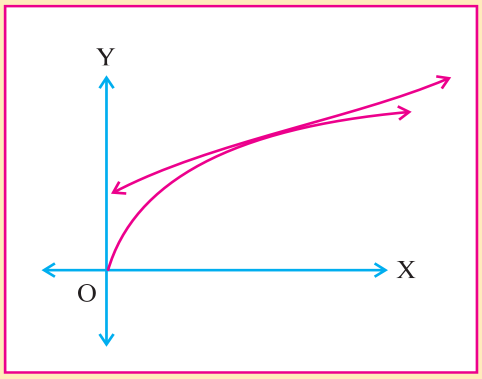
$\therefore \quad \sqrt{99} = f(99) \simeq f(100) + f'(100)\,\Delta x$

$= \sqrt{100} + (0.05)(-1)$

$= 10 - 0.05 = 9.95$

x	Approximate Value	Actual Value
$\sqrt{101}$	10.05	10.0498756....
$\sqrt{99}$	9.95	9.94987437....
$\sqrt{102}$	10.1	10.0995049....
$\sqrt{98}$	9.9	9.89949493....

We observe that as $\Delta x \to 0$, actual value approaches true value. Here the actual value is smaller than the approximate value, as the tangent lies above the graph of $y = \sqrt{x}$ or $y^2 = x$.



Example 42 : Find approximate value of $(65)^{\frac{1}{3}}$.

[**Note :** We will henceforth not use the phrase 'using differentiation' but it is implied.]

Solution : $f(x) = x^{\frac{1}{3}}$.

$x = 64$, $x + \Delta x = 65$. So, $\Delta x = 1$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} = \frac{1}{3(64)^{\frac{2}{3}}} = \frac{1}{48}$. So $\Delta f(x) \simeq f'(x)\,\Delta x = \frac{1}{48}$

$\therefore \quad (65)^{\frac{1}{3}} = (64)^{\frac{1}{3}} + \Delta f(x) \simeq 4 + \frac{1}{48} = \frac{193}{48}$

Example 43 : Find *tan* 46°.

Solution : Let $f(x) = \tan x$ and $x = \frac{\pi}{4}$, $x \in R - \left\{(2k-1)\frac{\pi}{2} \,\middle|\, k \in Z\right\}$ **$\left(45^{\circ} = \frac{\pi}{4}^{R}\right)$**

$\therefore \quad \Delta x = 1 \cdot \frac{\pi}{180} = \frac{\pi}{180}^{R}$

$\therefore \quad f'(x) = \sec^2 x = (\sqrt{2})^2 = 2$

$\therefore \quad \Delta f(x) \simeq f'(x)\,\Delta x = 2 \cdot \frac{\pi}{180} = \frac{\pi}{90}$

$\therefore \quad tan46^o = tan45^o + \Delta f(x)$

$$\simeq 1 + \frac{\pi}{90}$$

$\therefore$ An approximate value of $tan\, 46^o$ is $1 + \frac{\pi}{90}$.

Example 44 : Find approximate value of **(1)** $cos^{-1}\,(-0.49)$ **(2)** $sec^{-1}\,(-2.01)$

Solution : (1) Let $f(x) = cos^{-1}x,\ x = -0.5,\ \Delta x = 0.01$

$$f'(x) = \frac{-1}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-\frac{1}{4}}} = -\frac{2}{\sqrt{3}},\quad \Delta f(x) \simeq f'(x)\,\Delta x = -\frac{1}{50\sqrt{3}}$$

$\therefore \quad cos^{-1}\,(-0.49) = cos^{-1}\,(-0.5) + \Delta f(x)$

$$\simeq \pi - cos^{-1}\,(0.5) - \frac{1}{50\sqrt{3}}$$

$$= \pi - \frac{\pi}{3} - \frac{1}{50\sqrt{3}}$$

$$= \frac{2\pi}{3} - \frac{1}{50\sqrt{3}}$$

Another method : Let $f(x) = cos^{-1}x,\ x = 0.5,\ \Delta x = -0.01$

$\therefore \quad cos^{-1}\,(-0.49) = \pi - cos^{-1}\,(0.49)$

$$\simeq \pi - (cos^{-1}\,(0.5) + f'(x)\,\Delta x)$$

$$= \pi - \frac{\pi}{3} - \left(-\frac{2}{\sqrt{3}}\right)(-0.01)$$

$$= \frac{2\pi}{3} - \frac{1}{50\sqrt{3}}$$

(2) Let $f(x) = sec^{-1}x,\ x = 2,\ \Delta x = 0.01$

$$f'(x) = \frac{1}{|x|\sqrt{x^2-1}} = \frac{1}{2\sqrt{3}},\ \Delta f(x) \simeq f'(x)\,\Delta x = \frac{1}{200\sqrt{3}}$$

$\therefore \quad sec^{-1}\,(-2.01) = \pi - sec^{-1}\,(2.01)$

$$\simeq \pi - (sec^{-1}2 + f'(x)\,\Delta x)$$

$$= \pi - \left(\frac{\pi}{3} + \frac{1}{200\sqrt{3}}\right)$$

$$= \frac{2\pi}{3} - \frac{1}{200\sqrt{3}}$$

Example 45 : Find approximate value of **(1)** $\log_e 10.01$ **(2)** $\log_{10} 10.1$ **(3)** $\log_e(e + 0.1)$

($\log_{10}e = 0.4343,\ \ \log_e 10 = 2.3026$)

Solution : (1) Let $f(x) = \log_e x$

Let $x = 10,\ \Delta x = 0.01,\ f'(x) = \frac{1}{x} = \frac{1}{10} = 0.1$

$\therefore \quad \Delta f(x) \simeq f'(x)\,\Delta x = 0.001$

$\therefore \quad \log_e(10.01) \simeq \log_e 10 + f'(x)\,\Delta x$

$$= 2.3026 + 0.001$$

$$= 2.3036$$

(Actually $\log_e 10.01 = 2.30358459....$)

(2) Let $f(x) = \log_{10}x = \frac{\log_e x}{\log_e 10} = \log_e x \cdot \log_{10} e$

$= (0.4343) \log_e x$

Let $x = 10$, $\Delta x = 0.1$

$\therefore \quad f'(x) = \frac{0.4343}{x} = \frac{0.4343}{10} = 0.04343$

$\therefore \quad \Delta f'(x) \simeq f'(x)\, \Delta x = (0.04343)\ (0.1) = 0.004343$

$\therefore \quad \log_{10}(10.1) \simeq \log_{10}10 + f'(x)\, \Delta x$

$= 1.004343$

(Actually $\log_{10}(10.1) = 1.00432137....$)

(3) Let $f(x) = \log_e x$, $x = e$, $\Delta x = 0.1$

$\therefore \quad f'(x) = \frac{1}{x} = \frac{1}{e}, \quad \Delta f(x) \simeq f'(x)\, \Delta x = \frac{(0.1)}{(e)} = \frac{1}{10e}$

$\therefore \quad \log_e(e + 0.1) \simeq \log_e e + f'(x)\, \Delta x$

$= 1 + \frac{1}{10e} = 1.03678794$

(Actually it is 1.0367879441....)

Example 46 : If there is an error of x % in the measurement of radius of a sphere, what is the approximate error in the measurement of volume and surface area ?

Solution : There is x % error in the radius.

$\therefore \quad \Delta r = \frac{xr}{100}.$

Volume of a sphere, $V = \frac{4}{3}\pi r^3$

$\therefore \quad \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

$\therefore$ Error in volume $\Delta V \simeq \frac{dV}{dr}\, \Delta r$

$= 4\pi r^2 \cdot \frac{xr}{100}$

$= \frac{4}{3}\pi r^3 \cdot \frac{3x}{100} = \frac{3xV}{100}$

$\therefore$ There is approximately $3x$ % error in the volume.

Surface area $S = 4\pi r^2$

$\therefore \quad \frac{dS}{dr} = 8\pi r$

$\therefore$ Error in surface area $\Delta S \simeq \frac{dS}{dr}\, \Delta r$

$= 8\pi r \cdot \frac{xr}{100}$

$= 2(4\pi r^2)\ \frac{x}{100}$

$= \frac{2xS}{100}$

$\therefore$ There is approximately $2x$ % error in surface area.

Example 47 : The radius of a sphere is measured as 7 m with error of 0.02 m. What is the approximate error in the volume ?

Solution : For a sphere, volume $V = \frac{4}{3}\pi r^3$

$r = 7\ m$, $\Delta r = 0.02\ m$

$\therefore \quad \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

$\therefore \quad \Delta V \simeq \frac{dV}{dr}\Delta r$

$= 4\pi r^2 \cdot \Delta r$

$= 4\pi(49)(0.02)$

$= 3.92\ \pi\ m^3$

$\therefore$ There is approximately $3.92\ \pi\ m^3$ error in the volume.

Example 48 : Find the approximate error in the surface area of a cube with edge x cm, when the edge is increased by 2 %.

Solution : $S = 6x^2$, $\Delta x = \frac{2x}{100}$

$\therefore \quad \frac{dS}{dx} = 12x$

$\therefore \quad \Delta S \simeq \frac{dS}{dx} \cdot \Delta x$

$\therefore \quad \Delta S \simeq 12x\ \Delta x$

$= 12x \cdot \frac{2x}{100}$

$= \frac{4(6x^2)}{100} = \frac{4S}{100}$

$\therefore$ There is approximately 4 % increase in the surface area.

Example 49 : Prove that for a triangle inscribed in a circle of constant radius, sides change according to $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$ in usual notation, if da, db, dc are small.

Solution : We have $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ according to *sine* rule.

$a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$, R constant.

$\therefore \quad \frac{da}{dA} = 2R\cos A, \ \frac{db}{dB} = 2R\cos B, \ \frac{dc}{dC} = 2R\cos C$

$\therefore \quad da = \frac{da}{dA}\Delta A = 2R\cos A\ \Delta A$ etc.

$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R(\Delta A + \Delta B + \Delta C)$

$= 2R(\Delta(A + B + C))$

$= 2R\ \Delta(\pi)$

$= 0$

$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0$

Example 50 : When a circular plate is heated, its radius increases by 0.1 cm. Find the approximate increase in area, when the radius is 5 cm.

Solution : For a circle, area $A = \pi r^2$

$\therefore \quad \frac{dA}{dr} = 2\pi r$

$\therefore \quad \Delta A \simeq \frac{dA}{dr} \Delta r = 2\pi r \, \Delta r = 2\pi(5)(0.1)$

$\therefore \quad \Delta A \simeq \pi \, cm^2$

$\therefore$ There is $\pi \, cm^2$ increase in area approximately.

Example 51 : If $f(x) = cosx$, find the differential dy and evaluate dy when $x = \frac{\pi}{6}$ and $\Delta x = 0.01$.

Solution : $y = f(x) = cosx$

$\therefore \quad f'(x) = -sinx$. So $f'\left(\frac{\pi}{6}\right) = -sin\frac{\pi}{6} = -\frac{1}{2} = -0.5$

$\therefore \quad dy = f'(x) \, \Delta x = (-0.5)(0.01)$

$\therefore \quad dy = -0.005$

Example 52 : Prove that if h is very small, $sinh \simeq h$.

Solution : Let $f(x) = sinx$, $x = 0$, $x + \Delta x = h$

$\therefore \quad f'(x) = cosx$, $f'(0) = cos0 = 1$

$\therefore \quad f(x + \Delta x) \simeq f(x) + f'(x) \, \Delta x$

$\therefore \quad f(h) \simeq f(0) + f'(0) \, h \qquad (h = \Delta x)$

$\therefore \quad sinh \simeq sin0 + cos0 \cdot h$

$\therefore \quad sinh \simeq h$, if h is small.

Exercise 1.4

Find approximate value (1 to 12) :

1. $\sqrt{0.37}$ **2.** $(0.999)^{\frac{1}{10}}$ **3.** $(80)^{\frac{1}{4}}$ **4.** $(255)^{\frac{1}{4}}$

5. $(399)^{\frac{1}{2}}$ **6.** $(32.1)^{\frac{1}{5}}$ **7.** $cos\,29^{\circ}$ **8.** $sin61^{\circ}$

9. $tan31^{\circ}$ **10.** $\log_e(100.1)$ **11.** $\log_{10}(10.01)$ **12.** $(16.1)^{\frac{1}{4}}$

13. If the radius of a cone is twice its height, find the approximate error in the calculation of its volume, when the radius is 10 cm and the error in the radius is 0.01 cm.

14. If there is an error in measuring its radius by Δr, what is the approximate error in the volume of a sphere?

15. Kinetic energy is given by $k = \frac{1}{2}mv^2$. For constant mass there is approximately 1 % increase in the energy. What increase in the velocity v which caused it ?

16. Area of a triangle is calculated using formula $A = \frac{1}{2}absinC$. If $C = \frac{\pi}{6}$ and there is an error in measuring C by x %, what is the percentage error in area approximately ? a, b are kept constant.

17. Find approximate value of $f(3.01)$ where $f(x) = x^3 - 2x^2 - 3x + 1$.

18. Find approximate value of $f(1.05)$ where $f(x) = 2x^2 - 3x + 5$.

19. Find the approximate increase in the volume of a cube when the length of its edge increases by 0.2 *cm* and its edge has length 10 *cm*.

20. Find the approximate increase in the total surface area of a cone when its height remains constant and the radius increases by 2 % at the time when its radius is 8 *cm* and the height is 6 *cm*.

21. Find approximate value of $\cos \frac{11\pi}{36}$, knowing the value of $\cos \frac{\pi}{3}$.

*

1.6 Maximum and Minimum Values

We have seen some applications of differential calculus. Now we will learn an important application of differential calculus to optimization problems.

We may wish to find maximum volume of a box, minimum cost of a can to contain fixed quantity of fruit juice or minimize the cost and maximize the profit etc.

Definition : A function f has an absolute or global maximum at c if $f(c) \geq f(x)$, $\forall x \in D_f$, $c \in D_f$ and a function has an absolute or global minimum at c if $f(c) \leq f(x)$, $\forall x \in D_f$, $c \in D_f$. The maximum and minimum values are also called the extreme values of f on D_f.

Definition : A function f defined on an interval I has a local maximum value at $c \in$ I, if for some $h > 0$, $(c - h, c + h) \subset$ I and $f(c) \geq f(x)$, $\forall x \in (c - h, c + h)$.

A function f defined on an interval I has a local minimum value at $c \in$ I, if for some $h > 0$, $(c - h, c + h) \subset$ I and $f(c) \leq f(x)$, $\forall x \in (c - h, c + h)$.

Note : If I is a closed interval local maximum or local minimum cannot occur at an end-point of the interval because of the condition $(c - h, c + h) \subset$ I.
However global maximum or global minimum may occur at an end-point.

$f(x) = \sin x$, $x \in$ R takes global maximum 1 for $x = (4n + 1)\frac{\pi}{2}$, $n \in$ Z and global minimum -1 for $x = (4n + 3)\frac{\pi}{2}$, $n \in$ Z. Consider $f(x) = x^2$, $x \in$ R. Since $x^2 \geq 0$ $\forall x \in$ R, $f(0) = 0$ is global minimum as well as local minimum but f has no global maximum. But if the domain of f is restricted to $[-3, 5]$, say, it has a global maximum $f(5) = 25$.

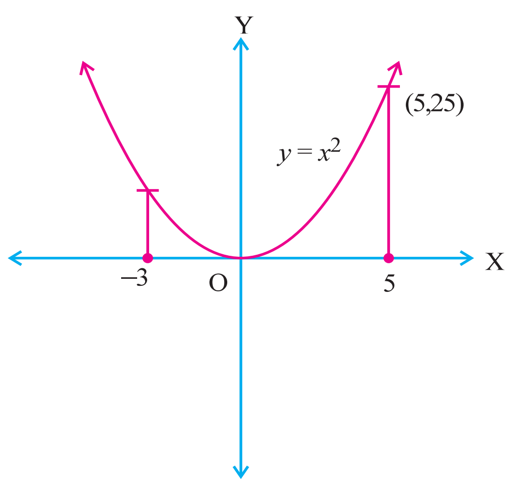


Figure 1.14

The function $f(x) = x^3$, $x \in$ R has no extreme value in R.

Look at following graph.

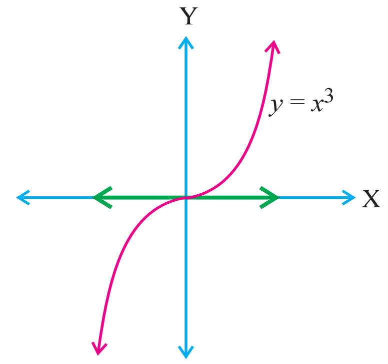


Figure 1.15

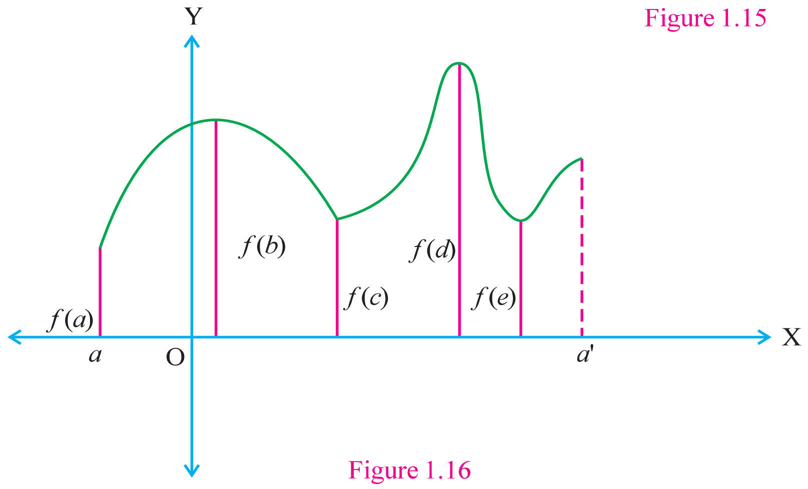
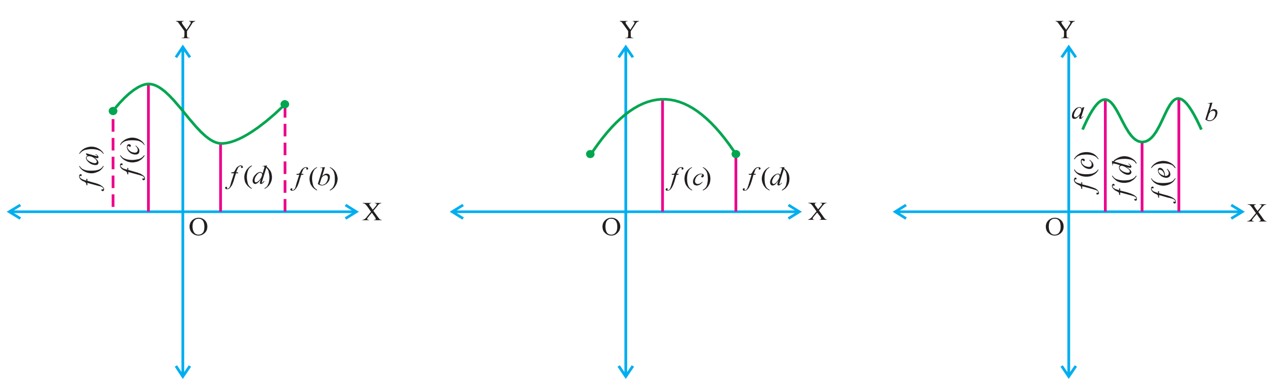


Figure 1.16

See that the global minimum occurs at $x = a$ in $[a, a']$ and the global maximum occurs at $x = d$. $f(b)$ is local maximum and $f(c)$ and $f(e)$ are local minimum values. Also global minimum occurs at an end-point of the interval but global maximum occurs at an interior point of the domain. Now we assume following result without giving proof.

The Extreme Value Theorem : If a function f is continuous on $[a, b]$, f attains its global maximum value at some $c \in [a, b]$ and global minimum value at some $d \in [a, b]$.

These are called extreme values of the function.



(a) (b) (c)

Figure 1.17

In figure 1.17(a) both maximum and minimum values of f occur at an interior point of $[a, b]$. In figure 1.17(b) the maximum occurs at $c \in (a, b)$ and minimum at $d = b$. In figure 1.17(c), there are two maxima (i.e. more than one).

Look at figure 1.18.

Here the domain of the function is [1, 4] but the function is discontinuous at $x = 2$. Its range is [0, 4). For no $x \in$ [1, 4], $f(x) = 4$. The function has no maximum.

Hence, we have kept the assumption that f is 'continuous' in the extreme value theorem.

But a discontinuous function could well have maximum and minimum value.

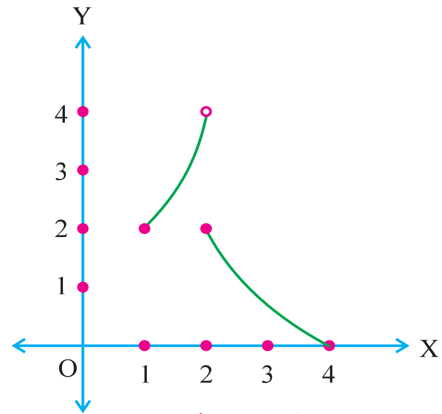


Figure 1.18

Look at figure 1.19.

The function is continuous on (0, 4), but it has neither maximum nor minimum value. The range is $(1, \infty)$. Hence, the condition 'closed interval' enters in the extreme value theorem.

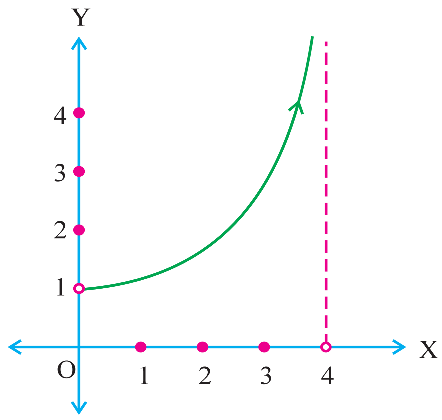


Figure 1.19

$f(x) = x$ in (0, 2) has no maximum or minimum but $f(x) = x$ in [0, 2] has maximum $f(2) = 2$ and minimum $f(0) = 0$. For $f(x) = x$, let $x_1 \in (0, 2)$. Then $x_1 < \left(\frac{x_1 + 2}{2}\right) < 2$ as $x_1 < 2$.

See that in figure 1.19(a), we get $f\left(\frac{x_1 + 2}{2}\right) = \frac{x_1 + 2}{2}$ which is larger than $f(x_1)$, where $x_1 \in (0, 2)$. No $f(x)$ can be maximum. Similarly $f\left(\frac{x_1}{2}\right) < f(x_1)$, so $f(x)$ has no minimum value.

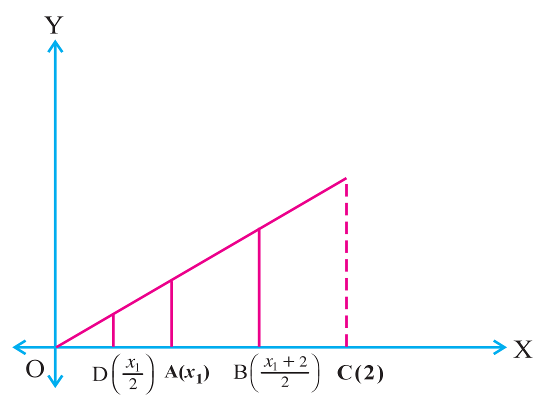


Figure 1.19(a)

Mid-point of $\overline{AC}$ is B and mid-point of $\overline{OA}$ is D. Thus we get a larger value $f\left(\frac{x_1 + 2}{2}\right)$ at B than any value $f(x_1)$ at A and a smallar value $f\left(\frac{x_1}{2}\right)$ at D than value at A.

$\therefore$ There is no maximum or no minimum.

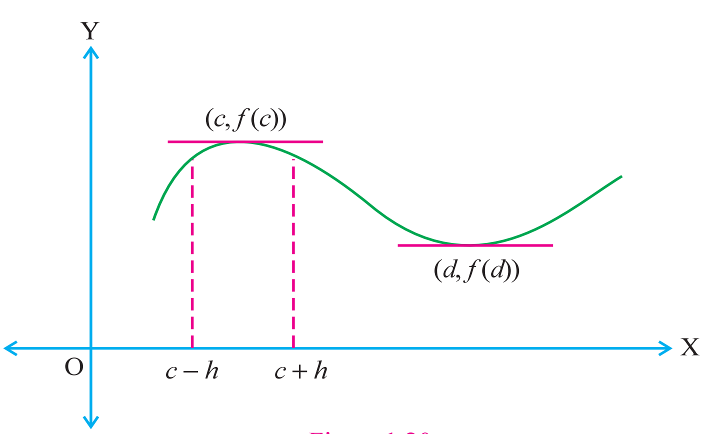


Figure 1.20

Look at the graph (figure 1.20). f has a local maximum at $x = c$. In $(c - h, c)$, f is increasing and therefore $f'(x) > 0$. In $(c, c + h)$, f is decreasing and so $f'(x) < 0$. As x takes values in $(c - h, c + h)$ and passes through c, $f'(x)$ changes from positive to negative. Also $f'(c) = 0$.

Similarly at $x = d$, f has a local minimum and f' changes sign from negative to positive and $f'(d) = 0$.

Thus we accept the following theorem without proof.

Theorem 1.2 (Fermat's Theorem) : If f has a local maximum or local minimum at c and if f is differentiable at c, then $f'(c) = 0$.

Although this is only a necessary condition and not sufficient. For $f(x) = x^3$, $f'(0) = 0$ but it does not have a maximum or minimum. Such a point where the graph crosses its horizontal tangent is called a point of inflexion. For $f(x) = x^3$, $(0, 0)$ is a point of inflexion. At $(0, 0)$ tangent is horizontal.

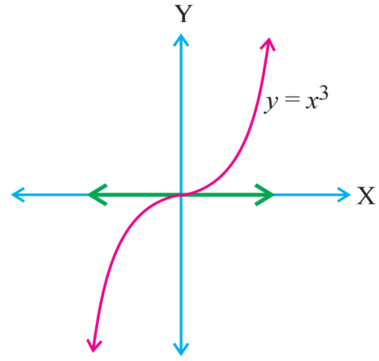


Figure 1.21

Fermat's theorem is named after **Pierre Fermat** (1601-1665). He was a French lawyer and mathematics was his hobby. He was one of the inventors of analytic geometry (the other being **Des Cartes**).

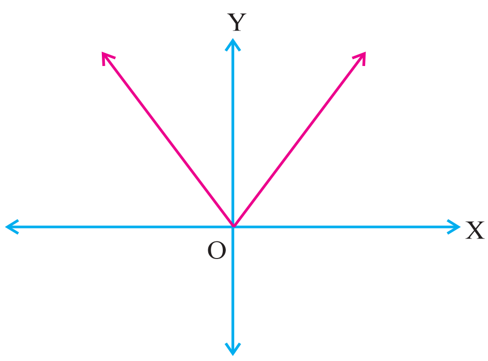


Figure 1.22

Also f may have an extreme value at $x = c$ and f may not be differentiable at c.

$f(x) = |x|$ has minimum at $x = 0$. $f(0) = |0| = 0$ is minimum value of $f(x) = |x|$ but f is not differentiable for $x = 0$.

Hence we define,

A Critical Number (Point) : A critical number (point) c of a function is a number $c \in D_f$ such that $f'(c) = 0$ or f is not differentiable at c.

Thus if f has a local maximum or local minimum at $x = c$, c is a critical number of f.

We now state following first derivative test from above discussion.

First Derivative Test : Let f be defined in an open interval $I = (a, b)$. $c \in I$ is a critical point of f and f is continuous at c.

(1) If there exists a positive number h such that $(c - h, c + h) \subset I$, $f'(x) > 0$ in $(c - h, c)$ and $f'(x) < 0$ in $(c, c + h)$, then f has a local maximum value at c.

(2) If there exists a positive number h such that $(c - h, c + h) \subset I$, $f'(x) < 0$ in $(c - h, c)$ and $f'(x) > 0$ in $(c, c + h)$, then f has a local minimum value at c.

(3) If $f'(x)$ does not change sign as x takes values in $(c - h, c + h)$ for any $h > 0$, f has neither maximum nor minimum value at $x = c$. Such a point is called a point of inflexion.

For some $h > 0$

$f'(x)$ changes from $+ve$ in $(c - h, c)$ to $-ve$ in $(c, c + h)$	$f(c)$ is a local maximum
$f'(x)$ changes from $-ve$ in $(c - h, c)$ to $+ve$ in $(c, c + h)$	$f(c)$ is a local minimum

Sometimes, it may not be easy to handle first derivative test. Then we may use following second derivative test.

Second Derivative Test : Let f be defined on an interval I = $[a, b]$. Let $c \in (a, b)$. Suppose $f''(c)$ exists. Then

(1) f has local maximum at $x = c$, if $f'(c) = 0$, $f''(c) < 0$.

(2) f has local minimum at $x = c$, if $f'(c) = 0$, $f''(c) > 0$.

(3) The test fails to give any conclusion if $f'(c) = f''(c) = 0$.

Note : $f''(c) < 0$, $f'(c) = 0$ means $f'(x)$ is decreasing at $x = c$ and since $f'(c) = 0$, $f'(x)$ changes from $+ve$ to $-ve$.

$\therefore$ $f(x)$ has a local maximum at $x = c$.

Similarly if $f''(c) > 0$, $f'(c) = 0$ we can conclude that $f(x)$ has a local minimum at $x = c$.

Example 53 : Find the critical points for $f(x) = x^{\frac{3}{5}}(4 - x)$, $x \in R^+ \cup \{0\}$.

Solution : $f(x) = 4x^{\frac{3}{5}} - x^{\frac{8}{5}}$

$$\therefore \quad f'(x) = \frac{12}{5}x^{-\frac{2}{5}} - \frac{8}{5}x^{\frac{3}{5}}$$

$$= \frac{4}{5}\left(\frac{3}{x^{\frac{2}{5}}} - 2x^{\frac{3}{5}}\right)$$

$$= \frac{4}{5}\left(\frac{3 - 2x}{x^{\frac{2}{5}}}\right)$$

$\therefore$ $f'(x) = 0$, if $x = \frac{3}{2}$ and $f'(x)$ does not exist at $x = 0$ but $0 \in D_f$.

$\therefore$ The cricital points are 0 and $\frac{3}{2}$.

Example 54 : Find local maximum or minimum values of $f(x) = |x|$. $x \in R$

Solution : f is not differentiable at $x = 0$, $0 \in D_f$. So 0 is a critical point and the second derivative of f does not exist at $x = 0$.

$$\therefore \quad f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$\therefore$ $f'(x) = 1$ if $x > 0$

and $f'(x) = -1$ if $x < 0$.

$\therefore$ $f'(x)$ changes from negative to positive as x passes through 0 and f is not differentiable at $x = 0$.

$\therefore$ $f'(x)$ changes from negative to positive as x changes from $(-h, 0)$ to $(0, h)$, $h > 0$.

$\therefore$ f has a local minimum value $f(0) = 0$ at $x = 0$. f has no maximum value.

Note : Obviously $f(x) = |x| \geq 0 \quad \forall x \in R$

$\therefore$ f has a local and global minimum at $x = 0$.

To find extreme values for a function defined on a closed interval [*a*, *b*].

(1) Find local maximum and local minimum values of f.

(2) Find values of f at end-points.

The largest of the values obtained in (1) and (2) is global maximum and the smallest of the values obtained in (1) and (2) is the global minimum value of f.

Example 55 : Examine for maximum and minimum values : $f(x) = 3x^4 - 16x^3 + 18x^2, x \in [-1, 4]$

Solution : $f(x) = 3x^4 - 16x^3 + 18x^2$

$\therefore$ $f'(x) = 12x^3 - 48x^2 + 36x$

$= 12x(x^2 - 4x + 3)$

$= 12x(x - 3)(x - 1)$

$\therefore$ $f'(x) = 0 \Rightarrow x = 0, 1$ or 3.

$\therefore$ $f''(x) = 36x^2 - 96x + 36$

$\therefore$ $f''(0) = 36 > 0, f''(1) = -24 < 0, f''(3) = 72 > 0$

$\therefore$ $f(0)$ is local minimum and $f(0) = 0$ is local minimum.

f has local maximum at $x = 1$ and $f(1) = 5$ is local maximum.

f has local minimum at $x = 3$ and $f(3) = -27$ is local minimum.

Local maximum or minimum values can occur only at an interior point of $[-1, 4]$.

For global maxinum and minimum values, consider $f(-1)$ and $f(4)$.

$f(-1) = 37, \quad f(4) = 32$

$f(0) = 0, \quad f(1) = 5, \quad f(3) = -27, \quad f(-1) = 37, \quad f(4) = 32$

$\therefore$ $f(-1) = 37$ is global maximum occuring at an end-point.

$\therefore$ $f(3) = -27$ is global minimum and it occurs at an interior point $3 \in (-1, 4)$.

Example 56 : Find maximum and minimum values of the function $f(x) = x^3 - 12x + 1, x \in [-3, 5]$

Solution : $f(x) = x^3 - 12x + 1$

$\therefore$ $f'(x) = 3x^2 - 12 = 3(x - 2)(x + 2)$

$\therefore$ $f'(x) = 0 \Rightarrow x = \pm 2$

$f''(x) = 6x$

$\therefore$ $f''(2) = 12 > 0$

$\therefore$ $f(2) = 8 - 24 + 1 = -15$ is local minimum value.

$\therefore$ $f''(-2) = -12 < 0$

$\therefore$ $f(-2) = -8 + 24 + 1 = 17$ is local maximum value.

Moreover, $f(-3) = -27 + 36 + 1 = 10, \quad f(5) = 125 - 60 + 1 = 66$

$f(2) = -15, f(-2) = 17$

$\therefore$ $f(5) = 66$ is global maximum and

$f(2) = -15$ is global minimum.

Example 57 : Find maximum and minimum values of the function $f(x) = 3x^5 - 5x^3 - 1$, $x \in [-2, 2]$

Solution : $f'(x) = 15x^4 - 15x^2$

$= 15x^2 (x^2 - 1)$

$= 15x^2 (x - 1)(x + 1)$

$\therefore$ $f'(x) = 0 \Rightarrow x = 0$ or $x = \pm 1$

$f''(x) = 60x^3 - 30x$

$f''(1) = 30 > 0$

$\therefore$ $f(1) = -3$ is local minimum value.

$f''(-1) = -30 < 0$

$\therefore$ $f(-1) = 1$ is local maximum value.

But $f''(0) = 0$

$\therefore$ Second derivative test fails.

$\therefore$ $f'(x) = 15x^2 (x - 1)(x + 1)$

$x^2 > 0$, if $x \neq 0$

If $-1 < x < 1$ then $x + 1 > 0$ and $x - 1 < 0$

$\therefore$ $f'(x) < 0$ for $-1 < x < 1$.

$\therefore$ $f'(x)$ does not change sign as x increases in $(-1, 1)$.

$\therefore$ 0 is a point of inflexion.

$\therefore$ $f(2) = 96 - 40 - 1 = 55$

$f(-2) = -96 + 40 - 1 = -57$. Also $f(1) = -3$, $f(-1) = 1$.

$\therefore$ $f(2) = 55$ is global maximum and $f(-2) = -57$ is global minimum value.

Example 58 : Determine maximum and minimum values of $f(x) = x - 2cosx$, $x \in [-\pi, \pi]$

Solution : $f(x) = x - 2cosx$

$\therefore$ $f'(x) = 1 + 2sinx$

$\therefore$ $f'(x) = 0 \Rightarrow sinx = -\frac{1}{2}$

$\therefore$ $x = -\frac{\pi}{6}, \frac{-5\pi}{6}$ $\qquad$ ***$x \in (-\pi, \pi)$***

Now $f''(x) = 2cosx$

$\therefore$ $f''\left(-\frac{\pi}{6}\right) = 2cos\left(-\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} > 0$

$\therefore$ $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2cos\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2 \times \frac{\sqrt{3}}{2} = -\frac{\pi}{6} - \sqrt{3}$

$\therefore$ $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - \sqrt{3}$ is local minimum value at $x = -\frac{\pi}{6}$.

$\therefore \quad f''\left(-\frac{5\pi}{6}\right) = 2cos\left(-\frac{5\pi}{6}\right) = 2cos\,\frac{5\pi}{6} = 2cos\left(\pi - \frac{\pi}{6}\right)$

$= -2cos\frac{\pi}{6}$

$= -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} < 0$

$\therefore \quad f\left(-\frac{5\pi}{6}\right) = -\frac{5\pi}{6} + 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} - \frac{5\pi}{6}$ is local maximum value.

$f(\pi) = \pi - 2cos\pi = \pi + 2$

$f(-\pi) = -\pi - 2cos(-\pi) = -\pi - 2cos\pi = -\pi + 2$

$\therefore \quad f(\pi) = \pi + 2$ is global maximum value.

$\therefore \quad f\left(-\frac{\pi}{6}\right) = -\sqrt{3} - \frac{\pi}{6}$ is global minimum value.

Example 59 : Find maximum and minimum values of $f(x) = 4x + cotx;\ x \in (0, \pi)$

Solution : Now $f'(x) = 4 - cosec^2x = 0$

$\therefore \quad cosec^2x = 4$

$\therefore \quad sin^2x = \frac{1}{4}$

$\therefore \quad sinx = \frac{1}{2}$ **$x \in (0, \pi)$**

$\therefore \quad x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$f''(x) = -2cosecx\ (-cosecx\ cotx)$

$= 2cosec^2x\ cotx$

$\therefore \quad f''\left(\frac{\pi}{6}\right) = 2(4)\sqrt{3} > 0,\quad f''\left(\frac{5\pi}{6}\right) = -8\sqrt{3} < 0$

$\therefore \quad f\left(\frac{\pi}{6}\right) = \frac{2\pi}{3} + \sqrt{3}$ is local minimum value at $x = \frac{\pi}{6}$.

and $f\left(\frac{5\pi}{6}\right) = \frac{10\pi}{3} - \sqrt{3}$ is local maximum value at $x = \frac{5\pi}{6}$.

[Why no global maximum or global minimum ?]

Example 60 : Prove that out of all rectangles with given area, the square has minimum perimeter.

Solution : Let the given area be A and the lengths of the sides of the rectangle be x and y.

$\therefore \quad A = xy$

Now perimeter of the rectangle, $p = 2x + 2y$

$= 2x + \frac{2A}{x}$

Now $\frac{dp}{dx} = 0 \Rightarrow 2 - \frac{2A}{x^2} = 0$

$\therefore \quad x^2 = A$

$\therefore \quad x = \sqrt{A}$ **(Since x is the side of a rectangle, $x > 0$)**

$\therefore \quad y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A}$

Since $x = y$ the rectangle becomes a square.

Also $\dfrac{d^2p}{dx^2} = 0 - 2A(-2x^{-3}) = \dfrac{4A}{x^3} > 0$

$\therefore$ The perimeter is minimum when the rectangle becomes a square.

Note : $(x + y)^2 = (x - y)^2 + 4xy = (x - y)^2 + 4A$

$\therefore$ $(x + y)^2$ is minimum when $x = y$ as $(x - y)^2 \geq 0$ and minimum value of $(x - y)^2 = 0$ if $x = y$ and A is a constant.

$\therefore$ The perimeter of a square is minimum.

Example 61 : Find a point P on $y^2 = 8x$ nearest to A(10, 4) and also find minimum distance AP.

Solution : Parametric equations of a parabola are $(at^2, 2at)$.

Here $4a = 8$. So $a = 2$.

$\therefore$ A typical point on the parabola is $P(2t^2, 4t)$.

Now $AP^2 = (2t^2 - 10)^2 + (4t - 4)^2$

$= 4t^4 - 40t^2 + 100 + 16t^2 - 32t + 16$

Let $f(t) = 4t^4 - 24t^2 - 32t + 116$

$f'(t) = 16t^3 - 48t - 32$

$= 16(t^3 - 3t - 2)$

$= 16(t + 1)(t^2 - t - 2)$

$= 16(t + 1)^2(t - 2)$

$\therefore$ $f'(t) = 0 \Rightarrow t = -1$ or $t = 2$

Let $t \in (-1 - h, -1 + h)$ where $h > 0$. Let $t = -1 + t_1$

$\therefore$ $-1 - h < -1 + t_1 < -1 + h$ i.e. $-h < t_1 < h$

$\therefore$ $f'(t) = 16(t + 1)^2(t - 2)$ **$(t = -1 + t_1)$**

$= 16t_1^2(-3 + t_1) > 0$ if $0 < t_1 < 3$

$\therefore$ $f'(t)$ does not change sign in $(-1 - h, -1 + h)$

$\therefore$ f has no maximum or minimum at $t = -1$.

$\therefore$ $f''(t) = 48t^2 - 48$

$\therefore$ $f''(2) = 192 - 48 = 144 > 0$

$\therefore$ $f(t)$ is minimum, if $t = 2$

$\therefore$ AP^2 is minimum for $t = 2$. For $t = 2$, P is (8, 8).

If P(8, 8), then $AP = \sqrt{(10-8)^2 + (8-4)^2}$

$= \sqrt{4 + 16}$

$= 2\sqrt{5}$ is minimum

$\therefore$ The point nearest to A(10, 4) and lying on $y^2 = 8x$ is P(8, 8).

Example 62 : Find the maximum area of a rectangle inscribed in a semi-circle of radius r.

Solution : Let us consider the semi-circle in upper half-plane of X-axis.

Let $A(x, y)$ be one vertex of the rectangle in the first quadrant. Obviously other vertices are $B(x, 0)$, $C(-x, 0)$ and $D(-x, y)$.

$\therefore$ AD = $2x$, AB = y

$\therefore$ The area of the rectangle $f(x) = 2xy$

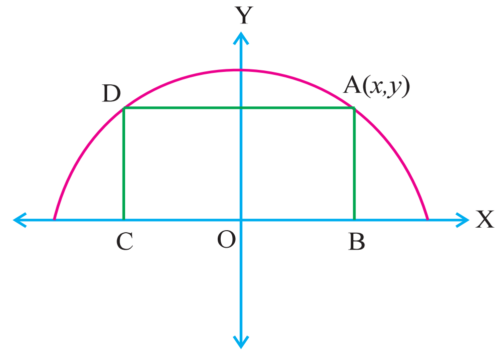


Figure 1.23

Also $x^2 + y^2 = r^2$

$\therefore$ $y = \sqrt{r^2 - x^2}$ **($y > 0$)**

$\therefore$ $f(x) = 2x\sqrt{r^2 - x^2}$

$\therefore$ $f'(x) = 2\sqrt{r^2 - x^2} + \dfrac{2x(-2x)}{2\sqrt{r^2 - x^2}}$

$= 2\sqrt{r^2 - x^2} - \dfrac{2x^2}{\sqrt{r^2 - x^2}}$

$= \dfrac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$

$\therefore$ $f'(x) = 0 \Rightarrow r^2 = 2x^2 \Rightarrow x = \dfrac{r}{\sqrt{2}}$

$\therefore$ $y = \sqrt{r^2 - x^2} = \sqrt{r^2 - \dfrac{r^2}{2}} = \dfrac{r}{\sqrt{2}}$

$\therefore$ $x = y = \dfrac{r}{\sqrt{2}}$

$$f''(x) = 2\left[(r^2 - 2x^2)\left(-\tfrac{1}{2}\right)(r^2 - x^2)^{-\frac{3}{2}}(-2x) + \dfrac{(-4x)}{\sqrt{r^2 - x^2}}\right]$$

$$f''\left(\dfrac{r}{\sqrt{2}}\right) = \dfrac{-8 \times \frac{r}{\sqrt{2}}}{\frac{r}{\sqrt{2}}} = -8 < 0$$

$\therefore$ Area is maximum for a square and maximum area is A $= 2xy = 2 \cdot \dfrac{r}{\sqrt{2}} \cdot \dfrac{r}{\sqrt{2}} = r^2$.

Note : **(1)** A $= 2xy$

Now $x^2 + y^2 = (x - y)^2 + 2xy$

$= (x - y)^2 + \text{A}$

$\therefore$ A $= r^2 - (x - y)^2$ is maximum if $(x - y)^2$ minimum. But $(x - y)^2 \geq 0$.

$\therefore$ $(x - y)^2$ has minimum value 0 when $x = y$. Hence maximum A $= r^2$.

(2) Let $x = r\cos\theta$, $y = r\sin\theta$ **(Parametric equations of $x^2 + y^2 = r^2$)**

$\therefore$ A $= 2xy = 2r^2 \sin\theta \cos\theta = r^2 \sin 2\theta$

$\therefore$ A is maximum when $\theta = \frac{\pi}{4}$ as $\sin 2\theta = 1$ is maximum for $\theta = \frac{\pi}{4}$.

$\therefore$ Maximum area $= r^2$

Example 63 : A cylinder is inscribed in a sphere of radius R. Prove that its volume is maximum if its height is $\frac{2R}{\sqrt{3}}$.

Solution : Let the radius and the height of the cylinder be r and h respectively.

Then $R^2 = r^2 + \frac{h^2}{4}$

Volume of the cylinder, $V = \pi r^2 h$

$\therefore \quad V = \pi\left(R^2 - \frac{h^2}{4}\right)h$

$= \pi R^2 h - \frac{\pi}{4}h^3$

$\frac{dV}{dh} = \frac{\pi}{4}(4R^2 - 3h^2)$

$\therefore \quad \frac{dV}{dh} = 0 \Rightarrow h = \frac{2R}{\sqrt{3}}$

Also $\frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) = \frac{-3\pi h}{2} = -\sqrt{3}\pi R < 0$

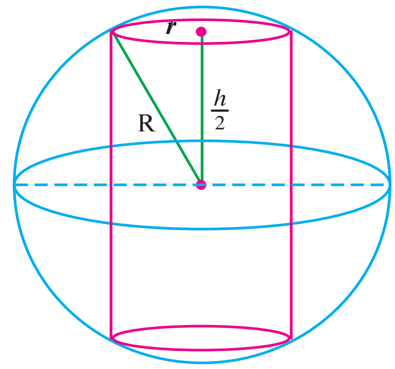


Figure 1.24

$\therefore$ The volume of the cylinder is maximum if $h = \frac{2R}{\sqrt{3}}$.

Maximum volume is $\pi r^2 h = \pi\left(R^2 - \frac{h^2}{4}\right)h$

$= \pi\left(R^2 - \frac{R^2}{3}\right)\frac{2R}{\sqrt{3}}$

$= \frac{4\pi R^3}{3\sqrt{3}}$

Example 64 : A cylindrical can is to be made to hold 1 l oil. Find its radius and height to minimize the cost.

Solution : The cost of making the can is minimum, if the metal used to make the can is minimum.

Its total surface area S is given by $S = 2\pi r^2 + 2\pi rh$

Now the volume $V = \pi r^2 h$ and 1 *litre* is 1000 cm^3.

$\therefore \quad V = \pi r^2 h = 1000$

$\therefore \quad h = \frac{1000}{\pi r^2}$

$\therefore \quad S = 2\pi r^2 + 2\pi r \times \frac{1000}{\pi r^2}$

$= 2\pi r^2 + \frac{2000}{r}$

$\therefore \quad \frac{dS}{dr} = 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow r^3 = \frac{500}{\pi}$

$\therefore \quad r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$

$\frac{d^2S}{dr^2} = 4\pi + \frac{4000}{r^3} > 0$

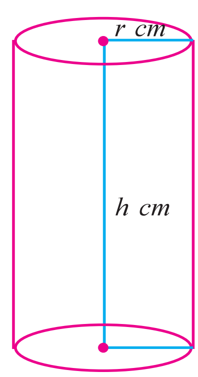


Figure 1.25

$\therefore$ Surface area and hence the cost is minimum if $r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}}$ cm and

$h = \frac{1000(\pi)^{\frac{2}{3}}}{\pi(500)^{\frac{2}{3}}} = 2\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$ $cm = 2r$.

Thus the height of the cylinder should equal its diameter for minimum cost.

Example 65 : Find the point on the line $y = 2x - 3$ nearest to origin.

Solution : Let M = $(x, 2x - 3)$ be any point on the given line.

$OM^2 = x^2 + (2x-3)^2$

$= 5x^2 - 12x + 9$

Let $f(x) = 5x^2 - 12x + 9$

$\therefore\ f'(x) = 10x - 12 = 0 \Rightarrow x = \frac{6}{5}$

Also $f''(x) = 10 > 0$

$\therefore$ Distance OM is minimum if $x = \frac{6}{5}$, $y = 2\left(\frac{6}{5}\right) - 3 = -\frac{3}{5}$

$M = \left(\frac{6}{5}, -\frac{3}{5}\right)$

$OM = \sqrt{\frac{36}{25} + \frac{9}{25}} = \sqrt{\frac{45}{25}} = \frac{3\sqrt{5}}{5} = \frac{3}{\sqrt{5}}$

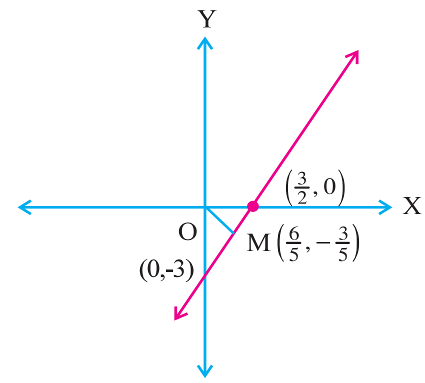


Figure 1.26

Note : $p = \frac{|c|}{\sqrt{a^2+b^2}} = \left|\frac{0+0-3}{\sqrt{4+1}}\right| = \frac{3}{\sqrt{5}}$

$\therefore$ OM is perpendicular distance of origin from $y = 2x - 3$ and M is the foot of perpendicular.

Exercise 1.5

Find the maximum and minimum values of following functions (1 to 15) :

1. $f(x) = 5 - 3x + 5x^2 - x^3$ $\quad x \in R$

2. $f(x) = x^4 - 6x^2$ $\quad x \in R$

3. $f(x) = x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$ $\quad x \in R^+$

4. $f(x) = 2cosx + sin^2x$ $\quad x \in R$

5. $f(x) = \log_e(1 + x^2)$ $\quad x \in R$

6. $f(x) = xe^{-x}$ $\quad x \in [0, 2]$

7. $f(x) = \frac{\log_e x}{x}$ $\quad x \in [1, 3]$

8. $f(x) = \sqrt{16 - x^2}$ $\quad |x| \le 4$

9. $f(x) = \frac{x}{x+1}$ $\quad x \in [1, 2]$

10. $f(x) = sinx + cosx$ $\quad x \in [0, 2\pi]$

11. $f(x) = \frac{cosx}{sinx + 2}$ $\quad x \in [0, 2\pi]$

12. $f(x) = x\sqrt{1-x}$ $\quad 0 < x < 1$

13. $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 125$ $\quad x \in [0, 3]$

14. $f(x) = \sin 2x$ $\quad x \in [0, 2\pi]$

15. $f(x) = 2x^3 - 24x + 107$ $\quad x \in [1, 3]$

16. A window is in the form of a rectangle surmounted by semicircular opening. The total perimeter of the window is 10 *m*. Find dimensions of the window for maximum air flow through the window.

17. Prove that the right circular cone of maximum volume inscribed in a sphere of radius r has altitude $\frac{4r}{3}$.

18. Find two positive numbers whose sum is 16 and the sum of cubes of them is minimum.

19. Find positive numbers x, y for which $x + y = 35$ and the product x^2y^5 is maximum.

20. Show that the semi-vertical angle of the cone having maximum volume and given slant height l is $\tan^{-1}\sqrt{2}$.

21. A open box with a square base is to be made. Its total surface area is c^2, a constant. Prove that its maximum volume is $\frac{c^3}{6\sqrt{3}}$.

22. Find a point on circle $x^2 + y^2 = 25$ whose distance from (12, 9) is minimum. Find also the point for which it is maximum. Explain geometrically.

23. Sum of circumference of a circle and perimeter of a square is constant. Prove that the sum of their areas is minimum when the ratio of the radius of the circle to a side of the square is 1:2.

24. An open tank with a square base is to be made to hold 4000 litres of water. What are the dimensions to make the cost minimum ?

25. $f(x) = x^3 + 3ax^2 + 3bx + c$ has a maximum at $x = -1$ and minimum zero at $x = 1$. Find a, b and c.

26. If a right triangle has hypotenuse having length 10 *cm*, what would be its largest area ?

*

Miscellaneous Examples :

Example 66 : Suppose we do not know formula for $g(x)$. But $g'(x) = \sqrt{x^2 + 12}$, $\forall x \in$ R. Also $g(2) = 4$. Find approximate value of $g(1.95)$.

Solution : Here $x = 2$. $\Delta x = 1.95 - 2 = -0.05$

$$g(x + \Delta x) \simeq g(x) + g'(x)\,\Delta x$$

$$\therefore \quad g(1.95) \simeq g(2) + g'(2)\,(-0.05)$$

$$= 4 - (0.05)4$$

$$= 4 - 0.2$$

$$= 3.8$$

Example 67 : Find the common tangents of $y = 1 + x^2$ and $y = -1 - x^2$. Also find their points of contact.

Solution : Let $\overleftrightarrow{PQ}$ touch $y = 1 + x^2$ at P and $y = -1 - x^2$ at Q. Let P have x-coordinate a.

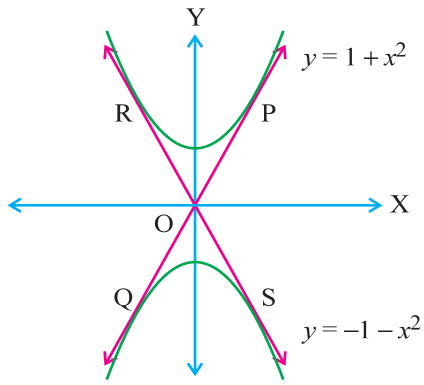


Figure 1.27

$\therefore$ $P(a,\ 1 + a^2)$, $Q = (-a,\ -(1 + a^2))$

Slope of $\overleftrightarrow{PQ} = \dfrac{1 + a^2 - (-(1 + a^2))}{a - (-a)}$

$= \dfrac{2(1 + a^2)}{2a} = \dfrac{1 + a^2}{a}$

Also $y = 1 + x^2 \Rightarrow \dfrac{dy}{dx} = 2x$

$\therefore$ Slope of tangent at P $= 2a$.

$\therefore$ $\dfrac{1 + a^2}{a} = 2a$ **($\overleftrightarrow{PQ}$ is a tangent)**

$\therefore$ $1 + a^2 = 2a^2$

$\therefore$ $a^2 = 1$

$\therefore$ $a = \pm 1$

$\therefore$ $P = (1, 2)$, $Q = (-1, -2)$

Similarly, $R = (-1, 2)$, $S(1, -2)$

The equation of $\overleftrightarrow{PQ}$ is $y - 2 = 2(x - 1)$

$\therefore$ $y - 2 = 2x - 2$

$\therefore$ $2x - y = 0$

Similarly, the equation of $\overleftrightarrow{RS}$ is $2x + y = 0$.

$\therefore$ The equations of common tangent are $2x - y = 0$ and $2x + y = 0$.

Example 68 : The position of a particle is given by $s = f(t) = t^3 - 6t^2 + 9t$, s is in meters, t is in seconds.

(1) Find the instantaneous velocity, when $t = 2$.

(2) When is the particle at rest ?

(3) Find the distance travelled in first 5 seconds.

Solution : $\dfrac{ds}{dt} = f'(t) = 3t^2 - 12t + 9$

(1) $[f'(t)]_{t\,=\,2} = 12 - 24 + 9 = -3\,m/sec$

(2) When the particle is at rest, its velocity at that time is zero.

$\therefore$ $3t^2 - 12t + 9$

$\therefore$ $t^2 - 4t + 3 = 0$

$\therefore$ $t = 1$ or 3

$\therefore$ The particle is at rest at $t = 1$ and $t = 3$.

(3) $f'(t) = 3(t - 1)(t - 3)$

$\therefore$ For $t < 1$ and $t > 3$, $f'(t) > 0$, and $f(t)$ is increasing and $f(t)$ is decreasing for $t \in (1, 3)$. The motion is divided into 3 parts $(0, 1)$, $(1, 3)$, $(3, 5)$.

$\therefore$ Total distance covered is $s_1 + s_2 + s_3$, where

$s_1 = | f(1) - f(0) | = 4$, $s_2 = | f(3) - f(1) | = | 0 - 4 | = 4$

$s_3 = | f(5) - f(3) | = 20$

$\therefore$ Total distance covered is $20 + 4 + 4 = 28$ *m*.

Note : $| f(5) - f(0) | = 20$ is not the total distance covered.

Example 69 : An exhibition is to be arranged in a rectangular ground. A fencing of 80 *m* is done on three sides of the plot and the fourth side is not to be covered by fencing. What should be the dimensions of the ground to cover maximum area ?

Solution : We have $2x + y = 80$

$A = xy = x(80 - 2x) = 80x - 2x^2$

$\therefore \frac{dA}{dx} = 0 \Rightarrow 80 - 4x = 0 \Rightarrow x = 20$

$\therefore \frac{d^2A}{dx^2} = -4 < 0$

$\therefore$ Largest area is covered if the length is

$y = 80 - 2x = 80 - 40 = 40$ *m*

and the breadth is $x = 20$ *m*.

$\therefore$ Maximum area covered is $40 \times 20 = 800$ m^2

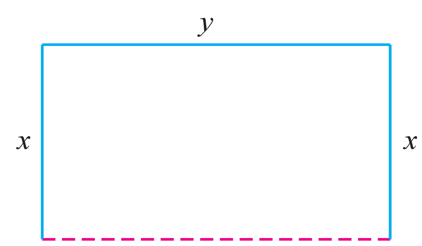


Figure 1.28

Only for information :

C(x) is the cost of producing x units. C(x) is the cost function.

C'(x) is the marginal cost.

$c(x) = \frac{C(x)}{x}$ is the cost per unit. $c(x)$ is average cost function.

$c'(x) = \frac{xC'(x) - C(x)}{x^2}$

For minimum of average cost $c'(x) = 0$.

$\therefore xC'(x) = C(x)$

$\therefore C'(x) = \frac{C(x)}{x} = c(x)$

If the average cost is minimum, marginal cost = average cost.

If the profit is maximum, marginal revenue $\frac{dR}{dx}$ = marginal cost $\frac{dC}{dx}$ and R''(x) < C''(x).

If $p(x)$ is the sale price per unit, if x units are sold, p is called demand function.

The total revenue is R(x) = $xp(x)$.

R(x) is called revenue function. R'(x) is marginal revenue function.

If P(x) is the profit function.

P(x) = R(x) − C(x)

For maximum profit P'(x) = 0

$\therefore$ R'(x) = C'(x)

$\therefore$ Marginal revenue = marginal cost for maximum profit.

Also P''(x) = R''(x) − C''(x) < 0

$\therefore$ R''(x) < C''(x) for maximum profit.

Example : A company estimates that the cost of producing x ball-pens is $C(x) = 3000 + 2x + 0.001x^2$.

(1) Find the cost, average cost and marginal cost of producing 1000 ball-pens.

(2) At what production level, will the average cost be minimum and what is that minimum average cost ?

Solution : (1) The average cost function is $c(x) = \frac{C(x)}{x}$

$$= \frac{3000 + 2x + 0.001x^2}{x}$$

$$= \frac{3000}{x} + 2 + 0.001x$$

Also marginal cost function is $C'(x) = 2 + 0.002x$

$\therefore$ For production of 1000 ball-pens, $C(1000) = 3000 + 2000 + \frac{1}{1000} \times (1000)^2$

$= ₹\ 6000$

$\therefore$ $c(x) = \frac{6000}{1000} = ₹\ 6$ per ball-pen.

$C'(x) = 2 + \frac{2}{1000} \times 1000 = ₹\ 4$

(2) For minimum average cost :

Marginal cost = Average cost

$C'(x) = c(x)$

$\therefore$ $2 + 0.002x = \frac{3000}{x} + 2 + 0.001x$

$\therefore$ $0.001x = \frac{3000}{x}$

$\therefore$ $x^2 = 3000 \times 1000$

$\therefore$ $x = \sqrt{3 \times 10^6} = \sqrt{3} \times 10^3 = 1730$

$\therefore$ Hence, 1730 ball-pens should be manufactured for minimum average cost.

Minimum average cost $= c(1730) = \frac{3000}{1730} + 2 + (0.001)(1730)$

$$= \frac{300}{173} + 2 + 1.73$$

$$= 1.73 + 2 + 1.73$$

$= ₹\ 5.46$

Example 70 : Find the point on $xy = 8$, nearest to P(3, 0) having integer coordinates and the minimum distance. **($x > 0$)**

Solution : Let the required point on $xy = 8$ be $Q\left(x, \frac{8}{x}\right)$

$\therefore$ $PQ^2 = (x - 3)^2 + \frac{64}{x^2}$

Let $f(x) = (x - 3)^2 + \frac{64}{x^2}$

$f'(x) = 2(x - 3) - \frac{128}{x^3} = 0 \Rightarrow x - 3 = \frac{64}{x^3}$

$\therefore$ $x^4 - 3x^3 - 64 = 0$

$\therefore$ $(x - 4)(x^3 + x^2 + 4x + 16) = 0$

$\therefore$ $x = 4$ (Verify that $x^3 + x^2 + 4x + 16 = 0$ has no integer solution !)

$\therefore$ $f''(x) = 2 - \frac{(128)(-3)}{x^4}$

$\therefore$ $f''(4) = 2 + \frac{(128)(3)}{256} = \frac{7}{2} > 0$

$\therefore$ $f(x)$ is minimum for $x = 4$.

$\therefore$ The point nearest to P(3, 0) and lying on $xy = 8$ is Q(4, 2).

PQ $= \sqrt{1+4} = \sqrt{5}$

Example 71 : Find a point on $y^2 = 2x$ nearest to (1, 4) and the minimum distance.

Solution : For $y^2 = 2x = 4ax$, $a = \frac{1}{2}$

$\therefore$ Let Q(1, 4) and P $\left(\frac{1}{2}t^2, t\right)$ be any point on parabola.

$\therefore$ $PQ^2 = \left(\frac{1}{2}t^2 - 1\right)^2 + (t - 4)^2$

$= \frac{1}{4}t^4 - t^2 + 1 + t^2 - 8t + 16$

$= \frac{1}{4}t^4 - 8t + 17$

Let $f(t) = \frac{1}{4}t^4 - 8t + 17$

$f'(t) = 0 \Rightarrow t^3 - 8 = 0 \Rightarrow t = 2$

$f''(t) = 3t^2 = 12 > 0$

$\therefore$ $f(t)$ is minimum, if $t = 2$

$\therefore$ P(2, 2), Q(1, 4)

$\therefore$ PQ $= \sqrt{1+4} = \sqrt{5}$ is the minimum distance.

Example 72 : A rectangular sheet of tin 45 *cm* × 24 *cm* is to be made into an open box by cutting off squares of the same size from each corner and folding up. Find the side of the square cut off from each corner for maximum volume of the box.

Solution : Let x *cm* be the side of the square removed from each corner.

$\therefore$ Length and breadth of the box are $(45 - 2x)$ *cm* and $(24 - 2x)$ *cm*. The height is x *cm*.

The volume $V = (45 - 2x)(24 - 2x)x$

$= 4x^3 - 138x^2 + 1080x$

$\frac{dV}{dx} = 0 \Rightarrow 12x^2 - 276x + 1080 = 0 \Rightarrow x^2 - 23x + 90 = 0$

$\therefore$ $x = 18$ or 5

But if $x = 18$, breadth $24 - 2x = 24 - 36 < 0$

$\therefore$ $x \neq 18$ and so $x = 5$

The length of the side of square removed is 5 *cm*.

$\frac{d^2V}{dx^2} = 24x - 276 = 120 - 276 < 0$

$\therefore$ V is maximum if $x = 5$ *cm*

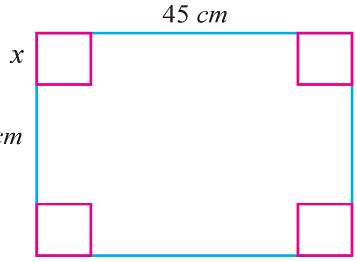


Figure 1.29

Exercise 1

1. Water is dripping out from conical funnel at the rate of 5 cm^3/sec. Slant height of the cone formed by water is 4 cm. Semi-vertical angle of the cone is $\frac{\pi}{6}$. Find the rate at which the slant height decreases.

2. Height of a kite is fixed at 40 m. The length of the string is 50 m at a moment. Velocity of the kite in horizontal direction is 25 m/sec at that time. Find the rate of slackening of the string at that time.

3. Altitude of a triangle increases at 2 cm/min. Its area increases at the rate 5 cm^2/min. Find the rate of change of length of base when the altitude is 10 cm and the area is 100 cm^2.

4. Find the intervals in which $f(x) = 2x^3 - 3x^2 - 36x + 25$ is (1) strictly increasing (2) strictly decreasing.

5. Find the intervals in which $f(x) = (x+1)^3(x-3)^3$ is (1) strictly increasing (2) strictly decreasing.

6. Prove $x^{101} + sinx - 1$ is increasing for $|x| > 1$.

7. Find the intervals where $f(x) = x^4 + 32x$ is increasing or decreasing. $x \in R$

8. Find the intervals in which $f(x) = x^2e^{-x}$ is increasing or decreasing. $x \in R$

9. Prove that curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each-other.

10. Find the equation of tangent to $y = be^{-\frac{x}{a}}$ where it intersects Y-axis.

11. Find the measure of the angle between $y^2 = 4ax$ and $x^2 = 4ay$.

12. Prove that $y = 6x^3 + 15x + 10$ has no tangent with slope 12.

13. Find points on the ellipse $x^2 + 2y^2 = 9$ at which tangent has slope $\frac{1}{4}$.

14. Find maximum and minimum values of $f(x) = x - 2sinx \quad x \in [0, 2\pi]$

15. Find maximum and minimum values of $f(x) = 1 - e^{-x} \quad x \geq 0$

16. Find maximum and minimum values of $f(x) = x^2 + \frac{2}{x} \quad x \neq 0$

17. Find where $f(x) = 4x - tanx$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is increasing or decreasing and find its maximum and minimum values.

18. Where does $f(x) = x + \sqrt{1-x}$, $0 < x < 1$ increase or decrease ? Find its maximum and minimum values.

19. Determine critical points for $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$, $x \in [0, 6]$ and determine where the function is increasing or decreasing. Find also maximum and minimum values.

20. Find the maximum and minimum values of $f(x) = sin^4x + cos^4x$. $x \in \left[0, \frac{\pi}{2}\right]$.

21. Show that $f(x) = \left(\frac{1}{x}\right)^x$ has local maximum at $x = \frac{1}{e}$.

22. Show that out of all rectangles with given area a square has minimum perimeter.

23. Show that out of all rectangles inscribed in a circle, the square has maximum area.

24. Prove that the area of a right angled triangle with given hypotenuse is maximum, if the triangle is isoceles.

25. A point on the hypotenuse of a right triangle is at distances a and b from the sides making right angle. (a, b constant). Prove that the hypotenuse has minimum length $\left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$

26. Show that the semi-vertical angle of a right circular cone with given surface and maximum volume is $sin^{-1}\frac{1}{3}$.

27. Find the measure of the angle between curves, if they intersect :

(1) $xy = 6$, $x^2y = 12$

(2) $y = x^2$, $x^2 + y^2 = 20$

(3) $2y^2 = x^3$, $y^2 = 32x$, $(x, y) \neq (0, 0)$

(4) $y^2 = 4ax$, $x^2 = 4by$

(5) $y^2 = 8x$, $x^2 = 27y$

(6) $x^2 + y^2 = 2x$, $y^2 = x$

28. (1) Prove $x^2 = 4y$, $x^2 + 4y = 8$ intersect orthogonally at (2, 1) and (−2, 1).

(2) Prove $x^2 = y$ and $x^3 + 6y = 7$ intersect at right angles at (1, 1).

29. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The side of an equilateral triangle expands at the rate of $\sqrt{3}$ *cm/sec*. When the side is 12 *cm*, the rate of increase of its area is .... ☐

(a) 12 cm^2/sec (b) 18 cm^2/sec (c) $3\sqrt{3}$ cm^2/sec (d) 10 cm^2/sec

(2) The distance s moved by a particle in time t is given by $s = t^3 - 6t^2 + 6t + 8$. When the acceleration is zero, the velocity is .... ☐

(a) 5 *cm/sec* (b) 2 *cm/sec* (c) 6 *cm/sec* (d) −6 *cm/sec*

(3) The volume of a sphere is increasing at the rate of π cm^3/sec. The rate at which the radius is increasing is ......, when the radius is 3 *cm*. ☐

(a) $\frac{1}{36}$ *cm/sec* (b) 36 *cm/sec* (c) 9 *cm/sec* (d) 27 *cm/sec*

(4) There is 4 % error in measuring the period of a simple pendulum. The approximate percentage error in length is .... (Hint : $T = 2\pi\sqrt{\frac{l}{g}}$) ☐

(a) 4 % (b) 8 % (c) 2 % (d) 6 %

(5) Approximate value of $(31)^{\frac{1}{5}}$ is .... ☐

(a) 2.01 (b) 2.1 (c) 2.0125 (d) 1.9875

(6) The height and radius of a cylinder are equal. An error of 2 % is made in measuring height. The approximate percentage error in volume is ...... ☐

(a) 6 % (b) 4 % (c) 3 % (d) 2 %

(7) The tangent to $(at^2, 2at)$ is perpendicular to X-axis at ☐

(a) $(4a, 4a)$ (b) $(a, 2a)$ (c) $(0, 0)$ (d) $(a, -2a)$

(8) The line $y = mx + 1$ touches $y^2 = 4x$, if $m = ...$

(a) 0 (b) 1 (c) -1 (d) 2

(9) The equation of normal to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is ......

(a) $2x + y = 0$ (b) $y = 1$ (c) $x = 0$ (d) $x = y$

(10) $f(x) = x^x$ decreases in ......

(a) $(0, e)$ (b) $\left(0, \frac{1}{e}\right)$ (c) $(0, 1)$ (d) $(0, \infty)$

(11) $f(x) = 2|x - 2| + 3|x - 4|$ is ...... in $(2, 4)$.

(a) decreasing (b) increasing (c) constant (d) cannot be decided

(12) $f(x) = x^7 + 5x^3 + 125$ is ....

(a) decreasing in $(0, \infty)$ (b) decreasing in $(-\infty, 0)$

(c) increasing on R (d) neither increasing nor decreasing in R

(13) The local maximum value of $f(x) = x + \frac{1}{x}$ is ......

(a) 2 (b) -2 (c) 4 (d) -4

(14) The local minimum value of $\frac{x}{\log x}$ is ......

(a) -1 (b) 0 (c) $\frac{1}{e}$ (d) e

(15) If $\log_e 4 = 1.3868$, then approximate value of $\log_e 4.01 = ...$

(a) 1.3867 (b) 1.3869 (c) 1.3879 (d) 1.3893

(16) The circumference of a circle is 20 *cm* and there is an error of 0.02 *cm* in its measurement. The approximate percentage error in area is ......

(a) 0.02 (b) 0.2 (c) π (d) $\frac{1}{\pi}$

(17) If the line $y = x$ touches the curve $y = x^2 + bx + c$ at $(1, 1)$, then ......

(a) $b = 1, c = 2$ (b) $b = -1, c = 1$ (c) $b = 1, c = 1$ (d) $b = 0, c = 1$

(18) $y = ae^x$, $y = be^{-x}$ intersect at right angles if ...... $(a \neq 0, b \neq 0)$

(a) $a = \frac{1}{b}$ (b) $a = b$ (c) $a = -\frac{1}{b}$ (d) $a + b = 0$

(19) Tangent to $y = 5x^5 + 10x + 15$...

(a) is always vertical

(b) is always horizontal

(c) makes acute angle with the positive X-axis

(d) makes obtuse angle with the positive X-axis

(20) $f(x) = 2x + cot^{-1}x - \log | x + \sqrt{1+x^2} |$ is ......

(a) decreasing on $(-\infty, 0)$ (b) decreasing on $(0, \infty)$

(c) constant (d) increasing on R

(21) The sum of two non-zero numbers in 12. The minimum sum of their reciprocals is ......

(a) $\frac{1}{10}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

(22) The local minimum value of $f(x) = x^2 + 4x + 5$ is ......

(a) 2 (b) 4 (c) 1 (d) -1

(23) The maximum value of $f(x) = 5cosx + 12sinx$ is ......

(a) 13 (b) 12 (c) 5 (d) 17

(24) The minimum value of $f(x) = 3cosx + 4sinx$ is ......

(a) 7 (b) 5 (c) -5 (d) 4

(25) $f(x) = x \log x$ has minimum value...

(a) 1 (b) 0 (c) e (d) $-\frac{1}{e}$

(26) $f(x) = \sqrt{3}\,cosx + sinx,\ x \in \left[0, \frac{\pi}{2}\right]$ is maximum for $x =$ ......

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) 0

(27) $f(x) = (x - a)^2 + (x - b)^2 + (x - c)^2$ has minimum value at $x =$ ......

(a) $\sqrt[3]{abc}$ (b) $a + b + c$ (c) $\frac{a+b+c}{3}$ (d) 0

(28) $f(x) = (x + 2)\,e^{-x}$ is increasing in ....

(a) $(-\infty, -1)$ (b) $(-1, -\infty)$ (c) $(2, \infty)$ (d) R^+

(29) The measure of the angle of intersection between $y^2 = x$ and $x^2 = y$ other than one at $(0, 0)$ is ......

(a) $tan^{-1}\frac{4}{3}$ (b) $tan^{-1}\frac{3}{4}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

(30) The point where normal to $y = x^2 - 2x + 3$ is parallel to Y-axis is ......

(a) (0, 3) (b) $(-1, 2)$ (c) (1, 2) (d) (3, 6)

(31) The slope of normal to $(3t^2 + 1, t^3 - 1)$ at $t = 1$ is ......

(a) $\frac{1}{2}$ (b) -2 (c) 2 (d) $-\frac{1}{2}$

(32) The equation of normal to $3x^2 - y^2 = 8$ at $(2, -2)$ is ......

(a) $x + 2y = -2$ (b) $x - 3y = 8$ (c) $3x + y = 4$ (d) $x + y = 0$

(33) The angle made by the tangent with the $+ve$ direction of X-axis to $x = e^t cost$, $y = e^t sint$ at $t = \frac{\pi}{4}$ is ......

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{3}$

(34) The equation of tangent to $y = cosx$ at (0, 1) is ......

(a) $x = 0$ (b) $y = 0$ (c) $x = 1$ (d) $y = 1$

(35) The equation of normal to $y = sinx$ at $\left(\frac{\pi}{2}, 1\right)$ is ......

(a) $x = 1$ (b) $x = 0$ (c) $y = \frac{\pi}{2}$ (d) $x = \frac{\pi}{2}$

(36) At ...... on circle $x^2 + y^2 - 2x - 3 = 0$, the tangent is horizontal.

(a) $(0, \pm\sqrt{3})$ (b) $(2, \pm\sqrt{3})$ (c) (1, 2), (1, -2) (d) (3, 0)

(37) The point on $y^2 = x$ where tangent makes angle of measure $\frac{\pi}{4}$ with the positive X-axis is ....

(a) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (b) (2, 1) (c) (0, 0) (d) (-1, 1)

(38) A cone with its height equal to the diameter of the base is expanding in volume at the rate of 50 cm^3/sec. If the base has area 1 m^2, the radius is increasing at the rate ......

(a) 0.0025 cm/sec (b) 0.25 cm/sec (c) 1 cm/sec (d) 4 cm/sec

(39) The rate of increase of $f(x) = x^3 - 5x^2 + 5x + 25$ is twice the rate of increase of x for $x =$ ....

(a) $-3, -\frac{1}{3}$ (b) $3, \frac{1}{3}$ (c) $-3, \frac{1}{3}$ (d) $3, -\frac{1}{3}$

(40) The radius of a cone increases at the rate of 4 cm/sec and the altitude is decreasing at the rate of 3 cm/sec. When the radius is 3 cm and altitude is 4 cm, the rate of change of lateral surface is ....

(a) 30π cm^2/sec (b) 10 cm^2/sec (c) 20π cm^2/sec (d) 22π cm^2/sec

(41) The rate of change of surface area of a sphere *w.r.t.* radius is ....

(a) 8π (diameter) (b) 3π (diameter) (c) 4π (radius) (d) 8π (radius)

(42) The rate of change of volume of a cylinder *w.r.t.* radius whose radius is equal to its height is ....

(a) 4 (area of base) (b) 3 (area of base) (c) 2 (area of base) (d) (area of base)

(43) $f(x) = tan^{-1} x - x$ is ....

(a) increasing on R (b) decreasing on R (c) increasing on R^+ (d) increasing on $(-\infty, 0)$

(44) $f(x) = tan\, x - x$, $x \in R - \{(2k - 1)\frac{\pi}{2} \mid k \in Z\}$ is ....

(a) increasing on its domain (b) decreasing on its domain

(c) increasing on $\left(0, \frac{\pi}{2}\right)$ (d) decreasing on $\left(0, \frac{\pi}{2}\right)$

(45) $f(x) = 2x - tan^{-1} x - \log | x + \sqrt{1+x^2} |$ is .... ($x \in$ R).

(a) increasing on R

(b) decreasing on R

(c) has a minimum at $x = 1$

(d) has a maximum at $x = 1$

(46) If ......, then $f(x) = x^2 - kx + 20$ is strictly increasing on [0, 3].

(a) $k < 0$ (b) $0 < k < 1$ (c) $1 < k < 2$ (d) $2 < k < 3$

(47) $f(x) = |x - 1| + |x - 2|$ is increasing if ....

(a) $x > 2$ (b) $x < 1$ (c) $x < 0$ (d) $x < -2$

(48) Normal to $9y^2 = x^3$ at ...... makes equal intercepts on axes.

(a) $\left(-4, -\frac{8}{3}\right)$ (b) $\left(4, \pm\frac{8}{3}\right)$ (c) $\left(\pm 4, \frac{8}{3}\right)$ (d) $\left(8, \frac{8}{3}\right)$

(49) $y = mx + 4$ touches $y^2 = 8x$, if m = ...

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 2 (d) -2

(50) The measure of the angle between the curves $y = 2sin^2 x$ and $y = cos\, 2x$ at $x = \frac{\pi}{6}$ is ....

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

(51) The normal to $x^2 = 4y$ passing through (1, 2) has equation ....

(a) $2x = y$ (b) $x + y - 3 = 0$ (c) $2x + 3y - 8 = 0$ (d) $x - y + 1 = 0$

(52) The local minimum value of $x^2 + \frac{16}{x}$ ($x \neq 0$) is ....

(a) 12 (b) 22 (c) -12 (d) 2

(53) The minimum value of $secx$, $x \in \left[\frac{2\pi}{3}, \pi\right]$ is ....

(a) 1 (b) -2 (c) 2 (d) π

(54) The maximum value of $cosecx$, $x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ is ....

(a) 2 (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

(55) If f is decreasing in $[a, b]$, its minimum and maximum values are respectively ...... and ....

(a) $f(a)$ and $f(b)$

(b) $f(b)$ and $f(a)$

(c) $f\left(\frac{a+b}{2}\right)$ and $f(a)$

(d) $f(b)$ and $f\left(\frac{a+b}{2}\right)$

Summary

We have studied the following points in this chapter :

1. Derivative as a rate measurer.

2. Increasing and decreasing functions.

3. Applications to Geometry : Tangents and normals

4. Angle between two curves.

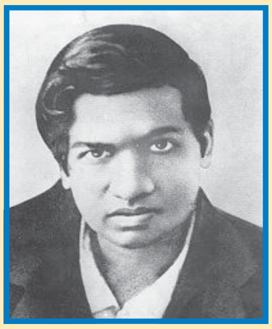
5. Differentials and approximate values.

6. Maximum and minimum values.

7. Application to optimization problems and practical applications.

●

RAMANUJAN



He was born on 22nd of December 1887 in a small village of Tanjore district, Madras.

He failed in English in Intermediate, so his formal studies were stopped but his self-study of mathematics continued.

He sent a set of 120 theorems to Professor Hardy of Cambridge. As a result he invited Ramanujan to England.

Ramanujan showed that any big number can be written as sum of not more than four prime numbers.

He showed that how to divide the number into two or more squares or cubes.

When Mr Littlewood came to see Ramanujan in taxi number 1729, Ramanujan said that 1729 is the smallest number which can be written in the form of sum of cubes of two numbers in two ways,

i.e. $1729 = 9^3 + 10^3 = 1^3 + 12^3$

since then the number 1729 is called Ramanujan's number.

INDEFINITE INTEGRATION 2

Science without religion is lame, religion without science is blind.
– Albert Einstein

◆

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.
– Tolstoy

2.1 Introduction

In semester III, we have studied about the definition of indefinite integral, working rules, standard forms and method of substitution for indefinite integrals. We have also studied trigonometric substitutions, an important substitution $tan\frac{x}{2} = t$, integrals of the type $\int sin^m x \cdot cos^n x \, dx$, $m, n \in$ N, integrals of the type $\int \frac{dx}{ax^2 + bx + c}$, $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, $\int \frac{Ax + B}{ax^2 + bx + c} dx$ and $\int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} dx$. Still there are functions for which integration using these methods is not possible or may be difficult. For example, log x, $sec^{-1}x$, $e^x sinx$, $\frac{x^2 + 1}{(x^2 + 2)(2x^2 + 1)}$ etc. are such functions. For integrating such functions, we have to develop some other techniques.

In this chapter, we will learn methods for obtaining integrals of such functions.We know the rule of differentiating the product of two functions. Now we will learn a method to find integral of product of two functions. It is known as rule of **integration by parts**.

2.2 Rule of Integration by Parts

If (1) f and g are differentiable on interval I = (a, b) and

(2) f' and g' are continuous on I, then $\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$

Proof : Here f and g are differentiable functions of x. So $f \cdot g$ is also differentiable and according to working rule for differentiation of a product,

$$\frac{d}{dx}[f(x)\ g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x) \qquad \textbf{(i)}$$

Now, f, g, f' and g' are continuous on I and hence they are integrable over I.

$\therefore$ fg' and gf' are also continuous and hence integrable.

$\therefore$ From (i), using definition of antiderivative,

$$f(x)\ g(x) = \int [f(x) \cdot g'(x) + g(x) \cdot f'(x)]\ dx$$

$$= \int f(x)\, g'(x)\ dx + \int f'(x) \cdot g(x)\ dx$$

$$\therefore \int f(x)\ g'(x)\, dx = f(x)\, g(x) - \int f'(x) \cdot g(x)\ dx \qquad \textbf{(ii)}$$

This rule is known as **Rule of Integration by Parts**.

Applications of Rule of Integration by Parts in Practice :

Rule of integration by parts is $\int f(x)\ g'(x)\, dx = f(x)\, g(x) - \int f'(x) \cdot g(x)\ dx$

If we take $f(x) = u$ and $g'(x) = v$ in this expression, then $f'(x) = \frac{du}{dx}$ and $g(x) = \int v\, dx$.

$\therefore$ The new form of this rule will be $\int uv\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \int v\, dx\right) dx$. **(iii)**

Note : **(1)** In the above formula, we have transformed the problem of integration of product of two functions into another problem of integration of product of two functions to make the integration simpler. The new product is the product of the derivative of one function $\frac{du}{dx}$ and integral of the other $\int v\, dx$. $\left(\text{i.e. } \frac{du}{dx} \int v\, dx\right)$. Thus we do not get the integral of the product $\int u \cdot v\, dx$ directly but the product is transformed into another possibly simpler integrable product $\int \left(\frac{du}{dx} \int v\, dx\right) dx$. Therefore, it is called the rule of integration by parts.

(2) While using this formula, we must select u and v properly. Let us understand this by an example.

Find : $\int x \cdot sinx\ dx$

If we take $u = x$ and $v = sinx$, then

$$\int x \cdot sinx\ dx = x \int sinx\ dx - \int \left(\frac{d}{dx} x \int sin\, x\, dx\right) dx$$

$$= -x\, cosx + \int (1 \cdot cos\, x)\, dx$$

$$= -x\, cosx + sin\, x + c$$

But, if we choose $u = sinx$, $v = x$, then

$$\int x \cdot sinx\ dx = sin\, x \int x\ dx - \int \left(\frac{d}{dx}(sin\, x) \int x\, dx\right) dx$$

$$= sin\, x \cdot \frac{x^2}{2} - \int \left(cos\, x \cdot \frac{x^2}{2}\right) dx$$

$$= \frac{x^2}{2} \cdot sin\, x - \frac{1}{2} \int cos\, x \cdot x^2\, dx$$

Thus, for this type of choice, power of x increases and the integrand is transformed into comparatively more complicated integrand having higher power of x. Therefore, the choice of u and v is very important. The success of this method depends on careful selection of u and v. We shall keep the following things in mind while using the rule.

(i) Integral of v is known.

(ii) It is simpler to integrate $\frac{du}{dx} \int v\, dx$.

Keeping these points in mind, we frame a rule.

L : Logarithmic function, **I :** Inverse trigonometric function, **A :** Algebraic function, **T :** Trigonometric function, **E :** Exponential function. First letters of above functions generate **LIATE**. The first function appearing in this order in product $u \cdot v$ to be integrated is taken as u. This order is formed keeping above two points in mind. This is a convention, not mandatory.

For example : (1) In the product $x \cdot sin^{-1}x$, x is algebraic and $sin^{-1}x$ is inverse trigonometric function. Now in LIATE rule, inverse trigonometric function precedes algebraic function. Hence we take $u = sin^{-1}x$ and $v = x$.

(2) In the product $x \cdot e^x$, x is algebraic and e^x is exponential function. Now in LIATE rule, algebraic function precedes exponential function, so we take $u = x$ and $v = e^x$.

(3) While using rule of integration by parts, when we integrate v we shall not add constant of integration. If we write the integration of $u = sinx$ as $-cos\,x + k$, where k is any constant,

$$\text{then } \int x\, sinx\ dx = x \int sin\,x\ dx - \int \left(\frac{d}{dx} x \int sin\,x\, dx\right) dx$$

$$= x\,(-cosx + k) - \int (-cosx + k)\ dx$$

$$= -x\ cosx + kx + \int cosx\ dx - \int k\ dx$$

$$= -x\ cosx + kx + sin\,x - kx + c$$

$$= -x\ cosx + sin\,x + c$$

This shows that, while integrating $u = sinx$ as $-cos\,x + k$, k is eliminated. Hence, we will add arbitrary a constant when we complete integration of product $\int \left(\frac{du}{dx} \int v\,dx\right)$.

(4) To integrate a function like $\log x$, $cosec^{-1}x$, $tan^{-1}x$ etc., we are unable to guess a function whose derivatives are $\log x$, $cosec^{-1}x$, $tan^{-1}x$. So, we take these functions as u and 1 as v. The integral of 1 is x.

For example, let $I = \int \log x\ dx$, we take

$$I = \int \log x \cdot 1\ dx$$

Here $u = \log x$ and $v = 1$ gives,

$$I = \log x \int 1\ dx - \int \left[\frac{d}{dx} \log x \int 1\ dx\right] dx$$

$$= \log x \cdot x - \int \left(\frac{1}{x} \cdot x\right) dx$$

$$= x \log x - \int 1\ dx$$

$$= x \log x - x + c$$

(5) Some times we have to use this rule repeatedly.

For example consider, $I = \int x^2\, e^{5x}\ dx$

Here, $u = x^2$ and $v = e^{5x}$ gives

$$I = x^2 \int e^{5x}\ dx - \int \left(\frac{d}{dx} x^2 \int e^{5x}\ dx\right) dx$$

$$= x^2 \cdot \frac{e^{5x}}{5} - \int \left(2x\, \frac{e^{5x}}{5}\right) dx$$

$$= \frac{x^2}{5}\, e^{5x} - \frac{2}{5} \int x\, e^{5x}\ dx$$

$$= \frac{x^2}{5}\, e^{5x} - \frac{2}{5}\left[x \int e^{5x}\ dx - \int \left(\frac{d}{dx} x \int e^{5x}\ dx\right) dx\right]. \qquad (u = x,\ v = e^{5x})$$

$$= \frac{x^2}{5} e^{5x} - \frac{2}{5}\left[x \cdot \frac{e^{5x}}{5} - \int \left(1 \cdot \frac{e^{5x}}{5}\right) dx\right]$$

$$= \frac{x^2}{5} e^{5x} - \frac{2}{5}\left[\frac{x}{5} e^{5x} - \frac{1}{5} \cdot \frac{e^{5x}}{5}\right] + c$$

$$= \frac{x^2}{5} e^{5x} - \frac{2x}{25} e^{5x} + \frac{2}{125} e^{5x} + c$$

$$= e^{5x}\left[\frac{1}{5} x^2 - \frac{2x}{25} + \frac{2}{125}\right] + c$$

Note : (i) In general $\int x^n e^{ax} dx = e^{ax}\left[\frac{1}{a} x^n - \frac{n}{a^2} x^{n-1} + \frac{n(n-1)x^{n-2}}{a^3} + \ldots + \frac{(-1)^n \cdot n!}{a^{n+1}}\right] + c$

(ii) Generally we will denote the integral by I.

Example 1 : Evaluate : $\int x \cos(3x + 5)\, dx$

Solution : Let $u = x$ and $v = \cos(3x + 5)$

$$I = \int x \cos(3x + 5)\, dx$$

$$= x \int \cos(3x + 5)\, dx - \int \left(\frac{d}{dx} x \int \cos(3x + 5)\, dx\right) dx$$

$$= x \frac{\sin(3x+5)}{3} - \int \left(1 \cdot \frac{\sin(3x+5)}{3}\right) dx$$

$$= \frac{x}{3} \sin(3x + 5) - \frac{1}{3} \int \sin(3x + 5)\, dx$$

$$= \frac{x}{3} \sin(3x + 5) + \frac{1}{3} \frac{\cos(3x+5)}{3} + c$$

$$= \frac{x}{3} \sin(3x + 5) + \frac{1}{9} \cos(3x + 5) + c$$

Example 2 : Evaluate : $\int \sec^{-1}x\, dx,\ x > 0$

Solution : Let $u = \sec^{-1}x$ and $v = 1$

$$I = \int \sec^{-1}x \cdot 1\ dx$$

$$= \sec^{-1}x \int 1\ dx - \int \left(\frac{d}{dx} \sec^{-1}x \int 1\ dx\right) dx$$

$$= \sec^{-1}x \cdot x - \int \left(\frac{1}{x\sqrt{x^2 - 1}} \cdot x\right) dx \qquad (|x| = x \text{ as } x > 0)$$

$$= x \sec^{-1}x - \int \frac{1}{\sqrt{x^2 - 1}}\, dx$$

$$= x \sec^{-1}x - \log |x + \sqrt{x^2 - 1}| + c$$

$$= x \sec^{-1}x - \log (x + \sqrt{x^2 - 1}) + c \qquad (x > 0)$$

Example 3 : Evaluate : $\int \frac{x \sin^{-1}x}{\sqrt{1 - x^2}}\, dx,\quad 0 < x < 1$

Solution : $I = \int \frac{x \sin^{-1}x}{\sqrt{1 - x^2}}\, dx,\quad 0 < x < 1$

Let $\sin^{-1}x = \theta,\ 0 < \theta < \frac{\pi}{2}$ as $0 < x < 1$

$\therefore \quad x = sin\theta,\ dx = cos\theta\ d\theta$

$\therefore \quad I = \int \frac{\theta\, sin\theta}{\sqrt{1 - sin^2\theta}} \cdot cos\theta\ d\theta$

$\therefore \quad I = \int \frac{\theta\, sin\theta}{cos\theta} \cdot cos\theta\ d\theta \qquad (cos\theta > 0)$

$= \int \theta\, sin\theta\ d\theta$

$= \theta \int sin\theta\ d\theta - \int \left(\frac{d}{d\theta}\theta \int sin\,\theta\ d\theta\right) d\theta$

$= -\theta\, cos\theta + \int (1 \cdot cos\theta)\ d\theta$

$= -\theta\, cos\theta + sin\theta + c$

$= -\theta \sqrt{1 - sin^2\theta} + sin\theta + c \qquad (cos\theta = \sqrt{1 - sin^2\theta})$

$= -sin^{-1}x \cdot \sqrt{1 - x^2} + x + c$

$= -\sqrt{1 - x^2} \cdot sin^{-1}x + x + c$

Second Method :

Let $u = sin^{-1}x$ and $v = \frac{x}{\sqrt{1 - x^2}}$

First we find integral of v, i.e., $\int \frac{x}{\sqrt{1 - x^2}}\ dx$.

$$\int \frac{x}{\sqrt{1 - x^2}}\ dx = \int (1 - x^2)^{-\frac{1}{2}} \cdot x\ dx$$

$$= -\frac{1}{2} \int (1 - x^2)^{-\frac{1}{2}} (-2x)\ dx$$

$$= -\frac{1}{2}\ \frac{(1 - x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$$= -(1 - x^2)^{\frac{1}{2}}$$

$$= -\sqrt{1 - x^2}$$

$$\therefore \quad \int \frac{x}{\sqrt{1 - x^2}}\ dx = -\sqrt{1 - x^2}$$

Now, $I = \int \frac{x\, sin^{-1}x}{\sqrt{1 - x^2}}\ dx$

$$= sin^{-1}x \int \frac{x}{\sqrt{1 - x^2}}\ dx - \int \left(\frac{d}{dx} sin^{-1}x \int \frac{x}{\sqrt{1 - x^2}}\ dx\right) dx$$

$$= (sin^{-1}x)(-\sqrt{1 - x^2}) - \int \frac{1}{\sqrt{1 - x^2}} \cdot (-\sqrt{1 - x^2})\ dx$$

$$= -\sqrt{1 - x^2}\ sin^{-1}x + x + c$$

Example 4 : Evaluate : $\int e^x \cos x \, dx$

Solution : $I = \int e^x \cos x \, dx$

Let $u = e^x$ and $v = \cos x$

$$\therefore \quad I = e^x \int \cos x \, dx - \int \left(\frac{d}{dx} e^x \int \cos x \, dx \right) dx$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

$$= e^x \sin x - \left[e^x \int \sin x \, dx - \int \left(\frac{d}{dx} e^x \int \sin x \, dx \right) dx \right] \qquad (u = e^x, v = \sin x)$$

$$= e^x \sin x - [-e^x \cos x - \int (e^x (-\cos x)) \, dx]$$

$$= e^x \sin x - [-e^x \cos x + \int e^x \cos x \, dx]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\therefore \quad I = e^x \sin x + e^x \cos x - I + c'$$

$$\therefore \quad 2I = e^x (\sin x + \cos x) + c'$$

$$\therefore \quad I = \frac{e^x}{2} (\sin x + \cos x) + \frac{c'}{2}$$

$$\therefore \quad I = \frac{e^x}{2} (\sin x + \cos x) + c \qquad \left(\frac{c'}{2} = c \right)$$

Note : In the product $e^x \cos x$, trigonometric function precedes exponential function as per LIATE rule. Hence, $u = \cos x$ and $v = e^x$ must be selected. But we have taken $u = e^x$ and $v = \cos x$. Remember earlier we stated that the rule LIATE is for convenience only. But we may take $u = \cos x$ and $v = e^x$ also and integrate.

Example 5 : Evaluate : $\int x^2 2^x \, dx$

Solution : Let $u = x^2$, $v = 2^x$

$$I = \int x^2 2^x \, dx$$

$$= x^2 \int 2^x \, dx - \int \left(\frac{d}{dx} x^2 \int 2^x \, dx \right) dx$$

$$= x^2 \frac{2^x}{\log_e 2} - \int \left(2x \frac{2^x}{\log_e 2} \right) dx$$

$$= \frac{x^2 2^x}{\log_e 2} - \frac{2}{\log_e 2} \int x 2^x \, dx$$

$$= \frac{x^2 2^x}{\log_e 2} - \frac{2}{\log_e 2} \left[x \int 2^x \, dx - \int \left(\frac{d}{dx} x \int 2^x \, dx \right) dx \right] \qquad (u = x, v = 2^x)$$

$$= \frac{x^2 2^x}{\log_e 2} - \frac{2}{\log_e 2} \left[x \frac{2^x}{\log_e 2} - \int \left(\frac{1 \cdot 2^x}{\log_e 2} \right) dx \right]$$

$$= \frac{x^2 2^x}{\log_e 2} - \frac{2}{\log_e 2} \left[\frac{x \cdot 2^x}{\log_e 2} - \frac{1}{\log_e 2} \cdot \frac{2^x}{\log_e 2} \right] + c$$

$$= \frac{x^2 2^x}{\log_e 2} - \frac{x \cdot 2^{x+1}}{(\log_e 2)^2} + \frac{2^{x+1}}{(\log_e 2)^3} + c$$

$$\int x^2 2^x \, dx = \int x^2 e^{x \log 2} \, dx$$

$$= e^{x \log 2} \left[\frac{x^2}{\log 2} - \frac{2x}{(\log 2)^2} + \frac{2}{(\log 2)^3} \right] + c = 2^x \left[\frac{x^2}{\log 2} - \frac{2x}{(\log 2)^2} + \frac{2}{(\log 2)^3} \right] + c$$

Note : We observe from above illustration that sometimes we have to use formula of integration by parts repeatedly.

Example 6 : Evaluate : $\int x\, sec^2x\ tan\, x\ dx$

Solution : $I = \int x\, sec^2x\ tan\, x\ dx$

Let $u = x$ and $v = sec^2x\, tan\, x$

First we find $\int v\ dx$, i.e. $\int tan\, x\, sec^2x\ dx$

$$\int tan\, x\, sec^2x\ dx = \int (tan\, x)\left(\frac{d}{dx}(tan\, x)\right) dx$$

$$= \frac{(tan\, x)^2}{2}$$

$$= \frac{tan^2x}{2}$$

$$\therefore \int tan\, x\, sec^2x\ dx = \frac{tan^2x}{2}$$

Now, $I = \int x\, sec^2x\ tan\, x\ dx$

$$= x\int tan\, x\, sec^2x\ dx - \int\left(\frac{d}{dx}x \int tan\, x \cdot sec^2x\ dx\right) dx$$

$$= x \cdot \frac{tan^2x}{2} - \int\left(1 \cdot \frac{tan^2x}{2}\right) dx$$

$$= \frac{x}{2}\ tan^2x - \frac{1}{2}\int (sec^2x - 1)\ dx$$

$$= \frac{x}{2}\ tan^2x - \frac{1}{2}\,[tan\, x - x] + c$$

$$= \frac{x}{2}\ tan^2x - \frac{1}{2}\ tan\, x + \frac{x}{2} + c$$

Example 7 : Evaluate : $\int cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx,\ x > 0$

Solution : $I = \int cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

Let $\theta = tan^{-1}x$, so that $x = tan\theta$ and $dx = sec^2\theta\, d\theta$, $0 < \theta < \frac{\pi}{2}$ as $x > 0$

$$\therefore\ I = \int cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right) sec^2\theta\ d\theta$$

$$= \int cos^{-1}(cos2\theta) \cdot sec^2\theta\ d\theta$$

$0 < \theta < \frac{\pi}{2}$

$\therefore\ 0 < 2\theta < \pi$

$\therefore\ cos^{-1}(cos2\theta) = 2\theta$ **(i)**

$\therefore\ I = 2\int \theta\ sec^2\theta\ d\theta$

$$= 2\left[\theta \int sec^2\theta\ d\theta - \int\left(\frac{d}{d\theta}\theta \int sec^2\theta\ d\theta\right) d\theta\right]$$

$$= 2\ [\theta \cdot tan\theta - \int 1 \cdot tan\theta\ d\theta]$$

$= 2\ [\theta \cdot tan\theta - \log |sec\theta|] + c$

Now, $\theta = tan^{-1}x$

$sec^2\theta = 1 + tan^2\theta = 1 + x^2$

$\therefore \quad sec\theta = \sqrt{1+x^2}$ $\left(sec\theta > 0 \text{ as } 0 < \theta < \frac{\pi}{2}\right)$

$\therefore \quad I = 2\ [x \cdot tan^{-1}x - \log \sqrt{1+x^2}\,] + c$

$= 2x\,tan^{-1}x - 2 \log (1 + x^2)^{\frac{1}{2}} + c$

$= 2x\,tan^{-1}x - \log (1 + x^2) + c$

Second Method : Let us transform $cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Let $x = tan\theta$, $0 < \theta < \frac{\pi}{2}$ as $x > 0$

$\therefore \quad cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)$

$= cos^{-1}\ (cos2\theta)$

$= 2\theta$ $(0 < 2\theta < \pi)$

$= 2\ tan^{-1}x$

Now, $\int cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$

$= \int 2\ tan^{-1}x\ dx$

$= 2\left[tan^{-1}x \int dx - \int \left(\frac{d}{dx} tan^{-1}x \int 1\ dx\right) dx\right]$

$= 2\left[tan^{-1}x \,.\, x - \int \left(\frac{1}{1+x^2} \,.\, x\right) dx\right]$

$= 2\left[x \,.\, tan^{-1}x - \frac{1}{2} \int \frac{2x}{1+x^2}\ dx\right]$

$= 2\left[x\,tan^{-1}x - \frac{1}{2} \log\ (1 + x^2)\right] + c$

$= 2x\,tan^{-1}x - \log\ (1 + x^2) + c$

Note : If $x < 0$, then $-\frac{\pi}{2} < \theta < 0$.

$\therefore \quad -\pi < 2\theta < 0$

$\therefore \quad 0 < -2\theta < \pi$

In step (i) $cos^{-1}(cos\ 2\theta) = cos^{-1}(cos\ (-2\theta)) = -2\theta$

$\therefore \quad I = -2\ [\theta\ tan\theta - \log |sec\theta|] + c$

$= -2x\,tan^{-1}x + \log(1 + x^2) + c$

Exercise 2.1

Find the integrals of the following functions with respect to x.

1. $x^2 \log x \quad x > 0$

2. $(3 + 5x) \cos 7x$

3. $\cos^{-1}x \quad x \in [-1, 1]$

4. $x^2 e^{3x}$

5. $x^2 \tan^{-1}x$

6. $\sin^{-1}\frac{1}{x},\ x > 1$

7. $\sin(\log x) \quad x > 0$

8. $\sec^3 x$

9. $\frac{x}{1 - \cos x} \quad x \neq 2n\pi,\ n \in Z$

10. $x^3 \sin x^2$

11. $\tan^{-1}\frac{2x}{1 - x^2},\ 0 < x < 1$

12. $x \cot x \operatorname{cosec}^2 x$

13. $x \cos^3 x$

14. $x^{2n-1} \cos x^n$

15. $(1 - x^2) \log x \quad x > 0$

16. $\frac{\log x}{(1+x)^2} \quad x > 0$

17. $\frac{\sin^{-1}x}{x^2} \quad x \in (0, 1)$

18. $\frac{\sin^{-1}\sqrt{x}}{\sqrt{1-x}} \quad 0 < x < 1$

*

2.3 Some More Standard Forms of Integration

Now we will obtain integrals of $\sqrt{x^2 \pm a^2}$, $\sqrt{a^2 - x^2}$, $e^{ax}\sin(bx + k)$, $e^{ax}\cos(bx + k)$ using integration by parts or trigonometric substitutions and accept them as standard forms.

(1) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c \qquad (x^2 > a^2)$

Proof : $I = \int \sqrt{x^2 - a^2}\, dx$

$$\therefore \quad I = \int \sqrt{x^2 - a^2} \cdot 1\, dx$$

$$= \sqrt{x^2 - a^2} \int 1\, dx - \int \left(\frac{d}{dx} \sqrt{x^2 - a^2} \int 1\, dx \right) dx$$

$$= x\sqrt{x^2 - a^2} - \int \left(\frac{2x}{2\sqrt{x^2 - a^2}} \cdot x \right) dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}}\, dx$$

$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\, dx - a^2 \int \frac{1}{\sqrt{x^2 - a^2}}\, dx$$

$$I = x\sqrt{x^2 - a^2} - I - a^2 \log |x + \sqrt{x^2 - a^2}| + c'$$

$\therefore \quad 2I = x\sqrt{x^2 - a^2} - a^2 \log|x + \sqrt{x^2 - a^2}| + c'$

$\therefore \quad I = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + c \qquad \left(\frac{c'}{2} = c\right)$

Second Method :

We can obtain the same standard form taking $x = a\,sec\theta$. $(x > a > 0)$

$I = \int \sqrt{x^2 - a^2}\, dx$

Proof : Let $x = a\,sec\theta$. So $dx = a sec\theta\ tan\theta\ d\theta$, $0 < \theta < \frac{\pi}{2}$ as $x > a > 0$.

$\therefore \quad I = \int \sqrt{a^2 sec^2\theta - a^2} \cdot a sec\theta\ tan\theta\ d\theta$

$= \int \sqrt{a^2 tan^2\theta} \cdot a sec\theta\ tan\theta\ d\theta$

$I = a^2 \int sec\theta \cdot tan^2\theta\ d\theta \qquad (a > 0 \text{ and } tan\theta > 0)$

$= a^2 \int sec\theta\,(sec^2\theta - 1)\, d\theta$

$= a^2 \int (sec^3\theta - sec\theta)\, d\theta$

$= a^2 \int sec^3\theta\ d\theta - a^2 \int sec\theta\, d\theta$

$= a^2 \int sec\theta \cdot sec^2\theta\ d\theta - a^2 \int sec\theta\, d\theta$

$= a^2 \left[sec\theta \int sec^2\theta\ d\theta - \int \left(\frac{d}{d\theta} sec\theta \int sec^2\theta\ d\theta\right) d\theta\right] - a^2 \int sec\theta\, d\theta$

$= a^2\ [sec\theta\ tan\theta - \int (sec\theta\ tan\theta \cdot tan\theta)\ d\theta] - a^2 \int sec\theta\, d\theta$

$= a^2\ [sec\theta\ tan\theta - \int sec\theta \cdot tan^2\theta\, d\theta] - a^2 \int sec\theta\, d\theta$

$= a^2\ sec\theta\ tan\theta - a^2 \int sec\theta\, tan^2\theta\, d\theta - a^2 \log|sec\theta + tan\theta| + c'$

$\therefore \quad I = a^2\ sec\theta\ tan\theta - I - a^2 \log|sec\theta + tan\theta| + c' \qquad \left(I = a^2 \int sec\theta\ tan^2\theta\ d\theta\right)$

$\therefore \quad 2I = a^2\ sec\theta\ tan\theta - a^2 \log|sec\theta + tan\theta| + c'$

$\therefore \quad I = \frac{a^2}{2}\ sec\theta\sqrt{sec^2\theta - 1} - \frac{a^2}{2} \log|sec\theta + \sqrt{sec^2\theta - 1}| + \frac{c'}{2} \qquad (tan\theta > 0)$

$= \frac{a^2}{2} \cdot \frac{x}{a}\sqrt{\frac{x^2}{a^2} - 1} - \frac{a^2}{2} \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + \frac{c'}{2}$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + \frac{c'}{2} \qquad (|a| = a)$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + \frac{c'}{2} + \frac{a^2}{2} \log a$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + c \qquad \left(\frac{c'}{2} + \frac{a^2}{2} \log a = c\right)$

$\therefore \quad \int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + c$

For example,

$$\int \sqrt{x^2 - 25}\, dx = \int \sqrt{x^2 - 5^2}\, dx$$

$$= \frac{x}{2}\sqrt{x^2 - 5^2} - \frac{5^2}{2}\log\left|x + \sqrt{x^2 - 5^2}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2 - 25} - \frac{25}{2}\log\left|x + \sqrt{x^2 - 25}\right| + c$$

(2) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c$

Proof : $I = \int \sqrt{x^2 + a^2} \cdot 1\, dx$

$$= \sqrt{x^2 + a^2}\int 1\, dx - \int \left(\frac{d}{dx}\sqrt{x^2 + a^2}\int 1\, dx\right) dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{2x}{2\sqrt{x^2 + a^2}}\, x\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{x^2}{\sqrt{x^2 + a^2}}\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}}\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2\int \frac{dx}{\sqrt{x^2 + a^2}}$$

$$I = x\sqrt{x^2 + a^2} - I + a^2\log|x + \sqrt{x^2 + a^2}| + c'$$

$$\therefore\ 2I = x\sqrt{x^2 + a^2} + a^2\log|x + \sqrt{x^2 + a^2}| + c'$$

$$\therefore\ I = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c \qquad \left(\frac{c'}{2} = c\right)$$

$$\therefore\ \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log|x + \sqrt{x^2 + a^2}| + c \qquad (a > 0)$$

This formula can also be obtained using substitution $x = a\tan\theta$ $(a > 0)$.

For example, $\int \sqrt{x^2 + 4}\, dx = \int \sqrt{x^2 + 2^2}\, dx$

$$= \frac{x}{2}\sqrt{x^2 + 2^2} + \frac{2^2}{2}\log\left|x + \sqrt{x^2 + 2^2}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2 + 4} + 2\log\left|x + \sqrt{x^2 + 4}\right| + c$$

(3) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$ **$(a > 0)$**

Proof : $I = \int \sqrt{a^2 - x^2} \cdot 1\, dx$

$$= \sqrt{a^2 - x^2}\int 1\, dx - \int \left(\frac{d}{dx}\sqrt{a^2 - x^2}\int 1\, dx\right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \left(\frac{1}{2\sqrt{a^2 - x^2}}(-2x) \cdot x \right) dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}} dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$I = x\sqrt{a^2 - x^2} - I + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c'$$

$$\therefore \quad 2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + c'$$

$$\therefore \quad I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \left(\frac{c'}{2} = c\right)$$

$$\therefore \quad \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$$

Remark : What difference will it make if $a < 0$?

For example,

$$\int \sqrt{9 - x^2}\, dx = \int \sqrt{3^2 - x^2}\, dx$$

$$= \frac{x}{2}\sqrt{3^2 - x^2} + \frac{3^2}{2} \sin^{-1}\left(\frac{x}{3}\right) + c$$

$$= \frac{x}{2}\sqrt{9 - x^2} + \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) + c$$

This formula can be proved using substitution $x = a \sin\theta$ also.

(4) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

Proof : $I = \int e^x [f(x) + f'(x)]\, dx$

$$= \int e^x f(x)\, dx + \int e^x f'(x)\, dx$$

$$= f(x) \int e^x\, dx - \int \left(\frac{d}{dx} f(x) \int e^x\, dx \right) dx + \int e^x \cdot f'(x)\, dx$$

$$= f(x)\, e^x - \int f'(x)\, e^x\, dx + \int f'(x)\, e^x\, dx$$

$$= e^x f(x) + c$$

For example,

(1) $\int e^x \sec x\,(1 + \tan x)\, dx = \int e^x (\sec x + \sec x \tan x)\, dx$

$$= \int e^x \left[\sec x + \frac{d}{dx}(\sec x) \right] dx$$

$$= e^x \sec x + c$$

(2) $\int e^x \left(\frac{x - 1}{x^2} \right) dx = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx$

$$= \int e^x \left[\frac{1}{x} + \frac{d}{dx}\left(\frac{1}{x}\right)\right] dx$$

$$= e^x \cdot \frac{1}{x} + c$$

(3) $\int x \cdot e^x\, dx = \int [(x - 1) + 1]\, e^x\, dx$

$$= \int \left[(x - 1) + \frac{d}{dx}(x - 1)\right] e^x\, dx$$

$$= e^x (x - 1) + c$$

(5) $\int e^{ax} \cdot \sin(bx + k)\, dx = \frac{e^{ax}}{a^2 + b^2}\, [a \sin(bx + k) - b \cos(bx + k)] + c, \quad a, b \neq 0$

Proof : $I = \int e^{ax} \cdot \sin(bx + k)\, dx$

$$= \sin(bx + k) \int e^{ax}\, dx - \int \left(\frac{d}{dx} \sin(bx + k) \int e^{ax}\, dx\right) dx$$

$$= \sin(bx + k) \cdot \frac{e^{ax}}{a} - \int \left(b \cos(bx + k) \cdot \frac{e^{ax}}{a}\right) dx$$

$$= \frac{e^{ax}}{a} \sin(bx + k) - \frac{b}{a} \int \cos(bx + k)\, e^{ax}\, dx$$

$$= \frac{e^{ax}}{a} \sin(bx + k) - \frac{b}{a}\left[\cos(bx + k) \int e^{ax}\, dx - \int \left(\frac{d}{dx} \cos(bx + k) \int e^{ax}\, dx\right) dx\right]$$

$$= \frac{e^{ax}}{a} \sin(bx + k) - \frac{b}{a}\left[\cos(bx + k) \frac{e^{ax}}{a} - \int \left(-b \sin(bx + k) \frac{e^{ax}}{a}\right) dx\right]$$

$$= \frac{e^{ax}}{a} \sin(bx + k) - \frac{b}{a^2}\, e^{ax} \cos(bx + k) - \frac{b^2}{a^2} \int e^{ax} \cdot \sin(bx + k)\, dx$$

$$\therefore \quad I = \frac{e^{ax}}{a^2} [a \sin(bx + k) - b \cos(bx + k)] - \frac{b^2}{a^2} I + c'$$

$$\therefore \quad I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a^2} [a \sin(bx + k) - b \cos(bx + k)] + c'$$

$$\therefore \quad (a^2 + b^2) I = e^{ax} [a \sin(bx + k) - b \cos(bx + k)] + a^2 c'$$

$$\therefore \quad I = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx + k) - b \cos(bx + k)] + c, \text{ where } c = \frac{a^2 c'}{a^2 + b^2} \qquad \textbf{(i)}$$

Now, we will express this result in another form.

$$I = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \left[\frac{a}{\sqrt{a^2 + b^2}} \sin(bx + k) - \frac{b}{\sqrt{a^2 + b^2}} \cos(bx + k)\right] + c$$

Here $a \neq 0,\ b \neq 0$. Hence,

$$0 < \left|\frac{a}{\sqrt{a^2 + b^2}}\right| < 1, \quad 0 < \left|\frac{b}{\sqrt{a^2 + b^2}}\right| < 1$$

Now $\left(\frac{a}{\sqrt{a^2 + b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2}}\right)^2 = 1.$

So, there exists $\alpha \in (0, 2\pi)$, such that

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}},\ \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

$$\therefore\quad I = \frac{e^{ax}}{\sqrt{a^2+b^2}}\ [\sin(bx+k)\cos\alpha - \cos(bx+k)\sin\alpha] + c$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\ \sin(bx+k-\alpha) + c, \text{ where } \cos\alpha = \frac{a}{\sqrt{a^2+b^2}},\ \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

$$\therefore\quad \int e^{ax}\cdot\sin(bx+k)\ dx = \frac{e^{ax}}{a^2+b^2}\ (a\sin(bx+k) - b\cos(bx+k)) + c,\quad a, b \neq 0$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\ \sin(bx+k-\alpha) + c$$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$. $\alpha \in (0, 2\pi)$

For example, $\int e^{2x}\cdot\sin 3x\ dx = \frac{e^{2x}}{2^2+3^2}\ (2\sin 3x - 3\cos 3x) + c = \frac{e^{2x}}{13}\ (2\sin 3x - 3\cos 3x) + c$

Another form for $\int e^{2x}\cdot\sin 3x\ dx$.

Let $\cos\alpha = \frac{2}{\sqrt{13}}$, $\sin\alpha = \frac{3}{\sqrt{13}}$, so $\tan\alpha = \frac{3}{2}$

$\therefore\quad \alpha = \tan^{-1}\frac{3}{2},\quad 0 < \alpha < \frac{\pi}{2}$

$$\therefore\quad \int e^{2x}\cdot\sin 3x\ dx = \frac{e^{2x}}{\sqrt{13}}\ \sin\left(3x - \tan^{-1}\frac{3}{2}\right) + c$$

(6) $$\int e^{ax}\cos(bx+k)\ dx = \frac{e^{ax}}{a^2+b^2}\ [a\cos(bx+k) + b\sin(bx+k)] + c,\quad a \neq 0, b \neq 0$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\ \cos(bx+k-\alpha) + c$$

where $\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$. $\alpha \in (0, 2\pi)$.

Proof : $I = \int e^{ax}\cos(bx+k)\ dx$

$$= \cos(bx+k)\int e^{ax}\ dx - \int\left(\frac{d}{dx}\cos(bx+k)\int e^{ax}\ dx\right) dx$$

$$= \cos(bx+k)\ .\ \frac{e^{ax}}{a} - \int\left(-b\ \sin(bx+k)\ .\ \frac{e^{ax}}{a}\right) dx$$

$$= \frac{e^{ax}}{a}\ \cos(bx+k) + \frac{b}{a}\int e^{ax}\ \sin(bx+k)\ dx$$

$$= \frac{e^{ax}}{a}\ \cos(bx+k) + \frac{b}{a}\left[\sin(bx+k)\int e^{ax}\,dx - \int\left(\frac{d}{dx}\sin(bx+k)\int e^{ax}\ dx\right) dx\right]$$

$$= \frac{e^{ax}}{a}\ \cos(bx+k) + \frac{b}{a}\left[\sin(bx+k)\ .\ \frac{e^{ax}}{a} - \int\left(b\ \cos(bx+k)\ .\ \frac{e^{ax}}{a}\right) dx\right]$$

$$= \frac{e^{ax}}{a}\ \cos(bx+k) + \frac{b}{a^2}\ e^{ax}\sin(bx+k) - \frac{b^2}{a^2}\int e^{ax}\cdot\cos(bx+k)\ dx$$

$\therefore \quad I = \frac{e^{ax}}{a} cos(bx + k) + \frac{b}{a^2} e^{ax} \cdot sin(bx + k) - \frac{b^2}{a^2} I + c'$

$\therefore \quad I + \frac{b^2}{a^2} I = \frac{e^{ax}}{a^2} [a\,cos(bx + k) + b\,sin(bx + k)] + c'$

$\therefore \quad (a^2 + b^2) I = e^{ax} [a\,cos(bx + k) + b\,sin(bx + k)] + a^2c'$

$\therefore \quad I = \frac{e^{ax}}{a^2 + b^2} [a\,cos(bx + k) + b\,sin(bx + k)] + c$, where $c = \frac{a^2c'}{a^2 + b^2}$ **(i)**

Another Form :

There exists $\alpha \in (0, 2\pi)$, such that $cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$.

$\therefore \quad I = \frac{e^{ax}}{\sqrt{a^2 + b^2}} [cos(bx + k) \cdot cos\alpha + sin(bx + k) \cdot sin\alpha] + c$

$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} cos(bx + k - \alpha) + c$

$\therefore \quad \int e^{ax} cos(bx + k)\, dx = \frac{e^{ax}}{\sqrt{a^2 + b^2}} cos(bx + k - \alpha) + c$

where $cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$

For example : $\int e^{-x} cos \frac{x}{2}\, dx = \frac{e^{-x}}{(1)^2 + \left(\frac{1}{2}\right)^2} \left(-1\, cos \frac{x}{2} + \frac{1}{2} sin \frac{x}{2}\right) + c$

$= \frac{4e^{-x}}{5} \left(-cos \frac{x}{2} + \frac{1}{2} sin \frac{x}{2}\right) + c$

Another form for $\int e^{-x} cos \frac{x}{2}\, dx$.

Here $cos\alpha = \frac{-2}{\sqrt{5}}$, $sin\alpha = \frac{1}{\sqrt{5}}$. So $tan\alpha = -\frac{1}{2}$, $\quad \frac{\pi}{2} < \alpha < \pi$

$\therefore \quad \alpha = \pi - tan^{-1}\left(\frac{1}{2}\right)$

$\therefore \quad \int e^{-x} cos \frac{x}{2}\, dx = \frac{2}{\sqrt{5}} e^{-x} \left[cos\left(\frac{x}{2} - \left(\pi - tan^{-1} \frac{1}{2}\right)\right)\right] + c$

$= \frac{2}{\sqrt{5}} e^{-x} cos\left(\frac{x}{2} + tan^{-1} \frac{1}{2} - \pi\right) + c$

$= -\frac{2}{\sqrt{5}} e^{-x} cos\left(\frac{x}{2} + tan^{-1} \frac{1}{2}\right) + c$

2.4 Integrals of the type : (1) $\int \sqrt{ax^2 + bx + c}\, dx$ (2) $\int (Ax + B) \sqrt{ax^2 + bx + c}\, dx$

(1) If we express $ax^2 + bx + c$ in the form of a perfect square, the integral can be obtained using standard forms (1), (2), (3).

(2) We will find out two constants m, n such that

$Ax + B = m(\text{derivative of } ax^2 + bx + c) + n$

$Ax + B = m\left(\frac{d}{dx}(ax^2 + bx + c)\right) + n$

$Ax + B = m(2ax + b) + n$

Comparing coefficient of x on both sides we get

$m = \frac{A}{2a}$ and $n = B - mb$

Now, $\int (Ax + B)\sqrt{ax^2+bx+c}\ dx = \int [m(2ax + b) + n]\sqrt{ax^2+bx+c}\ dx$

$$= m\int (2ax+b)\sqrt{ax^2+bx+c}\ dx + n\int \sqrt{ax^2+bx+c}\ dx$$

$$= mI_1 + nI_2$$

where $I_1 = \int (ax^2 + bx + c)^{\frac{1}{2}} (2ax + b)\, dx$

$$= \int (ax^2 + bx + c)^{\frac{1}{2}} \frac{d}{dx}(ax^2 + bx + c)\, dx$$

$$= \frac{(ax^2+bx+c)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c_1$$

$$= \frac{2}{3}(ax^2 + bx + c)^{\frac{3}{2}} + c_1$$

and $I_2 = \int \sqrt{ax^2+bx+c}\ dx$

I_2 can be obtained using method (1).

Example 8 : Evaluate : $\int x\sqrt{x^4-25}\ dx.$

Solution : $I = \int x\sqrt{x^4-25}\ dx$

Let $x^2 = t$. So $2x\ dx = dt$ i.e. $x\,dx = \frac{1}{2}dt$

$$\therefore \quad I = \int \sqrt{(x^2)^2-(5)^2}\cdot x\,dx$$

$$= \int \sqrt{t^2-5^2}\ \frac{1}{2}\,dt$$

$$= \frac{1}{2}\left[\frac{t}{2}\sqrt{t^2-5^2} - \frac{5^2}{2}\log|t + \sqrt{t^2-5^2}|\right] + c$$

$$= \frac{t}{4}\sqrt{t^2-25} - \frac{25}{4}\log|t + \sqrt{t^2-25}| + c$$

$$= \frac{x^2}{4}\sqrt{x^4-25} - \frac{25}{4}\log|x^2 + \sqrt{x^4-25}| + c$$

$$= \frac{x^2}{4}\sqrt{x^4-25} - \frac{25}{4}\log(x^2 + \sqrt{x^4-25}) + c, \text{ as } x^2 > 0$$

Example 9 : Evaluate : $\int \sqrt{(x-3)(7-x)}\ dx.$ $(3 < x < 7)$

Solution : $I = \int \sqrt{(x-3)(7-x)}\ dx$

$$= \int \sqrt{10x-x^2-21}\ dx$$

Now, $10x - x^2 - 21 = -[x^2 - 10x + 21]$

$$= -[x^2 - 10x + 25 - 4]$$

$$= -[(x-5)^2 - 4]$$

$$= 4 - (x-5)^2$$

$$\therefore \quad I = \int \sqrt{2^2 - (x-5)^2}\, dx$$

$$= \frac{x-5}{2}\sqrt{2^2 - (x-5)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-5}{2}\right) + c$$

$$= \frac{x-5}{2}\sqrt{(x-3)(7-x)} + 2\sin^{-1}\left(\frac{x-5}{2}\right) + c$$

Example 10 : Evaluate : $\int e^x\left(\frac{1+\sin x\cos x}{\cos^2 x}\right) dx$

Solution : $I = \int e^x\left(\frac{1+\sin x\cos x}{\cos^2 x}\right) dx$

$$= \int e^x\left(\frac{1}{\cos^2 x} + \frac{\sin x\cos x}{\cos^2 x}\right) dx$$

$$= \int e^x(\sec^2 x + \tan x)\, dx$$

$$= \int e^x\left(\tan x + \frac{d}{dx}(\tan x)\right) dx$$

$$= e^x \tan x + c$$

Example 11 : Evaluate : $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-\frac{x}{2}} dx,\ 0 < x < \frac{\pi}{2}$

Solution : $I = \int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-\frac{x}{2}} dx$

$$= \int \frac{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{2\cos^2\frac{x}{2}} e^{-\frac{x}{2}} dx$$

$$= \int \frac{\sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{2\cos^2\frac{x}{2}} e^{-\frac{x}{2}} dx$$

$$= \int \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{2\cos^2\frac{x}{2}} e^{-\frac{x}{2}} dx \qquad \left(\text{since } 0 < \frac{x}{2} < \frac{\pi}{4},\ \cos\frac{x}{2} > \sin\frac{x}{2}\right)$$

Let $-\frac{x}{2} = t$, $-dx = 2dt$. So $dx = -2dt$.

$$\therefore \quad I = -\int \frac{\cos t + \sin t}{2\cos^2 t} e^t \cdot (2dt)$$

$$= -\int \left(\frac{1}{\cos t} + \frac{\sin t}{\cos^2 t}\right) e^t\, dt$$

$= -\int (sec\, t + sec\, t\ tan\, t)\ e^t\, dt$

$= -\int \left(sec\, t + \frac{d}{dt}(sec\, t)\right) e^t\, dt$

$= -sec\, t \cdot e^t + c$

$= -e^{-\frac{x}{2}} \cdot sec\left(\frac{x}{2}\right) + c$ $\qquad \left(sec\left(-\frac{x}{2}\right) = sec\,\frac{x}{2}\right)$

Example 12 : Evaluate : $\int e^x\, sin^2 x\ dx$

Solution : $I = \int e^x\, sin^2 x\ dx$

$= \int e^x \frac{(1 - cos\,2x)}{2}\ dx$

$= \frac{1}{2}\int e^x\, dx - \frac{1}{2}\int e^x \cdot cos2x\ dx$

$= \frac{1}{2} e^x - \frac{1}{2}\left[\frac{e^x}{1^2 + 2^2}\ (cos2x + 2sin2x)\right] + c$

$= \frac{e^x}{2} - \frac{e^x}{10}(cos2x + 2sin2x) + c$

Example 13 : Evaluate : $\int 2^x\, cos^2 x\ dx$

Solution : $I = \int 2^x\, cos^2 x\ dx$

$= \int 2^x \left(\frac{1 + cos\,2x}{2}\right) dx$

$= \frac{1}{2}\int 2^x\, dx + \frac{1}{2}\int 2^x\, cos2x\ dx$

$= \frac{1}{2}\int 2^x\, dx + \frac{1}{2}\int e^{x \cdot \log_e 2}\, cos2x\ dx$

$= \frac{1}{2} \cdot \frac{2^x}{\log_e 2} + \frac{1}{2} \cdot \frac{e^{x \log_e 2}}{4 + (\log_e 2)^2}\ [(\log_e 2)\, cos2x + 2sin2x)] + c$

$\therefore \quad I = \frac{2^{x-1}}{\log_e 2} + \frac{1}{2} \cdot \frac{2^x}{4 + (\log_e 2)^2} \cdot [(\log_e 2)\, cos2x + 2sin2x)] + c$

Example 14 : Evaluate : $\int (x - 5)\sqrt{x^2 + x}\ dx$

Solution : Here, we find m and n such that,

$x - 5 = m\left[\frac{d}{dx}(x^2 + x)\right] + n$

$= m\,(2x + 1) + n$

$\therefore \quad x - 5 = 2mx + m + n$

Comparing coefficients of x and constant terms,

$2m = 1$ and $m + n = -5$

$\therefore \quad m = \frac{1}{2}$ and $n = -5 - \frac{1}{2} = -\frac{11}{2}$

$\therefore \quad x - 5 = \frac{1}{2}(2x + 1) - \frac{11}{2}$

$$\therefore \quad I = \int (x - 5)\sqrt{x^2 + x}\, dx$$

$$= \int \left[\tfrac{1}{2}(2x + 1) - \tfrac{11}{2}\right] \sqrt{x^2 + x}\, dx$$

$$= \tfrac{1}{2} \int (2x + 1)\sqrt{x^2 + x}\, dx - \tfrac{11}{2} \int \sqrt{x^2 + x}\, dx$$

$$= \tfrac{1}{2} \int (x^2 + x)^{\frac{1}{2}} \cdot \tfrac{d}{dx}(x^2 + x)\, dx - \tfrac{11}{2} \int \sqrt{\left(x + \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2}\, dx$$

$$= \tfrac{1}{2} \cdot \frac{(x^2 + x)^{\frac{3}{2}}}{\frac{3}{2}} - \tfrac{11}{2}\left[\frac{\left(x + \frac{1}{2}\right)}{2}\sqrt{x^2 + x} - \tfrac{1}{8}\log\left|\left(x + \tfrac{1}{2}\right) + \sqrt{x^2 + x}\right|\right] + c$$

$$= \tfrac{1}{3}(x^2 + x)^{\frac{3}{2}} - \tfrac{11}{2}\left[\frac{2x + 1}{4}\sqrt{x^2 + x} - \tfrac{1}{8}\log\left|x + \tfrac{1}{2} + \sqrt{x^2 + x}\right|\right] + c$$

Exercise 2.2

Integrate the following functions *w.r.t.* x considering them well defined over proper domains :

1. $\sqrt{9 - x^2}$

2. $\sqrt{2x^2 + 10}$

3. $\sqrt{5x^2 - 3}$

4. $\sqrt{4 - 3x - 2x^2}$

5. $\sqrt{4x^2 + 4x - 15}$

6. $x^2\sqrt{8 - x^6}$

7. $\cos x \sqrt{4 - \sin^2 x}$

8. $e^x(\log \sin x + \cot x)$

9. $e^x \dfrac{1 - \sin x}{1 - \cos x}$

10. $\dfrac{1 + \sin 2x}{1 + \cos 2x}\, e^{2x}$

11. $\dfrac{x^2 e^x}{(x + 2)^2}$

12. $\dfrac{x^2 - x + 1}{(x^2 + 1)^{\frac{3}{2}}}\, e^x$

13. $e^x \left(\dfrac{1 - x}{1 + x^2}\right)^2$

14. $x\sqrt{1 + x - x^2}$

15. $(3x - 2)\sqrt{x^2 + x + 1}$

16. $(2x - 5)\sqrt{2 + 3x - x^2}$

17. $e^{2x} \sin 4x$

18. $e^{-\frac{x}{2}} \cos^2 x$

19. $3^x \sin^2 x$

20. $e^{2x} \sin 3x \sin x$

*

2.5 Method of Partial Fractions

Now we shall study the method of integrating rational functions. If $p(x)$ and $q(x)$ are two polynomials, then $\dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ is called a rational algebraic function or a rational function of x. We know how to simplify algebraic operations on rational functions.

For example, $\dfrac{5}{x - 3} + \dfrac{1}{x - 2} = \dfrac{5(x - 2) + 1(x - 3)}{(x - 3)(x - 2)} = \dfrac{6x - 13}{(x - 3)(x - 2)}$

Let us think the other way round. Can we put $\dfrac{6x - 13}{(x - 3)(x - 2)}$ in the form $\dfrac{5}{x - 3} + \dfrac{1}{x - 2}$?

The method of expressing a rational function as a sum of other rational functions in this way is known as the **method of partial fractions**.

Expressing $\dfrac{6x-13}{(x-3)(x-2)}$ as $\dfrac{5}{x-3} + \dfrac{1}{x-2}$, its integration will become very simple.

Let us try to understand this method :

(1) If the degree of $p(x)$ < the degree of $q(x)$, then $\dfrac{p(x)}{q(x)}$ is called a **Proper Rational Function.**

For example, $\dfrac{5-3x}{x^3+3x+2}, \dfrac{2x^2+3x+7}{x^3-7x+2}, \dfrac{3x+2}{x^3-6x^2+11x-6}$ are proper rational functions.

(2) If the degree of $p(x) \geq$ the degree of $q(x)$, then $\dfrac{p(x)}{q(x)}$ is called an **Improper Rational Function.**

For example, $\dfrac{x^3+1}{x^2-2x+1}, \dfrac{x^2+x+1}{x^2+3x+2}, \dfrac{x^3-6x^2+10x-2}{x^2-5x+6}$ are improper rational functions.

If $\dfrac{p(x)}{q(x)}$ is an improper rational function, we divide $p(x)$ by $q(x)$ so that $p(x) = q(x)\ s(x) + r(x)$, where $r(x) = 0$ or degree of $r(x)$ is less then that of $q(x)$. The improper rational function $\dfrac{p(x)}{q(x)}$ is expressed in the form $s(x) + \dfrac{r(x)}{q(x)}$ where $r(x)$ and $s(x)$ are polynomials such that the degree of $r(x)$ is less than that of $q(x)$ or $r(x) = 0$. Thus, $\dfrac{r(x)}{q(x)}$ is a proper rational function or 0. For example, let us consider $\dfrac{4x^3-x^2+1}{x^2-2}$.

We should divide $p(x) = 4x^3 - x^2 + 1$ by $q(x) = x^2 - 2$.

$$\begin{array}{r|l}
 & 4x \quad - 1 \\ \hline
x^2 - 2 & 4x^3 - x^2 + 1 \\
 & 4x^3 \quad\quad - 8x \\
 & - \quad\quad\quad + \\ \hline
 & \quad -x^2 + 8x + 1 \\
 & \quad -x^2 \quad\quad + 2 \\
 & \quad + \quad\quad\quad - \\ \hline
 & \quad\quad\quad 8x - 1
\end{array}$$

$\therefore$ Quotient $s(x) = 4x - 1$ and remainder $r(x) = 8x - 1$

Thus, $\dfrac{4x^3-x^2+1}{x^2-2} = (4x-1) + \dfrac{8x-1}{x^2-2}$.

Here, the quotient $4x - 1$ is a polynomial function and $\dfrac{8x-1}{x^2-2}$ is a proper rational function. Now we study the method of integrating a proper rational function.

Suppose $\dfrac{p(x)}{q(x)}$ is a proper rational function. The resolution of $\dfrac{p(x)}{q(x)}$ into partial fraction depends mainly upon the nature of the factors of $q(x)$ as discussed below.

Case 1 : Real, Linear and Non-repeated Factors :

Let $q(x)$ have n real, linear and non-repeated factors $x - \alpha_1, x - \alpha_2, ..., x - \alpha_n$. i.e.

$q(x) = (x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$. **$(\alpha_i \neq \alpha_j$ for $i \neq j)$**

Then we can express $\frac{p(x)}{q(x)}$ as

$\frac{p(x)}{q(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + ... + \frac{A_n}{x-\alpha_n}$, where $A_1, A_2, ..., A_n$ are constants. We can always determine A_i, $i = 1, 2, ..., n$ uniquely and integrate function on the right hand side easily. Let us take an example to understand this method.

Example 15 : Evaluate : $\int \frac{2x-3}{(x-1)(x-2)(x-3)} dx$

Solution : $I = \int \frac{2x-3}{(x-1)(x-2)(x-3)} dx$

We can see that given rational function is a proper rational function and in the denominator, we have real, linear and non-repeated factors.

Let $\frac{2x-3}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$. **(i)**

where A, B, C are constants. Multiplying both sides by $(x - 1)(x - 2)(x - 3)$ we get

$2x - 3 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$ **(ii)**

Now we can find constants A, B, C by any one of the following three methods.

First Method :

Denominator of the rational function $(x - 1)(x - 2)(x - 3)$ has three zeros 1, 2, 3.

Let $x = 1, 2, 3$ in equation (ii) by turn and we get the values of A, B, C.

$x = 1$ gives $2(1) - 3 = A(-1)(-2)$. Hence $A = -\frac{1}{2}$.

$x = 2$ gives $2(2) - 3 = B(1)(-1)$. Hence $B = -1$.

$x = 3$ gives $2(3) - 3 = C(2)(1)$. Hence $C = \frac{3}{2}$.

Second Method :

We have $\frac{2x-3}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$. **(ii)**

To find A, we select the factor $x - 1$ in the denominator of A and put that factor equal to zero (i.e. $x - 1 = 0$) and obtain the value of x (i.e. $x = 1$). Replace x by that value in $\frac{2x-3}{(x-2)(x-3)}$, obtained after removing $x - 1$ from L.H.S. Then $A = \frac{2(1)-3}{(1-2)(1-3)} = -\frac{1}{2}$. Similarly to obtain the value of B, we substitute $x = 2$ in $\frac{2x-3}{(x-1)(x-3)}$. So $B = \frac{2(2)-3}{(2-1)(2-3)} = -1$. To obtain value of C, we substitute $x = 3$ in $\frac{2x-3}{(x-1)(x-2)}$. So $C = \frac{2(3)-3}{(3-1)(3-2)} = \frac{3}{2}$.

Thus, $A = -\frac{1}{2}$, $B = -1$ and $C = \frac{3}{2}$.

Third Method :

From (ii) we have,

$(2x - 3) = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$

$\therefore \quad 2x - 3 = A(x^2 - 5x + 6) + B(x^2 - 4x + 3) + C(x^2 - 3x + 2)$

$\therefore \quad 2x - 3 = (A + B + C)\,x^2 + (-5A - 4B - 3C)x + (6A + 3B + 2C)$

Comparing the coefficients of x^2 coefficients of x and constant terms on both sides we get,

$A + B + C = 0,\ -5A - 4B - 3C = 2,\ 6A + 3B + 2C = -3$

Solving these equations, we get $A = -\frac{1}{2}$, $B = -1$ and $C = \frac{3}{2}$.

We can use any of the above three methods, whichever seems simple for a particular problem.

Now, substituting values of A, B and C in (i) we get,

$$\frac{2x-3}{(x-1)(x-2)(x-3)} = \frac{-\frac{1}{2}}{x-1} + \frac{-1}{x-2} + \frac{\frac{3}{2}}{x-3}.$$

$$\therefore \int \frac{2x-3}{(x-1)(x-2)(x-3)}\,dx = -\frac{1}{2}\int \frac{1}{x-1}\,dx - \int \frac{1}{x-2}\,dx + \frac{3}{2}\int \frac{1}{x-3}\,dx.$$

$$= -\frac{1}{2}\log|x-1| - \log|x-2| + \frac{3}{2}\log|x-3| + c$$

Case 2 : Real, Linear Repeated and Non-repeated Factors :

If $q(x) = (x - \alpha)^k (x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$, then let

$$\frac{p(x)}{q(x)} = \frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + ... + \frac{A_k}{(x-\alpha)^k} + \frac{B_1}{x-\alpha_1} + \frac{B_2}{x-\alpha_2} + ... + \frac{B_n}{(x-\alpha_n)}$$

Corresponding to non-repeated linear factors we assume as in case (1) and for each repeated factor $(x - \alpha)^k$,, we assume partial fractions,

$\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \frac{A_3}{(x-\alpha)^3} + ... + \frac{A_k}{(x-\alpha)^k}$, where $A_1, A_2, A_3, ..., A_k$ are constants. Let us take an example to understand this method.

Example 16 : Evaluate : $\int \frac{x}{(x-1)^2(x+2)}\,dx$

Solution : $I = \int \frac{x}{(x-1)^2(x+2)}\,dx$

Let $\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$ **(i)**

Multiplying both sides by $(x - 1)^2(x + 2)$, we get

$x = A(x - 1)(x + 2) + B(x + 2) + C(x - 1)^2$

Now, $x = 1$ gives $1 = B(3)$. So $B = \frac{1}{3}$

$x = -2$ gives $-2 = C(9)$. So $C = -\frac{2}{9}$

Comparing coefficient of x^2. $A + C = 0$. So $A = -C$.

$\therefore \quad A = \frac{2}{9}$

Substituting values of A, B, C in expression (i),

$$\frac{x}{(x-1)^2(x+2)} = \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}$$

$$\therefore \quad \int \frac{x\,dx}{(x-1)^2(x+2)} = \frac{2}{9}\int \frac{1}{x-1}\,dx + \frac{1}{3}\int \frac{1}{(x-1)^2}\,dx - \frac{2}{9}\int \frac{1}{x+2}\,dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\frac{(x-1)^{-1}}{-1} - \frac{2}{9}\log|x+2| + c$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + c$$

Case 3 : One Real Quadratic and Other Linear non-repeated factors :

If $q(x) = (ax^2 + bx + c)(x - \alpha_1)(x - \alpha_2)... (x - \alpha_n)$, then let

$$\frac{p(x)}{q(x)} = \frac{Ax + B}{ax^2 + bx + c} + \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + ... + \frac{A_n}{x - \alpha_n}$$

where $A_1, A_2, A_3, ..., A_n$ are constants to be determined. Let us take an example to understand this method.

Example 17 : Evaluate : $\int \frac{x\,dx}{(3x^2+2)(x-2)}$

Solution : $I = \int \frac{x\,dx}{(3x^2+2)(x-2)}$

Let $\frac{x}{(3x^2+2)(x-2)} = \frac{A}{x-2} + \frac{Bx + C}{3x^2+2}$

Multiplying by $(3x^2 + 2)(x - 2)$ on both the sides,

$$x = A(3x^2 + 2) + (Bx + C)(x - 2)$$

$$\therefore \quad x = A(3x^2 + 2) + Bx(x - 2) + C(x - 2)$$

$x = 2$ gives $2 = 14A$. So $A = \frac{1}{7}$.

Comparing coefficients of x^2 on both sides,

$3A + B = 0$. So $B = -3A$

$\therefore \quad B = -\frac{3}{7}$

Comparing coefficients of x on both sides,

$C - 2B = 1$. So $C = 1 + 2B = 1 - \frac{6}{7} = \frac{1}{7}$

$\therefore \quad C = \frac{1}{7}$

$$\therefore \quad \int \frac{x\,dx}{(3x^2+2)(x-2)} = \int \frac{\frac{1}{7}\,dx}{x-2} + \int \frac{\left(-\frac{3}{7}x + \frac{1}{7}\right)dx}{3x^2+2}$$

$$= \frac{1}{7}\int \frac{dx}{x-2}\,dx - \frac{1}{7}\int \frac{(3x-1)\,dx}{3x^2+2}$$

$$= \frac{1}{7}\int \frac{1}{x-2}\,dx - \frac{1}{7}\int \frac{3x\,dx}{3x^2+2} + \frac{1}{7}\int \frac{dx}{3x^2+2}$$

$$= \frac{1}{7}\int \frac{1}{x-2}\,dx - \frac{1}{14}\int \frac{6x\,dx}{3x^2+2} + \frac{1}{7}\int \frac{dx}{(\sqrt{3}x^2)+(\sqrt{2})^2}\,dx$$

$$= \frac{1}{7}\log|x-2| - \frac{1}{14}\log|3x^2+2| + \frac{1}{7\sqrt{6}}\,tan^{-1}\left(\frac{\sqrt{3}x}{\sqrt{2}}\right) + c$$

$$= \frac{1}{7}\log|x-2| - \frac{1}{14}\log(3x^2+2) + \frac{1}{7\sqrt{6}}\,tan^{-1}\frac{\sqrt{3}x}{\sqrt{2}} + c \text{ as } x^2 \geq 0$$

Example 18 : Evaluate : $\int \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

Solution : $I = \int \frac{x^2\,dx}{(x^2+1)(x^2+4)}$

Here all the indices of x are even. Write $x^2 = t$ in the integrand. (It is not a substitution).

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{t}{(t+1)(t+4)}$$

Let $\frac{t}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4}$

$\therefore \quad t = A(t+4) + B(t+1)$

Taking $t = -1$, we get $-1 = 3A$. So $A = -\frac{1}{3}$.

Taking $t = -4$, we get $-4 = -3B$. So $B = \frac{4}{3}$.

Substituting values of A and B in (i)

$$\frac{t}{(t+1)(t+4)} = \frac{-\frac{1}{3}}{t+1} + \frac{\frac{4}{3}}{t+4}$$

Now, $t = x^2$ thus, $\frac{x^2}{(x^2+1)(x^2+4)} = \frac{-\frac{1}{3}}{x^2+1} + \frac{\frac{4}{3}}{x^2+4}$

$$\therefore \quad \int \frac{x^2}{(x^2+1)(x^2+4)}\,dx = -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

$$= -\frac{1}{3}\,tan^{-1}x + \frac{4}{3} \times \frac{1}{2}\,tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\therefore \quad I = -\frac{1}{3}\,tan^{-1}x + \frac{2}{3}\,tan^{-1}\left(\frac{x}{2}\right) + c$$

Example 19 : Evaluate : $\int \frac{x^2}{(x^3+2)(x^3-5)}\,dx$

Solution : $I = \int \frac{x^2}{(x^3+2)(x^3-5)}\,dx$

Let $x^3 = t$, So $3x^2\,dx = dt$. Hence $x^2\,dx = \frac{1}{3}\,dt$

$\therefore \quad I = \frac{1}{3}\int \frac{dt}{(t+2)(t-5)}$.

Let $\frac{1}{(t+2)(t-5)} = \frac{A}{t+2} + \frac{B}{t-5}$

$1 = A(t-5) + B(t+2)$

$t = -2$ gives, $1 = -7A$. So $A = -\frac{1}{7}$

$t = 5$ gives, $1 = 7B$. So $B = \frac{1}{7}$

$$\therefore \quad \frac{1}{(t+2)(t-5)} = \frac{-\frac{1}{7}}{t+2} + \frac{\frac{1}{7}}{t-5}.$$

$$\therefore \quad I = \frac{1}{3}\int \frac{dt}{(t+2)(t-5)}$$

$$= -\frac{1}{21}\int \frac{1}{t+2}\,dt + \frac{1}{21}\int \frac{1}{t-5}\,dt$$

$$= -\frac{1}{21}\log|t+2| + \frac{1}{21}\log|t-5| + c$$

$$= \frac{1}{21}\log\left|\frac{t-5}{t+2}\right| + c$$

$$= \frac{1}{21}\log\left|\frac{x^3-5}{x^3+2}\right| + c$$

Example 20 : Evaluate : $\int \frac{x^2+x+1}{(x-1)^3}\,dx$

Solution : $I = \int \frac{x^2+x+1}{(x-1)^3}\,dx$

Let $x - 1 = t$, $dx = dt$.

$$I = \int \frac{(t+1)^2+(t+1)+1}{t^3}\,dt$$

$$= \int \frac{t^2+3t+3}{t^3}\,dt$$

$$= \int \left(\frac{1}{t} + \frac{3}{t^2} + \frac{3}{t^3}\right) dt$$

$$= \int \frac{1}{t}\,dt + 3\int t^{-2}\,dt + 3\int t^{-3}\,dt$$

$$= \log|t| + 3\left(\frac{-1}{t}\right) + 3\left(\frac{1}{-2t^2}\right) + c$$

$$= \log|t| - \frac{3}{t} - \frac{3}{2t^2} + c$$

$$= \log|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + c$$

Note : This sum can also be done using partial fractions.

$$\frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Example 21 : Evaluate : $\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$

Solution : $I = \int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$

$$= \int \frac{\tan\theta\,(1+\tan^2\theta)}{1+\tan^3\theta}\,d\theta$$

$$= \int \frac{\tan\theta \cdot \sec^2\theta}{1+\tan^3\theta}\,d\theta$$

Let $tan\theta = t$. So $sec^2\theta \, d\theta = dt$

$$I = \int \frac{t\,dt}{1+t^3}$$

$$= \int \frac{t\,dt}{(t+1)(t^2-t+1)}$$

Let $\frac{t}{(t+1)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$

$\therefore \quad t = A(t^2 - t + 1) + (Bt + C)(t + 1)$

$\therefore \quad t = A(t^2 - t + 1) + Bt(t + 1) + C(t + 1)$

$t = -1$ gives $-1 = 3A$. So $A = -\frac{1}{3}$

Comparing the coefficients of t^2 on both sides, we get $A + B = 0$. So $B = -A$.

$\therefore \quad B = \frac{1}{3}$

Comparing the constant terms on both sides, we get $A + C = 0$. So $C = -A$.

$\therefore \quad C = \frac{1}{3}$

$$\therefore \quad \frac{t}{(t+1)(t^2-t+1)} = \frac{-\frac{1}{3}}{t+1} + \frac{\frac{1}{3}t+\frac{1}{3}}{t^2-t+1}$$

$$\therefore \quad I = -\frac{1}{3}\int \frac{1}{t+1}\,dt + \frac{1}{3}\int \frac{t+1}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\int \frac{1}{t+1}\,dt + \frac{1}{6}\int \frac{2t+2}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\int \frac{1}{t+1}\,dt + \frac{1}{6}\int \frac{(2t-1)+3}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\int \frac{dt}{t+1} + \frac{1}{6}\int \frac{(2t-1)\,dt}{t^2-t+1} + \frac{3}{6}\int \frac{dt}{t^2-t+1}$$

$$= -\frac{1}{3}\int \frac{dt}{t+1} + \frac{1}{6}\int \frac{(2t-1)\,dt}{t^2-t+1} + \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\times\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\,tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$= -\frac{1}{3}\log|t+1| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{\sqrt{3}}\,tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + c$$

$$\therefore \quad I = -\frac{1}{3}\log|tan\theta+1| + \frac{1}{6}\log|tan^2\theta - tan\theta + 1| + \frac{1}{\sqrt{3}}\,tan^{-1}\left(\frac{2tan\theta-1}{\sqrt{3}}\right) + c$$

Exercise 2.3

Integrate the following functions defined over a proper domain *w.r.t.* x :

1. $\dfrac{x^2 + 4x - 1}{x^3 - x}$

2. $\dfrac{3x + 2}{(x-1)(x-2)(x-3)}$

3. $\dfrac{x^3 - 6x^2 + 10x - 2}{x^2 - 5x + 6}$

4. $\dfrac{x^2}{(2x^2 + 1)(x^2 - 1)}$

5. $\dfrac{x^2 + 1}{(x^2 + 2)(2x^2 + 1)}$

6. $\dfrac{x^3}{(x^2 + 2)(x^2 + 5)}$

7. $\dfrac{x^2 + x + 1}{(x+1)^2(x+2)}$

8. $\dfrac{5x}{(x+1)(x^2 + 9)}$

9. $\dfrac{1}{6e^{2x} + 5e^x + 1}$

10. $\dfrac{sec^2\theta}{tan^2\theta - 4tan\,\theta + 3}$

11. $\dfrac{1}{(x+1)^2(x^2+1)}$

12. $\dfrac{x^2}{(x-1)^3(x+1)}$

13. $\dfrac{1}{sin\ x - sin\ 2x}$

14. $\dfrac{1}{sin\ x(3 + 2cosx)}$

*

Miscellaneous Examples :

Example 22 : Evaluate : $\int (x+1)\sqrt{\dfrac{x+2}{x-2}}\,dx$ $\quad x > 2$ **(If $x < -2$?)**

Solution : $I = \int (x+1)\sqrt{\dfrac{x+2}{x-2}}\,dx$

$$= \int (x+1)\sqrt{\frac{x+2}{x-2} \times \frac{x+2}{x+2}}\,dx \qquad (x > 2)$$

$$= \int \frac{(x+1)(x+2)}{\sqrt{x^2-4}}\,dx$$

$$= \int \frac{x^2 + 3x + 2}{\sqrt{x^2-4}}\,dx$$

$$= \int \frac{(x^2-4) + 3x + 6}{\sqrt{x^2-4}}\,dx$$

$$= \int \sqrt{x^2-4}\,dx + 3\int \frac{x}{\sqrt{x^2-4}}\,dx + 6\int \frac{dx}{\sqrt{x^2-4}}$$

$$= \int \sqrt{x^2-4}\,dx + \frac{3}{2}\int (x^2-4)^{-\frac{1}{2}}(2x)\,dx + 6\int \frac{dx}{\sqrt{x^2-4}}$$

$$= \frac{x}{2}\sqrt{x^2-4} - \frac{4}{2}\log\left|x + \sqrt{x^2-4}\right| + \frac{3}{2}\,\frac{(x^2-4)^{\frac{1}{2}}}{\frac{1}{2}} + 6\log\left|x + \sqrt{x^2-4}\right| + c$$

$$= \frac{x}{2}\sqrt{x^2-4} + 4 \log |x + \sqrt{x^2-4}| + 3\sqrt{x^2-4} + c$$

$$= \left(\frac{x}{2} + 3\right)\sqrt{x^2-4} + 4 \log |x + \sqrt{x^2-4}| + c$$

Example 23 : Evaluate : $\int \frac{(1 + sinx)\, dx}{sinx\,(1 + cosx)}$

Solution : $I = \int \frac{(1 + sinx)\, dx}{sinx\,(1 + cosx)}$

$$I = \int \frac{dx}{sinx\,(1 + cosx)} + \int \frac{dx}{1 + cosx}$$

Let $I = I_1 + I_2$ where $I_1 = \int \frac{dx}{sin\,x\,(1 + cosx)}$, $I_2 = \int \frac{dx}{1 + cosx}$

$$I_1 = \int \frac{dx}{sinx\,(1 + cosx)}$$

$$= \int \frac{sin\,x\,dx}{sin^2 x\,(1 + cosx)}$$

$$= \int \frac{sin\,x\,dx}{(1 - cosx)(1 + cosx)^2}$$

Now, $cosx = t$ gives $sinx\ dx = -dt$

$$I_1 = \int \frac{-dt}{(1 - t)(1 + t)^2}$$

Let $\frac{-1}{(1 - t)(1 + t)^2} = \frac{A}{1 - t} + \frac{B}{1 + t} + \frac{C}{(1 + t)^2}$

$-1 = A(1 + t)^2 + B(1 - t)(1 + t) + C(1 - t)$

$t = 1$ gives $-1 = A(4)$. So $A = -\frac{1}{4}$

$t = -1$ gives $-1 = C(2)$. So $C = -\frac{1}{2}$

$t = 0$ gives (or any convenient value of t can be taken)

$-1 = A + B + C$

$\therefore\ B = -1 + \frac{1}{4} + \frac{1}{2}$

$\therefore\ B = -\frac{1}{4}$

$$\therefore\ \frac{-1}{(1 - t)(1 + t)^2} = \frac{-\frac{1}{4}}{1 - t} + \frac{-\frac{1}{4}}{1 + t} + \frac{-\frac{1}{2}}{(1 + t)^2}$$

$$I_1 = -\frac{1}{4}\int \frac{1}{1 - t}\, dt - \frac{1}{4}\int \frac{1}{1 + t}\, dt - \frac{1}{2}\int (1 + t)^{-2}\, dt$$

$$= \frac{1}{4} \log \left| \frac{t - 1}{t + 1} \right| + \frac{1}{2(t + 1)} + c_1$$

$$\therefore \quad I_1 = \frac{1}{4}\log\left|\frac{\cos x - 1}{\cos x + 1}\right| + \frac{1}{2(\cos x + 1)} + c_1$$

$$\text{Now, } I_2 = \int \frac{1}{1+\cos x}\,dx = \int \frac{1}{2\cos^2\frac{x}{2}}\,dx = \frac{1}{2}\int \sec^2\frac{x}{2}\,dx$$

$$= \frac{1}{2}\,\frac{\tan\frac{x}{2}}{\frac{1}{2}} + c_2$$

$$\therefore \quad I_2 = \tan\frac{x}{2} + c_2$$

$$I = I_1 + I_2$$

$$\therefore \quad I = \frac{1}{4}\log\left|\frac{\cos x - 1}{\cos x + 1}\right| + \frac{1}{2(\cos x + 1)} + \tan\frac{x}{2} + c \qquad (c_1 + c_2 = c)$$

$$= \frac{1}{4}\log\left|\tan^2\frac{x}{2}\right| + \frac{1}{4\cos^2\frac{x}{2}} + \tan\frac{x}{2} + c$$

$$= \frac{1}{2}\log\left|\tan\frac{x}{2}\right| + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + c$$

Second Method :

Let $\tan\frac{x}{2} = t$, so $\sec^2\frac{x}{2}\cdot\frac{1}{2}\,dx = dt$

Hence $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$

$$I = \int \frac{(1+\sin x)\,dx}{\sin x\,(1+\cos x)}$$

$$= \int \frac{1+\frac{2t}{1+t^2}}{\left(\frac{2t}{1+t^2}\right)\left(1+\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$$

$$= \int \frac{1+t^2+2t}{2t\,(1+t^2+1-t^2)}\cdot 2dt$$

$$= \int \frac{1+2t+t^2}{2t}\,dt$$

$$= \frac{1}{2}\int\left(\frac{1}{t}+2+t\right)dt$$

$$= \frac{1}{2}\left[\log|t|+2t+\frac{t^2}{2}\right] + c$$

$$= \frac{1}{2}\log|t|+t+\frac{1}{4}t^2 + c$$

$$= \frac{1}{2}\log\left|\tan\frac{x}{2}\right| + \tan\frac{x}{2} + \frac{1}{4}\tan^2\frac{x}{2} + c'$$

$$\text{Observe that } I = \frac{1}{2}\log\left|\tan\frac{x}{2}\right| + \tan\frac{x}{2} + \frac{1}{4}\left(\sec^2\frac{x}{2}-1\right) + c'$$

$$= \frac{1}{2}\log\left|\tan\frac{x}{2}\right| + \tan\frac{x}{2} + \frac{1}{4}\sec^2\frac{x}{2} - \frac{1}{4} + c'$$

$$= \frac{1}{2}\log\left|\tan\frac{x}{2}\right| + \tan\frac{x}{2} + \frac{1}{4}\sec^2\frac{x}{2} + c \qquad \left(c = c'-\frac{1}{4}\right)$$

Thus, we can see that answers obtained by both the methods are same.

Example 24 : Evaluate : $\int \left(\log(\log x) + \frac{1}{(\log x)^2}\right) dx, \quad x > 1$

Solution : $I = \int \left(\log(\log x) + \frac{1}{(\log x)^2}\right) dx$

Let $\log x = t$. So $x = e^t$

$\therefore \quad dx = e^t \, dt$

$\therefore \quad I = \int \left(\log t + \frac{1}{t^2}\right) e^t \, dt$

$= \int \left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right) e^t \, dt$

$= \int \left[\left(\log t + \frac{1}{t}\right) - \left(\frac{1}{t} - \frac{1}{t^2}\right)\right] e^t \, dt$

$= \int \left(\log t + \frac{1}{t}\right) e^t \, dt - \int \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t \, dt$

$= e^t \log t - e^t \frac{1}{t} + c$

$= x \log(\log x) - \frac{x}{\log x} + c$

Example 25 : Evaluate : $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} \, dx$

Solution : $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} \, dx$

$= \int \frac{\sin^{-1}\sqrt{x} - (\frac{\pi}{2} - \sin^{-1}\sqrt{x})}{\frac{\pi}{2}} \, dx \qquad \left(\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}\right)$

$= \int \frac{2\sin^{-1}\sqrt{x} - \frac{\pi}{2}}{\frac{\pi}{2}} \, dx$

$= \frac{4}{\pi} \int \sin^{-1}\sqrt{x} \, dx - \int dx$

Let $I_1 = \int \sin^{-1}\sqrt{x} \, dx$

Let $\sin^{-1}\sqrt{x} = \theta$. So $x = \sin^2\theta, \quad 0 < \theta < \frac{\pi}{2} \qquad \left(\sqrt{x} > 0. \text{ So, } 0 < \theta < \frac{\pi}{2}\right)$

$\therefore \quad dx = 2\sin\theta \cdot \cos\theta \, d\theta$

$\therefore \quad I_1 = \int \theta \, 2\sin\theta \, \cos\theta \, d\theta$

$= \int \theta \sin 2\theta \, d\theta$

$= -\frac{\theta \cos 2\theta}{2} + \frac{1}{2} \int \cos 2\theta \, d\theta$

$= -\frac{\theta}{2} \cos 2\theta + \frac{\sin 2\theta}{4}$

$= -\frac{\theta}{2} (1 - 2\sin^2\theta) + \frac{1}{2} \sin\theta \cdot \cos\theta$

$$= -\frac{1}{2}\, sin^{-1}\sqrt{x}\ (1 - 2x) + \frac{1}{2}\sqrt{x}\,\sqrt{1-x}$$

$$= -\frac{1}{2}\, sin^{-1}\sqrt{x} + x\, sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x - x^2}$$

$$\therefore \quad I = \frac{4}{\pi}\int sin^{-1}\sqrt{x}\, dx - \int dx$$

$$= \frac{4}{\pi}\left[-\frac{1}{2}\, sin^{-1}\sqrt{x} + x sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x - x^2}\right] - x + c$$

Exercise 2

Integrate the following functions defined on proper domain *w.r.t.* x :

1. $x^2\ sin^{-1}x$

2. $tan^{-1}\sqrt{\frac{1-x}{1+x}}$

3. $\frac{x - sin\ x}{1 - cos\ x}$

4. $\frac{\sqrt{sin\ x}}{cos\ x}$

5. $\log\ (x + \sqrt{x^2 + a^2})$

6. $sin^{-1}\sqrt{\frac{x}{x+a}}$

7. $\frac{sin^{-1}\sqrt{x}}{\sqrt{1-x}}$

8. $\frac{\sqrt{1 + sin\, 2x}}{1 + cos\, 2x}\ e^x$

9. $\frac{\log x - 1}{(\log x)^2}$

10. $\log\ (\log x) + \frac{1}{\log x}$

11. $x\sqrt{2ax - x^2}$

12. $(x - 5)\sqrt{x^2 + x}$

13. $\frac{1}{cos\ x\ cos\ 2x}$

14. $\frac{1}{sin\ x + sin\ 2x}$

15. $\frac{sin\ x}{sin\ 4x}$

16. $cot^{-1}(1 - x + x^2)$ **$(0 < x < 1)$**

17. $\frac{1}{sin\ x\sqrt{cos^3 x}}$

18. $\frac{sec\ x}{1 + cosec\ x}$

19. $\frac{1 + sin\ x}{sin\ x\ (1 + cos\ x)}$

20. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) $\int cos\ (\log x)\ dx = ...\ + c$ ☐

(a) $\frac{x}{2}\ [cos\ (\log x) + sin\ (\log x)]$

(b) $\frac{x}{4}\ [cos\ (\log x) + sin\ (\log x)]$

(c) $\frac{x}{2}\ [cos\ (\log x) - sin\ (\log x)]$

(d) $\frac{x}{2}\ [sin\ (\log x) - cos\ (\log x)]$

(2) $\int e^x\ sin\ x\ \ cos\ x\ \ dx = ...\ + c$ ☐

(a) $\frac{e^x}{2\sqrt{5}}\ cos\ (2x - tan^{-1}2)$

(b) $\frac{e^x}{2\sqrt{5}}\ sin\ (2x - tan^{-1}2)$

(c) $\frac{e^2}{2\sqrt{5}}\ sin\ (2x + tan^{-1}2)$

(d) $\frac{e^{2x}}{2\sqrt{5}}\ sin\ (2x + \pi - tan^{-1}2)$

(3) $\int e^x \sec x \,(1 + \tan x)\, dx = \ldots\ldots + c$

(a) $e^x \sec x \ \tan x$ (b) $e^x \tan x$ (c) $e^x \sec x$ (d) $-e^x \sec x$

(4) $\int \frac{(5 + \log x)\, dx}{(6 + \log x)^2} = \ldots\ldots + c$

(a) $\frac{x}{\log_e x + 6}$ (b) $\frac{1}{5 + \log_e x}$ (c) $\frac{x}{\log_e x + 5}$ (d) $\frac{e^x}{\log_e x + 6}$

(5) $\int \frac{e^{\tan^{-1} x}}{1 + x^2} (1 + x + x^2)\, dx = \ldots\ldots + c$

(a) $e^{\tan^{-1} x}$ (b) $\frac{e^{\tan^{-1} x}}{1 + x^2}$ (c) $x \cdot e^{\tan^{-1} x}$ (d) $\frac{x}{1 + x}\ e^{\tan^{-1} x}$

(6) $\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx = \ldots\ldots + c$

(a) $e^x \cot x$ (b) $e^x \cot \frac{x}{2}$ (c) $e^x \tan \frac{x}{2}$ (d) $e^{\frac{x}{2}} \cdot \tan \frac{x}{2}$

(7) $\int e^x \left(\frac{1 + x \log x}{x}\right) dx = \ldots\ldots + c$

(a) $e^x \log x$ (b) $x \cdot e^x$ (c) $\frac{1}{x} \log x$ (d) $e^{-x} \log x$

(8) $\int \left(\log x + \frac{1}{x^2}\right) e^x\, dx = \ldots\ldots + c$

(a) $e^x \left(\log x + \frac{1}{x^2}\right)$ (b) $e^x \left(\log x + \frac{1}{x}\right)$ (c) $e^x \left(\log x - \frac{1}{x^2}\right)$ (d) $e^x \left(\log x - \frac{1}{x}\right)$

(9) $\int \left(\frac{x - 1}{x^2}\right) e^x\, dx = \ldots\ldots + c$

(a) $\frac{1}{x^2}\ e^x$ (b) $\frac{1}{x}\ e^x$ (c) $-\frac{1}{x^2}\ e^x$ (d) $-\frac{1}{x}\ e^x$

(10) $\int (x^6 + 7x^5 + 6x^4 + 5x^3 + 4x^2 + 3x + 1)\, e^x\, dx = \ldots\ldots + c$

(a) $\sum_{i=1}^{7} x^i e^x$ (b) $\sum_{i=1}^{6} x^i e^x$ (c) $\sum_{i=0}^{6} i\, e^x$ (d) $\sum_{i=0}^{6} (xe)^i$

(11) $\int \tan^{-1} x\, dx = \ldots\ldots + c$

(a) $x \tan^{-1} x - \frac{1}{2} \log |1 + x^2|$ (b) $x \tan^{-1} x + \frac{1}{2} \log \frac{\tan^{-1} x}{1 + x^2}$

(c) $x \tan^{-1} x + \frac{1}{2} \log |x^2 + 1|$ (d) $\frac{1}{1 + x^2}$

Summary

We have studied the following points in this chapter :

1. Rule of Integration by Parts :

If (1) f and g are differentiable on I = (a, b) and

(2) f' and g' are continuous on I, then $\int f(x) \cdot g'(x)\ dx = f(x)\, g(x) - \int f'(x)\, g(x)\ dx$

If we take $f(x) = u$ and $g'(x) = v$, then $f'(x) = \frac{du}{dx}$ and $g(x) = \int v\ dx$

Then the new form is $\int uv\ dx = u \int v\ dx - \int \left(\frac{du}{dx} \int v\ dx\right) dx$.

2. Standard Forms of Integration :

(1) $\int \sqrt{x^2 - a^2}\ dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c$

(2) $\int \sqrt{x^2 + a^2}\ dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + c$

(3) $\int \sqrt{a^2 - x^2}\ dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + c$ **$(a > 0)$**

(4) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c$

(5) $\int e^{ax} \cdot \sin(bx + k)\ dx = \frac{e^{ax}}{a^2 + b^2} [a \sin(bx + k) - b \cos(bx + k)] + c$ **$(a \neq 0, b \neq 0)$**

$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin(bx + k - \alpha) + c$

where $\cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$. $\alpha \in (0, 2\pi)$

(6) $\int e^{ax} \cos(bx + k)\ dx = \frac{e^{ax}}{a^2 + b^2} [a \cos(bx + k) + b \sin(bx + k)] + c$ **$(a \neq 0, b \neq 0)$**

$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos(bx + k - \alpha) + c$

where $\cos\alpha = \frac{a}{\sqrt{a^2 + b^2}}$, $\sin\alpha = \frac{b}{\sqrt{a^2 + b^2}}$. $\alpha \in (0, 2\pi)$.

3. Integrals of the type : (1) $\int \sqrt{ax^2 + bx + c}\ dx$ (2) $\int (\mathrm{A}x + \mathrm{B}) \sqrt{ax^2 + bx + c}\ dx$

4. Method of Partial Fractions.

3 DEFINITE INTEGRATION

Calculus required continuity and continuity was supposed to require the infinitely little; but nobody could discover what the infinitely little might be.

– Bertrand Russell

◆

All great theorems were discovered after midnight.

– Adrian Mathesis

3.1 Introduction

We have already studied integration (antiderivation) as an operation inverse to differentiation. From the historical point of view, the concept of integration originated earlier than the concept of differentiation. Infact the concept of integration owes its origin to the problem of finding areas of plane regions, surface areas and volumes of solid bodies etc. Firstly the definite integral was expressed as a limit of a certain sum expressing the area of some region. The word integration has originated from 'addition' and the verb 'to integrate' means 'to merge'. Later on, link between apparently two different concepts of differentiation and integration was established by well known mathematicians **Newton** and **Leibnitz** in 17th century. This relation is known as fundamental theorem of integral calculus and we will learn it in this chapter.

The calculations of area, volume are done using integration. In the 19th century, **Cauchy** and **Riemann** developed the concept of Riemann integration.

Now in this chapter we shall understand the idea of definite integration as the limit of a sum and how it is helpful to find out the area as well as how it can be linked with differentiation.

3.2 Definite Integral as the Limit of a Sum

You have studied in std. XI that restoring force acting on spring-mass system is given by $F = -kx$, where k is force constant of the spring. If we consider only magnitude, we may consider $F = kx$. If $k = 10$, then $F = 10x$. Here we would find the work done, if displacement occurs due to the force. As per definition of work, work done by the system at a particular moment is,

w = Force acting at a particular moment × displacement due to force.

Now $F = 10x$ shows that force changes with displacement. So, how would we find the work done during the displacement of 10 units ?

As per a common estimate for work done during the displacement,

Initial force × displacement $\leq w \leq$ final force × displacement

Let us calculate w for the above mentioned example. First displacement occurs in [0, 10]. In this case for $x = 10$, force is maximum i.e. 100 units and for $x = 0$, it is minimum i.e. zero. So in this interval, work w satisfies,

$0 \times 0 \leq w \leq 100 \times 10$ **($w \times d = 0 \times 0$ and $w \times d = 100 \times 10$)**

$\therefore$ For work done in interval [0, 10], $0 \leq w \leq 1000$ **(i)**

Now to get a better estimate of work (w), let us divide the interval [0, 10] into two congruent subintervals i.e. [0, 5] and [5, 10]. Suppose in the interval [0, 5], the work done is w_1, then since maximum force is 50 units and minimum force is 0 unit in this interval, so for interval [0, 5], work done w_1 satisfies,

$$0 \le w_1 \le 50 \times 5$$

$$\therefore \quad 0 \le w_1 \le 250$$

Similarly, if the work done in the second interval is w_2, $250 \le w_2 \le 500$

$\therefore$ Total work done $w = w_1 + w_2$

$$250 \le w_1 + w_2 \le 750$$

$$\therefore \quad 250 \le w \le 750 \qquad \textbf{(ii)}$$

Here it can be seen that result (ii) gives a better estimate than result (i). If the interval [0, 10] is divided into three subintervals $\left[0, \frac{10}{3}\right], \left[\frac{10}{3}, \frac{20}{3}\right], \left[\frac{20}{3}, 10\right]$, work done in each interval would be as follows :

Taking $x = \frac{10}{3}$ in F = $10x$, we get maximum work $w = \frac{100}{3} \times \frac{10}{3} = \frac{1000}{9}$

$$0 \le w_1 \le \frac{1000}{9}$$

Similarly $\frac{1000}{9} \le w_2 \le \frac{2000}{9}$

and $\frac{2000}{9} \le w_3 \le \frac{3000}{9}$

As $w = w_1 + w_2 + w_3$, so, $\frac{3000}{9} \le w \le \frac{6000}{9}$

$$\therefore \quad 333\tfrac{1}{3} \le w \le 666\tfrac{2}{3} \qquad \textbf{(iii)}$$

It is seen that result (iii) is still a better estimate than result (ii). Thus more and more divisions of the intervals lead to better estimates of the work. If [0, 10] is divided into n equal intervals viz, $\left[0, \frac{10}{n}\right]$, $\left[\frac{10}{n}, \frac{20}{n}\right], \left[\frac{20}{n}, \frac{30}{n}\right], \ldots, \left[\frac{10(n-1)}{n}, 10\right]$.

ith interval in this partition would satisfy $\left[\frac{10(i-1)}{n}, \frac{10i}{n}\right]$.

Taking $x = \frac{10i}{n}$ in F = $10x$, we get maximum work $w = 10 \times \frac{10i}{n} \times \frac{10}{n} = \frac{1000i}{n^2}$

The work done in this subinterval would satisfy $\frac{1000(i-1)}{n^2} \le w_i \le \frac{1000i}{n^2}$

$\therefore$ Total work will satisfy $\frac{1000}{n^2}\sum_{i=1}^{n}(i-1) \le w \le \frac{1000}{n^2}\sum_{i=1}^{n} i$.

Here, the difference between the maximum and minimum values of work is

$$\frac{1000}{n^2}\sum_{i=1}^{n} i - \frac{1000}{n^2}\sum_{i=1}^{n}(i-1) = \frac{1000}{n^2}\sum_{i=1}^{n}(1) = \frac{1000}{n^2} \times n = \frac{1000}{n}$$

As value of n increases, this decreases and the difference tends to zero. In other words

$$\lim_{n\to\infty}\frac{1000}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{1000}{n^2}\sum_{i=1}^{n}(i-1)$$

Since the value of w lies between these two, as per sandwich theorem, true value of w will be the value of this limit.

$$\therefore \quad w = \lim_{n \to \infty} \frac{1000}{n^2} \sum_{i=1}^{n} i = \lim_{n \to \infty} \frac{1000}{n^2} \left(\frac{n(n+1)}{2}\right).$$

$$= \lim_{n \to \infty} 500 \left(1 + \frac{1}{n}\right) = 500$$

Thus $w = 500$ which is the correct value of work done. Thus we have carried out integration in the interval [0, 10] *w.r.t.* x, which is known as $\int_0^{10} f(x)\,dx = \int_0^{10} 10x\,dx$.

Here we are using the concept of the limit of a sequence. If (S_n) is a sequence and as n increases indefinitely $|S_n - l|$ becomes arbitrarily small for a definite real number l, we say the sequence is approching l as n tends to infinity and write $\lim_{n \to \infty} S_n = l$. We had intuitively seen this concept in the introduction of e in semester III. We will not study this concept in detail.

Generally, to evaluate $\int_a^b f(x)\,dx$, $[a, b]$ is divided into n congruent sub-intervals. Each interval will have length $h = \left(\frac{b-a}{n}\right)$. Now $[a, b]$ can be partitioned into $[a, a + h]$, $[a + h, a + 2h]$,..., $[a + (n - 1)h, a + nh]$.

$$\frac{b-a}{n} \sum_{i=1}^{n} f[a + (i - 1)h] \le \int_a^b f(x)\,dx \le \frac{b-a}{n} \sum_{i=1}^{n} f(a + ih)$$

and we can take $\int_a^b f(x)\,dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + ih)$. From these concepts and understanding, this conclusion will be accepted as a definition.

Definition : Let $f : [a, b] \to R$ be a continuous function. For positive interger n, let $h = \frac{b-a}{n}$. If we partition $[a, b]$ into n sub-intervals of equal length, then the dividing points are a, $a + h$, $a + 2h$,..., $a + nh = b$.

Figure 3.1

Let $S_n = \frac{b-a}{n} \sum_{i=1}^{n} f(a + ih)$

Thus we get a sequence $\{S_n\}$ based on function f and partition of $[a, b]$. We assume that for a continuous function, this sequence has a limit and this limit is called definite integral of f over $[a, b]$. It is denoted by $\int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} f(a + ih) \qquad \textbf{(i)}$$

a is called the lower limit and b is called the upper limit of definite integration.

Also, we can prove that $\lim_{n \to \infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f(a + ih)$ is also equal to $\int_a^b f(x)\,dx$.

Above definition is called the definition of definite integral as the limit of a sum. The above process of linking a function f with its definite integral is called evaluation of definite integral as a limit of a sum.

Note : $\int_a^b f(x)\,dx$ can be defined for certain functions which may not be continuous. But at present we will not discuss them.

Symbol :

Integrand ↓

Upper limit of integration → b

$\int_a^b f(x)\,dx$ ← dx suggests integration is carried out *w.r.t.* x.

Lower limit of integration → a

Integration of f from a to b.

3.3 Some Important Results

(1) $1 + 2 + 3 + ... + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(2) $1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $1^3 + 2^3 + 3^3 + ... + n^3 = \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

(4) $a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1} \quad (r \neq 1)$

(5) Let $S_n = sin(a + h) + sin(a + 2h) + ... + sin(a + nh)$, where $h \neq 2n\pi$. $n \in Z$

To find this sum let us multiply both sides by $2sin\frac{h}{2}$. So, we have

$$2sin\frac{h}{2} \cdot S_n = \left[2sin(a + h)\,sin\frac{h}{2} + 2sin(a + 2h)\,sin\frac{h}{2} + 2sin(a + 3h)\,sin\frac{h}{2} + ... + 2sin(a + nh)\,sin\frac{h}{2}\right]$$

$$= \left[cos\left(a + \frac{h}{2}\right) - cos\left(a + \frac{3h}{2}\right)\right] + \left[cos\left(a + \frac{3h}{2}\right) - cos\left(a + \frac{5h}{2}\right)\right] + \left[cos\left(a + \frac{5h}{2}\right) - cos\left(a + \frac{7h}{2}\right)\right] + ... + \left[cos\left(a + nh - \frac{h}{2}\right) - cos\left(a + nh + \frac{h}{2}\right)\right]$$

$$2sin\frac{h}{2} \cdot S_n = \left[cos\left(a + \frac{h}{2}\right) - cos\left(a + nh + \frac{h}{2}\right)\right]$$

$$\therefore \quad S_n = \frac{cos\left(a + \frac{h}{2}\right) - cos\left(a + nh + \frac{h}{2}\right)}{2sin\frac{h}{2}} \qquad \left(sin\frac{h}{2} \neq 0\right)$$

If $h = 2n\pi$, $S_n = n\,sin\,na$

(6) Let $S_n = cos(a + h) + cos(a + 2h) + cos(a + 3h) + ... + cos(a + nh)$, where $h \neq 2n\pi$. $n \in Z$

To find this sum let us multiply both the sides by $2sin\frac{h}{2}$. So, we have

$$2sin\frac{h}{2} \cdot S_n = \left[2cos(a + h)\,sin\frac{h}{2} + 2cos(a + 2h)\,sin\frac{h}{2} + 2cos(a + 3h)\,sin\frac{h}{2} + ... + 2cos(a + nh)\,sin\frac{h}{2}\right]$$

$$= \left[\sin\left(a + \frac{3h}{2}\right) - \sin\left(a + \frac{h}{2}\right)\right] + \left[\sin\left(a + \frac{5h}{2}\right) - \sin\left(a + \frac{3h}{2}\right)\right] + \left[\sin\left(a + \frac{7h}{2}\right) - \sin\left(a + \frac{5h}{2}\right)\right] + ... + \left[\sin\left(a + nh + \frac{h}{2}\right) - \sin\left(a + nh - \frac{h}{2}\right)\right]$$

$$2\sin\frac{h}{2} \cdot S_n = \left[\sin\left(a + nh + \frac{h}{2}\right) - \sin\left(a + \frac{h}{2}\right)\right]$$

$$\therefore \quad S_n = \frac{\sin\left(a + nh + \frac{h}{2}\right) - \sin\left(a + \frac{h}{2}\right)}{2\sin\frac{h}{2}} \qquad \left(\sin\frac{h}{2} \neq 0\right)$$

If $h = 2n\pi$, $S_n = n\cos na$

Example 1 : Obtain $\int_1^3 x\,dx$ as the limit of a sum.

Solution : Here, $f(x) = x$ is continuous on [1, 3]. Divide [1, 3] into n congruent sub-intervals and the length of each sub-interval is given by $h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$.

Here, $a = 1$, $b = 3$ and $f(a + ih) = f(1 + ih) = 1 + ih$

According to the definition,

$$\int_1^3 x\,dx = \lim_{n \to \infty} h \sum_{i=1}^{n} f(a + ih)$$

$$= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} f(1 + ih)$$

$$= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} (1 + ih)$$

$$= \lim_{n \to \infty} \frac{2}{n} \left[\sum_{i=1}^{n} 1 + h \sum_{i=1}^{n} i\right]$$

$$= \lim_{n \to \infty} \frac{2}{n} \left[n + \frac{2}{n} \frac{n(n+1)}{2}\right]$$

$$= \lim_{n \to \infty} \left[2 + 2\left(1 + \frac{1}{n}\right)\right]$$

$$= 2 + 2(1 + 0)$$

$$= 4$$

Example 2 : Obtain $\int_0^2 (3x^2 - 2x + 4)dx$ as the limit of a sum.

Solution : Here, $f(x) = 3x^2 - 2x + 4$ is continuous on [0, 2]. Divide [0, 2] into n congruent sub-intervals and the length of each sub-interval is given by $h = \frac{b-a}{n}$.

$$\therefore \quad h = \frac{2-0}{n} = \frac{2}{n}$$

$$\therefore \quad h = \frac{2}{n}$$

Here $a = 0$, $b = 2$, $f(x) = 3x^2 - 2x + 4$

$f(a + ih) = f(0 + ih)$
$= f(ih)$
$= 3i^2h^2 - 2ih + 4$

According to the definition,

$$\int_0^2 (3x^2 - 2x + 4)dx = \lim_{n \to \infty} h \sum_{i=1}^{n} f(a + ih)$$
$$= \lim_{n \to \infty} \frac{2}{n} \sum_{i=1}^{n} (3i^2h^2 - 2ih + 4)$$
$$= \lim_{n \to \infty} \frac{2}{n} \left[3h^2 \sum_{i=1}^{n} i^2 - 2h \sum_{i=1}^{n} i + \sum_{4=1}^{n} 1\right]$$
$$= \lim_{n \to \infty} \frac{2}{n} \left[3 \cdot \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} - 2 \cdot \frac{2}{n} \frac{n(n+1)}{2} + 4n\right]$$
$$= \lim_{n \to \infty} \left[4\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 4\left(1 + \frac{1}{n}\right) + 8\right]$$
$$= 4(1 + 0)(2 + 0) - 4(1 + 0) + 8$$
$$= 8 - 4 + 8$$
$$= 12$$

Example 3 : Obtain $\int_{-1}^{1} a^x\, dx$ as the limit of a sum. $(a > 0)$

Solution : Here, $f(x) = a^x$ is continuous on $[-1, 1]$. Divide $[-1, 1]$ into n congruent sub-intervals. Length of each sub-interval is $h = \frac{b-a}{n} = \frac{1+1}{n} = \frac{2}{n}$. So $nh = 2$.

Here, $a = -1$, $b = 1$, $f(x) = a^x$

$f(a + ih) = f(-1 + ih)$
$= a^{-1 + ih}$
$= a^{-1} \cdot a^{ih}$

$f(a + ih) = \frac{a^{ih}}{a}$

As $n \to \infty$, $h \to 0$

$$\text{Now, } \int_{-1}^{1} a^x\, dx = \lim_{h \to 0} h \sum_{i=1}^{n} f(a + ih)$$
$$= \lim_{h \to 0} h \sum_{i=1}^{n} \frac{a^{ih}}{a}$$
$$= \lim_{h \to 0} \frac{h}{a} [a^h + a^{2h} + a^{3h} + \ldots + a^{nh}]$$
$$= \lim_{h \to 0} \frac{h}{a} \left[\frac{a^h(a^{nh} - 1)}{a^h - 1}\right]$$
$$= \lim_{h \to 0} \frac{1}{a} \frac{a^h(a^2 - 1)}{\left(\frac{a^h - 1}{h}\right)} \qquad (nh = 2)$$

$$= \frac{1}{a} \cdot \frac{a^0(a^2 - 1)}{\log_e a}$$

$$= \left(\frac{a^2 - 1}{a}\right) \log_a e$$

$$= \left(a - \frac{1}{a}\right) \log_a e$$

Example 4 : Obtain $\int_a^b sin\, x\, dx$ as the limit of a sum.

Solution : Here, $f(x) = sin\, x$ is a continuous function on $[a, b]$. Divide $[a, b]$ into n congruent sub-intervals. Length of each sub-interval is $h = \frac{b - a}{n}$.

$\therefore \quad nh = b - a, \; a + nh = b$

Also $f(a + ih) = sin\,(a + ih)$

As $n \to \infty$, $h \to 0$.

$$\text{Now, } \int_a^b sin\, x\, dx = \lim_{h \to 0} h \sum_{i=1}^{n} f(a + ih)$$

$$= \lim_{h \to 0} h \sum_{i=1}^{n} sin\,(a + ih)$$

$$= \lim_{h \to 0} h\, [sin(a + h) + sin(a + 2h) + sin(a + 3h) + ... + sin(a + nh)]$$

$$= \lim_{h \to 0} h \left[\frac{cos\left(a + \frac{h}{2}\right) - cos\left(a + nh + \frac{h}{2}\right)}{2 sin \frac{h}{2}}\right]$$

$$= \lim_{h \to 0} \frac{cos\left(a + \frac{h}{2}\right) - cos\left(b + \frac{h}{2}\right)}{\left(\frac{sin \frac{h}{2}}{\frac{h}{2}}\right)} \qquad (a + nh = b)$$

$$= \frac{cos\, a - cos\, b}{1} \qquad \text{(as } cosine \text{ is continuous)}$$

$$= cosa - cosb$$

Note : Since $h \to 0$, we can have $|h| < 2\pi < 2|k|\pi$, $k \in Z - \{0\}$.

Exercise 3.1

Obtain the following definite integrals as the limit of a sum :

1. $\int_0^2 (x + 3)dx$ **2.** $\int_2^4 (2x - 1)dx$ **3.** $\int_1^3 (2x^2 + 7)dx$

4. $\int_1^3 (x^2 + x)dx$ **5.** $\int_{-1}^1 e^x\, dx$ **6.** $\int_0^1 e^{2 - 3x}\, dx$

7. $\int_1^2 3^x\, dx$ **8.** $\int_{\log_e 2}^{\log_e 5} e^x\, dx$ **9.** $\int_0^2 (e^x - x)\, dx$

10. $\int_{\log_a 2}^{\log_a 4} a^x \, dx$ **11.** $\int_0^2 (6x^2 - 2x + 7) \, dx$ **12.** $\int_a^b \cos x \, dx$

13. $\int_0^{\pi} \sin x \, dx$ **14.** $\int_0^{\frac{\pi}{2}} \cos x \, dx$ **15.** $\int_1^3 x^3 \, dx$

*

3.4 Fundamental Principle of Definite Integration

From what we have learnt, we can definitely say that to obtain definite integral as the limit of a sum is not so simple. In fact it is tedious. We will see that this task becomes very simple using fundamental principle of definite integration.

The following principle is called **fundamental principle of definite integration.**

Principle : If function f is continuous on $[a, b]$ and F is a differentiable function on (a, b) such that $\forall x \in (a, b)$, $\frac{d}{dx}[F(x)] = f(x)$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Here, $F(x)$ is a primitive of $f(x)$. $F(b) - F(a)$ is expressed as $[F(x)]_a^b$.

With the help of this result, we can obtain definite integral by taking difference of values of its primitive at the end-points of given interval. **Newton** and **Leibnitz** independently obtained this result. This principle establishes a relation between the process of differentiation and integration. This result is accepted without proof.

Note : (1) Here $\forall x \in (a, b)$, $\frac{d}{dx}[F(x)] = f(x)$.

So, $\int f(x) \, dx = F(x) + c$, where c is an arbitrary constant.

But $\int_a^b f(x) \, dx = [F(x) + c]_a^b$

$= [F(b) + c] - [F(a) + c]$

$= F(b) + c - F(a) - c$

$= F(b) - F(a)$

Thus, in definite integration arbitrary constant is eliminated and we get the definite value of integral.

$\therefore$ Definite integral is a finite definite real number. Hence the process of obtaining such an integral is called definite integration.

(2) If $a > b$, then we define $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$

Also, we will accept that for $a = b$,

$$\int_a^b f(x) \, dx = \int_a^a f(x) \, dx = 0$$

(3) $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$, where f is continuous on $[a, b]$.

Let F(x) be a primitive of $f(x)$. Then by fundamental principle of definite integration,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a) \text{ and}$$

$$\int_a^b f(t)\,dt = [F(t)]_a^b = F(b) - F(a).$$

Hence, $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

Thus, the value of definite integral does not depend upon variable with respect to which integration is carried out.

Earlier in this chapter, we have learnt how to obtain value of definite integral as the limit of a sum. Now we will see how easily we can obtain the value of definite integral using the fundamental principle of definite integration.

Now, we will review examples 1 to 4 using the fundamental principle of definite integration.

(1) $\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3 = \left[\frac{3^2}{2} - \frac{1^2}{2}\right] = \left[\frac{9}{2} - \frac{1}{2}\right] = \frac{8}{2} = 4$

(2) $\int_0^2 (3x^2 - 2x + 4)\,dx = \left[\frac{3x^3}{3} - \frac{2x^2}{2} + 4x\right]_0^2 = [8 - 4 + 8] = 12$

(3) $\int_{-1}^1 a^x\,dx = \left[\frac{a^x}{\log_e a}\right]_{-1}^1 = \frac{1}{\log_e a}(a^1 - a^{-1}) = \left(a - \frac{1}{a}\right)\log_a e.$

(4) $\int_a^b \sin x\,dx = [-\cos x]_a^b = -[\cos b - \cos a] = \cos a - \cos b$

3.5 Working Rules of Definite Integration

(1) If functions f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof : Let F(x) and G(x) be primitives of $f(x)$ and $g(x)$ respectively on $[a, b]$.

∴ F(x) + G(x) is a primitive $f(x) + g(x)$.

∴ According to the fundamental principle of definite integration,

$$\begin{aligned}\int_a^b [f(x) + g(x)]\,dx &= [F(x) + G(x)]_a^b \\ &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

(2) If f is continuous on $[a, b]$ and k is a constant, then $\int_a^b k f(x)\, dx = k \int_a^b f(x)\, dx$.

Proof : Let F(x) be a primitive of $f(x)$ on $[a, b]$ and k is any constant,

$\therefore$ kF(x) is a primitive of $kf(x)$.

$\therefore$ According to the fundamental principle of definite integration.

$$\int_a^b k f(x)\, dx = [k\text{F}(x)]_a^b$$

$$= k\text{F}(b) - k\text{F}(a)$$

$$= k[\text{F}(b) - \text{F}(a)]$$

$$= k \int_a^b f(x)\, dx$$

(3) If function f is continuous on $[a, b]$ and $a < c < b$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

Proof : Let F(x) be a primitive of $f(x)$ over $[a, b]$. Then by the fundamental principle of definite integration,

$$\int_a^b f(x)\, dx = [\text{F}(x)]_a^b = \text{F}(b) - \text{F}(a)$$

$$\int_a^c f(x)\, dx = [\text{F}(x)]_a^c = \text{F}(c) - \text{F}(a)$$

$$\int_c^b f(x)\, dx = [\text{F}(x)]_c^b = \text{F}(b) - \text{F}(c)$$

Now, $\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \text{F}(c) - \text{F}(a) + \text{F}(b) - \text{F}(c)$

$$= \text{F}(b) - \text{F}(a)$$

$$= \int_a^b f(x)\, dx$$

Thus, if $a < c < b$, then $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

The same result holds for any finite partition of $[a, b]$. For instance, if $a < c < d < b$, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^d f(x)\, dx + \int_d^b f(x)\, dx.$$

Even if c is not in between a and b, and $a < c$ and f is continuous on $[a, c]$, then also this result is true. If $a < b < c$, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 5 : Evaluate : **(1)** $\int_0^{\frac{\pi}{2}} cos^3 x \; dx$ **(2)** $\int_0^{\frac{\pi}{4}} \sqrt{1 - sin\,2x} \; dx$

Solution : (1) $I = \int_0^{\frac{\pi}{2}} cos^3 x \; dx$

$$= \int_0^{\frac{\pi}{2}} \frac{cos3x + 3cos\,x}{4} \, dx$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{2}} (cos\ 3x + 3cos\ x) \; dx$$

$$= \frac{1}{4} \left[\frac{sin\,3x}{3} + 3sin\ x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4} \left[\left(\frac{1}{3}sin\ \frac{3\pi}{2} + 3sin\ \frac{\pi}{2}\right) - \left(\frac{1}{3}sin\ 0 + 3sin\ 0\right)\right]$$

$$= \frac{1}{4} \left[\left(-\frac{1}{3} + 3\right) - (0 + 0)\right]$$

$$= \frac{1}{4} \left(\frac{8}{3}\right) = \frac{2}{3}$$

(2) $I = \int_0^{\frac{\pi}{4}} \sqrt{1 - sin\,2x} \; dx$

$$= \int_0^{\frac{\pi}{4}} \sqrt{sin^2 x + cos^2 x - 2\,sinx\,cosx} \; dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{(cos\,x - sin\,x)^2} \; dx$$

$$= \int_0^{\frac{\pi}{4}} |\,cos\ x - sin\ x\,| \; dx$$

$$= \int_0^{\frac{\pi}{4}} (cos\ x - sin\ x) \; dx \qquad \left(\text{As, } 0 < x < \frac{\pi}{4};\ \ cos\ x > sin\ x\right)$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$

$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1) = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

Example 6 : Evaluate : **(1)** $\int_0^3 \frac{1}{\sqrt{x^2+2x+3}}\,dx$ **(2)** $\int_0^2 \frac{5x+2}{x^2+4}\,dx$

Solution : (1) $I = \int_0^3 \frac{1}{\sqrt{x^2+2x+3}}\,dx$

$$= \int_0^3 \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}\,dx$$

$$= \left[\log\left|x + 1 + \sqrt{(x+1)^2+(\sqrt{2})^2}\right|\right]_0^3$$

$$= \left[\log\left(x + 1 + \sqrt{x^2+2x+3}\right)\right]_0^3 \qquad (x \in (0, 3))$$

$$= \log\left(4 + \sqrt{9+6+3}\right) - \log\left(1 + \sqrt{3}\right)$$

$$= \log\left(4 + 3\sqrt{2}\right) - \log\left(\sqrt{3} + 1\right)$$

$$= \log\left(\frac{4+3\sqrt{2}}{\sqrt{3}+1}\right)$$

(2) $I = \int_0^2 \frac{5x+2}{x^2+4}\,dx$

$$= \int_0^2 \frac{5x}{x^2+4}\,dx + \int_0^2 \frac{2}{x^2+4}\,dx$$

$$= \frac{5}{2}\int_0^2 \frac{2x}{x^2+4}\,dx + 2\int_0^2 \frac{1}{x^2+2^2}\,dx$$

$$= \frac{5}{2}\left[\log(x^2+4)\right]_0^2 + \frac{2}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= \frac{5}{2}\left[\log 8 - \log 4\right] + \left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{5}{2}\log\left(\frac{8}{4}\right) + \left[\frac{\pi}{4} - 0\right]$$

$$= \left(\frac{5}{2}\log 2 + \frac{\pi}{4}\right)$$

Example 7 : Evaluate : $\int_0^{2\pi} f(x)\,dx$, where $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 1+\cos x, & \pi \le x \le 2\pi \end{cases}$

Solution : (1) $\int_0^{2\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx$

$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (1 + \cos x)\,dx$

$= [-\cos x]_0^{\pi} + [x + \sin x]_{\pi}^{2\pi}$

$= -[\cos\pi - \cos 0] + [(2\pi + \sin 2\pi) - (\pi + \sin \pi)]$

$= -[-1 - 1] + [(2\pi + 0) - (\pi + 0)]$

$= 2 + \pi = \pi + 2$

3.6 Method of Substitution for Definite Integration

We have learnt the method of substitution for indefinite integration. We have seen that if the integrand is not in standard form, then the method of substitution is very useful to obtain certain integrals.

We can use it in combination with the fundamental principle for definite integration. Let us see the rule of substitution for definite integration.

Rule of substitution for definite integration :

Let $f : [a, b] \to R$ be a continuous function and $g : [\alpha, \beta] \to [a, b]$ be increasing or decreasing (monotonic) function. $x = g(t)$ is continuous in $[\alpha, \beta]$ and differentiable in (α, β). $g'(t)$ is continuous in (α, β) and $g'(t) \ne 0$, $\forall\, t \in (\alpha, \beta)$. $a = g(\alpha)$ and $b = g(\beta)$.

Then, $\int_a^b f(x)\,dx = \int_{\alpha}^{\beta} f(g(t))\,g'(t)\,dt$

Let us understand this method by some illustrations.

Example 8 : Evaluate : **(1)** $\int_1^9 \frac{dx}{x+\sqrt{x}}$ **(2)** $\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$ **(3)** $\int_0^{\frac{\pi}{2}} \frac{\sin 2t}{\sin^4 t + \cos^4 t}\,dt$

Solution : (1) $I = \int_1^9 \frac{dx}{x+\sqrt{x}}$

Let $x = t^2$ $(t \ge 1)$, $dx = 2t\,dt$

When, $x = 1$, $t = 1$ since $x = t^2$ and $t \ge 1$ and if $x = 9$, $t = 3$ as $x = t^2$

$x = g(t) = t^2$ is increasing for $t \ge 1$. It is continuous in $[1, 3]$ and differentiable in $(1, 3)$.

$g'(t) = 2t \ne 0$ in $(1, 3)$

$\therefore$ $\alpha = 1$, $\beta = 3$

$$\therefore \quad I = \int_1^9 \frac{dx}{x+\sqrt{x}}$$

$$= \int_1^3 \frac{2t\,dt}{t^2+t} \qquad (\sqrt{x} = t \geq 1 \text{ as } t \geq 1)$$

$$= 2\int_1^3 \frac{1}{t+1}\,dt \qquad (t \neq 0)$$

$$= 2[\log (t+1)]_1^3$$

$$= 2[\log 4 - \log 2]$$

$$= 2 \log 2$$

(2) $$I = \int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$$

Let $tan\,\frac{x}{2} = t$, $dx = \frac{2dt}{1+t^2}$, $cos\,x = \frac{1-t^2}{1+t^2}$, $sin\,x = \frac{2t}{1+t^2}$

When, $x = 0$, $t = tan\,0 = 0$ and when $x = \frac{\pi}{2}$, $t = tan\,\frac{\pi}{4} = 1$ $\quad (\alpha = 0, \beta = 1)$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x + 4\sin x}$$

$$= \int_0^1 \frac{1}{2\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{dt}{1-t^2+4t}$$

$$= \int_0^1 \frac{dt}{5-(t^2-4t+4)}$$

$$= \int_0^1 \frac{dt}{(\sqrt{5})^2-(t-2)^2}$$

$$= \frac{1}{2\sqrt{5}}\left[\log\left|\frac{\sqrt{5}+(t-2)}{\sqrt{5}-(t-2)}\right|\right]_0^1$$

$$= \frac{1}{2\sqrt{5}}\left[\log\left|\frac{\sqrt{5}-1}{\sqrt{5}+1}\right| - \log\left|\frac{\sqrt{5}-2}{\sqrt{5}+2}\right|\right]$$

$$= \frac{1}{2\sqrt{5}}\log\left(\frac{\sqrt{5}-1}{\sqrt{5}+1} \times \frac{\sqrt{5}+2}{\sqrt{5}-2}\right) \qquad (\sqrt{5} - 2 > 0)$$

$$= \frac{1}{2\sqrt{5}} \log \left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right)$$

$$= \frac{1}{2\sqrt{5}} \log \left(\frac{3+\sqrt{5}}{3-\sqrt{5}} \times \frac{3+\sqrt{5}}{3+\sqrt{5}}\right)$$

$$= \frac{1}{2\sqrt{5}} \log \left(\frac{3+\sqrt{5}}{2}\right)^2$$

$$= \frac{1}{\sqrt{5}} \log \left(\frac{3+\sqrt{5}}{2}\right)$$

Note : $I = \frac{1}{\sqrt{5}} \log \left(\frac{6+2\sqrt{5}}{4}\right)$

$$= \frac{1}{\sqrt{5}} \log \left(\frac{\sqrt{5}+1}{2}\right)^2$$

$$= \frac{2}{\sqrt{5}} \log \left(\frac{\sqrt{5}+1}{2}\right)$$

(3) $I = \int_0^{\frac{\pi}{2}} \frac{\sin 2t}{\sin^4 t + \cos^4 t} \, dt$

Let $\sin^2 t = x$, $2\sin t \cos t \, dt = dx$. So $\sin 2t \, dt = dx$

When, $t = 0$, $x = 0$ and when $t = \frac{\pi}{2}$, $x = 1$ **($\alpha = 0, \beta = 1$)**

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\sin 2t}{\sin^4 t + \cos^4 t} \, dt$$

$$= \int_0^1 \frac{dx}{x^2 + (1-x)^2}$$

$$= \int_0^1 \frac{dx}{2x^2 - 2x + 1}$$

$$= \frac{1}{2} \int_0^1 \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2} \left[2 \tan^{-1} \left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$= [\tan^{-1}(2x - 1)]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

3.7 Method of Integration by Parts for Definite Integration

We have studied the method of integration by parts to obtain indefinite integral of product of two functions. We can also use integration by parts for definite integration.

To use method of integration by parts in definite integration, we use following formula.

$f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ all are continuous on $[a, b]$.

$$\int_a^b f(x)\ g'(x)\,dx = [f(x)\ g(x)]_a^b - \int_a^b f'(x)\ g(x)\,dx$$

$$\therefore \int_a^b f(x)\ g'(x)\,dx = [f(b)\ g(b) - f(a)\ g(a)] - \int_a^b f'(x)\ g(x)\,dx$$

Now, we will understand this method by some examples.

Example 9 : Evaluate : **(1)** $\int_0^1 x\,tan^{-1}x\,dx$ **(2)** $\int_0^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}\,dx$ **(3)** $\int_0^1 \frac{x\,dx}{(1+x^2)(2+x^2)}$

Solution : (1) $I = \int_0^1 x\,tan^{-1}x\,dx$

$$= \left[tan^{-1}x \cdot \frac{x^2}{2}\right]_0^1 - \int_0^1 \left(\frac{1}{1+x^2} \cdot \frac{x^2}{2}\right) dx$$

$$= \left(tan^{-1}(1) \cdot \frac{1}{2} - 0\right) - \frac{1}{2}\int_0^1 \frac{x^2}{x^2+1}\,dx$$

$$= \left(\frac{\pi}{4} \cdot \frac{1}{2} - 0\right) - \frac{1}{2}\int_0^1 \frac{(x^2+1)-(1)}{x^2+1}\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{x^2+1}\right) dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\,[x - tan^{-1}x]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2}\,[(1 - tan^{-1}1) - (0 - tan^{-1}0)]$$

$$= \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

(2) $I = \int_0^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}} dx$

Let $sin^{-1}x = t$, $x = sin\, t$, $dx = cos\, t\, dt$, $x \in \left[0, \frac{1}{\sqrt{2}}\right]$, $t \in \left[0, \frac{\pi}{4}\right]$

When, $x = 0$, $t = sin^{-1}0 = 0$ and when $x = \frac{1}{\sqrt{2}}$, $t = sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$ $\left(\alpha = 0, \beta = \frac{\pi}{4}\right)$

$$\therefore \quad I = \int_0^{\frac{1}{\sqrt{2}}} \frac{sin^{-1}x}{(1-x^2)^{\frac{3}{2}}} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{t}{(1-sin^2 t)^{\frac{3}{2}}} \cdot cost\ dt$$

$$= \int_0^{\frac{\pi}{4}} t\, sec^2 t\, dt \qquad \left(cos\, t > 0 \text{ in } \left[0, \frac{\pi}{4}\right]\right)$$

$$= [t \cdot tan\, t]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} tan\, t\, dt$$

$$= [t \cdot tan\, t]_0^{\frac{\pi}{4}} + [\log |cos\, t|]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} tan\frac{\pi}{4} - 0\right) + \left[\log\left(cos\frac{\pi}{4}\right) - \log(cos\, 0)\right]$$

$$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} - \log 1$$

$$= \frac{\pi}{4} - \frac{1}{2}\log 2$$

(3) $I = \int_0^1 \frac{x\, dx}{(1+x^2)(2+x^2)}$

For $x \geq 0$, let $x^2 = t$, $2x\ dx = dt$. So $x\, dx = \frac{1}{2}\ dt$

When, $x = 0$, $t = 0$ and when $x = 1$, $t = 1$

$$\therefore \quad I = \int_0^1 \frac{x\, dx}{(1+x^2)(2+x^2)} = \frac{1}{2}\int_0^1 \frac{dt}{(t+1)(t+2)}$$

Now let $\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

$\therefore$ $1 = A(t+2) + B(t+1)$

If $t = -2$, $1 = -B$. So $B = -1$

If $t = -1$, $1 = A$. So $A = 1$

$\therefore$ $\frac{1}{(t+1)(t+2)} = \frac{1}{t+1} + \frac{-1}{t+2}$

$$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{dt}{(t+1)(t+2)} = \frac{1}{2}\int_0^1 \left(\frac{1}{t+1} + \frac{-1}{t+2}\right) dt$$

$$= \frac{1}{2}\left[\log|t+1| - \log|t+2|\right]_0^1$$

$$= \frac{1}{2}\left[\log\left|\frac{t+1}{t+2}\right|\right]_0^1$$

$$= \frac{1}{2}\left[\log\frac{2}{3} - \log\frac{1}{2}\right]$$

$$= \frac{1}{2}\log\left(\frac{4}{3}\right)$$

Example 10 : Evaluate : $\int_0^{2\pi} \sin ax \cdot \sin bx \, dx, \quad a, b \in N$

Solution : $I = \int_0^{2\pi} \sin ax \cdot \sin bx \, dx$

Case 1 : $a \neq b$

$$I = \frac{1}{2}\int_0^{2\pi} 2\sin ax \cdot \sin bx \, dx$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(a-b)x - \cos(a+b)x] \, dx$$

$$= \frac{1}{2}\left[\frac{\sin(a-b)x}{a-b} - \frac{\sin(a+b)x}{a+b}\right]_0^{2\pi}$$ **($a \neq b$ and $a + b \neq 0$ as $a, b \in N$)**

$$= \frac{1}{2}(0 - 0)$$ **(Why ?)**

$\therefore$ $I = 0$

Case 2 : $a = b$

$$I = \int_0^{2\pi} \sin^2 ax \, dx$$

$$= \int_0^{2\pi} \left(\frac{1-\cos 2ax}{2}\right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2ax}{2a}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[\left(2\pi - \frac{\sin 4\pi a}{2a}\right) - (0 - 0)\right]$$

$$= \frac{1}{2}(2\pi)$$ **(Why $\sin 4\pi a = 0$?)**

$\therefore \quad I = \pi$

$$\therefore \int_0^{2\pi} \sin ax \cdot \sin bx \; dx = \begin{cases} 0 & \text{if } a \neq b \\ \pi & \text{if } a = b \end{cases}$$

Example 11 : For $\alpha > 0$, if $f(x + \alpha) = f(x)$, $\forall x \in$ R i.e. if f has period α, prove that $\int_0^{n\alpha} f(x)\,dx = n\int_0^{\alpha} f(x)\,dx$, where $n \in$ N and hence obtain $\int_0^{10\pi} |\sin x|\; dx$.

Solution : $I = \int_0^{n\alpha} f(x)\,dx$, $n \in$ N

$$= \int_0^{\alpha} f(x)\,dx + \int_{\alpha}^{2\alpha} f(x)\,dx + \int_{2\alpha}^{3\alpha} f(x)\,dx + ... + \int_{k\alpha}^{(k+1)\alpha} f(x)\,dx + ... + \int_{(n-1)\alpha}^{n\alpha} f(x)\,dx$$

We shall prove that each of these integrals is equal to $\int_0^{\alpha} f(x)\,dx$

Now let $I_k = \int_{k\alpha}^{(k+1)\alpha} f(x)\,dx$ **[k = 1, 2,..., (n − 1)]**

Let $x = k\alpha + t$, $dx = dt$

When $x = k\alpha$, $t = 0$ and $x = (k + 1)\alpha$, $t = \alpha$

$\therefore \quad I_k = \int_0^{\alpha} f(k\alpha + t)\,dt$

If α is a period of f, then $k\alpha$ are periods of f. **($k \in$ N)**

$\therefore \quad f(k\alpha + t) = f(t)$

$\therefore \quad I_k = \int_0^{\alpha} f(t)\,dt = \int_0^{\alpha} f(x)\,dx$

i.e. $\int_{k\alpha}^{(k+1)\alpha} f(x)\,dx = \int_0^{\alpha} f(x)\,dx$ **[k = 1, 2, 3,..., n − 1]**

$\therefore \quad I = \int_0^{\alpha} f(x)\,dx + \int_0^{\alpha} f(x)\,dx + ... + \int_0^{\alpha} f(x)\,dx$ (n times)

$$= n\int_0^{\alpha} f(x)\,dx$$

Now $I = \int_0^{10\pi} |\sin x| \, dx.$

$= 10 \int_0^{\pi} |\sin x| \, dx$ **(period of $|\sin x|$ is π)**

$= 10 \int_0^{\pi} \sin x \, dx$ **(for $0 \le x \le \pi, \sin x \ge 0$)**

$= 10 [-\cos x]_0^{\pi}$

$= -10 [\cos\pi - \cos 0]$

$= -10 (-1 - 1)$

$= -10 (-2)$

$= 20$

Example 12 : Evaluate $\int_{-1}^{3} |2x - 1| \, dx$

Solution : $2x - 1 \ge 0 \Leftrightarrow x \ge \frac{1}{2}$

$\therefore \quad |2x - 1| = \begin{cases} 2x - 1 & x \ge \frac{1}{2} \\ 1 - 2x & x < \frac{1}{2} \end{cases}$

Now $-1 < \frac{1}{2} < 3$

$\therefore \quad I = \int_{-1}^{3} |2x - 1| \, dx$

$= \int_{-1}^{\frac{1}{2}} |2x - 1| \, dx + \int_{\frac{1}{2}}^{3} |2x - 1| \, dx$

$= \int_{-1}^{\frac{1}{2}} (1 - 2x) \, dx + \int_{\frac{1}{2}}^{3} (2x - 1) \, dx$

$= [x - x^2]_{-1}^{\frac{1}{2}} + [x^2 - x]_{\frac{1}{2}}^{3}$

$= \left[\left(\frac{1}{2} - \frac{1}{4}\right) - (-1 - 1)\right] + \left[(9 - 3) - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$

$= \left(\frac{1}{4} + 2\right) + \left(6 + \frac{1}{4}\right)$

$= \frac{17}{2}$

Exercise 3.2

Evaluate (1 to 35) :

1. $\int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}}\,dx$ **2.** $\int_0^{\frac{\pi}{4}} tan^2x\;dx$ **3.** $\int_0^{\frac{\pi}{2}} sin^2x\;dx$

4. $\int_0^{\frac{\pi}{4}} tan\,x\;dx$ **5.** $\int_0^{\frac{\pi}{2}} \sqrt{1-cos\,2x}\;dx$ **6.** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1-sin\,2x}\;dx$

7. $\int_0^{\sqrt{2}} \sqrt{2-x^2}\;dx$ **8.** $\int_2^5 \frac{2x}{5x^2+1}\,dx$ **9.** $\int_0^1 \frac{2x+3}{5x^2+1}\,dx$

10. $\int_0^{\frac{\pi}{2}} \frac{sin^2x}{(1+cosx)^2}\,dx$ **11.** $\int_0^1 \frac{dx}{x^2+x+1}$ **12.** $\int_0^9 \frac{dx}{1+\sqrt{x}}$

13. $\int_0^{\frac{\pi}{4}} \frac{cosx}{\sqrt{2-sin^2x}}\,dx$ **14.** $\int_0^1 \frac{dx}{e^x+e^{-x}}$ **15.** $\int_0^1 tan^{-1}x\;dx$

16. $\int_0^4 \frac{dx}{\sqrt{12+4x-x^2}}$ **17.** $\int_0^{\frac{\pi}{2}} x^2cos\,2x\,dx$ **18.** $\int_0^{\frac{1}{2}} \frac{xsin^{-1}x}{\sqrt{1-x^2}}\,dx$

19. $\int_0^{\frac{\pi}{2}} \frac{dx}{3+2sinx+cosx}$ **20.** $\int_0^{\frac{\pi}{4}} \frac{dx}{2+3cos^2x}$ **21.** $\int_0^1 sin^{-1}\sqrt{\frac{x}{x+1}}\;dx$

22. $\int_0^1 \sqrt{\frac{1-x}{1+x}}\;dx$ **23.** $\int_0^{\frac{\pi}{2}} \frac{cos\,x}{(1+sinx)(2+sinx)(3+sinx)}\,dx$

24. $\int_0^{\frac{\pi}{4}} \frac{sin\,x+cos\,x}{1+cos2x}\,dx$ **25.** $\int_1^2 \frac{1}{x(1+x^2)}\,dx$ **26.** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos\,2x \cdot \log\;sin\,x\;dx$

27. $\int_0^{\frac{\pi}{2}} \frac{1}{3+2cosx}\,dx$ **28.** $\int_0^{\frac{\pi}{4}} \frac{1}{4sin^2x+5cos^2x}\,dx$ **29.** $\int_0^{2\pi} |\,cosx\,|\;dx$

30. $\int_1^4 f(x)\,dx$, where $f(x) = \begin{cases} 2x+8 & 1 \le x \le 2 \\ 6x & 2 < x \le 4 \end{cases}$

31. $\int_0^9 f(x)\,dx$, where $f(x) = \begin{cases} \sin x & 0 \le x \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < x \le 5 \\ e^{x-5} & 5 < x \le 9 \end{cases}$

32. $\int_0^1 |5x - 3|\,dx$

33. $\int_0^2 |x^2 + 2x - 3|\,dx$

34. $\int_0^{2\pi} \sin ax \cos bx\,dx \quad \forall a, b \in Z - \{0\}$

35. $\int_0^{2\pi} \sin mx \cos nx\,dx \quad \forall m, n \in N$

36. If $\int_{\sqrt{2}}^{k} \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{12}$, obtain k.

37. If $\int_0^k \frac{dx}{2+8x^2} = \frac{\pi}{16}$, then find k.

38. If $\int_0^a \sqrt{x}\,dx = 2a \int_0^{\frac{\pi}{2}} \sin^3 x\,dx$, then obtain $\int_a^{a+1} x\,dx$.

*

3.8 Some Useful Results about Definite Integration

Theorem 3.1 : If f is continuous on $[0, a]$, then $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Proof : Let $I = \int_0^a f(x)\,dx$

Let $x = a - t$. So $dx = -dt$

$x = g(t)$ is monotonic decreasing and continuous in $[0, a]$.

Now, $\frac{dx}{dt} = -1$ is continuous on $(0, a)$.

Also, $x = 0 \Rightarrow t = a$ and $x = a \Rightarrow t = 0$. So $\alpha = a$, $\beta = 0$

$$\therefore \quad I = \int_a^0 f(a-t)(-dt)$$

$$= -\int_a^0 f(a-t)\,dt$$

$$= \int_0^a f(a-t)\,dt$$

$$= \int_0^a f(a-x)\,dx$$

i.e. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Now to understand this theorem, let us take an example.

Evaluate : $\int_0^{2\pi} cos^3x\, sin^5x\ dx$

$$I = \int_0^{2\pi} cos^3x\, sin^5x\ dx$$

$$= \int_0^{2\pi} cos^3(2\pi - x)\, sin^5(2\pi - x)\ dx$$

$$= \int_0^{2\pi} (cos^3x)\ (-sin^5x)\ dx$$

$$= -\int_0^{2\pi} cos^3x\ sin^5x\ dx = -I$$

$\therefore\ I = -I$

$\therefore\ 2I = 0$

$\therefore\ I = 0$

Theorem 3.2 : If f is continuous over $[a, b]$, then $\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$

Proof : Let $I = \int_a^b f(x)\ dx$

Let $x = a + b - t$. So $dx = -dt$

$\therefore\ x = g(t) = a + b - t$ is decreasing and continuous in $[a, b]$.

Also, $\frac{dx}{dt} = -1$ is continuous on (a, b).

Here, $x = a \Rightarrow t = b$ and $x = b \Rightarrow t = a$. So $\alpha = b$ and $\beta = a$

$$\therefore\ I = \int_b^a f(a+b-t)(-dt)$$

$$= -\int_b^a f(a+b-t)\ dt$$

$$= \int_a^b f(a+b-t)\ dt$$

$$= \int_a^b f(a+b-x)\ dx$$

i.e. $\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$

(See that in theorem 3.2, if $a = 0$ and b is replaced by a, we get theorem 3.1)

Now, to understand this theorem, let us take an example.

Evaluate : $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$

$$I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx \qquad \textbf{(i)}$$

$$= \int_1^2 \frac{\sqrt{(1+2)-x}}{\sqrt{3-(1+2-x)}+\sqrt{1+2-x}}\,dx \qquad \textbf{(}\boldsymbol{a+b=1+2=3}\textbf{)}$$

$$= \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx \qquad \textbf{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\,dx$$

$$= \int_1^2 1\,dx = [x]_1^2 = 2 - 1 = 1$$

$\therefore \quad 2I = 1$

$\therefore \quad I = \frac{1}{2}$

Theorem 3.3 : If the function f is continuous over $[0, 2a]$, then

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

Proof : Here $0 < a < 2a$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad \textbf{(i)}$$

Now, let $I = \int_a^{2a} f(x)\,dx$

Let $x = g(t) = 2a - t$. So $dx = -dt$

$x = g(t)$ is decreasing in $[a, 2a]$. $\frac{dx}{dt} = -1$ is continuous in $(a, 2a)$.

Now, if $x = a$, $t = a$ and if $x = 2a$, $t = 0$ $\qquad$ **($\alpha = a$ and $\beta = 0$)**

$$I = \int_a^{0} f(2a-t)(-dt)$$

$$= -\int_a^{0} f(2a-t)\,dt$$

$$= \int_0^{a} f(2a-t)\,dt$$

$$I = \int_0^{a} f(2a-x)\,dx$$

Substituting the value of I in (i), we get

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$$

Corollary : If $\forall x \in [0, 2a]$, $f(2a - x) = f(x)$, then $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

If $\forall x \in [0, 2a]$, $f(2a - x) = -f(x)$, then $\int_0^{2a} f(x)\,dx = 0$

Proof : $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$ (i)

Now, taking $f(2a - x) = f(x)$ in (i), we get

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

Now, if $f(2a - x) = -f(x)$, we get

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx - \int_0^{a} f(x)\,dx = 0$$

i.e. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$

Note : (1) We will see in chapter 4 that the area enclosed by the curve $y = f(x)$, lines $x = a$, $x = b$ and X-axis is $\left| \int_a^{b} f(x)\,dx \right|$. With this reference we interpret the above theorems as follows.

(2) If $f(2a - x) = f(x)$, then the graph of $f(x)$ is symmetric about $x = a$ as shown in figure 3.2.

$\therefore \int_a^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx$

$\therefore \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

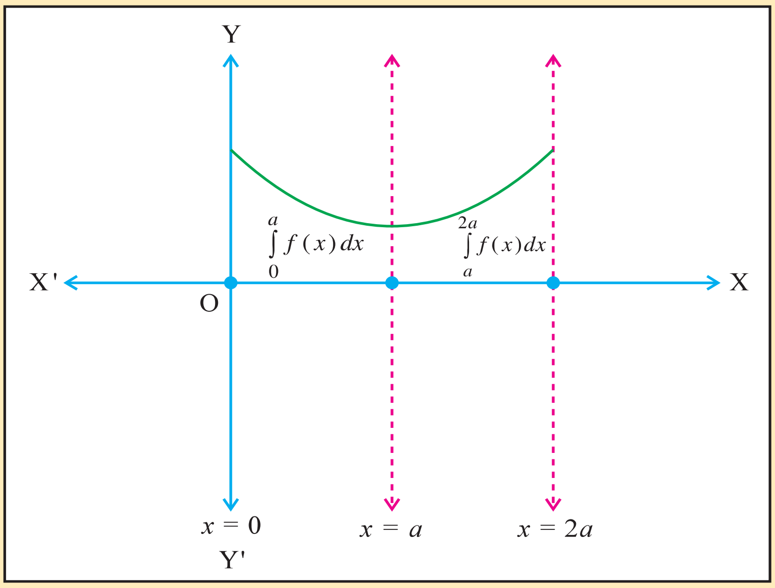


Figure 3.2

If $f(2a - x) = -f(x)$, then the graph of $f(x)$

$\therefore \int_0^{\frac{\pi}{2}} f(x)\, dx = -\int_{\frac{\pi}{2}}^{\pi} f(x)\, dx$

$\therefore \int_0^{\frac{\pi}{2}} f(x)\, dx + \int_{\frac{\pi}{2}}^{\pi} f(x)\, dx = 0$

$\therefore \int_0^{\pi} f(x)\, dx = 0$

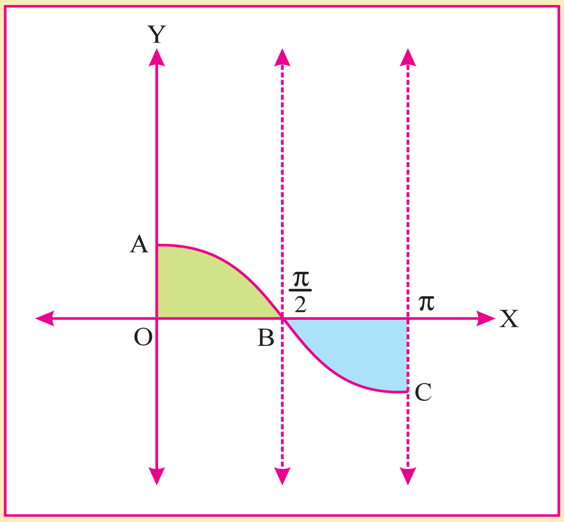


Figure 3.3

Now, to understand this theorem, let us take an example.

Evaluate : $\int_0^{2\pi} cos^3x\, dx.$

$$I = \int_0^{2\pi} cos^3x\, dx.$$

Let $f(x) = cos^3x$. Then

$$f(2\pi - x) = cos^3(2\pi - x) = cos^3x = f(x)$$

$\therefore \int_0^{2\pi} cos^3x\, dx = 2\int_0^{\pi} cos^3x\, dx$ **$(a = \pi, f(2a - x) = f(x))$**

Now, $f(\pi - x) = cos^3(\pi - x) = -cos^3x = -f(x)$ **$\left(a = \frac{\pi}{2}, f(2a - x) = -f(x)\right)$**

$\therefore \int_0^{\pi} cos^3x\, dx = 0$

Hence, $\int_0^{2\pi} cos^3x\, dx = 2\int_0^{\pi} cos^3x\, dx = 2 \times 0 = 0$

We have studied about even and odd functions. Let us recall them. Let $f : A \to R$ be a real function of a real variable. Let $\forall x \in A, -x \in A$.

(i) If $f(-x) = f(x)$, $\forall x \in A$, then f is called an even function.

(ii) If $f(-x) = -f(x)$, $\forall x \in A$, then f is called an odd function.

For example $cosx$, $secx$, x^2 are even functions and $sinx$, $tanx$, x^3 are odd functions.

Theorem 3.4 : If f is an even continuous function defined on $[-a, a]$ $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$

Proof : Here $-a < 0 < a$.

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx \qquad \text{(i)}$$

Let $I = \int_{-a}^{0} f(x)\, dx$

Let $x = -t$, $dx = -dt$

Also when $x = -a$, $t = a$ and when $x = 0$, $t = 0$

Here $\frac{dx}{dt} = -1$ is continuous and non-zero on $(-a, 0)$

$$\therefore \quad I = \int_{a}^{0} f(-t)(-dt)$$

$$= -\int_{a}^{0} f(-t)\, dt$$

$$= \int_{0}^{a} f(-t)\, dt$$

$$= \int_{0}^{a} f(t)\, dt, \text{ as } f \text{ is an even function.}$$

$$\therefore \quad I = \int_{0}^{a} f(x)\, dx$$

Substituting the value of I in (i), we get

$$\int_{-a}^{a} f(x)\, dx = \int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

$$= 2\int_{0}^{a} f(x)\, dx$$

Now, let us understand this by an example.

$y = \cos x$ is continuous even function in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx = [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) = 1 + 1 = 2$$

$$2\int_0^{\frac{\pi}{2}} \cos x \; dx = 2[\sin x]_0^{\frac{\pi}{2}} = 2\left[\sin \frac{\pi}{2} - \sin 0\right] = 2(1) = 2$$

Thus, $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \; dx = 2\int_0^{\frac{\pi}{2}} \cos x \; dx$

Theorem 3.5 : If f is an odd continuous function on $[-a, a]$, $\displaystyle\int_{-a}^{a} f(x)\, dx = 0$.

Proof : Here $-a < 0 < a$.

$$\therefore \quad \int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx \qquad \text{(i)}$$

Let $\displaystyle I = \int_{-a}^{0} f(x)\, dx$

Let $x = -t$, $dx = -dt$

Also, when $x = -a$, $t = a$ and when $x = 0$, $t = 0$

Here $\dfrac{dx}{dt} = -1$ is continuous and non-zero on $(-a, 0)$

$$\therefore \quad I = \int_{-a}^{0} f(x)\, dx$$

$$= \int_{a}^{0} f(-t)\,(-dt)$$

$$= -\int_{a}^{0} f(-t)\, dt$$

$$= \int_{0}^{a} f(-t)\, dt$$

$$= -\int_{0}^{a} f(t)\, dt, \text{ since } f \text{ is an odd function.}$$

$$= -\int_{0}^{a} f(x)\, dx$$

Substituting the value of I in (i), we get,

$$\int_{-a}^{a} f(x)\, dx = -\int_{0}^{a} f(x)\, dx + \int_{0}^{a} f(x)\, dx$$

$$\therefore \quad \int_{-a}^{a} f(x)\, dx = 0$$

Now, let us understand this by an example.

$y = \sin x$ is an odd continuous function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\left[\cos\frac{\pi}{2} - \cos\left(-\frac{\pi}{2}\right)\right] = -\left[\cos\frac{\pi}{2} - \cos\frac{\pi}{2}\right] = -(0 - 0) = 0$$

Hence $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx = 0$

Example 13 : Evaluate **(i)** $\int_{-1}^{1} \sin^3 x \cos^4 x \, dx$ **(ii)** $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} \, dx$ **$(a > 0)$**

Solution : (i) $I = \int_{-1}^{1} \sin^3 x \cos^4 x \, dx$

Here $f(x) = \sin^3 x \cos^4 x$

$\therefore \quad f(-x) = \sin^3(-x) \cos^4(-x)$

$= -\sin^3 x \cdot \cos^4 x$

$= -f(x)$

$\therefore \quad f(x) = \sin^3 x \cos^4 x$ is an odd function on $[-1, 1]$

$\therefore \quad \int_{-1}^{1} \sin^3 x \cos^4 x \, dx = 0$

(ii) $I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} \, dx$

$$= \int_{-a}^{a} \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}} \, dx$$

$$= \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}} \, dx \qquad \left(\textbf{Since } x < a, \sqrt{(a-x)^2} = |x-a| = a-x\right)$$

$$= \int_{-a}^{a} \frac{a}{\sqrt{a^2-x^2}} \, dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}} \, dx$$

$I = aI_1 - I_2$, where $I_1 = \int_{-a}^{a} \frac{1}{\sqrt{a^2-x^2}} \, dx$ and $I_2 = \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}} \, dx$

Let $f(x) = \frac{1}{\sqrt{a^2-x^2}}$ and $g(x) = \frac{x}{\sqrt{a^2-x^2}}$

Then $f(-x) = \frac{1}{\sqrt{a^2-(-x)^2}} = \frac{1}{\sqrt{a^2-x^2}} = f(x)$ and

$g(-x) = \frac{-x}{\sqrt{a^2-(-x)^2}} = \frac{-x}{\sqrt{a^2-x^2}} = -g(x)$

$\therefore \quad f(x)$ is an even function and $g(x)$ is an odd function.

$\therefore \quad I_1 = 2\int_{0}^{a} \frac{1}{\sqrt{a^2-x^2}} \, dx$ and $I_2 = 0$

$$\therefore \quad I = 2a \int_0^a \frac{1}{\sqrt{a^2 - x^2}} dx$$

$$= 2a \left[\sin^{-1} \frac{x}{a}\right]_0^a$$

$$= 2a\,[\sin^{-1} 1 - \sin^{-1} 0]$$

$$= 2a\left[\frac{\pi}{2}\right]$$

$$= a\pi$$

Example 14 : Evaluate **(i)** $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ **(ii)** $\int_0^1 x^2(1 - x)^{\frac{1}{2}} dx$ **(iii)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \log \tan x \, dx$

Solution : (i) $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$

$$= \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx \qquad \text{(i)}$$

Replace x by $\pi - x$ in (i)

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \sin(\pi - x)} dx$$

$$= \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \sin x} dx$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1 + \sin x} dx - \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$$

$$\therefore \quad I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx - I \qquad \text{(by (i))}$$

$$\therefore \quad 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi} \frac{1 + \sin x - 1}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi} dx - \pi \int_0^{\pi} \frac{dx}{1 + \sin x}$$

$$= \pi\,[x]_0^{\pi} - \pi \int_0^{\pi} \frac{dx}{1 + \sin x}$$

$$= \pi^2 - \pi \int_0^{\pi} \frac{dx}{1 + \sin x}$$

Now let, $I_1 = \int_0^{\pi} \frac{dx}{1 + \sin x}$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sin x}$$ **(since $f(2a - x) = f(\pi - x) = f(x)$)**

Let, $\tan\frac{x}{2} = t$, $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ in $[0, \pi]$

when $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$ and when $x = 0$, $t = 0$.

$$\therefore \quad I_1 = 2 \int_0^1 \frac{\frac{2dt}{1+t^2}}{1 + \frac{2t}{1+t^2}}$$

$$= 4 \int_0^1 \frac{dt}{(1+t)^2}$$

$$= 4 \left[-\frac{1}{1+t}\right]_0^1$$

$$= 4\left[-\frac{1}{2} + 1\right] = 4\left(\frac{1}{2}\right) = 2$$

$$\therefore \quad 2I = \pi^2 - 2\pi = \pi(\pi - 2)$$

$$\therefore \quad I = \frac{\pi}{2}(\pi - 2)$$

Note : Multiplying and dividing I_1 by $1 - \sin x$, the calculation seems to become simpler but at $x = \frac{\pi}{2}$, $1 - \sin x = 0$

(ii) $I = \int_0^1 x^2(1 - x)^{\frac{1}{2}}\, dx$

Replace x by $1 - x$.

$$\therefore \quad I = \int_0^1 (1 - x)^2\,[1 - (1 - x)]^{\frac{1}{2}}\, dx$$

$$= \int_0^1 (1 - 2x + x^2) \cdot x^{\frac{1}{2}}\, dx$$

$$= \int_0^1 \left(x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}}\right) dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right]_0^1$$

$$= \left(\frac{2}{3} - \frac{4}{5} + \frac{2}{7}\right) = \frac{16}{105}$$

(iii) $\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \ \log \ \tan x \ dx$ **(i)**

Replace x by $\frac{\pi}{6} + \frac{\pi}{3} - x = \frac{\pi}{2} - x$ in (i).

$$\therefore \quad \mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2\left(\frac{\pi}{2} - x\right) \log \ \tan\left(\frac{\pi}{2} - x\right) dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \log \ \cot x \ dx \qquad \textbf{(ii)}$$

Adding (i) and (ii), we get,

$$2\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \left[\log \ \tan x + \log \ \cot x\right] dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \log \ (\tan x \cdot \cot x) \ dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \cdot \log 1 \ dx$$

$\therefore \quad 2\mathrm{I} = 0$

$\therefore \quad \mathrm{I} = 0$

Exercise 3.3

1. Evaluate :

(1) $\int_{-1}^{1} \frac{x}{\sqrt{a^2 - x^2}} dx \quad (a > 1)$ **(2)** $\int_{-a}^{a} \frac{x}{2 + x^8} dx$ **(3)** $\int_{-\pi}^{\pi} \sqrt{5 + x^4} \ \sin^3 x \ dx$

(4) $\int_{-1}^{1} \log\left(\frac{3 - x}{3 + x}\right) dx$ **(5)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \ dx$ **(6)** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \ dx$

2. Evaluate :

(1) $\int_0^{\pi} \sin^2x \cos^3x \, dx$ **(2)** $\int_0^{2\pi} \sin^3x \cos^2x \, dx$

Prove the following (3 to 15)

3. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx = \frac{\pi}{4}$ **4.** $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x+\cos^n x} dx = \frac{\pi}{4}$ $(n \in N)$ **5.** $\int_1^4 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} dx = \frac{3}{2}$

6. $\int_0^1 x(1-x)^{\frac{3}{2}} dx = \frac{4}{35}$ **7.** $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}} dx = \frac{\pi}{2}$ **8.** $\int_0^3 x^2(3-x)^{\frac{1}{2}} dx = \frac{144\sqrt{3}}{35}$

9. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\cot x}} dx = \frac{\pi}{12}$ **10.** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \log\left(\frac{1+\sin x}{1+\cos x}\right) dx = 0$ **11.** $\int_0^{\pi} \frac{x\, dx}{1+\sin x} = \pi$

12. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx = \frac{\pi}{8}\log 2$ **13.** $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$

14. $\int_0^{\pi} x \sin^3 x \, dx = \frac{2\pi}{3}$ **15.** $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x+\cos x} dx = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

*

Miscellaneous Examples :

Example 15 : Prove that $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x+\cos x} dx = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$

Solution : $I = \int_0^{\frac{\pi}{2}} \frac{x}{\cos x+\sin x} dx$ **(i)**

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)}{\cos x+\sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2}}{\cos x+\sin x} dx - \int_0^{\frac{\pi}{2}} \frac{x}{\cos x+\sin x} dx$$

$$\therefore \quad I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sin x}\, dx - I$$

$$\therefore \quad 2I = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + \sin x}\, dx$$

$$\therefore \quad I = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)}\, dx$$

$$\therefore \quad I = \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\left(\cos x \cos\frac{\pi}{4} + \sin x \sin\frac{\pi}{4}\right)}\, dx$$

$$= \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{1}{\cos\left(x - \frac{\pi}{4}\right)}\, dx$$

$$= \frac{\pi}{4\sqrt{2}}\int_0^{\frac{\pi}{2}} \sec\left(x - \frac{\pi}{4}\right)\, dx$$

$$= \frac{\pi}{4\sqrt{2}}\left[\log\left|\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right|\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4\sqrt{2}}\left[\log\left|\sec\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \tan\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right| - \log\left|\sec\left(-\frac{\pi}{4}\right) + \tan\left(-\frac{\pi}{4}\right)\right|\right]$$

$$= \frac{\pi}{4\sqrt{2}}\left[\log\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \log\left|\sec\frac{\pi}{4} - \tan\frac{\pi}{4}\right|\right]$$

$$= \frac{\pi}{4\sqrt{2}}\left(\log|\sqrt{2} + 1| - \log|\sqrt{2} - 1|\right)$$

$$= \frac{\pi}{4\sqrt{2}}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}\right) \qquad (\sqrt{2} > 1)$$

$$= \frac{\pi}{4\sqrt{2}}\log(\sqrt{2} + 1)^2$$

$$= \frac{\pi}{2\sqrt{2}}\log(\sqrt{2} + 1)$$

Example 16 : Prove that $\int_0^{\frac{\pi}{4}} \tan^n x\, dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx = \frac{1}{n-1}$, $n \in N - \{1\}$.

Solution : $I = \int_0^{\frac{\pi}{4}} \tan^n x\, dx + \int_0^{\frac{\pi}{4}} \tan^{n-2} x\, dx$

$$= \int_0^{\frac{\pi}{4}} (\tan^n x + \tan^{n-2} x)\, dx$$

$$= \int_0^{\frac{\pi}{4}} tan^{n-2}x \ (tan^2x + 1) \ dx$$

$$= \int_0^{\frac{\pi}{4}} tan^{n-2}x \ (sec^2x) \ dx$$

$$= \int_0^{\frac{\pi}{4}} (tan\,x)^{n-2} \ \frac{d}{dx}(tan\,x) \ dx$$

$$= \left[\frac{(tan\,x)^{n-1}}{n-1}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{n-1}\left[\left(tan\,\frac{\pi}{4}\right)^{n-1} - (tan\,0)^{n-1}\right]$$

$$= \frac{1}{n-1}$$

Example 17 : Evaluate : $\int_0^1 cot^{-1}(1 - x + x^2) \ dx$

Solution : $I = \int_0^1 cot^{-1}(1 - x + x^2) \ dx$

$\therefore \quad 0 < x < 1$

$\therefore \quad 0 < 1 - x < 1$

$\therefore \quad 0 < x(1 - x) < 1$

$\therefore \quad 0 < x - x^2 < 1$

$\therefore \quad 0 < 1 - x + x^2$

$$\therefore \quad I = \int_0^1 tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx \qquad \left(cot^{-1}x = tan^{-1}\frac{1}{x} \ \text{for } x > 0\right)$$

$$= \int_0^1 tan^{-1}\left(\frac{1}{1-x(1-x)}\right) dx$$

$$= \int_0^1 tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right) dx$$

$$= \int_0^1 (tan^{-1}x + tan^{-1}(1 - x)) \ dx \qquad (0 < x < 1, \ 0 < 1 - x < 1, \ 0 < x(1 - x) < 1)$$

$$= \int_0^1 tan^{-1}x \ dx + \int_0^1 tan^{-1}(1 - x) \ dx$$

$$= \int_0^1 tan^{-1}x \ dx + \int_0^1 tan^{-1}(1 - (1 - x)) \ dx$$

$$= \int_0^1 tan^{-1}x \; dx + \int_0^1 tan^{-1}x \; dx$$

$$= 2\int_0^1 tan^{-1}x \cdot 1 \; dx$$

$$= 2\left[x\, tan^{-1}x\right]_0^1 - 2\int_0^1 \frac{x}{1+x^2}\, dx$$

$$= 2\left[x\, tan^{-1}x\right]_0^1 - \int_0^1 \frac{2x}{x^2+1}\, dx$$

$$= 2\left[x\, tan^{-1}x\right]_0^1 - \left[\log|x^2+1|\right]_0^1$$

$$= 2\left[tan^{-1}1 - 0\right] - \left[\log(1+1) - \log(0+1)\right]$$

$$= 2\left(\frac{\pi}{4}\right) - (\log 2 - \log 1)$$

$$= \frac{\pi}{2} - \log 2$$

Example 18 : Evaluate : $\displaystyle\int_{-\pi}^{\pi} \frac{2x\,(1+sin\,x)}{1+cos^2 x}\, dx$

Solution : $\displaystyle I = \int_{-\pi}^{\pi} \frac{2x\,(1+sin\,x)}{1+cos^2 x}\, dx$

$$= \int_{-\pi}^{\pi} \frac{2x}{1+cos^2 x}\, dx + \int_{-\pi}^{\pi} \frac{2x\,sin\,x}{1+cos^2 x}\, dx$$

$\therefore$ $\displaystyle I = I_1 + I_2$, where $\displaystyle I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+cos^2 x}\, dx$ and $\displaystyle I_2 = \int_{-\pi}^{\pi} \frac{2x\,sin\,x}{1+cos^2 x}\, dx$

Let $\displaystyle f(x) = \frac{2x}{1+cos^2 x}$ and $\displaystyle g(x) = \frac{2x\,sin\,x}{1+cos^2 x}$

Then $\displaystyle f(-x) = \frac{2\,(-x)}{1+cos^2(-x)} = \frac{-2x}{1+cos^2 x} = -f(x)$ and

$$g(-x) = \frac{2\,(-x)\,sin\,(-x)}{1+cos^2(-x)} = \frac{2x\,sin\,x}{1+cos^2 x} = g(x)$$

$\therefore$ $f(x)$ is an odd function and $g(x)$ is an even function.

$\therefore$ $\displaystyle I_1 = 0$ and $\displaystyle I_2 = 2\int_0^{\pi} \frac{2x\,sin\,x}{1+cos^2 x}\, dx$

$$\therefore \quad I_2 = 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx \qquad \text{(i)}$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx$$

$$= 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx$$

$$I_2 = 4\int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}\, dx - 4\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$$

$$\therefore \quad I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx - I_2 \qquad \text{(Re (i))}$$

$$\therefore \quad 2I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx$$

Let $\cos x = t$, $-\sin x\ dx = dt$, $\sin x\ dx = -dt$. When $x = 0$, $t = 1$ and when $x = \pi$, $t = -1$

$$\therefore \quad 2I_2 = 4\pi\int_1^{-1} \frac{-dt}{1+t^2}$$

$$= 4\pi\int_{-1}^{1} \frac{dt}{1+t^2}$$

$$= 4\pi\left[\tan^{-1} t\right]_{-1}^{1}$$

$$= 4\pi\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$= 4\pi\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$\therefore \quad 2I_2 = 2\pi^2$

$\therefore \quad I_2 = \pi^2$

Now, $I = I_1 + I_2$

$\therefore \quad I = 0 + \pi^2$

$\therefore \quad I = \pi^2$

Example 19 : Prove that : $\int_0^{\frac{\pi}{2}} \log \sin x\ dx = -\frac{\pi}{2}\log 2.$

Solution : $I = \int_0^{\frac{\pi}{2}} \log \sin x\ dx \qquad \text{(i)}$

Then, $I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx$

$$\therefore \quad I = \int_0^{\frac{\pi}{2}} \log \cos x \, dx \qquad \textbf{(ii)}$$

Adding (i) and (ii) we get

$$2I = \int_0^{\frac{\pi}{2}} \log \sin x \, dx + \int_0^{\frac{\pi}{2}} \log \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin x + \log \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log (\sin x \cdot \cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \left(\frac{2 \sin x \cos x}{2} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log \left(\frac{\sin 2x}{2} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx - \int_0^{\frac{\pi}{2}} \log 2 \, dx$$

Let $I_1 = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx$

$$\therefore \quad 2I = I_1 - \log 2 \int_0^{\frac{\pi}{2}} dx \qquad \textbf{(iii)}$$

Now, $I_1 = \int_0^{\frac{\pi}{2}} \log \sin 2x \, dx$

Let $2x = t$, we get $dx = \frac{1}{2} dt$

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$.

$$\therefore \quad I_1 = \int_0^{\pi} \log \sin t \cdot \frac{1}{2} dt$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin t \, dt$$

$$= \frac{1}{2} \cdot 2 \cdot \int_0^{\frac{\pi}{2}} \log \sin t \, dt$$

$$\left(\log \sin (\pi - t) = \log \sin t. \text{ So } \int_0^{\pi} \log \sin t \, dt = 2 \int_0^{\frac{\pi}{2}} \log \sin t \, dt \right)$$

$$\therefore \quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin t \, dt$$

$$\therefore \quad I_1 = \int_0^{\frac{\pi}{2}} \log \sin x \, dx = I$$ **(Definite integral does not depend upon variable)**

So, from (iii) we get,

$$2I = I - \frac{\pi}{2} \log 2$$

$$\therefore \quad I = -\frac{\pi}{2} \log 2$$

Not for examination :

Infact $\int_0^{\frac{\pi}{2}} \log \sin x \, dx$ is not a definite integral in usual sense. The function $\log \sin x$ is unbounded near end point 0 of $\left[0, \frac{\pi}{2}\right]$. Such integrals are called improper integrals.

Actually $\lim\limits_{t \to 0+} \int_t^{\frac{\pi}{2}} \log sinx \, dx = \int_0^{\frac{\pi}{2}} \log \sin x \, dx$.

It is improper integral of first kind. Integrals like $\int_0^{\infty} \frac{sinx}{x} dx$ are called improper integrals of second kind.

If either function is unbounded in $[a, b]$, $a \in R$, $b \in R$ or interval is unbounded like $(-\infty, a)$, (a, ∞), $(-\infty, \infty)$ the integral is an improper definite integral as against definite integral studied in the chapter.

Sometimes regarding an improper integral as a definite integral would give incorrect results.

We could get $\int_{-2}^{3} \frac{dx}{x} = [\log|x|]_{-2}^{3} = \log3 - \log2 = \log\frac{3}{2}$

But $\frac{1}{x}$ is unbounded at $x = 0$.

$$\therefore \quad \int_{-2}^{3} \frac{dx}{x} = \int_{-2}^{0} \frac{dx}{x} + \int_{0}^{3} \frac{dx}{x}$$

$$= \lim_{t_1 \to 0-} \int_{-2}^{t_1} \frac{dx}{x} + \lim_{t_2 \to 0+} \int_{t_2}^{3} \frac{dx}{x}$$

does not exist.

$\int_0^{\pi} sec^2x \, dx = [tanx]_0^{\pi} = 0 - 0 = 0$ is incorrect.

sec is unbounded at $x = \frac{\pi}{2}$

Exercise 3

1. If $I_n = \int_0^{\frac{\pi}{4}} tan^n x\, dx$, then prove that $n\,(I_{n-1} + I_{n+1}) = 1$

2. If $f(x) = f(a + b - x)$, prove that $\int_a^b x\, f(x)\, dx = \frac{a+b}{2} \int_a^b f(x)\, dx$.

3. Prove that $\int_0^{\pi} x\, f(sin\, x)\, dx = \frac{\pi}{2} \int_0^{\pi} f(sin\ x)\, dx$ and using this evaluate

(i) $\int_0^{\pi} x\ sin^2 x\ dx$ (ii) $\int_0^{\pi} \frac{x}{1 + sin x}\ dx$

4. Prove that $\int_0^n f(x)\ dx = \sum_{r=1}^{n} \int_0^1 f(t + r - 1)\ dt$

5. If $\int_n^{n+1} f(x)\ dx = n^3$, then find $\int_{-4}^{4} f(x)\ dx,\ n \in Z$

6. Prove that : $\int_0^{\frac{\pi}{2}} \log\ cos x\ dx = -\frac{\pi}{2} \log 2$

7. Prove that : $\int_0^a x^2 (a - x)^n\ dx = \frac{2a^{n+3}}{(n+1)(n+2)(n+3)}$

Evaluate (8 to 17) :

8. $\int_0^{\log 2} xe^{-x}\, dx$

9. $\int_0^{\frac{\pi}{4}} \frac{dx}{a^2 cos^2 x - b^2 sin^2 x}$ **($a > b > 0$)**

10. $\int_1^2 \frac{x^2+1}{x^4+1}\, dx$

11. $\int_0^{\frac{\pi}{2}} \left(\frac{\pi x}{2} - x^2\right) cos 2x\ \ dx$

12. $\int_0^{\frac{\pi}{4}} \frac{sin^2 x}{1 + sin x\, cos x}\, dx$

13. $\int_0^{\frac{\pi}{2}} \frac{cos\ x}{1 + cos x + sin x}\, dx$

14. $\int_0^1 \frac{\log\,(1+t)}{1+t^2}\, dt$

15. $\int_1^3 \frac{dx}{x^2 (x + 1)}$

16. $\int_0^{\pi} \frac{x^2\, sin x}{(2x - \pi)(1 + cos^2 x)}\, dx$

17. $\int_0^{\frac{\pi}{2}} |\, sin x - cos x\,|\ dx$

18. Evaluate : $\int_1^3 (x^2 + x)\, dx$ as the limit of a sum.

19. Evaluate : $\int_0^4 (x + e^{2x})\, dx$ as the limit of a sum.

20. Prove that $\int_0^{\frac{\pi}{2}} \log \tan x \, dx = 0$

21. Prove that $\int_0^{\frac{\pi}{2}} (2 \log \sin x - \log \sin 2x)\, dx = -\frac{\pi}{2} \log 2$

22. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{1+\sqrt{\tan x}}\, dx = ...$. ☐

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{\pi}{2}$

(2) $\int_1^e \log x \, dx = ...$. ☐

(a) 1 (b) $e + 1$ (c) $e - 1$ (d) 0

(3) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cot x}\, dx = ...$. ☐

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π

(4) If $\int_0^a \frac{1}{1+4x^2}\, dx = \frac{\pi}{8}$, then $a = ...$. ☐

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{1}{2}$ (d) 1

(5) $\int_0^3 \frac{3x+1}{x^2+9}\, dx = ...$. ☐

(a) $\frac{\pi}{12} + \log(2\sqrt{2})$ (b) $\frac{\pi}{3} + \log(2\sqrt{2})$ (c) $\frac{\pi}{12} + \log\sqrt{2}$ (d) $\frac{\pi}{6} + \log(2\sqrt{2})$

(6) $\int_{-1}^1 |1 - x|\, dx = ...$. ☐

(a) -2 (b) 2 (c) 0 (d) 4

(7) If $\int_0^1 (3x^2 + 2x + k)\, dx = 0$, then $k = ...$. ☐

(a) 1 (b) 2 (c) -2 (d) 4

(8) If $\int_1^a (3x^2 + 2x + 1)\, dx = 11$, then $a = ...$. ☐

(a) 2 (b) 3 (c) -3 (d) $\frac{2}{3}$

(9) $\int_{-1}^{0} |x|\, dx =$ ....

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 2

(10) $\int_{-1}^{1} \log\left(\frac{7-x}{7+x}\right) dx =$ ....

(a) 1 (b) 0 (c) 2 (d) -2

(11) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cotx\ dx =$ ....

(a) $\frac{1}{2}\log\left(\frac{3}{2}\right)$ (b) $\log\left(\frac{3}{2}\right)$ (c) $\frac{1}{2}\log\frac{\sqrt{3}}{2}$ (d) $2\log\frac{3}{2}$

(12) $\int_{1}^{k} f(x)\, dx = 47,\ f(x) = \begin{cases} 2x+8 & 1 \le x \le 2 \\ 6x & 2 < x \le k \end{cases}$, then k ....

(a) 4 (b) -4 (c) 2 (d) -2

(13) $\int_{1}^{\sqrt{3}} \frac{dx}{1+x^2} =$ ....

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

(14) $\int_{1}^{4} \left(\frac{x^2+1}{x}\right)^{-1} =$ ....

(a) $\log\left(\frac{17}{2}\right)$ (b) $\frac{1}{2}\log\left(\frac{17}{2}\right)$ (c) $2\log(17)$ (d) $\log(17)$

(15) $\int_{0}^{\sqrt{2}} \sqrt{2-x^2}\, dx =$ ....

(a) $-\frac{\pi}{2}$ (b) π (c) 0 (d) $\frac{\pi}{2}$

(16) $\int_{0}^{2a} \frac{f(x)\,dx}{f(x)+f(2a-x)} =$ ....

(a) $-a$ (b) a (c) $\frac{a}{2}$ (d) $-\frac{a}{2}$

(17) $\int_{0}^{\pi} sin^3x\ cos^3x\ \ dx =$ ....

(a) π (b) $-\pi$ (c) $\frac{\pi}{2}$ (d) 0

(18) If $\int_{2}^{k} (2x + 1)\, dx = 6$, then $k =$ ....

(a) 3 (b) 4 (c) -4 (d) -2

(19) $\int_{0}^{1} \frac{dx}{x+\sqrt{x}} =$ ....

(a) log 2 (b) log 4 (c) log 3 (d) $-$ log 2

(20) If $\int_0^{\frac{\pi}{3}} \frac{\cos x}{3 + 4\sin x}\, dx = k \log\left(\frac{3 + 2\sqrt{3}}{3}\right)$, then $k = ...$.

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

•

Summary

We have studied the following points in this chapter :

1. $f : [a, b] \to R$ is a continuous function. Divide $[a, b]$ into n sub-intervals of equal length given by $h = \frac{b-a}{n}$. Then $\int_a^b f(x)\, dx = \lim_{n \to \infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + ih)$.

2. Fundamental theorem of integral calculus : If the function f is continuous on $[a, b]$ and F is a differentiable in (a, b) such that

$\forall x \in (a, b)$, $\frac{d}{dx}[F(x)] = f(x)$, then $\int_a^b f(x)\, dx = F(b) - F(a)$

3. $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$, i.e. definite integral is independent of variable.

4. $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$ $(a > b)$

5. If f is continuous on $[a, b]$ and $a < c < b$, then $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

6. If f is continuous on $[a, b]$, then $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

7. If f is continuous on $[a, b]$, then $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

8. If f is continuous on $[0, 2a]$, then $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$

If $f(2a - x) = f(x)$, $\forall x \in [0, 2a]$, then $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$

If $f(2a - x) = -f(x)$, $\forall x \in [0, 2a]$, then $\int_0^{2a} f(x)\, dx = 0$

9. If f is an even continuous function on $[-a, a]$, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$

10. If f is an odd continuous function on $[-a, a]$, then $\int_{-a}^{a} f(x)\, dx = 0$.

•

AN APPLICATION OF INTEGRALS

4

Music is the pleasure the human soul experiences from counting without being aware that it is counting.
– Gottfried Wilhelm

◆

There are no deep theorems - only theorems that we have not understood very well.
– Nicholas Goodman

4.1 Introduction

Integration and differentiation are basic operations of calculus having numerous applications in science and engineering. Integrals appear in many practical applications.

If the archways of a building has triangular shape or semi-circular shape or rectangular shape and we need to fix glass in the archways, then we can use formulae of elementary geometry to decide how much glass material is needed. But if the archways are in section of an elliptic shape, then we have to resort to integration to find out the quantity of glass material needed.

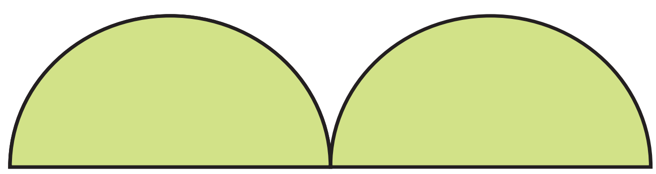


Figure 4.1

We need to know the area under a curve for this purpose. Before integration was developed, one could only approximate the area. Method of approximation was known to the ancient Greeks. A Greek mathematician **Archimedes**, worked-out good approximation to the area of a circle. Finding the area of a region is one of the most fundamental applications of the definite integral. The concept of integration was developed by **Newton** and **Leibnitz**.

4.2 Area Under Simple Curves

In the previous chapter, we have studied how to find the value of a definite integral as the limit of a sum. Let us study how integration is useful to find the area enclosed by simple curves, area between lines and arcs of circles, parabolas and ellipses. We shall also discuss how to find the area between two curves.

We will assume following property of a continuous function defined on a closed interval : A continuous function defined on a closed interval attains maximum value at some point of interval as well as minimum value at some point of interval.

Case 1 : Curves which are entirely above X-axis :

Let f be a continuous function defined over $[a, b]$. Assume that $f(x) \geq 0$ for all $x \in [a, b]$. We want to find the area A enclosed by the curve $y = f(x)$, the X-axis and the lines $x = a$ and $x = b$. (The coloured region in the figure 4.2(a) and 4.2(b).)

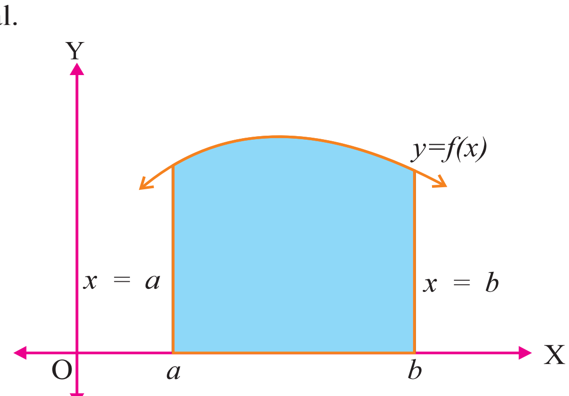


Figure 4.2(a)

We first divide the interval $[a, b]$ into n subintervals determined by the end-points $a = x_0, x_1, x_2, \ldots, x_n = b$. Since $f(x)$ is continuous on each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, there exists a point $x_i' \in [x_{i-1}, x_i]$ such that $f(x_i')$ is minimum value of $f(x)$ in this subinterval. Also, there exists a point $x_i^* \in [x_{i-1}, x_i]$ such that $f(x_i^*)$ is maximum value of $f(x)$ in this subinterval. Let $\Delta x_i = x_i - x_{i-1}$. We construct a rectangle with $f(x'_i)$ as its height and Δx_i $(i = 1, 2, \ldots, n)$ as its breadth. (as in the figure 4.3). The sum of the areas of these rectangles is clearly less than the area A we are trying to find.

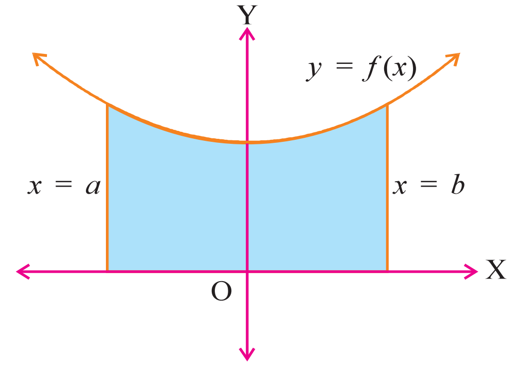


Figure 4.2(b)

i.e., $\sum_{i=1}^{n} f(x'_i)\,\Delta x_i \leq A$ **(i)**

This sum $\sum_{i=1}^{n} f(x'_i)\,\Delta x_i$ is called a **lower sum**.

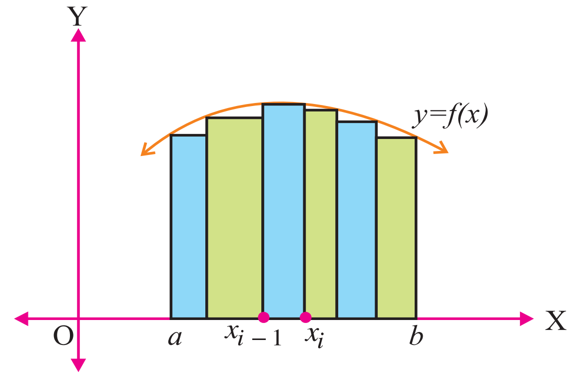


Figure 4.3

We construct a rectangle with $f(x_i^*)$ as its height and $\Delta x_i = x_i - x_{i-1}$ $(i = 1, 2, \ldots, n)$ as its breadth. (as in the figure 4.4)

The sum of the areas of these rectangles is clearly greater than the area A we are trying to find.

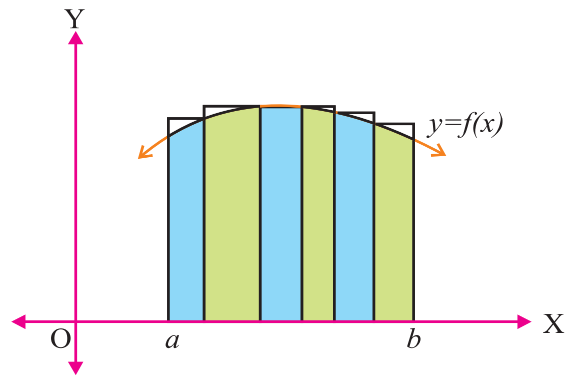


Figure 4.4

i.e., $\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \geq A$ **(ii)**

This sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$ is called an **upper sum**.

Thus, from (i) and (ii) we have

$$\sum_{i=1}^{n} f(x'_i)\,\Delta x_i \leq A \leq \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The area is equal to the limit of the lower sum or of the upper sum as the number of subdivisions tend to infinity and maximum of $\Delta x_i \to 0$ provided upper sums and lower sums tend to a common limit and can be written as follows :

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x'_i)\, \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x_i$$

As discussed in previous chapter, the above expression is $\int_a^b f(x)\, dx$.

Thus, area $A = \int_a^b f(x)\, dx$.

Case 2 : Curves which are entirely below the X-axis

If the curve under consideration lies below the X-axis, then $f(x) < 0$ from $x = a$ to $x = b$ as shown in figure 4.5. Then the sum defined in (i) and (ii) will be negative but the area bounded by the curve $y = f(x)$, X-axis and the lines $x = a$, $x = b$ is positive. In this case we take absolute value of the integral i.e., $\left| \int_a^b f(x)\, dx \right|$ as the area enclosed.

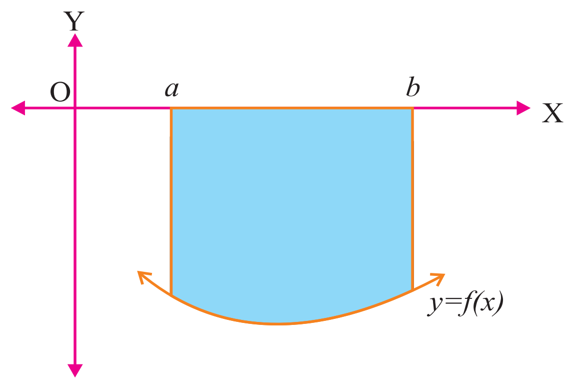


Figure 4.5

Thus, area $A = |I|$ where $I = \int_a^b f(x)\, dx$.

Case 3 : Curves which intersect X-axis at one point :

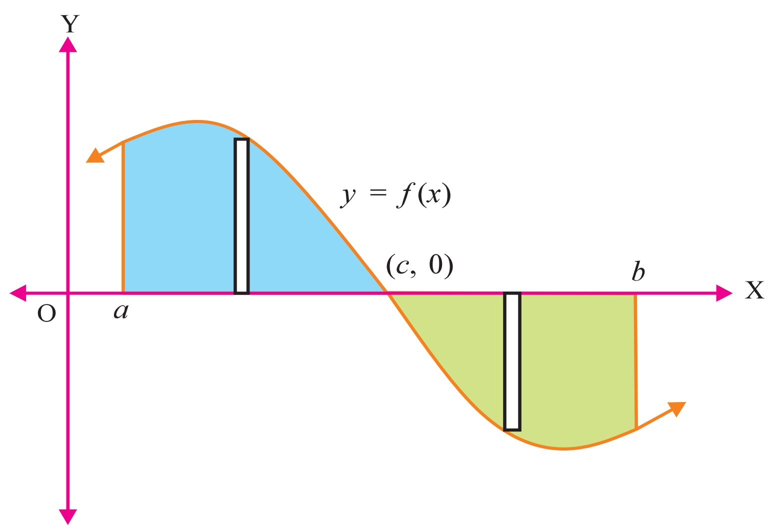


Figure 4.6

Let the graph of $y = f(x)$ intersect X-axis at $(c, 0)$ only and $a < c < b$. Let $f(x) \geq 0\ \forall x \in [a, c]$, $f(x) \leq 0\ \forall x \in [c, b]$. Then the area of the region bounded by $y = f(x)$, $x = a$, $x = b$ and X-axis is given by $A = |I_1| + |I_2|$.

where $I_1 = \int_a^c f(x)\, dx$, $I_2 = \int_c^b f(x)\, dx$.

Even if the curve intersects X-axis at finite number of points $c_1, c_2, ..., c_n$, we can have

$$I_k = \int_{c_k}^{c_{k+1}} f(x)dx \text{ and Area } = \sum_{k=0}^{n} |I_k|. \ (c_0 = a, c_{n+1} = b)$$

As above,

(1) Let $x = g(y)$ be continuous function of y over $[c, d]$ and $g(y) \geq 0$ or $g(y) \leq 0$, $\forall y \in [c, d]$. Then the area of the region bounded by $x = g(y)$, $y = c$, $y = d$ and Y-axis is A = | I |.

$$\text{where } I = \int_c^d x dy = \int_c^d g(y)\, dy.$$

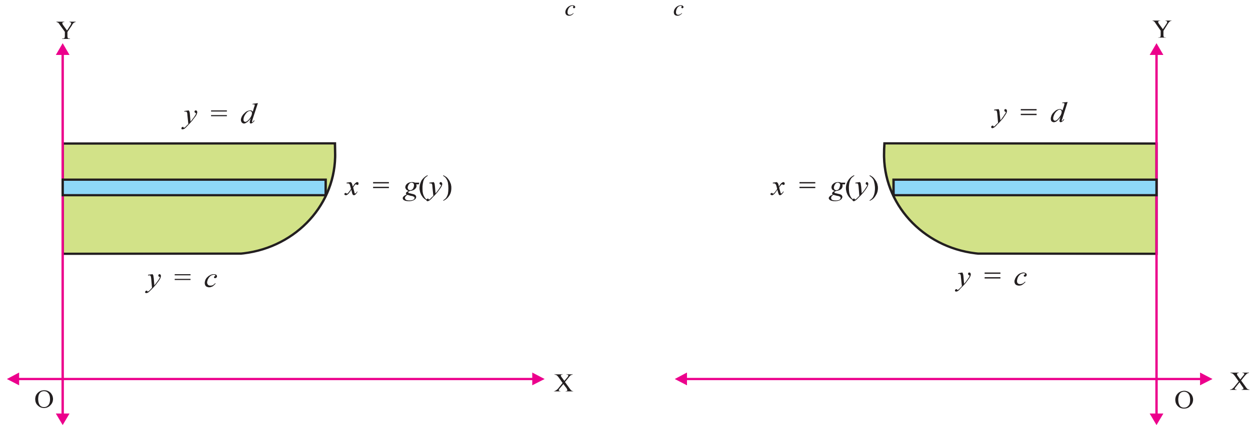


Figure 4.7

(2) Let the graph of $x = g(y)$ intersect Y-axis at only $(0, e)$ and $c < e < d$. Then the area of the region bounded by $x = g(y)$, $y = c$, $y = d$ and Y-axis is given by $A = |I_1| + |I_2|$,

$$\text{where } I_1 = \int_c^e g(y)\, dy \quad \text{and}$$

$$I_2 = \int_e^d g(y)\, dy.$$

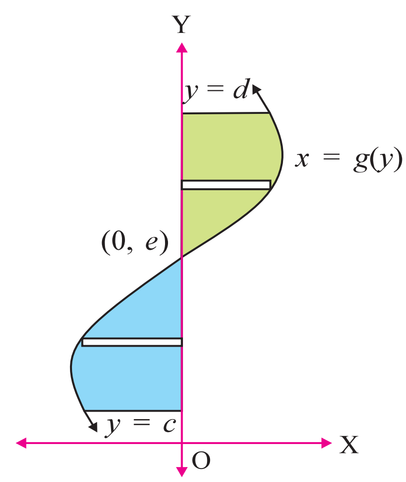


Figure 4.8

(3) If the curve and the region bounded by the curve are symmetric about X-axis and if one part of the area is in upper semi-plane of X-axis and the second one is in the lower semi-plane of X-axis, then the total area of the region will be two times the area in the upper semi-plane. This method can also be applied to calculate the area of a region symmetric about Y-axis.

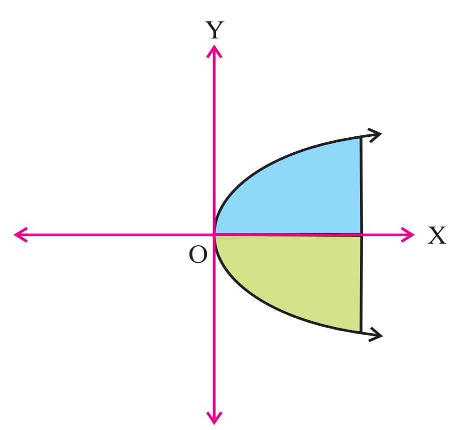


Figure 4.9

(4) If the curve and the region bounded by the curve are symmetric about both the axes, then its area can be calculated by considering the area of the region in the first quadrant and multiplying the same by four.

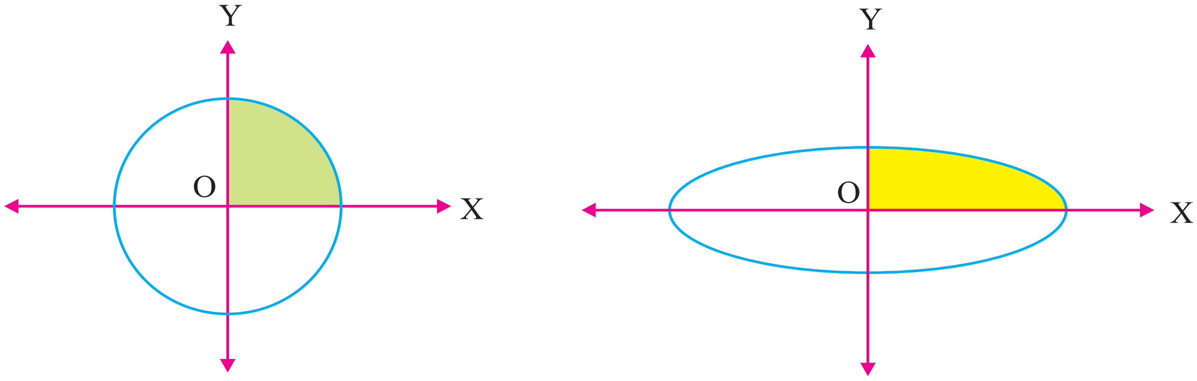


Figure 4.10

Region bounded by circle, ellipse are examples of this type.

Example 1 : Using integration, find the area of the region bounded by the line $2y = -x + 8$, X-axis and the lines $x = 2$ and $x = 4$.

Solution : Required area = | I |, where

$$I = \int_2^4 y\,dx$$

$$= \int_2^4 \left(\frac{-x}{2} + 4\right) dx$$

$$= \left[\frac{-x^2}{4} + 4x\right]_2^4$$

$$= \left[\frac{-(4)^2}{4} + 16\right] - \left[\frac{-(2)^2}{4} + 8\right]$$

$$= (-4 + 16) - (-1 + 8)$$

$$= 12 - 7$$

$$= 5$$

$\therefore$ Required area = 5

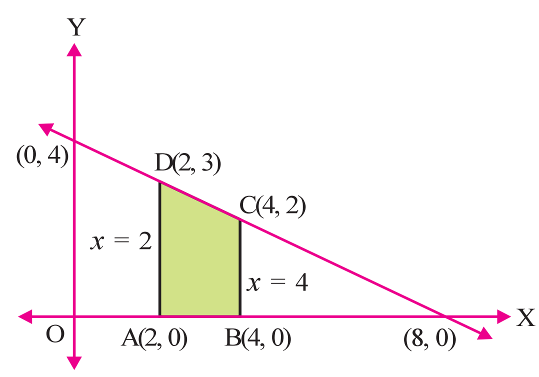


Figure 4.11

Note : Area of trapezium ABCD

$= \frac{1}{2}$ (Distance between parallel sides)(Sum of lengths of parallel sides)

$= \frac{1}{2}(4 - 2)(3 + 2) = 5$

Example 2 : Find the area of the region bounded by the curve $y = 4 - x^2$, X-axis and the lines $x = 0$ and $x = 2$.

Solution : Here $y = 4 - x^2$

$\therefore \quad x^2 = -(y - 4)$ which represents a parabola. Its vertex is (0, 4). Parabola opens downwards. Required area A = | I |, where

$$I = \int_0^2 y dx$$

$$= \int_0^2 (4 - x^2)\, dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 8 - \frac{8}{3} = \frac{16}{3}$$

$$\therefore \quad A = \frac{16}{3}$$

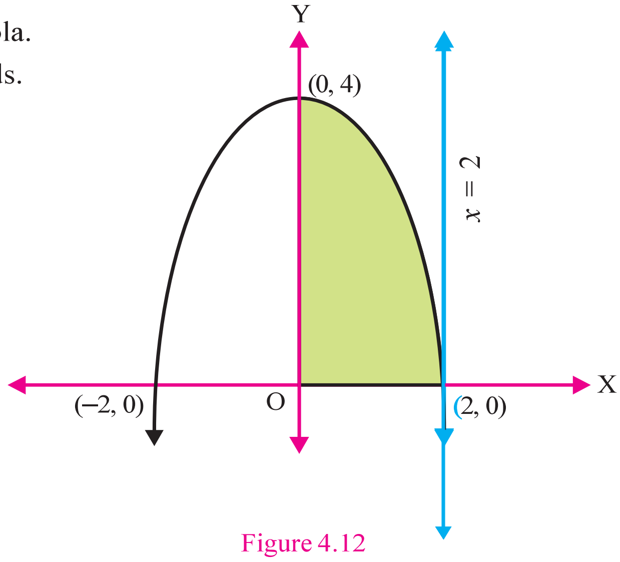


Figure 4.12

Example 3 : Find the area of the region bounded by $y = x^2 - 1$, X-axis and $y = 8$.

Solution : Here the curve $y = x^2 - 1$ is symmetric about Y-axis. So its area can be calculated by calculating the area enclosed by the arc in the first quadrant and then multiplying the same by 2.

Now, $y = x^2 - 1$. So $x^2 = y - (-1)$

This is a parabola whose vertex is (0, −1) and it opens upwards. The limits of the region bounded by the curve in the first quadrant and Y-axis are $y = 0$ and $y = 8$.

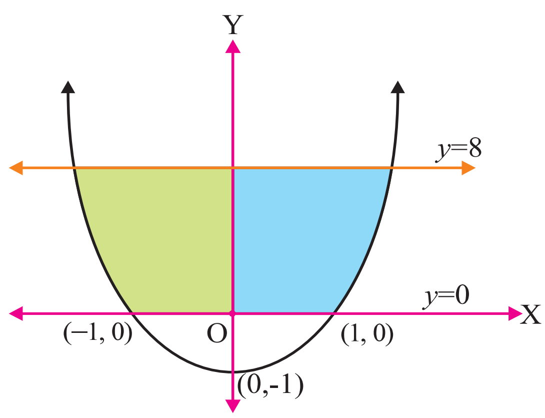


Figure 4.13

$\therefore$ Area A = 2 | I |

where $I = \int_0^8 x\, dy$

$$= \int_0^8 \sqrt{y+1}\, dy \qquad (x > 0 \text{ in the first quadrant})$$

$$= \frac{2}{3}\left[(y + 1)^{\frac{3}{2}}\right]_0^8$$

$$= \frac{2}{3}\left((9)^{\frac{3}{2}} - 1\right) = \frac{52}{3}$$

$$\therefore \quad A = 2|I| = 2\left(\frac{52}{3}\right) = \frac{104}{3}$$

Example 4 : Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution : The ellipse is symmetrical about both X-axis and Y-axis.

Required area = 4 × Area OAB in the 1st quadrant

$$= 4\,|\,I\,|, \text{ where } I = \int_0^a y dx$$

Now, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$

$\therefore \quad y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$

In the first quadrant, $y > 0$

$\therefore \quad y = \frac{b}{a}\sqrt{a^2 - x^2}$

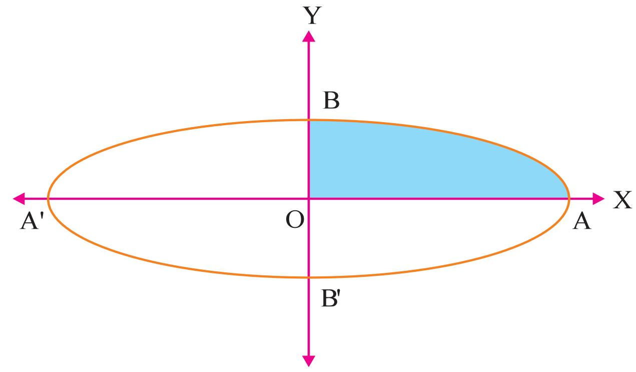


Figure 4.14

$$\therefore \quad \text{I} = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\left(\frac{a}{2} \times 0 + \frac{a^2}{2} sin^{-1} 1\right) - (0 + 0)\right]$$

$$= \frac{b}{a}\left[\frac{a^2}{2} sin^{-1} 1\right] = \frac{b}{a}\left[\frac{a^2}{2} \cdot \frac{\pi}{2}\right] = \frac{\pi ab}{4}$$

$\therefore$ Required area $= 4 \times \frac{\pi ab}{4} = \pi ab$

Remain : If we consider $x^2 + y^2 = r^2$ in this question then we get well known formula πr^2 for area of a circle.

Exercise 4.1

1. Find the area bounded by the parabola $y = x^2 + 2$, X-axis and the lines $x = 1$ and $x = 2$.
2. Find the area bounded by the parabola $y = x^2 - 4$, the X-axis and the lines $x = -1$ and $x = 2$.
3. What is the area bounded by the parabola $y = x^2$ and the lines $x = -2$ and $x = 1$?
4. Find the area of the region bounded by the curve $y = \sqrt{x-1}$, the Y-axis and the lines $y = 1$ and $y = 5$.
5. Find the area bounded by the X-axis the parabola $y = -x^2 + 4$.
6. Find the area bounded by the curve $y = 9 - x^2$ and the X-axis.
7. Find the area enclosed by the circle $x^2 + y^2 = a^2$.
8. Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 4$.

*

4.3 Area Between Two Curves

In this section, we will find the area of the region bounded by a line and a circle, a line and a parabola, a line and an ellipse, a circle and a parabola, two circles etc.

Let us try to get intuitive idea of how area between two intersecting curves may be obtained. As discussed earlier, area of the region bounded by $y = f_1(x)$, $x = a$, $x = b$

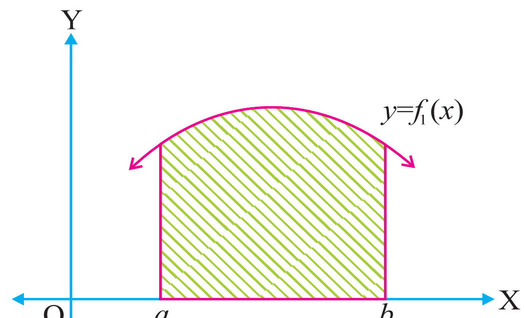


Figure 4.15

and X-axis is given by $A_1 = |I_1|$ where $I_1 = \int_a^b f_1(x)\, dx$.

Here, $I_1 \geq 0$ as we have assumed that $f_1(x) \geq 0$. (See figure 4.15)

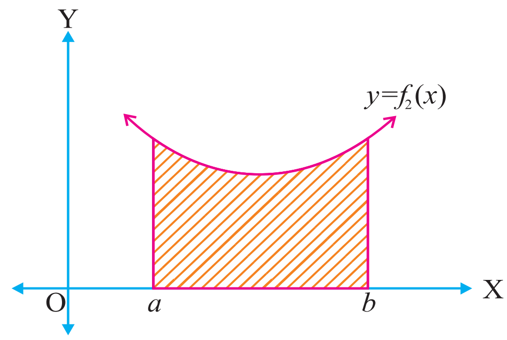


Figure 4.16

As shown in figure 4.16 area of the region bounded by $y = f_2(x)$, $x = a$, $x = b$ and X-axis is given by $A_2 = |I_2|$ where $I_2 = \int_a^b f_2(x)\, dx$.

Since $f_2(x) \geq 0$ we have $I_2 \geq 0$.

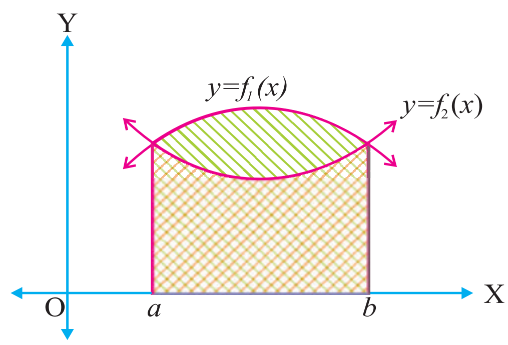


Figure 4.17

If two curves $y = f_1(x)$ and $y = f_2(x)$ intersect each other at only two points for which their x-coordinates are a and b $(a \neq b)$, then the area enclosed by them is given by

$A = |I|$

where $I = I_1 - I_2 = \int_a^b f_1(x)\, dx - \int_a^b f_2(x)\, dx$

$$= \int_a^b (f_1(x) - f_2(x))\, dx$$

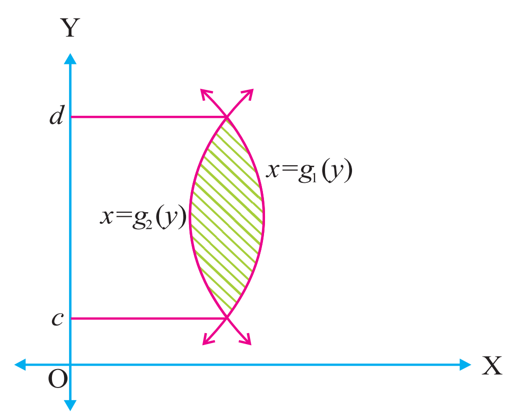


Figure 4.18

If two curves $x = g_1(y)$ and $x = g_2(y)$ intersect each other at only two points for which their y-coordinates are c and d $(c \neq d)$ then the area enclosed by them is given by $A = |I|$.

where $I = \int_c^d (g_1(y) - g_2(y))\, dy$.

Here we have assumed that $g_1(y) \geq 0$, $g_2(y) \geq 0$.

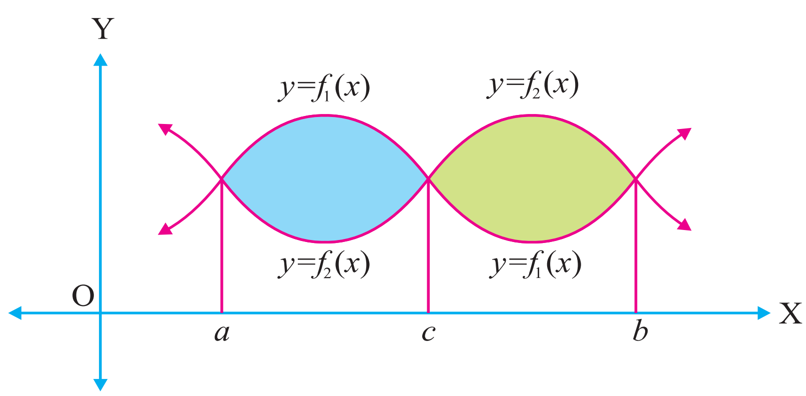


Figure 4.19

If the curves intersect once within the region being considered then as shown in the figure 4.19, the interval of integration will have to be split up. Suppose we wish to find the area between the curves

$y = f_1(x)$ and $y = f_2(x)$ and the lines $x = a$ and $x = b$. Suppose that the curves intersect each other at some point c between a and b then A = | I_1 | + | I_2 |.

where $I_1 = \int_a^c (f_1(x) - f_2(x))\,dx, \quad I_2 = \int_c^b (f_1(x) - f_2(x))\,dx$

Example 5 : Find the smaller of the two areas enclosed between the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$.

Solution : The given line is $\frac{x}{a} + \frac{y}{b} = 1 \qquad$ **(i)**

and the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad$ **(ii)**

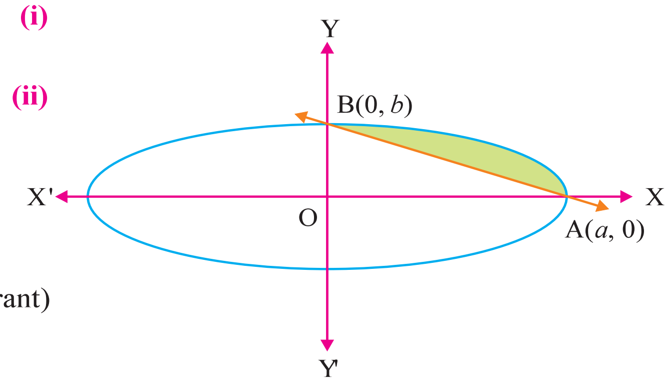


Figure 4.20

Clearly, the line intersects the ellipse at A(a, 0) and B(0, b). The required area is shown as in the figure 4.20 as coloured region.

For the ellipse $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (First quadrant)

Now, area of ΔAOB $= \frac{1}{2}$OA · OB

$= \frac{1}{2}ab \qquad$ **(iii)**

Also, area enclosed by the ellipse in the first quadrant is

$$\int_0^a y\,dx = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\sin^{-1} 1\right] = \frac{\pi ab}{4} \qquad \textbf{(iv)}$$

$\therefore$ By (iii) and (iv)

Required area $= \left|\frac{\pi ab}{4} - \frac{1}{2}ab\right| = \left|\frac{(\pi - 2)ab}{4}\right| = \frac{(\pi - 2)ab}{4}$ as $\pi > 2$.

Second Method : Required area = | I |

where $I = \int_0^a (f_1(x) - f_2(x))\,dx$, where $f_1(x) = \frac{b}{a}\sqrt{a^2 - x^2}$ and $f_2(x) = b\left(1 - \frac{x}{a}\right)$

$$= \int_0^a \left[\frac{b}{a}\sqrt{a^2 - x^2} - b\left(1 - \frac{x}{a}\right)\right] dx$$

$$= \left[\frac{b}{a}\left(\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right) - b\left(x - \frac{x^2}{2a}\right)\right]_0^a$$

$$= \left[\frac{b}{a}\left(0 + \frac{a^2}{2}\sin^{-1} 1\right) - b\left(a - \frac{a}{2}\right)\right] - (0)$$

$$= \frac{\pi ab}{4} - \frac{ab}{2}$$

$$= \frac{(\pi - 2)ab}{4}$$

$\therefore$ Required area $= \left| \frac{(\pi - 2)ab}{4} \right| = \frac{(\pi - 2)ab}{4}$ as $\pi > 2$.

Example 6 : Using integration, find the area of the region bounded by the circle $x^2 + y^2 = 4$, line $x - y\sqrt{3} = 0$ and X-axis in the first quadrant.

Solution : Here the given curves are $x^2 + y^2 = 4$ and $x - y\sqrt{3} = 0$.

Substitute $y = \frac{x}{\sqrt{3}}$ in $x^2 + y^2 = 4$.

$$x^2 + \frac{x^2}{3} = 4$$

$\therefore$ $4x^2 = 12$

$\therefore$ $x = \pm\sqrt{3}$

In the first quadrant $x = \sqrt{3}$ and so $y = \frac{x}{\sqrt{3}} = 1$.

$\therefore$ In the first quadrant the point of intersection of the line and the circle is P($\sqrt{3}$, 1).

$\overline{PM} \perp$ X-axis and M($\sqrt{3}$, 0) is the foot of the perpendicular.

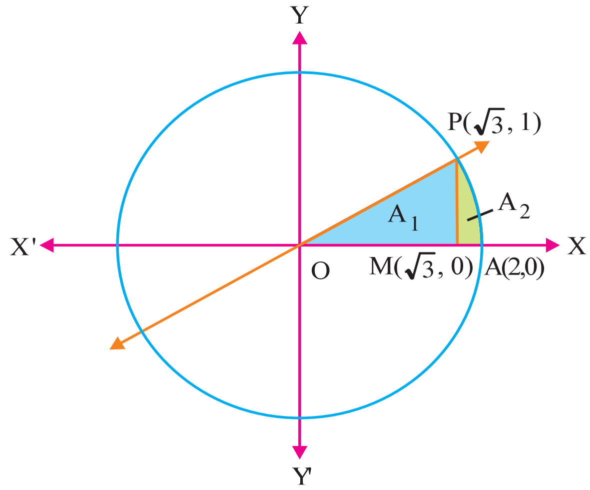


Figure 4.21

Now, area of the sector OPA.

= area of ΔOPM + Area of the region bounded by the circle $x^2 + y^2 = 4$, X-axis and the lines $x = \sqrt{3}$ and $x = 2$.

$\therefore$ Required area $= A_1 + A_2$

A_1 = Area of ΔOPM

$= \frac{1}{2}$OM $\times$ PM

$= \frac{1}{2}\sqrt{3} \times 1 = \frac{\sqrt{3}}{2}$ **(i)**

$A_2 = |\,I\,|$

where $I = \int_{\sqrt{3}}^{2} y\,dx = \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}\,dx$ **($y > 0$ in the first quadrant)**

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2} sin^{-1} \frac{x}{2}\right]_{\sqrt{3}}^{2}$$

$$= \left(0 + 2sin^{-1} 1\right) - \left(\frac{\sqrt{3}}{2} + 2sin^{-1} \frac{\sqrt{3}}{2}\right)$$

$$= \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$\therefore$ $A_2 = \left|\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right| = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$ **(ii)**

$\left[\frac{\pi}{3} > 1 \text{ as } \pi > 3 \text{ and } \sqrt{3} < 2 \text{ so } \frac{\sqrt{3}}{2} < 1. \text{ So, } \frac{\pi}{3} - \frac{\sqrt{3}}{2} > 0\right]$

$\therefore$ Required area $= \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

Second Method : Required area A = | I |

where $I = \int_0^1 (g_1(y) - g_2(y))\, dy$, where $g_1(y) = \sqrt{4-y^2}$ and $g_2(y) = \sqrt{3}y$

$$= \int_0^1 \left(\sqrt{4-y^2} - \sqrt{3}y\right) dy$$

$$= \left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\frac{y}{2} - \frac{\sqrt{3}}{2}y^2\right]_0^1$$

$$= \frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{1}{2} - \frac{\sqrt{3}}{2} = 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}$$

∴ Required area = $\frac{\pi}{3}$

Note : $y = \frac{x}{\sqrt{3}}$ means $y = mx$, where $m = \tan\theta = \frac{1}{\sqrt{3}}$ and $\theta = m\angle POM$.

So $m\angle POM = \frac{\pi}{6}$.

∴ Area of sector = $\frac{1}{2}r^2\theta = \frac{1}{2} \cdot 4 \cdot \frac{\pi}{6} = \frac{\pi}{3}$

We may feel that it is easy to find area using geometry than using calculus. But we have to use integration to derive formula $\frac{1}{2}r^2\theta$ for area of a sector.

Example 7 : Find the area of the region bounded by the parabola $y = x^2$ and the rays $y = |x|$.

Solution : Consider the curves $y = x^2$ **(i)**

and $y = |x|$ **(ii)**

The two curves intersect where $x^2 = |x|$

∴ $|x|^2 - |x| = 0$ **($x^2 = |x|^2$)**

∴ $|x|(|x| - 1) = 0$

∴ $x = 0$ or $x = \pm 1$

For $x = 0$, $y = 0$

For $x = \pm 1$, $y = 1$

Hence, the two curves intersect at the points (−1, 1), (0, 0) and (1, 1).

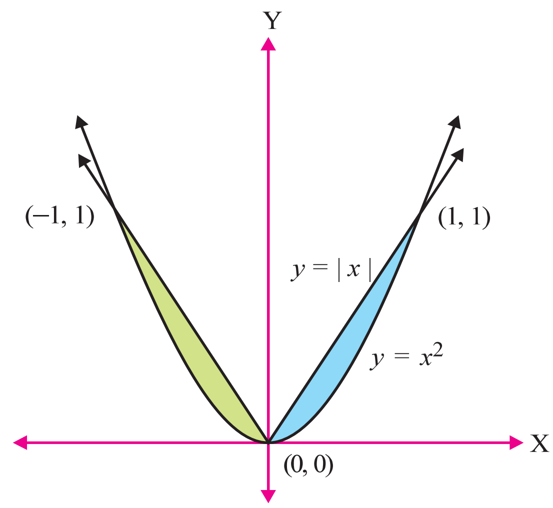


Figure 4.22

We have to find area of the region enclosed between given curves and is shown as coloured region in the figure 4.22.

As both the curves are symmetrical about Y-axis,

required area A = 2(area of the region in the first quadrant)

$= 2|I|$ where $I = \int_0^1 (f_1(x) - f_2(x))\, dx$, where $f_1(x) = |x|$ and $f_2(x) = x^2$

$$I = \int_0^1 (|x| - x^2)\, dx$$

$$= \int_0^1 (x - x^2)\, dx \qquad \textbf{(|x| = x in [0, 1])}$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6}$$

$\therefore$ Required area A $= 2 \times \frac{1}{6} = \frac{1}{3}$

Example 8 : Find the area of the region bounded by the circle $x^2 + y^2 = \frac{9}{4}$ and the parabola $x^2 = 4y$.

Solution : Given circle has equation $x^2 + y^2 = \frac{9}{4}$. **(i)**

and parabola has equation $x^2 = 4y$ **(ii)**

The two curves intersect at points where $4y = \frac{9}{4} - y^2$ **(each x^2)**

$\therefore$ $16y = 9 - 4y^2$

$\therefore$ $4y^2 + 16y - 9 = 0$

$\therefore$ $(2y - 1)(2y + 9) = 0$

$\therefore$ $y = \frac{1}{2}$ or $-\frac{9}{2}$

But $y \nless 0$, (Why ?) therefore the two curves intersect when $y = \frac{1}{2}$.

$\therefore$ $x^2 = 4y = 4 \times \frac{1}{2} = 2$

$\therefore$ $x = \pm\sqrt{2}$

$\therefore$ The two curves intersect at $\left(-\sqrt{2}, \frac{1}{2}\right)$ and $\left(\sqrt{2}, \frac{1}{2}\right)$.

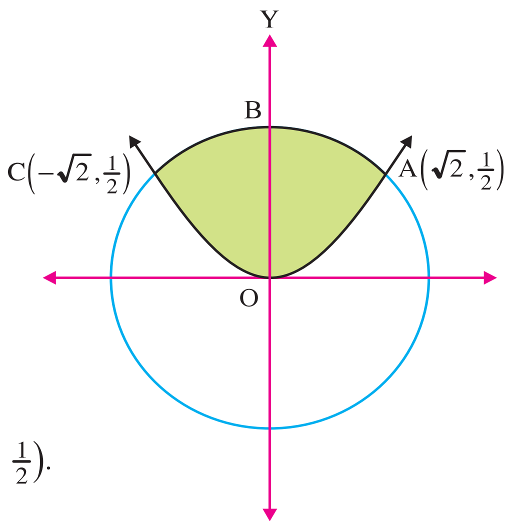


Figure 4.23

Since both the curves are symmetrical about Y-axis,

required area $= 2$(Area of region OABO)

$= 2\,|\,I\,|$

where I $= \int_0^{\sqrt{2}} (f_1(x) - f_2(x))\, dx$, where $f_1(x) = \sqrt{\frac{9}{4} - x^2}$ and $f_2(x) = \frac{x^2}{4}$.

$$= \int_0^{\sqrt{2}} \left(\sqrt{\frac{9}{4} - x^2} - \frac{x^2}{4}\right) dx$$

$$= \left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{\left(\frac{3}{2}\right)^2}{2} \sin^{-1}\frac{x}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^{\sqrt{2}}$$

$$= \left[\frac{\sqrt{2}}{2}\sqrt{\frac{9}{4} - 2} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{12}\right]$$

$$= \left[\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{\sqrt{2}}{6}\right]$$

$$= \frac{\sqrt{2}}{12} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\therefore \quad \text{Required area A} = 2\left[\frac{\sqrt{2}}{12} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]$$

$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Example 9 : Using integration, find the area of the triangular region whose vertices are (4, 1), (6, 6) and (8, 4).

Solution : Let A(4, 1), B(6, 6) and C(8, 4) be the vertices of a triangle ABC. (See figure 4.24)

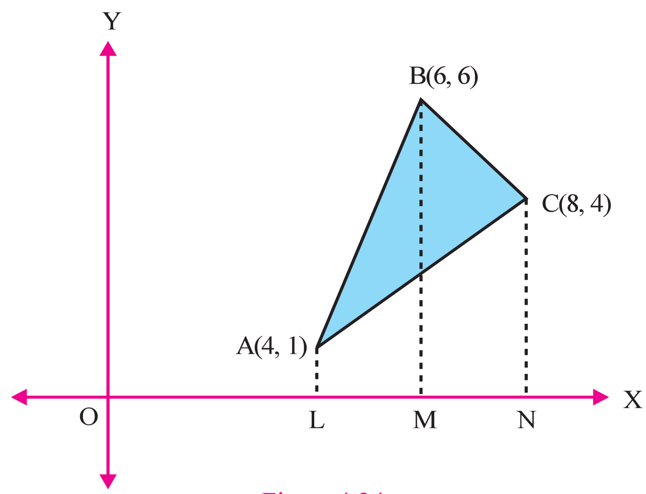


Figure 4.24

The equation of $\overleftrightarrow{AB}$ is $\frac{y-1}{6-1} = \frac{x-4}{6-4}$

$\therefore \quad y - 1 = \frac{5}{2}(x - 4)$

$\therefore \quad y - 1 = \frac{5}{2}x - 10$

$\therefore \quad y = \frac{5}{2}x - 9$

Similarly, the equation of $\overleftrightarrow{BC}$ is $y = -x + 12$ and the equation of $\overleftrightarrow{AC}$ is $y = \frac{3}{4}x - 2$

Let L, M, N be the feet of perpendiculars from A, B, C to X-axis respectively.

Now, area of ΔABC = area of region ALMB + area of region BMNC − area of region ALNC.

$$= |I_1| + |I_2| - |I_3|$$

$$= \left|\int_4^6 \left(\frac{5}{2}x - 9\right)dx\right| + \left|\int_6^8 (-x + 12)\,dx\right| - \left|\int_4^8 \left(\frac{3}{4}x - 2\right)dx\right|$$

$$= \left|\left[\frac{5x^2}{4} - 9x\right]_4^6\right| + \left|\left[-\frac{x^2}{2} + 12x\right]_6^8\right| - \left|\left[\frac{3x^2}{8} - 2x\right]_4^8\right|$$

$$= \left|\left[\left(\frac{5}{4}(36) - 54\right) - \left(\frac{5}{4}(16) - 36\right)\right]\right| + \left|\left[\left(-\frac{64}{2} + 96\right) - \left(-\frac{36}{2} + 72\right)\right]\right|$$

$$- \left|\left[\left(\frac{3}{8}(64) - 16\right) - \left(\frac{3}{8}(16) - 8\right)\right]\right|$$

$= |(-9 + 16)| + |(64 - 54)| - |(8 + 2)|$

$= 7 + 10 - 10$

$\therefore$ Required area $= 7$

Note : Area of the triangle $\Delta = \frac{1}{2}|D|$

where $D = \begin{vmatrix} 4 & 1 & 1 \\ 6 & 6 & 1 \\ 8 & 4 & 1 \end{vmatrix}$

$= 4(2) - 1(-2) + 1(-24) = -14$

$\therefore \quad \Delta = \frac{1}{2}|-14| = 7$

Example 10 : Find the area of the region bounded by the circle $x^2 + y^2 - 2ax = 0$ and the parabola $y^2 = ax$, $a > 0$ in the first quadrant.

Solution : The equation $x^2 + y^2 - 2ax = 0$ can be written as $(x - a)^2 + y^2 = a^2$ which represents a circle whose centre is $(a, 0)$ and radius is a. $y^2 = ax$ is a parabola whose vertex is $(0, 0)$ and its axis is X-axis.

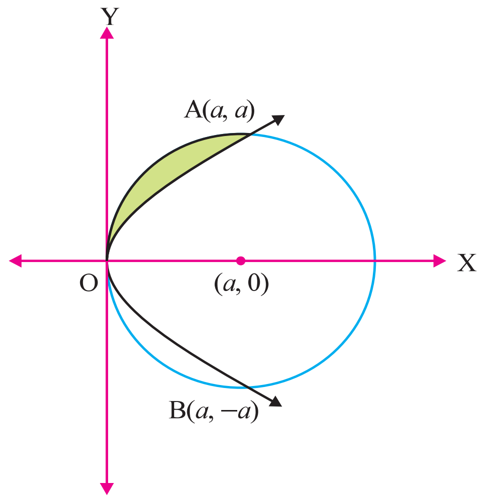


Figure 4.25

Substituting $y^2 = ax$ in $x^2 + y^2 - 2ax = 0$,

$x^2 + ax - 2ax = 0$

$\therefore \quad x^2 - ax = 0$

$\therefore \quad x(x - a) = 0$

$\therefore \quad x = 0$ or $x = a$

Since $y^2 = ax$,

$y = 0$ or $y = \pm a$

$\therefore$ Both the curves intersect at O(0, 0), A(a, a) and B(a, $-a$)

$\therefore \quad x^2 + y^2 = 2ax$ gives $y = \sqrt{2ax - x^2}$, $y^2 = ax$ gives $y = \sqrt{ax}$ **(as $y \geq 0$)**

Required area $= |I|$

where $I = \int_0^a (f_1(x) - f_2(x))\, dx$, where $f_1(x) = \sqrt{2ax - x^2}$ and $f_2(x) = \sqrt{ax}$.

$= \int_0^a \left(\sqrt{2ax - x^2} - \sqrt{ax}\right) dx$ **(First quadrant)**

$= \int_0^a \left(\sqrt{a^2 - (x-a)^2} - \sqrt{a}\sqrt{x}\right) dx$

$= \left[\left(\frac{x-a}{2}\right)\sqrt{a^2 - (x-a)^2} + \frac{a^2}{2} sin^{-1}\left(\frac{x-a}{a}\right) - \sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$

$$= \left[-\frac{2}{3}\sqrt{a}\cdot a^{\frac{3}{2}} - \frac{a^2}{2}\sin^{-1}(-1)\right]$$

$$\text{I} = -\frac{2}{3}a^2 + \frac{a^2\pi}{4} = \left(\frac{3\pi - 8}{12}\right)a^2$$

$\therefore$ Required area $= \left(\frac{3\pi - 8}{12}\right)a^2$

Example 11 : Find the area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 3$ and $x = 0$.

Solution : Here $y = x^2 + 2$

$\therefore$ $x^2 = y - 2$, which is a parabola whose vertex is (0, 2) and it opens upwards.

Let us draw a graph of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 3$ and $x = 0$.

Required area A = | I |

where $\text{I} = \int_0^3 (f_1(x) - f_2(x))\,dx,$

where $f_1(x) = x^2 + 2$ and $f_2(x) = x$.

$$= \int_0^3 (x^2 + 2 - x)\,dx$$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$$

$$= 9 + 6 - \frac{9}{2}$$

$$= \frac{21}{2}$$

$\therefore$ $\text{A} = \frac{21}{2}$

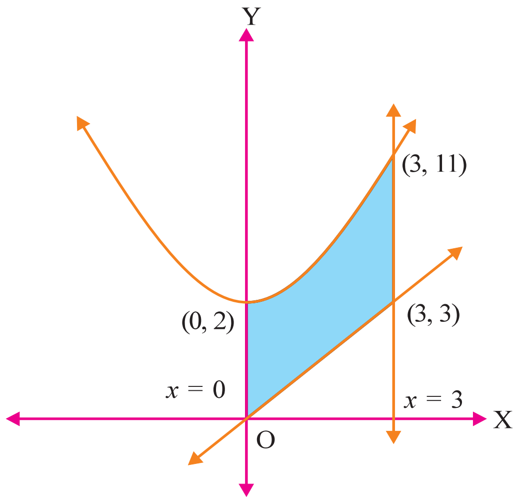


Figure 4.26

Example 12 : Find the area of the region bounded by the curves $y = 4 - x^2$, $x = 0$, $x = 3$ and X-axis.

Solution : Here $y = 4 - x^2$

So $x^2 = 4 - y$

$\therefore$ $x^2 = -(y - 4)$, which is the equation of a parabola. Its vertex is (0, 4) and opens downwards. To find its point of intersection with X-axis, let us take $y = 0$.

$\therefore$ $4 - x^2 = 0$

$\therefore$ $x = \pm 2$

$\therefore$ The points of intersection of the curve with X-axis are (2, 0) and (−2, 0).

Here, the limits of the region bounded by the curve and the X-axis are $x = 0$ and $x = 3$. The curve intersects X-axis at (2, 0) between (0, 0) and (3, 0).

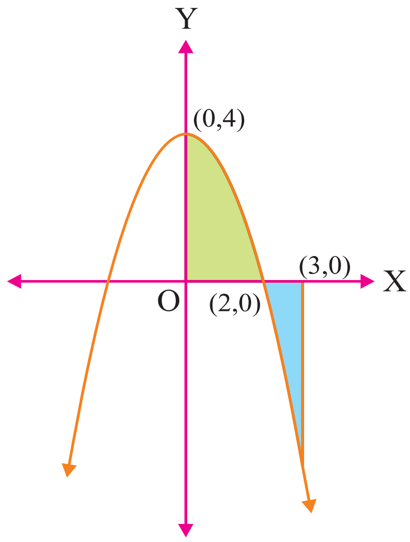


Figure 4.27

So, $A = |I_1| + |I_2|$

where $I_1 = \int_0^2 y\,dx, \quad I_2 = \int_2^3 y\,dx$

$$I_1 = \int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

$$I_2 = \int_2^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_2^3 = (12 - 9) - \left(8 - \frac{8}{3}\right)$$

$$= 3 - \frac{16}{3} = -\frac{7}{3}$$

$\therefore$ Required area $A = \left|\frac{16}{3}\right| + \left|-\frac{7}{3}\right| = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$

Example 13 : Find the area bounded by the curve $y = cosx$ between $x = 0$ and $x = 2\pi$.

Solution :

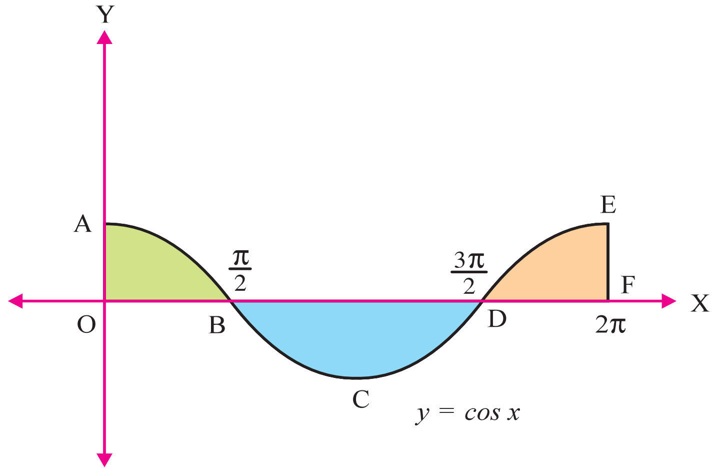


Figure 4.28

From the figure 4.28, the required area = area of the region OABO + area of the region BCDB + area of the region DEFD

$\therefore$ Required area $= \left|\int_0^{\frac{\pi}{2}} cosx\;dx\right| + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} cosx\;dx\right| + \left|\int_{\frac{3\pi}{2}}^{2\pi} cosx\;dx\right|$

$$= \left|[sin\,x]_0^{\frac{\pi}{2}}\right| + \left|[sin\,x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| + \left|[sin\,x]_{\frac{3\pi}{2}}^{2\pi}\right|$$

$$= |(1 - 0)| + |(-1 - 1)| + |(0 - 1)|$$

$$= 1 + 2 + 1 = 4$$

Example 14 : Determine the area of the region enclosed by $y = sinx$, $y = cosx$, $x = \frac{\pi}{2}$ and the Y-axis.

Solution : First let us draw the graph of the region.

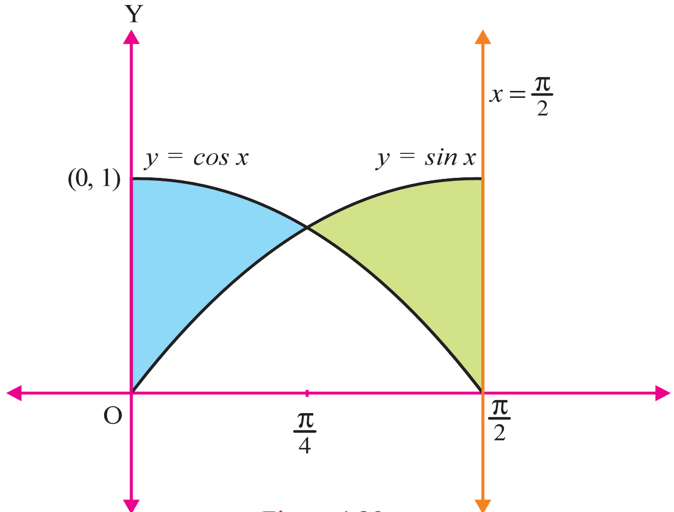


Figure 4.29

Now, from the figure it is clearly seen that we have a situation where we will need to evaluate two integrals to get the area. The point of intersection of $y = \sin x$ and $y = \cos x$ will be where $\sin x = \cos x$ in $\left[0, \frac{\pi}{2}\right]$.

This gives $x = \frac{\pi}{4}$. **(Why ?)**

The required area $A = |I_1| + |I_2|$

where $I_1 = \int_0^{\frac{\pi}{4}} (f_1(x) - f_2(x))\, dx$, where $f_1(x) = \cos x$ and $f_2(x) = \sin x$.

$$= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx$$

$$= [\sin x + \cos x]_0^{\frac{\pi}{4}}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0 + 1)\right] = \sqrt{2} - 1 \qquad \textbf{(i)}$$

$$I_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (f_1(x) - f_2(x))\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos x - \sin x)\, dx$$

$$= [\sin x + \cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[(1 + 0) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\right]$$

$$= 1 - \sqrt{2} < 0 \qquad \textbf{(ii)}$$

$\therefore \quad |I_2| = \sqrt{2} - 1$

From (i) and (ii) required area $A = |I_1| + |I_2| = \sqrt{2} - 1 + \sqrt{2} - 1 = 2(\sqrt{2} - 1)$

Exercise 4.2

1. Find the area of the region enclosed by parabola $4y = 3x^2$ and the line $2y = 3x + 12$.
2. Find the area of the region bounded by curves $y = 2x - x^2$ and the line $y = -x$.
3. Find the area of the region bounded by the curves $f(x) = cos\pi x$ and X-axis where $x \in [0, 2]$.
4. Find the area of the region bounded by the curves $f(x) = 4 - x^2$ and $g(x) = x^2 - 4$.
5. Find the area of the region bounded by the curves $y = x$, $y = 1$ and $y = \frac{x^2}{4}$ lying in the first quadrant.
6. Find the area of the region enclosed by the curves $y = x^2 + 5x$ and $y = 3 - x^2$ and bounded by $x = -2$ and $x = 0$.
7. Find the area bounded by the curves $y = x^2$, $y = 2 - x$ and above the line $y = 1$.
8. Determine the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$.
9. Using integration, find the area of the triangular region whose sides lie along the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.
10. Using integration, find the area of the triangular region formed by $(-1, 1)$, $(0, 5)$ and $(3, 2)$.
11. Find the area of the region in the first quadrant enclosed by the X-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.
12. Find the area of the region bounded by $y = 5 - x^2$, $x = 2$, $x = 3$ and X-axis.

*

Region Represented by Inequalities :

Consider $\{(x, y) \mid 0 \le y \le x^2\}$.

As shown in the figure 4.30, if we consider any point $P(x, y)$ on $\overline{AB}$, then $y \ge 0$ and $y \le x^2$.

So if B is any point on the parabola and A is on X-axis such that $\overline{AB} \perp$ X-axis then any point $P(x, y) \in \overline{AB}$ will satisfy $0 \le y \le x^2$.

Now, consider $\{(x, y) \mid 0 \le y \le x^2, 0 \le y \le x + 2, x \ge 0\}$

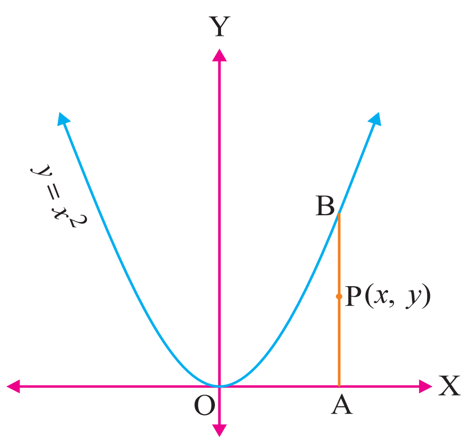


Figure 4.30

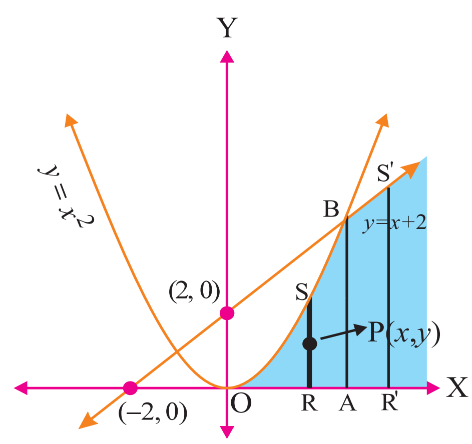


Figure 4.31

As shown in the figure 4.31, if we consider any point $P(x, y)$ on $\overline{RS}$, then $y \ge 0$, $y \le x^2$ and $y \le x + 2$. Similarly for any point on $\overline{R'S'}$ also conditions satisfied.

All such points P form a set satisfying given conditions. The region represented by the given set is coloured in the figure 4.31.

Miscellaneous Examples :

Example 15 : Find the area of the region : $\{(x, y) \mid 0 \le y \le x^2, 0 \le y \le x + 2, 0 \le x \le 3\}$.

Solution : Let us first sketch the region whose area is to be found out.

We have $0 \le y \le x^2$ **(i)**

$0 \le y \le x + 2$ **(ii)**

$0 \le x \le 3$ **(iii)**

Draw the curve $y = x^2$, a parabola with origin as vertex.

The line $y = x + 2$ intersects the parabola $y = x^2$, where $x + 2 = x^2$

$\therefore \quad x^2 - x - 2 = 0$

$\therefore \quad (x - 2)(x + 1) = 0$

$\therefore \quad x = 2, -1$

For $x = 2$, $y = 4$ and for $x = -1$, $y = 1$

The points of intersection of $y = x^2$ and $y = x + 2$ are P(2, 4) and M(−1, 1).

Since $0 \le x \le 3$ the above region is as shown as coloured region OPQRSO in figure 4.32.

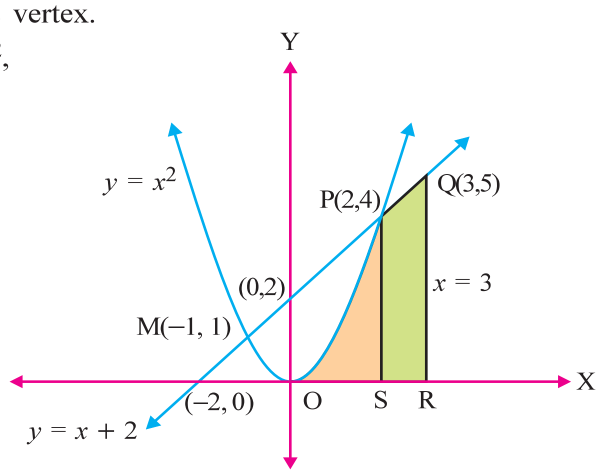


Figure 4.32

The required area A = area of region OPSO + area of the region SPQRS

The area of the region OPSO is bounded by the curve $y = x^2$, $x = 0$, $x = 2$ and X-axis.

The area of the region SPQRS is bounded by $y = x + 2$, $x = 2$, $x = 3$ and X-axis.

$$\therefore \quad \text{Required area} = \int_0^2 x^2 dx + \int_2^3 (x + 2)\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{x^2}{2} + 2x\right]_2^3$$

$$= \left(\frac{8}{3} - 0\right) + \left(\frac{9}{2} + 6\right) - (2 + 4)$$

$$= \frac{43}{6}$$

Example 16 : Find the area of the region enclosed by two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

Solution : Here, $x^2 + y^2 = 1$

$\therefore \quad y^2 = 1 - x^2$

$(x - 1)^2 + y^2 = 1$

$\therefore \quad y^2 = 1 - (x - 1)^2$

For points of intersection, $1 - x^2 = 1 - (x - 1)^2$

$\therefore \quad -x^2 = -x^2 + 2x - 1$

$\therefore \quad x = \frac{1}{2}$

$$\therefore \quad y = \pm\sqrt{1 - x^2} = \pm\sqrt{1 - \frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$$

Hence the circles intersect at the points $A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $B\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

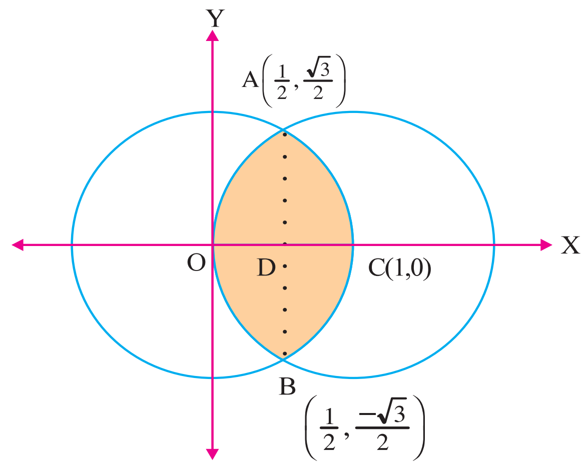


Figure 4.33

Required area = area of the region OACBO.

Since both the circles are symmetric about X-axis, the required area,

= 2(area of the region OACDO)

= 2[area of the region OADO + area of the region DACD)

The area of the region OADO is bounded by the circle $(x - 1)^2 + y^2 = 1$

i.e., $y = \sqrt{1-(x-1)^2}$ (first quadrant), $x = 0$, $x = \frac{1}{2}$ and X-axis, while the area of the region DACD is bounded by the circle $x^2 + y^2 = 1$. i.e. $y = \sqrt{1-x^2}$, $x = \frac{1}{2}$, $x = 1$ and X-axis.

The required area is sum of the two areas. **(Why not $|I_1| + |I_2|$?)**

$$\text{Required area} = 2\left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\, dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\, dx\right]$$

$$= 2\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}} + 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$= 2\left[\frac{1}{2}\left(-\frac{1}{2}\right)\frac{\sqrt{3}}{2} + \frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right) - 0 - \frac{1}{2}\sin^{-1}(-1)\right] +$$

$$2\left[0 + \frac{1}{2}\sin^{-1}1 - \frac{1}{4}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left(-\frac{\sqrt{3}}{8} - \frac{\pi}{12} + \frac{\pi}{4}\right) + 2\left(\frac{\pi}{4} - \frac{\sqrt{3}}{8} - \frac{\pi}{12}\right)$$

$$= 2\left(-\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2}\right) = 2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$

Second Method :

Required area = | I |,

$$I = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} (g_1(y) - g_2(y))\, dy$$

where $g_1(y) = \sqrt{1-y^2}$ and $g_2(y) = 1 - \sqrt{1-y^2}$ **(Why ?)**

$$I = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left[\sqrt{1-y^2} - \left(1 - \sqrt{1-y^2}\right)\right] dy$$

$$= 2\int_{0}^{\frac{\sqrt{3}}{2}} \left(2\sqrt{1-y^2} - 1\right) dy$$

$$= 4\int_{0}^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-y^2} - \frac{1}{2}\right) dy$$

$$= 4\left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}y - \frac{y}{2}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= 4\left[\frac{\sqrt{3}}{4}\sqrt{1-\frac{3}{4}} + \frac{1}{2}\sin^{-1}\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\right]$$

$$= 4\left[\frac{\sqrt{3}}{4}\cdot\frac{1}{2} + \frac{1}{2}\times\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = 2\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$

$\therefore$ Required area $= 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$

Note : From figure 4.34, $OM = \frac{1}{2}$, $AM = \frac{\sqrt{3}}{2}$

Therefore $m\angle AOM = \frac{\pi}{3}$

$\therefore$ Area of sector OAC $= \frac{1}{2}(1)^2\frac{\pi}{3} = \frac{\pi}{6}$

$\therefore$ Area of $\Delta AOM = \frac{1}{2} \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{8}$

$\therefore$ $A_2 = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

Similarly, $A_1 = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$

$\therefore$ Required area $= 2\left[\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) + \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right)\right]$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

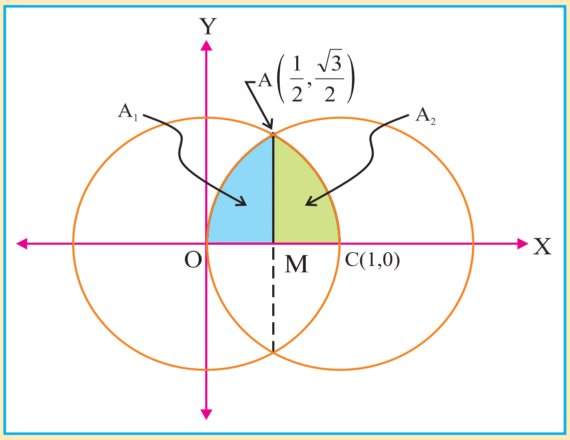


Figure 4.34

Exercise 4

1. Find the area of the region bounded by the curve $y = x^2 - x - 6$ and the X-axis.
2. Find the area of the region bounded by the Y-axis, the line $y = 3$ and the curve $y = x^2 + 2$ in the first quadrant.
3. Calculate the area bounded by the curve $y = (x - 1)(x - 2)$ and the X-axis.
4. Find the area of the region bounded by the circle $x^2 + y^2 = 3$, line $x - y\sqrt{3} = 0$ and the X-axis in the first quadrant.
5. Determine the area enclosed between the two curves $y^2 = x + 1$ and $y^2 = -x + 1$.

6. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

7. Find the area lying in the first quadrant enclosed by X-axis, the circle $x^2 + y^2 = 8x$ and parabola $y^2 = 4x$.

8. Find the area of the region bounded by the line $y = 3x + 2$, the X-axis and the lines $x = -1$ and $x = 1$.

9. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three congruent parts.

10. Find the area of the region $\{(x, y) \mid 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

11. Find the area of the region bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$.

12. Find the area of the region enclosed by $y^2 = 8x$ and $x + y = 0$.

13. Using integration, find the area of the region bounded by the curve $|x| + |y| = 1$.

14. Using integration, find the area of the given region : $\{(x, y) \mid |x - 1| \le y \le \sqrt{5 - x^2}\}$.

15. Find the area of the region enclosed by the parabola $y^2 = x$ and the line $x + y = 2$.

16. Find the area of the region bounded by $y = x^2 + 1$, $y = x$, $x = 0$ and $y = 2$.

17. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

(1) The area enclosed by $y = x$, $y = 1$, $y = 3$ and the Y-axis is .... ☐

(a) 2 (b) $\frac{9}{2}$ (c) 4 (d) $\frac{3}{2}$

(2) The area enclosed by the curve $y = 2x - x^2$ and the X-axis is .... ☐

(a) $\frac{8}{5}$ (b) 2 (c) 8 (d) $\frac{4}{3}$

(3) The area enclosed by $y = \cos x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and the X-axis is .... ☐

(a) 1 (b) 4 (c) 2 (d) π

(4) The area bounded by the curve $y = \sin x$, $\pi \le x \le 2\pi$ and the X-axis is .... ☐

(a) π (b) 2 (c) -2 (d) 0

(5) The area enclosed by $y = x^2$, the X-axis and the line $x = 4$ is divided into two congruent halves by the line $x = a$. The value of a is .... ☐

(a) 2 (b) $2^{\frac{4}{3}}$ (c) $2^{\frac{5}{3}}$ (d) 4

(6) The area of the region bounded by the lines $x = 2y + 3$, $y = 1$, $y = -1$ and Y-axis is .... ☐

(a) 4 (b) $\frac{3}{2}$ (c) 6 (d) 8

(7) The area bounded by the parabola $y^2 = 4ax$ and its latus rectum is .... ☐

(a) $\frac{4}{3}a^2$ (b) $\frac{8}{3}a^2$ (c) $\frac{16}{3}a^2$ (d) $\frac{32}{3}a^2$

(8) Area bounded by the curve $y = 2x^2$, the X-axis and the line $x = 1$ is ....

(a) 2 (b) 1 (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

(9) The area bounded by the curve $y = x|x|$, X-axis and the lines $x = -1$ and $x = 1$ is ......

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

(10) The area bounded by the curves $y = cosx$, $y = sinx$, Y-axis and $0 \le x \le \frac{\pi}{4}$ is ....

(a) $2(\sqrt{2} - 1)$ (b) $\sqrt{2} - 1$ (c) $\sqrt{2} + 1$ (d) $\sqrt{2}$

(11) Area bounded by the line $y = 3 - x$ and the X-axis on the interval [0, 3] is ....

(a) $\frac{9}{2}$ (b) 4 (c) 5 (d) $\frac{11}{2}$

(12) Area bounded by the curves $y = x^2$ and $x = y^2$ is ....

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{12}$ (d) 1

(13) Area bounded by the curve $y = sinx$ bounded by $x = 0$ and $x = 2\pi$ is ....

(a) 1 (b) 2 (c) 3 (d) 4

(14) The area bounded by the curve $y = 3\,cosx$, $0 \le x \le \frac{\pi}{2}$, $y = 0$ is ....

(a) 3 (b) 1 (c) $\frac{3}{2}$ (d) $\frac{1}{2}$

(15) The area under the curve $y = cos^2x$ between $x = 0$ and $x = \pi$ is ....

(a) π (b) $\frac{\pi}{2}$ (c) 2π (d) 2

(16) The area under the curve $y = 2\sqrt{x}$ bounded by the lines $x = 0$ and $x = 1$ is ....

(a) $\frac{4}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{8}{3}$

(17) The area bounded by $y = 2x - x^2$ and X-axis is ....

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) 1 (d) $\frac{4}{3}$

(18) The area bounded by the curve $y = 3x$, X-axis and the lines $x = 1$, $x = 3$ is ....

(a) 3 (b) 6 (c) 12 (d) 36

(19) The area bounded by the curve $y = |x - 5|$, X-axis and the lines $x = 0$, $x = 1$ is ....

(a) $\frac{9}{2}$ (b) $\frac{7}{2}$ (c) 9 (d) 5

(20) The area of the region between the curve $y^2 = 4x$ and the line $x = 3$ is ....

(a) $4\sqrt{3}$ (b) $8\sqrt{3}$ (c) $16\sqrt{3}$ (d) $5\sqrt{3}$

Summary

We have studied the following points in this chapter :

1. The area A of the region bounded by the curve $y = f(x)$, X-axis and the lines $x = a$, $x = b$ is given by $A = |I|$, where $I = \int_a^b f(x)\,dx$.

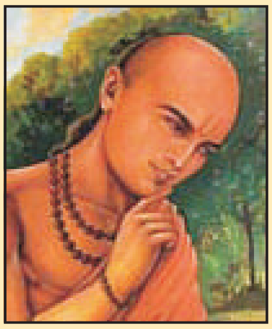
2. The area A of the region bounded by the curve $x = g(y)$, Y-axis and the lines $y = c$, $y = d$ is given by $A = |I|$, where $I = \int_c^d g(y)\,dy$.

3. If the graph of $y = f(x)$ intersects X-axis at $(c, 0)$ only and $a < c < b$, then the area of the region bounded by $y = f(x)$, $x = a$, $x = b$ and X-axis is given by $A = |I_1| + |I_2|$, where $I_1 = \int_a^c f(x)\,dx$, $I_2 = \int_c^b f(x)\,dx$.

4. If the two curves $y = f_1(x)$ and $y = f_2(x)$ intersect each other at only two points for $x = a$ and $x = b$ $(a \neq b)$, then the area enclosed by them is given by $A = |I|$, where $I = \int_a^b (f_1(x) - f_2(x))\,dx$.

5. If the two curves $x = g_1(y)$ and $x = g_2(y)$ intersect each other at only two points for $y = c$ and $y = d$ $(c \neq d)$, then the area enclosed by them is given by $A = |I|$, where $I = \int_c^d (g_1(y) - g_2(y))\,dy$.

BHASKARACHARYA



He was born in a village of Mysore district.

He was the first to give that any number divided by 0 gives infinity.

He has written a lot about zero, surds, permutation and combination.

He wrote, "The hundredth part of the circumference of a circle seems to be straight. Our earth is a big sphere and that's why it appears to be flat."

He gave the formulae like $\sin(A \pm B) = \sin A \cdot \cos B \pm \cos A \cdot \sin B$

DIFFERENTIAL EQUATIONS 5

Mathematics is the art of giving the same name to different things.
– Jules Henri

5.1 Introduction

If y is a function of x, then we denote it as $y = f(x)$. Here x is called **an independent variable** and y is called **a dependent variable**. We have already learnt various methods to find $\frac{dy}{dx}$ or $f'(x)$. Also we know how to find f using indefinite integration when we are given an equation like $f'(x) = g(x)$ (Primitive) i.e. $\frac{dy}{dx} = g(x)$

Here the equation $\frac{dy}{dx} = g(x)$ contains the variable x and derivative of y *w.r.t.* x. This type of an equation is known as **a differential equation**. We will give a formal definition later.

Differential equations play an important role in the solution of problems of Physics, Chemistry, Biology, Engineering etc. Here we will study the basic concepts of differential equations, the solution of a first order - first degree differential equation and also simple applications of differential equations.

Note : If the function $y = f(x)$ is a differentiable function of x, then its first order derivative is denoted by $\frac{dy}{dx}$, y_1, y' or $f'(x)$. If $f'(x)$ is also a differentiable function of x, then the second order derivative of the function $y = f(x)$ is denoted by $\frac{d^2y}{dx^2}$, y_2, y'' or $f''(x)$. Similarly we may get third order, fourth order derivatives of the function $y = f(x)$ etc. In general nth order derivative of the function $y = f(x)$ is denoted by the symbols $\frac{d^ny}{dx^n}$, y_n, $y^{(n)}$ or $f^{(n)}(x)$. Here, $y_n = \frac{d}{dx}(y_{n-1})$.

5.2 Differential Equation

An equation containing an independent variable and a dependent variable and the derivatives of the dependent variable with respect to the independent variable is called an ordinary differential equation.

If x is an independent variable, y is a dependent variable depending upon x i.e. $y = f(x)$ or $G(x, y) = 0$ and the derivatives of y *w.r.t.* x are $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3},...$ then the functional equation $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3},..., \frac{d^ny}{dx^n}\right) = 0$ is called an ordinary differential equation (Derivatives must occur in this equation)

For instance, (1) $\frac{dy}{dx} + y \cos x = \sin x$ (2) $\frac{d^2y}{dx^2} = 2x$

(3) $\frac{dy}{dx} + y = x^2$ (4) $2y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

(5) $2x^2\left(\frac{d^2y}{dx^2}\right)^3 + 5y\frac{dy}{dx} = 2xy$ (6) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 5\frac{d^2y}{dx^2}$

(7) $e^{\frac{dy}{dx}} + \frac{dy}{dx} = ky$ (8) $\log\left|\frac{dy}{dx}\right| = kx$

5.3 Order and Degree of a Differential Equation

Order of the highest order derivative of the dependent variable with respect to the independent variable occurring in a given differential equation is called the order of differential equation.

(1) $\frac{dy}{dx} + y \cos x = \sin x$

The order of the highest order derivative is 1. So it is a differential equation of order 1.

(2) $2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = e^x$

The order of the highest order derivative is 2. So it is a differential equation of order 2.

(3) $\left(\frac{dy}{dx}\right)^2 + 6y + x = 0.$

The order of the highest order derivative is 1. So it is a differential equation of order 1.

(4) $\frac{d^4y}{dx^4} - 6\left(\frac{dy}{dx}\right)^6 - 4y = 0.$

The order of the highest order derivative is 4. So it is a differential equation of order 4.

(5) $\frac{d^2y}{dx^2} = \sqrt{\frac{dy}{dx}+5}.$

The order of the highest order derivative is 2. So it is a differential equation of order 2.

Degree of a Differential Equation :

When a differential equation is in a polynomial form in derivatives, the highest power of the highest order derivative occurring in the differential equation is called the degree of the differential equation.

Obviously to obtain the degree of a differential equation, we should make the equation free from radicals and fractional powers.

The degree of a differential equation is a positive integer.

(1) $\left(\frac{dy}{dx}\right)^2 + 2y = \sin x.$

In this equation the highest power of the highest order derivative is 2. So the degree of the differential equation is 2.

(2) $\frac{d^3y}{dx^3} + 7\left(\frac{dy}{dx}\right)^4 - 4y = 0$

In this equation the highest power of the highest order derivative is 1. So its degree is 1. (Why not 4?)

(3) $x = y\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Convert this equation in a polynomial form in derivatives.

We get, $(y^2 - 1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + x^2 - 1 = 0$

In this equation, the power of highest order derivative is 2. So the differential equation has degree 2.

Note : To determine the degree, the differential equation has to be expressed in a polynomial form. If the differential equation cannot be expressed in a polynomial form in the derivatives, the degree of the differential equation is not defined.

For example,

(1) $x\frac{dy}{dx} + sin\left(\frac{dy}{dx}\right) = 0$ is a given differential equation. Its order is 1 and degree is not defined because the equation is not in a polynomial form in derivatives.

(2) $\frac{d^2y}{dx^2} = \log\left(\frac{dy}{dx}\right) + y$, the order of the equation is 2 and the degree is not defined because we cannot express this equation in a polynomial form in derivatives.

Example 1 : Obtain the order and degree (if possible) of the following differential equation :

(1) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y = x^2$

(2) $\frac{d^2y}{dx^2} = \sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}$

(3) $xe^{\frac{dy}{dx}} + \frac{dy}{dx} + 2 = 0$

(4) $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + xy = 0$

(5) $\left(\frac{d^2y}{dx^2}\right)^3 = siny + 3x$

Solution : (1) The highest order derivative is $\frac{d^3y}{dx^3}$ and its power is 1.

$\therefore$ The differential equation has order 3 and degree 1.

(2) $\frac{d^2y}{dx^2} = \sqrt[3]{1+\left(\frac{dy}{dx}\right)^2}$

To make it radical free, we cube both the sides.

$\therefore$ $\left(\frac{d^2y}{dx^2}\right)^3 = 1 + \left(\frac{dy}{dx}\right)^2$

This differential equation has order 2 and degree 3.

(3) The highest order derivatives is $\frac{dy}{dx}$. Hence the differential equation has order 1. But we can not express the differential equation in a polynomial form in derivatives. So the degree is not defined.

(4) The highest order derivative is $\frac{d^2y}{dx^2}$ and its power is 1, so the differential equation has order 2 and degree 1.

(5) The highest order derivative is $\frac{d^2y}{dx^2}$ and its power is 3, so the differential equation has order 2 and degree 3.

Exercise 5.1

Obtain the order and degree (if possible) of the following differential equations :

1. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2$

2. $x + \left(\frac{dy}{dx}\right)^2 = \sqrt{1+y}$

3. $\frac{d^2y}{dx^2} + sin\left(\frac{dy}{dx}\right) + y = 0$

4. $y^{\frac{dy}{dx}} = x$

5. $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^4 + x \log y = 0$

6. $\sqrt[3]{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx}}$

7. $\left(\frac{dy}{dx}\right) + \frac{x}{\left(\frac{dy}{dx}\right)} = 0$

8. $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 = 0$

9. $\frac{d^2y}{dx^2} = 3sin\,3x$

10. $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^5 - 5y = 0$

*

5.4 Formation of a Differential Equation

Now let us try to understand a family of curves. Consider the equation $x^2 + y^2 = r^2$ **(i)**
and assign different values to r.

If $r = 1$, then $x^2 + y^2 = 1$

If $r = 2$, then $x^2 + y^2 = 4$

If $r = 3$, then $x^2 + y^2 = 9$

If $r = 4$, then $x^2 + y^2 = 16$

From the above equations, it is clear that equation (i) represents **a family of concentric circles** having center at origin and having different radii.

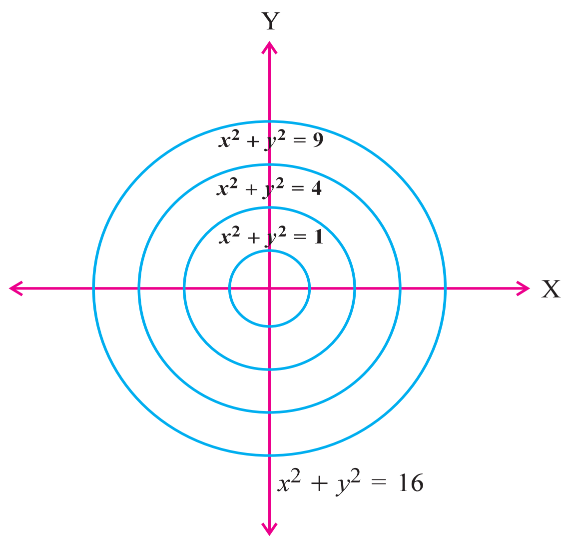


Figure 5.1

Now we are interested to find the differential equation which is satisfied by each member of the family irrespective of radius. The above equation has one arbitrary constant. i.e. r. We should find an equation which is free from r.

Differentiate $x^2 + y^2 = r^2$ *w.r.t.* x

So $2x + 2y\frac{dy}{dx} = 0$

$$x + y\frac{dy}{dx} = 0$$

This is the required differential equation satisfied by all the members of the family of concentric circles $x^2 + y^2 = r^2$ and note that it does not contain arbitrary constant r.

Example 2 : Obtain the differential equation of the family of parabolas having vertex at origin and having Y-axis as axis.

Solution : We know that the equation of the family of parabolas having vertex at origin and axis along positive direction of Y-axis is $x^2 = 4by$.

Let S(0, b) be the focus of one of these parabolas where b is an arbitrary constant.

Now differentiating both the sides of the equation $x^2 = 4by$ *w.r.t.* x we get,

$\therefore \quad 2x = 4b \dfrac{dy}{dx}$

$\therefore \quad 2xy = 4by \dfrac{dy}{dx}$

But $4by = x^2$

$\therefore \quad x^2 \dfrac{dy}{dx} = 2xy \quad$ or $\quad x^2 \dfrac{dy}{dx} - 2xy = 0$

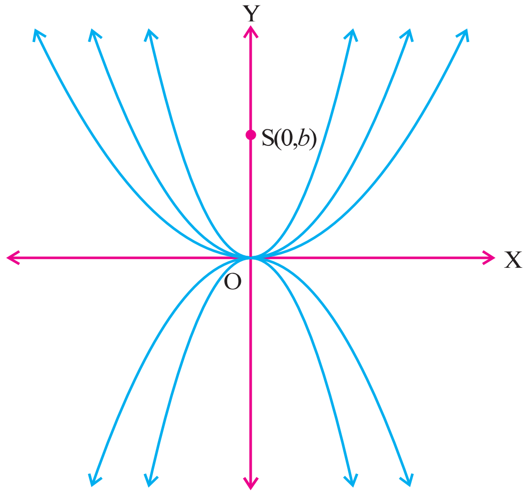


Figure 5.2

$\therefore \quad x \dfrac{dy}{dx} = 2y \qquad$ **($x \neq 0$)**

This is the differential equation of the given family of parabolas.

Note : If $x = 0$, then $y = 0$, since $x^2 = 4by$.

$\therefore \quad$ (0, 0) also satisfies $x \dfrac{dy}{dx} = y$.

Example 3 : Obtain the differential equation of family of all the parallel lines represented by $y = 2x + c$ having slope 2. (c is an arbitrary constant).

Solution : $y = 2x + c$ is the given equation of line where c is an arbitrary constant.

For distinct values of c we get different lines. All the lines are parallel to each other.

So, $y = 2x + c$, (c abitrary constant) is a family of parallel lines.

Now we shall find an equation not containing the arbitrary constant and which is satisfied by all such members of the family of parallel lines.

Hence differentiating $y = 2x + c$ with respect to x.

$$\frac{dy}{dx} = 2$$

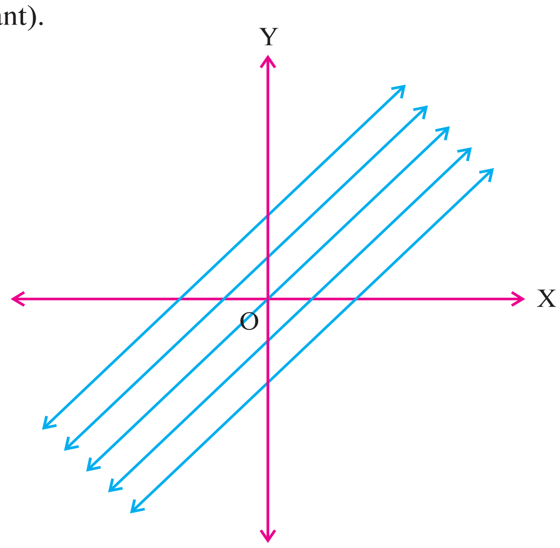


Figure 5.3

This equation not containing arbitrary constant represents the differential equation of family of lines.

Example 4 : Obtain the differential equation of the family of curves $y = a\, sin(x + b)$, (a and b are arbitrary constants).

Solution : $y = a\, sin(x + b)$ is a given family curves.

Differentiating *w.r.t.* x, $\dfrac{dy}{dx} = a\, cos(x + b)$

Again differentiating *w.r.t.* x,

$$\frac{d^2y}{dx^2} = -a\, sin(x + b)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -y \quad \text{or} \quad \frac{d^2y}{dx^2} + y = 0$$ is the differential equation representing the given family.

From examples 2 and 3, we can say that the differential equation of a family of curves having one arbitray constant is of order one. From example 4, we can say that the differential equation of a family of curves having two arbitrary constants is of order two. From these examples let us understand the formation of a differential equation as under.

(a) If the family of curves has only one arbitrary constant c, then it can be represented by the equation $f(x, y, c) = 0$. Differentiating above equation *w.r.t.* x, we get a new functional relation showing relation among x, y, y' and c. Let this functional relation be $g(x, y, y', c) = 0$

Now eliminating c from the equations $f(x, y, c) = 0$ and $g(x, y, y', c) = 0$, we get an equation $F(x, y, y') = 0$ representing differential equation of the family $f(x, y, c) = 0$.

(b) If the family of curves has two arbitrary constants c_1 and c_2, then it can be represented by the equation $f(x, y, c_1, c_2) = 0$.

Differentiating *w.r.t.* x, we get a new functional relation showing relation among x, y, y', c_1 and c_2. Let this functional relation be the equation $g(x, y, y', c_1, c_2) = 0$ relating x, y, y', c_1 and c_2. But both arbitrary constants c_1 and c_2 can not be eliminated from only these two equations. Differentiating equation $g(x, y, y', c_1, c_2) = 0$ again *w.r.t.* x,

the equation $h(x, y, y', y'', c_1, c_2) = 0$ is obtained relating x, y, y', y'', c_1 and c_2.

Now eliminating arbitrary constants c_1 and c_2 from $f(x, y, c_1, c_2) = 0$ and $g(x, y, y', c_1, c_2) = 0$ and $h(x, y, y', y'', c_1, c_2) = 0$ we get an equation $F(x, y, y', y'') = 0$ which represents the differential equation of given family $f(x, y, c_1, c_2) = 0$.

In short differentiating n times, the functional relation $f(x, y, c_1, c_2, ..., c_n) = 0$ containing n arbitrary constants, we get $(n + 1)$ equations including given equation.

Eliminating $c_1, c_2, ..., c_n$; we get the differential equation of the given family. Remember that, if the number of arbitrary constants is n, then the order of the differential equation so obtained is also n.

Example 5 : Obtain the differential equation representing the family of ellipses having focii on X-axis or Y-axis and centre at the origin.

Solution : We have the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ where, a and b are arbitrary constant. **($a \neq b$) (i)**

This equation represents a family of ellipses.

Differentiating equation (i) *w.r.t.* x,

We get $$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\therefore \quad y\frac{dy}{dx} = -\frac{b^2}{a^2}x \qquad \textbf{(ii)}$$

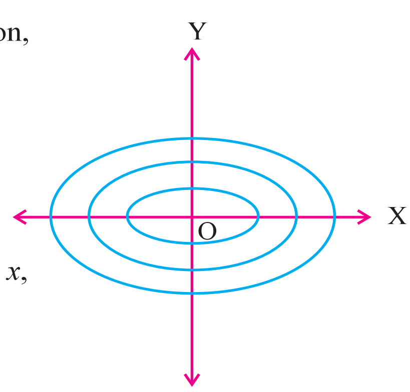


Figure 5.4(a)

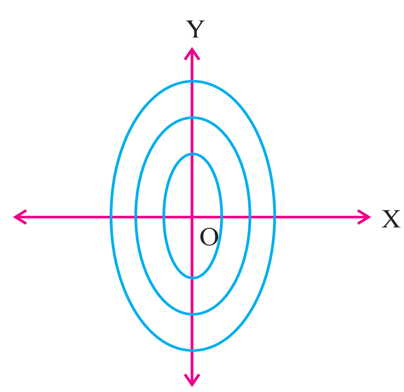


Figure 5.4(b)

Differentiating both the sides of equation (ii) *w.r.t.* x,

We get, $\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}$

Multiply by x on both sides

$$x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}x$$

$$\therefore \quad x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} = y\frac{dy}{dx} \qquad \textbf{(using (ii))}$$

$$\therefore \quad x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0$$

This is the required differential equation representing the family of ellipses.

Note : There are two arbitrary constants. So we have differentiated twice. The differential equation is of order 2.

Example 6 : Find the differential equation of the family of circles having centre on X-axis and radius 1 unit.

Solution :

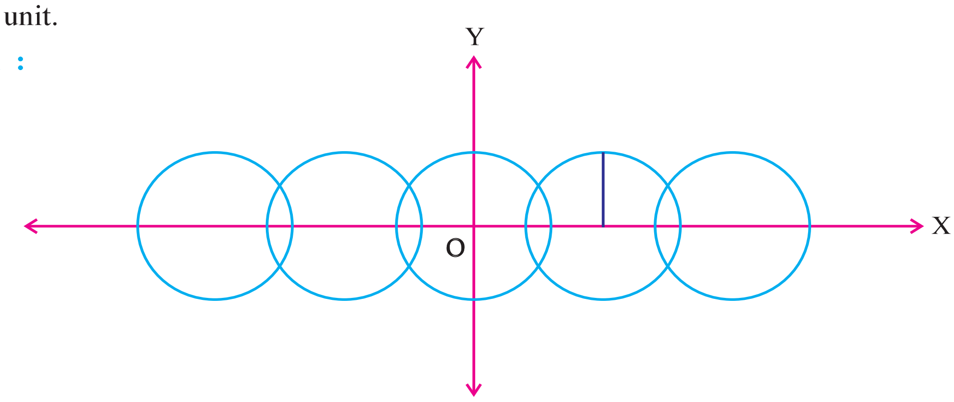


Figure 5.4

Here the centres of the circles in the family are on X-axis. Let the centre of a circle be $(a, 0)$, $(a \in \mathbb{R})$ and let these circles have radius 1.

$\therefore$ The equation of this family of circles is

$$(x - a)^2 + y^2 = 1 \qquad \textbf{(i)}$$

Differentiating *w.r.t.* x,

$$\therefore \quad 2(x - a) + 2y\frac{dy}{dx} = 0$$

$$\therefore \quad (x - a) + y\frac{dy}{dx} = 0$$

$$\therefore \quad (x - a) = -y\frac{dy}{dx} \qquad \textbf{(ii)}$$

To remove the arbitrary constant a, substitute the value of $(x - a)$ in equation (i),

$$\left(-y\frac{dy}{dx}\right)^2 + y^2 = 1$$

$\therefore \quad y^2\left(\frac{dy}{dx}\right)^2 + y^2 - 1 = 0$ is the differential equation of the given family of circles.

Note : There is only one arbitrary constant. So we have differentiated only once. We get first order differential equation.

5.5 Solution of a Differential Equation

The solution of a differential equation is a function $y = f(x)$ or functions obtained from functional relation $f(x, y) = 0$ which independent of derivatives and shows relation between variables and satisfies the given differential equation along with all its derivatives.

If for a function $y = f(x)$, defined on some interval, there exist derivatives of f upto order n and if the function f and its derivatives together satisfy the given differential equation, then this function $y = f(x)$ is called a solution of given differential equation.

In order that a function $y = f(x)$ is a solution of a given differential equation it is necessary that some conditions regarding domain and continuity of functions are satisfied. In other words if solution of a differential equation can be obtained, we discuss how to obtain the solution under some favourable conditions. We will not discuss the existence of a solution of a differentiable equation. We will study some methods to obtain the solution, when it exists and we will not mention the conditions or circumstances under which the solution exists.

Solution of a differential equation :

$y = 2x + c$ is a solution of $\frac{dy}{dx} = 2$. (Example 3) because $y = 2x + c$, satisfies the differential equation $\frac{dy}{dx} = 2$.

Let us see another example.

$y = sinx$, $x \in$ R is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$

because differentiating $y = sinx$ *w.r.t.* x, $\frac{dy}{dx} = cosx$

$$\therefore \quad \frac{d^2y}{dx^2} = -sinx = -y$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

Now $y = cosx$, $x \in$ R is also a solution of $\frac{d^2y}{dx^2} + y = 0$.

Here $y = cosx$

Differentiating *w.r.t.* x

$$\frac{dy}{dx} = -sin\,x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -cosx = -y$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

From the above examples, we say that in general there can be more than one solution of a differential equation.

General and Particular Solution :

The general solution of a differential equation is a function $y = f(x, c_1, c_2,..., c_n)$ or $f(x, y, c_1, c_2,..., c_n) = 0$ with arbitrary constants whose number is equal to the order of the differential equation.

In general, there are n arbitrary constants in the solution of the differential equation

$$\mathbf{F}\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2},\ldots, \frac{d^ny}{dx^n}\right) = 0.$$

This solution is denoted by $G(x, y, c_1, c_2,\ldots, c_n) = 0$ where $c_1, c_2,\ldots, c_n$ are arbitrary constants.

If we can find definite values of the arbitrary constants occurring in the general solution of the differential equation under some conditions on the given variables x, y and derivaties $\frac{dy}{dx}, \frac{d^2y}{dx^2},\ldots$ etc, then the solution of the differential equation with definite values of arbitrary constants is called a particular solution and the given conditions are called initial conditions or boundary conditions.

If a solution other than general solution of a differential equation cannot be obtained as a particular solution from the general solution, then such a solution of the differential equation is called a singular solution.

Example 7 : Verify that the function $y = A\cos x + B\sin x$, where A and B are arbitrary constants, is a general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

Solution : Here $y = A\cos x + B\sin x$ is the given function.

Differentiating both sides of the equation *w.r.t.* x,

we get, $\frac{dy}{dx} = -A\sin x + B\cos x$

$$\therefore \quad \frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$$\therefore \quad \frac{d^2y}{dx^2} = -(A\cos x + B\sin x)$$

$$\therefore \quad \frac{d^2y}{dx^2} = -y$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

Therefore, the given function $y = A\cos x + B\sin x$ is the general solution of the given differential equation $\frac{d^2y}{dx^2} + y = 0$, because there are two arbitrary constants in this solution of the differential equation.

Example 8 : Verify that $y = cx + \frac{1}{c}$ is a solution of the differential equation $y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + 1$, where c is an arbitrary constant.

Solution : Here $y = cx + \frac{1}{c}$ (c is an arbitrary constant)

Differentiating *w.r.t.* x, $\frac{dy}{dx} = c$

Substituting $c = \frac{dy}{dx}$ in the equation $y = cx + \frac{1}{c}$,

we get, $y = \left(\frac{dy}{dx}\right)x + \frac{1}{\left(\frac{dy}{dx}\right)}$

$$\therefore \quad y\left(\frac{dy}{dx}\right) = \left(\frac{dy}{dx}\right)^2 x + 1$$

Therefore, the function $cx + \frac{1}{c}$ is a solution of the given differential equation.

Example 9 : Verify $y = cx^4$ is a solution of the differential equation $x\frac{dy}{dx} - 4y = 0$, where c is an arbitrary constant.

Solution : Here given relation is $y = cx^4$ **(i)**

Differentiating (i) *w.r.t.* x,

we get $\frac{dy}{dx} = 4cx^3$ **(ii)**

$$\therefore \quad x\frac{dy}{dx} - 4y = x(4cx^3) - 4cx^4$$
$$= 4cx^4 - 4cx^4$$
$$= 0$$

Hence, $y = cx^4$ is a solution of $x\frac{dy}{dx} - 4y = 0$.

Example 10 : Verify that $y = ax + a^2$ (a is an arbitrary constant) is the general solution of the differential equation $\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = y$. Find a particular solution, when $a = 3$. Also show that a singular solution of this differential equation is $x^2 + 4y = 0$.

Solution : Here $y = ax + a^2$ (a is an arbitrary constant)

$$\therefore \quad \frac{dy}{dx} = a$$

Substituting $a = \frac{dy}{dx}$ in $y = ax + a^2$, we get the given differential equation

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx}$$

Because of presence of one arbitary constant $y = ax + a^2$ is the general solution of $\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = y$.

Now substitute $a = 3$ in the general solution.

We get $y = 3x + 9$, which is a particular solution of the given differential equation.

Now consider $x^2 + 4y = 0$

$$\therefore \quad 4y = -x^2$$

$$\therefore \quad 4\frac{dy}{dx} = -2x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{2}$$

Substituting this value of $\frac{dy}{dx}$ in the given differential equation, we get,

$\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) = \frac{x^2}{4} + x\left(-\frac{x}{2}\right) = -\frac{x^2}{4} = y$, which shows that $x^2 + 4y = 0$ satisfies given differential equation.

Thus $x^2 + 4y = 0$ satisfies the given differential equation. This is a solution of the differential equation. But this solution cannot be obtained by substituting any value of a in the general solution. Hence this solution is a singular solution of the differential equation.

Note : General solution represents a family of lines. A singular solution $x^2 + 4y = 0$ represents a parabola.

Exercise 5.2

1. Find the differential equation of all the circles which touch the coordinate axes in the first quadrant.

2. Obtain the differential equation representing family of lines $y = mx + c$ (m and c are arbitrary constant)

3. Form the differential equation representing family of curves $y^2 = m(a^2 - x^2)$ (m and a are arbitrary constants).

4. Find the differential equation of the family of all the circles touching X-axis at the origin.

5. Show that the differential equation $\frac{dy}{dx} + 2xy = 4x^3$ has the solution $y = 2(x^2 - 1) + ce^{-x^2}$, where c is an arbitrary constant.

6. Verify that $y^2 = 4b(x + b)$ is a solution of the differential equation $y\left[1 - \left(\frac{dy}{dx}\right)^2\right] = 2x\frac{dy}{dx}$.

7. Prove $y = a\,cos(\log x) + b\,sin(\log x)$ is a solution of the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, where a and b are arbitrary constants.

8. Verify that differential equation $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$ has solution $y = a\,cos^{-1}x + b$. (where a and b are arbitrary constants.)

9. Find the differential equation of the following family of curves, where a and b are arbitrary constants :

   (1) $\frac{x}{a} + \frac{y}{b} = 1$ (2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (3) $(y - b)^2 = 4(x - a)$ (4) $y = \left(ax + \frac{b}{x}\right)$

   (5) $y = ax^3$ (6) $y = e^{2x}(a + bx)$ (7) $y^2 = a(b^2 - x^2)$

10. Verify that $y = 5sin4x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 16y = 0$.

11. Show that $Ax^2 + By^2 = 1$ is the general solution of the differential equation

    $x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = y\left(\frac{dy}{dx}\right)$. (A, B are arbitrary constants)

12. Show that $y = \frac{a}{x} + b$ is a solution of $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$.

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5.6 Solution of Differential Equation of First Order and First Degree :

A first order and first degree differential equation is represented by $\frac{dy}{dx} = F(x, y)$, $x \in I$ (I is any interval). If we let $F(x, y) = \frac{-f(x, y)}{g(x, y)}$

$f(x, y)dx + g(x, y)dy = 0$ is also another form of first order and first degree differential equation. The first order and first degree differential equation may not be always solvable but we will discuss particular forms of these equations which have a general solution.

Now we shall discuss some methods to solve a first order and first degree differential equation.

(1) Method of Variables Separable : In the differential equation $f(x, y)dx + g(x, y)dy = 0$ of first order and first degree, if $f(x, y)$ is a function $p(x)$ of x only and $g(x, y)$ is a function $q(y)$ of y only, then the general form of first order and first degree differential equation is $p(x)dx + q(y)dy = 0$. Such an equation is said to be in variable-separable form.

Now $\int p(x)dx + \int q(y)dy = c$ (c is an arbitrary constant) is the general solution.

Note : In the general solution of a differential equation, we can take arbitrary constant in a form according to our convenience.

Example 11 : Solve the differential equation, $x(1 + y^2)dx - y(1 + x^2)dy = 0$.

Solution : Here $x(1 + y^2)dx = y(1 + x^2)dy$

$\therefore \quad \frac{x}{1+x^2}\,dx = \frac{y}{1+y^2}\,dy$ **(Variables Separable form)**

$\therefore \quad \frac{2x}{1+x^2}\,dx = \frac{2y}{1+y^2}\,dy$

Integrating on both the sides,

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{2y}{1+y^2}\,dy$$

$\therefore \quad \log|1 + x^2| = \log|1 + y^2| + \log c$ (Instead of c, let $\log c$ be the arbitrary constant, $c > 0$)

$\therefore \quad \log\left(\frac{1+x^2}{1+y^2}\right) = \log c \quad (c > 0)$ **($1 + x^2 > 0$, $1 + y^2 > 0$)**

$\therefore \quad \frac{1+x^2}{1+y^2} = c$

$\therefore \quad (1 + x^2) = c(1 + y^2)$

This is the general solution and c is an arbitrary positive constant.

Example 12 : Solve the differential equation $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

Solution : Here $(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}$

$\therefore \quad dy = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$ **(Variables Separable)**

Integrating on both the sides,

$$\int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$$

$y = \log|e^x + e^{-x}| + c$

which is the required general solution of the given equation.

We may write $y = \log(e^x + e^{-x}) + c$ as $e^x + e^{-x} > 0$.

Example 13 : Find the particular solution of the differential equation $\frac{dy}{dx} = y \tan x$ given that $y = 1$ when $x = 0$. $(y \neq 0)$

Solution : $\frac{dy}{dx} = y \tan x$

$\therefore \quad \frac{1}{y} dy = \tan x \; dx$ **(i)**

Integrating on both sides of equation (i),

we get, $\int \frac{1}{y} dy = \int \tan x \; dx$

$\therefore \quad \log |y| = \log |\sec x| + \log |c|$ **(log | c | arbitrary constant)**

$\therefore \quad \log |y| = \log |c \sec x|$

$\therefore \quad y = c \sec x$ **(ii)**

This is the general solution.

Substituting $y = 1$ and $x = 0$ in equation (ii), we get value of arbitrary constant c which gives a particular solution

$1 = \sec 0 \cdot c$

$1 = 1 \cdot c$

$c = 1$

$\therefore \quad y = \sec x$ is the required particular solution.

Note : Sometimes if y is a function of x, we express it as $y = y(x)$. Thus if $y(x) = x^2$, $y(1) = 1$, $y(2) = 4$ etc. Find $y(2)$ means find $y(x)$, when $x = 2$. In this example we can say $y(0) = 1$.

Example 14 : Solve the differential equation $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

Solution : Here we have $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$.

$\therefore \quad \frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$

$\therefore \quad \frac{dy}{dx} = \frac{e^x + x^2}{e^y}$

$\therefore \quad e^y dy = (e^x + x^2) dx$

Integrating on both the sides,

$\int e^y dy = \int (e^x + x^2) dx$

$\therefore \quad e^y = e^x + \frac{x^3}{3} + c$ **(c arbitrary constant)**

is the general solution of the given differential equation.

Example 15 : Solve : $\frac{dy}{dx} = (x + y)^2$

Solution : This differential equation cannot be expressed in the form $p(x)\,dx + q(y)\,dy = 0$. So at first sight this differential equation does not seem to be of variables separable form. But we can transform it into that form.

Here $\frac{dy}{dx} = (x + y)^2$

Substitute $x + y = z$ in the equation.

$\therefore \quad 1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$

So the equation will become

$\therefore \quad \frac{dz}{dx} - 1 = z^2$

$\therefore \quad \frac{dz}{dx} = 1 + z^2$

$\therefore \quad \frac{dz}{1+z^2} = dx$ **(Variables Separable form)**

Integrating on both the sides

$$\int \frac{dz}{1+z^2} = \int dx$$

$\therefore \quad tan^{-1}z = x + c$ **(c arbitrary constant)**

$\therefore \quad tan^{-1}(x + y) = x + c$ is the general solution.

Example 16 : Solve $cos(x - y)dy = dx$

Solution : Here $\frac{dy}{dx} = \frac{1}{cos\,(x-y)}$ **(i)**

Substituting $x - y = t,$ **(ii)**

$$1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$\therefore \quad \frac{dy}{dx} = 1 - \frac{dt}{dx}$ **(iii)**

From (i), (ii) and (iii)

$$1 - \frac{dt}{dx} = \frac{1}{cos\,t}$$

$\therefore \quad 1 - \frac{1}{cos\,t} = \frac{dt}{dx}$

$\therefore \quad \frac{cost - 1}{cost} = \frac{dt}{dx}$

$\therefore \quad \frac{-(1 - cost)}{cost} = \frac{dt}{dx}$

$\therefore \quad -dx = \frac{cost}{1 - cost}dt$

Integrating on both the sides,

$\therefore \quad -\int dx = \int \frac{cos\,t}{1 - cos\,t} \times \frac{1 + cost}{1 + cost}\,dt$

$\therefore \quad -\int dx = \int \frac{cos\,t + cos^2 t}{sin^2 t}\,dt$

$\therefore \quad -\int dx = \int cosec\,t \cdot cot\,t\,dt + \int cot^2 t\,dt$

$\therefore \quad -\int dx = \int cosec\,t \cdot cot\,t\,dt + \int (cosec^2 t - 1)\,dt$

$\therefore \quad -x + c = -cosec\, t - cot\, t - t$

$\therefore \quad -x + c = -cosec(x - y) - cot(x - y) - (x - y)$

$\therefore \quad cosec(x - y) + cot(x - y) + c = y$

Exercise 5.3

1. Solve the following differential equations. Also find particular solution where initial conditions are given :

(1) $xy(y + 1)\, dy = (x^2 + 1)\, dx$

(2) $y(1 + e^x)\, dy = (y + 1)\, e^x\, dx$

(3) $\frac{dy}{dx} = -tanx\; tany$

(4) $\frac{dy}{dx} - y\; tanx = -y\; sec^2x$

(5) $(e^y + 1)\, cosx\; dx + e^y\; sin\, x\; dy = 0$

(6) $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

(7) $y \log y\; dx - x\; dy = 0$

(8) $\frac{dy}{dx} = -4xy^2;\; y(0) = 1$

(9) $x\, dy = (2x^2 + 1)\, dx \quad (x \neq 0);\; y(1) = 1$

(10) $xy\frac{dy}{dx} = y + 2;\quad y(2) = 0$

(11) $\frac{dy}{dx} = 2e^x\, y^3;\; y(0) = \frac{1}{2}$

(12) $x\frac{dy}{dx} + cot\; y = 0;\quad y(\sqrt{2}) = \frac{\pi}{4}$

(13) $e^{\frac{dy}{dx}} = x + 1;\quad y(0) = 3,\; x > -1$

(14) $sin\left(\frac{dy}{dx}\right) = a$ when $x = 0,\; y = 1,\; (a \in \mathrm{R})$

(15) $\frac{dy}{dx} = y\; tanx,\; y(0) = 1$

(16) $(x + 1)^2\frac{dy}{dx} = xe^x$

2. Solve the following differential equations :

(1) $\frac{dy}{dx} = sin(x + y)$

(2) $\frac{dy}{dx} = \frac{(x - y) + 3}{2(x - y) + 5}$

(3) $(x + y + 1)\frac{dy}{dx} = 1$

(4) $\frac{dy}{dx} = e^{x + y}$

(5) $(x + y)^2\frac{dy}{dx} = a^2$

*

5.7 Homogeneous Differential Equations :

Let $f(x,\; y) = 3x^2 + 2xy + y^2$

$$= x^2\left(3 + 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2\right)$$

$$= x^2\;\phi\left(\frac{y}{x}\right)$$

$$\therefore \quad f(x,\; y) = x^2\;\phi\left(\frac{y}{x}\right)$$

Here we have expressed $f(x,\; y)$ in the form of $x^n\;\phi\left(\frac{y}{x}\right)$. **If a two variable function $f(x, y)$ can be written as $f(x,\; y) = x^n\;\phi\left(\frac{y}{x}\right)$ form, then the function $f(x,\; y)$ is called a homogeneous function of degree n.**

Now let us see a method to solve a differential equation of first order and first degree.

In place of x and y substitute λx and λy respectively in $f(x, y)$. (where $\lambda \neq 0$ is constant)

We get, $f(\lambda x, \lambda y) = 3(\lambda x)^2 + 2(\lambda x)(\lambda y) + (\lambda y)^2$

$$= 3\lambda^2 x^2 + 2\lambda^2 xy + \lambda^2 y^2$$

$$= \lambda^2 (3x^2 + 2xy + y^2)$$

$$= \lambda^2 f(x, y)$$

Here we have expressed the relation in the form $f(\lambda x, \lambda y) = \lambda^n f(x, y)$. Such a function $f(x, y)$ is called a homogeneous function of degree n and λ is a non-zero constant.

$f(x, y) = \tan x + \tan y$. This type of function cannot be written in the form $f(x, y) = x^n \phi\left(\frac{y}{x}\right)$. So it is not a homogeneous function.

Homogeneous Differential Equation : If in a differential equation $f(x, y)\,dx + g(x, y)\,dy = 0$, $f(x, y)$ and $g(x, y)$ are homogeneous functions with same degree, then this differential equation is called homogeneous differential equation.

Note : $\phi\left(\frac{y}{x}\right)$ type of functions are always homogeneous.

Solution of homogeneous Differential Equation :

Let the homogeneous differential equation $f(x, y)\,dx + g(x, y)\,dy = 0$ be in the form of $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$.

Let $\frac{y}{x} = v$, so $y = vx$

Differentiating *w.r.t.* 'x',

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore \quad v + x\frac{dv}{dx} = \phi(v)$ $\qquad$ $\left(\frac{dy}{dx} = \phi\left(\frac{y}{x}\right) = \phi(v)\right)$

$\therefore \quad x\frac{dv}{dx} = \phi(v) - v$

$\therefore \quad \frac{dv}{\phi(v) - v} = \frac{dx}{x}$ $\qquad$ **(Variables Separable form)**

Integrating on both the sides, we get,

$$\int \frac{dv}{\phi(v) - v} = \int \frac{1}{x}\,dx$$

$\therefore \quad \int \frac{dv}{\phi(v) - v} = \log|x| + c$ $\qquad$ **($x \neq 0$)**

This is the general solution of a homogeneous differential equation and c is an arbitrary constant.

Example 17 : Solve $\frac{dy}{dx} + \frac{y(x+y)}{x^2} = 0$

Solution : $\dfrac{dy}{dx} = -\dfrac{y(x+y)}{x^2} = -\left[\dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2\right]$ **(i)**

Let $\dfrac{y}{x} = v$

$\therefore \quad y = vx$ **(ii)**

So, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ **(iii)**

$\therefore \quad v + x\dfrac{dv}{dx} = -v - v^2$ **(using (i), (ii) and (iii))**

$\therefore \quad x\dfrac{dv}{dx} = -(2v + v^2)$

$\therefore \quad \dfrac{dv}{2v + v^2} = -\dfrac{dx}{x}$ **(Variables Separable form)**

$\therefore \quad \displaystyle\int \frac{1}{v(v+2)}\, dv = \int -\frac{1}{x}\, dx$ **(Integrating both the sides)**

$\therefore \quad \displaystyle\frac{1}{2}\int \frac{v+2-v}{(v+2)v}\, dv = -\int \frac{1}{x}\, dx$

$\therefore \quad \displaystyle\frac{1}{2}\int \frac{1}{v}\, dv - \frac{1}{2}\int \frac{1}{v+2}\, dv = -\int \frac{1}{x}\, dx$

$\therefore \quad \dfrac{1}{2}\log|v| - \dfrac{1}{2}\log|v+2| = -\log|x| + \dfrac{1}{2}\log|c|$ **(c is an arbitrary constant)**

$\therefore \quad \log|v| - \log|v+2| = -2\log|x| + \log|c|$

$\therefore \quad \log\left|\dfrac{v}{v+2}\right| = \log\left|\dfrac{c}{x^2}\right|$

$\therefore \quad \log\left|\dfrac{y}{y+2x}\right| = \log\left|\dfrac{c}{x^2}\right|$

$x^2y = c(2x + y)$ $\left(v = \dfrac{y}{x}\right)$

This is the general solution.

Example 18 : Solve $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$.

Solution : $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

$\therefore \quad \dfrac{dy}{dx} = 1 + \dfrac{y}{x} + \left(\dfrac{y}{x}\right)^2$ **(i)**

Let $\dfrac{y}{x} = v$, so $y = vx$ **(ii)**

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ **(iii)**

From equations (i), (ii) and (iii),

$v + x\dfrac{dv}{dx} = 1 + v + v^2$

$\therefore \quad x\dfrac{dv}{dx} = 1 + v^2$

$\therefore \quad \dfrac{dv}{1+v^2} = \dfrac{dx}{x}$ **($x \neq 0$) (Variables Separable form)**

Integrating both the sides, we get,

$$\int \frac{1}{1+v^2}\, dv = \int \frac{1}{x}\, dx$$

$tan^{-1} v = \log|x| + \log|c|$ **(c arbitrary constant)**

$tan^{-1} v = \log|xc|$

$tan^{-1}\left(\dfrac{y}{x}\right) = \log|xc|$ is the general solution of the given differential equation.

Example 19 : Solve $x \sin\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} + x - y\sin\left(\dfrac{y}{x}\right) = 0$. Find the particular solution, if the initial condition is $y(1) = \dfrac{\pi}{2}$.

Solution : Here $x \sin\left(\dfrac{y}{x}\right)\dfrac{dy}{dx} + x - y\sin\left(\dfrac{y}{x}\right) = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - x}{x\sin\left(\frac{y}{x}\right)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{y}{x}\sin\left(\frac{y}{x}\right) - 1}{\sin\left(\frac{y}{x}\right)} \qquad \text{(i)}$$

Let $\dfrac{y}{x} = v$ **(ii)**

So, $y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{(iii)}$$

From equations (i), (ii) and (iii),

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v}$$

$$\therefore \quad v + x\frac{dv}{dx} = v - \frac{1}{\sin v}$$

$$\therefore \quad x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\therefore \quad \sin v\; dv = -\frac{dx}{x}$$

Integrating both the sides,

$$\int \sin v\, dv = -\int \frac{dx}{x}$$

$$\therefore \quad -\cos v = -\log|x| - \log|c|$$

$$\therefore \quad \cos\left(\frac{y}{x}\right) = \log|x| + \log|c|$$

$$\therefore \quad \cos\frac{y}{x} = \log|cx| \qquad \text{(iv)}$$

This is the general solution.

Now we are given $y(1) = \frac{\pi}{2}$ i.e. when $x = 1$ and $y = \frac{\pi}{2}$

So, from equation (iv),

$$cos\,\frac{\pi}{2} = \log|c|$$

$$\therefore \quad \log|c| = 0$$

$$\therefore \quad |c| = 1$$

$\therefore \quad cos\left(\frac{y}{x}\right) = \log|x|$ $(x \neq 0)$ is the required particular solution.

Example 20 : Solve $\left[x\,sin^2\left(\frac{y}{x}\right) - y\right] dx + x\,dy = 0$. Find the particular solution, if the initial condition is $y(1) = \frac{\pi}{4}$.

Solution : Here $\left[x\,sin^2\left(\frac{y}{x}\right) - y\right] dx + x\,dy = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x} - sin^2\frac{y}{x} \qquad \text{(i)}$$

Let $\frac{y}{x} = v$, so $y = vx$ **(ii)**

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{(iii)}$$

From equations (i), (ii) and (iii) we get

$$v + x\frac{dv}{dx} = v - sin^2 v$$

$$x\frac{dv}{dx} = -sin^2 v$$

$$\therefore \quad \frac{1}{sin^2 v}\,dv = -\frac{dx}{x} \qquad \text{(Variables Separable form)}$$

Integrating both the sides

$$\int cosec^2 v\,dv = -\int \frac{1}{x}\,dx$$

$$-cot\,v = -\log|x| - \log|c|$$

$cot\left(\frac{y}{x}\right) = \log|cx|$ which is general solution.

Now we are given $y(1) = \frac{\pi}{4}$ i.e. when $x = 1$ $y = \frac{\pi}{4}$

$$cot\,\frac{\pi}{4} = \log|c|$$

$$\therefore \quad \log|c| = 1$$

$$\therefore \quad |c| = e$$

$\therefore \quad cot \frac{y}{x} = \log |ex| = \log |x| + \log e$ **$(x \neq 0)$**

$= \log |x| + 1$

This is the required particular solution.

Example 21 : Solve $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$. Also find the particular solution for $y(1) = 2$.

Solution : Here $2xy + y^2 - 2x^2 \frac{dy}{dx} = 0$

$\therefore \quad \frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2$ **(i)**

Let $\frac{y}{x} = v$ **(ii)**

$\therefore \quad y = vx$

$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$ **(iii)**

From equations (i), (ii) and (iii)

$$v + x\frac{dv}{dx} = v + \frac{1}{2}v^2$$

$$x\frac{dv}{dx} = \frac{1}{2}v^2$$

$\frac{2}{v^2}\,dv = \frac{dx}{x}$ **(Variables Separable form)**

Integrating both the sides,

$$2\int \frac{1}{v^2}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{2}{v} = \log |x| + c$$

$-\frac{2x}{y} = \log |x| + c$ is the general solution.

Now $y(1) = 2$. So if $x = 1$, $y = 2$

$\therefore \quad -\frac{2}{2} = \log |1| + c$

$\therefore \quad c = -1$

$$-\frac{2x}{y} = \log |x| - 1$$

$y = \frac{2x}{1 - \log |x|}$ **$(x \neq 0, x \neq e)$**

Exercise 5.4

1. Solve the following differential equations :

(1) $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$

(2) $\left(x\cos\frac{y}{x}+y\sin\frac{y}{x}\right)y=\left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)x\frac{dy}{dx}$

(3) $x\frac{dy}{dx}-y+x\sin\left(\frac{y}{x}\right)=0$

(4) $y\,e^{\frac{x}{y}}\,dx=(x\,e^{\frac{x}{y}}+y^2)\,dy$

(5) $x\sin\left(\frac{y}{x}\right)\frac{dy}{dx}=y\sin\left(\frac{y}{x}\right)+x$

(6) $y+2ye^{\frac{x}{y}}\frac{dx}{dy}=2xe^{\frac{x}{y}}$

(7) $x^2\frac{dy}{dx}=x^2-2y^2+xy$

(8) $(1+e^{\frac{x}{y}})\,dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$

(9) $x\frac{dy}{dx}=x+y$

(10) $y\,dx+x\log\left(\frac{y}{x}\right)dy=2x\,dy$

(11) $(xe^{\frac{y}{x}}+y)\,dx=x\,dy$

(12) $\frac{dy}{dx}+\frac{y(x+y)}{x^2}=0$

(13) $\frac{dy}{dx}=\frac{y}{x}+\tan\left(\frac{y}{x}\right)$

2. Find the particular solution of the given differential equations under given initial condition :

(1) $(x^2+y^2)\,dx+xy\,dy=0;\ y(1)=1$

(2) $x\,e^{\frac{y}{x}}-y+x\frac{dy}{dx}=0;\ y(e)=0$

(3) $\frac{dy}{dx}-\frac{y}{x}+\operatorname{cosec}\frac{y}{x}=0;\ y(1)=0$

(4) $(x^2-2y^2)\,dx+2xy\,dy=0;\ y(1)=1$

(5) $2xy+y^2-2x^2\frac{dy}{dx}=0;\ y(1)=2$

(6) $(x^2+3xy+y^2)\,dx-x^2\,dy=0;\ y(1)=0$

*

5.8 Linear Differential Equation :

If P(x) and Q(x) are functions of variable x, then the differential equation $\frac{dy}{dx}+P(x)\,y=Q(x)$ is called a Linear Differential Equation.

For example, (1) $\frac{dy}{dx}+xy=\cos x$ $\quad$ $P(x)=x,\ Q(x)=\cos x$

(2) $\frac{dy}{dx}-\frac{y}{x}=e^x$ $\quad$ $P(x)=-\frac{1}{x},\ Q(x)=e^x$

(3) $x\frac{dy}{dx}+2y=x^3$ $\quad$ $P(x)=\frac{2}{x},\ Q(x)=x^2$

(4) $\frac{dy}{dx}+y=x$ $\quad$ $P(x)=1,\ Q(x)=x$

Method of solving a linear differential equation :

Let $\frac{dy}{dx}+P(x)\,y=Q(x)$ be a given linear differential equation.

If we multiply both the sides by $e^{\int P(x)\,dx}$, we get $\frac{dy}{dx}e^{\int P(x)\,dx}+y\,e^{\int P(x)\,dx}\cdot P(x)=Q(x)\,e^{\int P(x)\,dx}$

$\therefore\ \frac{d}{dx}\left[ye^{\int P(x)\,dx}\right]=Q(x)\,e^{\int P(x)\,dx}$

Integrating *w.r.t.* x, we get

$$ye^{\int P(x)\,dx}=\int\left[Q(x)\,e^{\int P(x)\,dx}\right]dx$$

Note : **Here the linear differential equation is multiplied on both the sides by $e^{\int P(x)\,dx}$ to make it easily integrable. So $e^{\int P(x)\,dx}$ is called an Integrating Factor - I.F.**

The first order linear differential equation is $\frac{dy}{dx} + P(x)y = Q(x)$.

If we multiply both the sides by $h(x)$, a function of x, we get

$$h(x)\frac{dy}{dx} + h(x)\ P(x)y = h(x)Q(x) \qquad \text{(i)}$$

Choose a function $h(x)$ in such a way that $h(x)Q(x)$ becomes a derivative of $y\,h(x)$.

$\therefore\ h(x)\frac{dy}{dx} + h(x)\ P(x)y = \frac{d}{dx}\,yh(x)$

$\therefore\ h(x)\frac{dy}{dx} + h(x)\ P(x)y = h(x)\frac{dy}{dx} + yh'(x)$

$\therefore\ h(x)\cdot P(x)y = yh'(x)$

$\therefore\ h(x)\cdot P(x) = h'(x)$

$\therefore\ P(x) = \frac{h'(x)}{h(x)}$

Integrating both the sides with respect to x,

$\therefore\ \int P(x)\,dx = \int \frac{1}{h(x)}\ h'(x)\,dx$

$\therefore\ \int P(x)\,dx = \log|(h(x))|$

$\therefore\ h(x) = e^{\int P(x)\,dx}$

In the equation (i) substitute the value of $h(x)$,

$\therefore\ e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}\ P(x)y = e^{\int P(x)\,dx}\ Q(x)$

$\therefore\ \frac{d}{dx}\left(e^{\int P(x)\,dx}\,y\right) = e^{\int P(x)\,dx}\,Q(x)$

$\therefore\ e^{\int P(x)\,dx}\,y = \int e^{\int P(x)\,dx}\,Q(x)\ dx.$

In this way we get the solution of a linear differential equation.

The function $h(x) = e^{\int P(x)\,dx}$ is an Integrating Factor.

Example 22 : Solve $\frac{dy}{dx} + \frac{y}{x} = x^2$. The given linear differential equation of the type $\frac{dy}{dx} + P(x)\,y = Q(x)$.

Solution : The given differential equation is linear.

Here $P(x) = \frac{1}{x}$, $Q(x) = x^2$

$$\begin{aligned}\therefore\ \text{I.F.} &= e^{\int P(x)\,dx}\\ &= e^{\int \frac{1}{x}\,dx}\\ &= e^{\log x}\\ &= x\end{aligned}$$

We can take I.F. as x because if we multiply both sides of the differential equation by x, then there will no change.

Multiply by x on both the sides.

$$x\frac{dy}{dx} + y = x^3$$

$$\therefore \quad \frac{d}{dx}(xy) = x^3$$

$$\therefore \quad xy = \int x^3\, dx$$

$$\therefore \quad xy = \frac{x^4}{4} + c$$ **(c is an arbitrary constant)**

This is the general solution of the given differential equation.

Example 23 : Solve $\frac{dy}{dx} + y\sec x = \tan x$.

Solution : $\frac{dy}{dx} + y\sec x = \tan x$.

This is a linear differential equation.

Here, $P(x) = \sec x$, $Q(x) = \tan x$

$$\therefore \quad \text{I.F.} = e^{\int P(x)\, dx}$$

$$= e^{\int \sec x\, dx}$$

$$= e^{\log|\sec x + \tan x|}$$

$$= |\sec x + \tan x|$$

We can take I.F. $= \sec x + \tan x$

Multiply both the sides of given equation by I.F., we get,

$$(\sec x + \tan x)\frac{dy}{dx} + \sec x\,(\sec x + \tan x)\,y = \tan x\,(\sec x + \tan x)$$

$$\frac{d}{dx}[y(\sec x + \tan x)] = \tan x\,(\sec x + \tan x)$$

$$\therefore \quad y(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)\, dx$$

$$\therefore \quad y(\sec x + \tan x) = \int \sec x \tan x\, dx + \int \tan^2 x\, dx$$

$$\therefore \quad y(\sec x + \tan x) = \int \sec x \tan x\, dx + \int (\sec^2 x - 1)\, dx$$

$$\therefore \quad y(\sec x + \tan x) = \sec x + \tan x - x + c$$ **(c is an arbitrary constant)**

is the general solution.

Example 24 : Solve $\frac{dy}{dx} = y\tan x + e^x$

Solution : $\frac{dy}{dx} = y\tan x + e^x$ is a linear differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Here $P(x) = -\tan x$ and $Q(x) = e^x$

Now, I.F. $= e^{\int P(x)\, dx}$

$$= e^{\int -\tan x\, dx}$$

$$= e^{-\log|\sec x|}$$

$$= e^{\log|\cos x|}$$

$$= |\cos x|$$

We can take I.F. $= \cos x$

$\therefore$ General solution of this linear equation is,

$y \ cos\, x = \int e^x cos\, x \ dx$ $\left(ye^{\int P(x)\, dx} = \int Q(x) \ e^{\int P(x)\, dx} dx \right)$

$\therefore$ $y \ cos\, x = \frac{e^x}{2} (cos\, x + sin\, x) + c$ is the general solution. **(c arbitrary constant)**

Example 25 : Solve $\frac{dy}{dx} + \frac{y}{x} = \log x$

Solution : This is a linear differential equation in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

Here $P(x) = \frac{1}{x}$ and $Q(x) = \log x$

$$\text{I.F.} = e^{\int P(x)\, dx}$$
$$= e^{\int \frac{1}{x} dx}$$
$$= e^{\log |x|}$$
$$= x$$

We can take I.F. $= x$

According to the general solution,

$$ye^{\int P dx} = \int Q(x) \cdot e^{\int P(x)\, dx} dx$$

$$yx = \int x \log x \ dx$$

$$\therefore \quad yx = \log x \int x \ dx - \int \left(\frac{d}{dx} (\log x) \int x \ dx \right) dx$$

$$\therefore \quad yx = \log x \cdot \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} \ dx$$

$\therefore \quad yx = \frac{x^2}{2} \log x - \frac{1}{4} x^2 + c$ **(c is an arbitrary constant)**

This is the general solution.

Exercise 5.5

Solve the following differential equations :

1. $\frac{dy}{dx} + 2y = sin\, x$

2. $x \frac{dy}{dx} - y = (1 + x) e^{-x}$

3. $x \frac{dy}{dx} = x + y$

4. $\frac{dy}{dx} - \frac{2xy}{1 + x^2} = x^2 + 1$

5. $\frac{dy}{dx} = x + y$

6. $\frac{dy}{dx} + \frac{2y}{x} = e^x$

7. $4 \frac{dy}{dx} + 8y = 5e^{-3x}$

8. $(1 + x^2) \frac{dy}{dx} + 2xy - 4x^2 = 0$

9. $(1 + y^2)\, dx = (tan^{-1}y - x)\, dy$

10. $x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x, \ x > 0$

11. $sin^2 x \frac{dy}{dx} + y = cot\, x$

12. $y \ dx - (x + 2y^2)\, dy = 0$

*

5.9 Applications of differention Equations :

As we know the study of differential equations began in order to solve the problems that originated from different branches of mathematics, physics, biological sciences etc.

(1) Physics (RL circuit) : Let us consider RL circuit. This circuit contains resistor (R) and Inductor (L). So it is known as RL circuit. At $t = 0$, the switch is closed and current does not pass throuch the circuit. When switch is on, the current passes through the circuit. As per the electricity law, when voltage across a resistor of resistance R is equal to Ri, the voltage across an inductor is given by $L\frac{di}{dt}$, where i is the current.

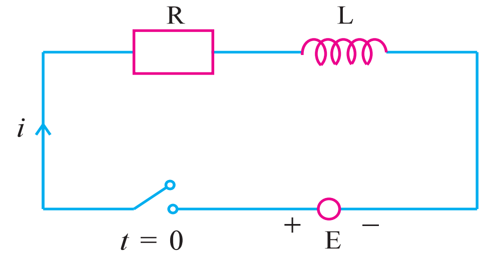

Figure 5.5

Example 26 : The equation of electromotive force (*e.m.f.*) is $E = Ri + L\frac{di}{dt}$, where R is resistance, L is the self inductance and i is electric current. Find the equation relating time (t) and electric current (i).

Solution : The given equation can be written as $L\frac{di}{dt} = E - Ri$

$\therefore \quad \frac{1}{E - Ri}\, di = \frac{1}{L}\, dt$

$\therefore \quad \frac{-R}{E - Ri}\, di = \frac{-R}{L}\, dt$ **(Variable Separable form)**

Now integrating both the sides,

$$\int \frac{-R}{E - Ri}\, di = \int \frac{-R}{L}\, dt$$

$\therefore \quad \log (E - Ri) = \frac{-R}{L} t + \log c$

$\therefore \quad \log \frac{(E - Ri)}{c} = \frac{-R}{L} t$

$\therefore \quad E - Ri = ce^{\frac{-R}{L} t}$

$Ri = E - ce^{\frac{-R}{L} t}$

$\therefore \quad i = \frac{E}{R} - \frac{ce^{-\frac{R}{L}t}}{R}$ is the required equation.

Another Method :

Given equation is $L\frac{di}{dt} = E - Ri$

$\therefore \quad \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$

This is a linear differential equation. I.F. $= e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}$

Multiplying both the sides by I.F., $e^{\frac{R}{L} t} \frac{di}{dt} + e^{\frac{R}{L} t} \frac{R}{L} i = \frac{E}{L} e^{\frac{R}{L} t}$

$\therefore \quad \frac{d}{dt}\left(e^{\frac{R}{L} t} i\right) = \frac{E}{L} e^{\frac{R}{L} t}$

Integrating both the sides *w.r.t.* t,

$$e^{\frac{R}{L}t} \cdot i = \int \frac{E}{L} e^{\frac{R}{L}t} dt$$

$$\therefore \quad e^{\frac{R}{L}t} \cdot i = \frac{\frac{E}{L} e^{\frac{R}{L}t}}{\frac{R}{L}} - \frac{c}{R} \qquad \left(-\frac{c}{R} \text{ arbitrary constant}\right)$$

$$\therefore \quad e^{\frac{R}{L}t} \cdot i = \frac{E}{R} e^{\frac{R}{L}t} - \frac{c}{R}$$

$$\therefore \quad i = \frac{E}{R} - \frac{c}{R} e^{-\frac{R}{L}t}$$

This is the general solution.

(2) Application in Geometry :

$y = f(x)$ is a given curve.

If $y = f(x)$ is differentiable at (x_0, y_0) then, slope of the tangent at the point (x_0, y_0)

is given by $m = \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$

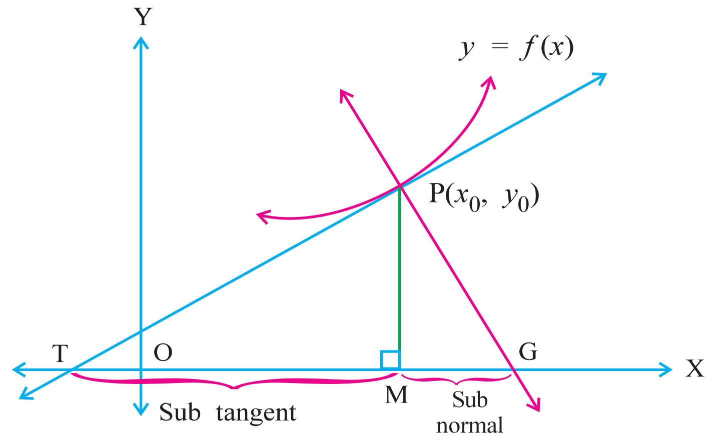

Figure 5.6

(1) The equation of the tangent to the curve at point (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

(2) The equation of the normal to the curve at point (x_0, y_0) is

$$y - y_0 = -\left(\frac{dx}{dy}\right)_{(x_0, y_0)} (x - x_0) \qquad \left(\frac{dy}{dx} \neq 0\right)$$

Let $M(x_0, 0)$ be the foot of perpendicular from $P(x_0, y_0)$ on the X-axis. Suppose tangent at P intersects X-axis at T, then $\overline{TM}$ is called the **subtangent**.

$$\text{Length of subtangent TM} = \left| \frac{y_0}{\left(\frac{dy}{dx}\right)_{(x_0, y_0)}} \right|$$

Suppose the normal at P intersects X-axis at G, then $\overline{MG}$ is called the **subnormal**.

$$\text{Length of subnormal MG} = \left| y_0 \left(\frac{dy}{dx}\right)_{(x_0, y_0)} \right|$$

Example 27 : The slope of the tangent to the curve at any point is reciprocal of the y-coordinate of that point $(y \neq 0)$ and the curve passes through $(-1, 2)$. Find the equation of the curve.

Solution : Let $P(x, y)$ be any point on the curve.

Slope of the tangent to the curve at the point $P(x, y)$ is $\frac{dy}{dx}$.

But the slope of the tangent to the curve at point $P(x, y) = \frac{1}{y}$.

$\therefore \quad \frac{dy}{dx} = \frac{1}{y}$

$\therefore \quad y\,dy = dx$

Integrating both the sides,

$$\int y\,dy = \int dx$$

$$\frac{y^2}{2} = x + \frac{c}{2} \qquad \textbf{(c is an arbitrary constant)}$$

$\therefore \quad y^2 = 2x + c,$

It passes through $(-1, 2)$

$\therefore \quad 4 = -2 + c$

$\therefore \quad c = 6$

$\therefore \quad y^2 = 2x + 6$ is the equation of the curve.

(3) Exponential Growth :

Let $p(t)$ be a quantity which increases with time t. Suppose at time $t = 0$, $p(t) = p_0$.

So the rate of increase of the quantity is proportional to the given quantity $p(t)$.

i.e. $\frac{d\,p(t)}{dt} \propto p(t)$

$$\frac{d\,p(t)}{dt} = kp(t) \qquad \textbf{(k > 0)}$$

$$\frac{1}{P(t)}\frac{d\,p(t)}{dt} = k$$

Integrating both the sides, we get

$$\int \frac{d\,p(t)}{dt} = \int k\,dt$$

$$\log p(t) = kt + \log c$$

$\therefore \quad \log p(t) - \log c = kt$

$\therefore \quad \log \frac{p(t)}{c} = kt$

$\therefore \quad p(t) = ce^{kt}$, where c is an arbitrary constant.

Suppose at $t = 0$, $p(t) = p_0$.

Then $p(0) = ce^0$

$\therefore \quad c = p(0)$

$\therefore \quad p(t) = p(0)e^{kt}$

Using this solution, we can find the growth of quantity $p(t)$ at any time t.

Example 28 : The population of a city increases at the rate of 2 % per year. How many years will it take to double the population ?

Solution : Let the p_0 be the population at present and after t years suppose it will be $p(t)$.

Now population increases at the rate of 2 %.

So, $\frac{dp}{dt} = \frac{2}{100} p$

$$\int \frac{dp}{p} = \frac{1}{50} \int dt$$

$\therefore$ $\log p = \frac{1}{50} t + \log c$

$\therefore$ $p = ce^{\frac{1}{50} t}$

At $t = 0$, $p = p_0$

So $p_0 = ce^0$

$\therefore$ $c = p_0$

$\therefore$ $p = p_0 e^{\frac{1}{50} t}$

Now if the population doubles, then $p = 2p_0$.

$\therefore$ $2p_0 = p_0 e^{\frac{1}{50} t}$

$\therefore$ $\log_e 2 = \frac{1}{50} t$

$\therefore$ $t = 50 \log_e 2 = 34.65 \cong 35$ years

(4) Exponential Decay :

Let $m(t)$ be the mass of a product which decreases with time t.

The rate of decrease is proportional to the given mass m.

So, $\frac{dm}{dt} = -km$ **$(k > 0)$**

Using the above method, we can find the decay.

Example 29 : A certain radioactive material has a half life of 2000 years. (This is called half life period of the substance.) Find the time required for a given amount to become one tenth of its original mass.

Solution : Let initial mass of the material be m_0 grams.

If the mass of the material is m grams after time t, then from the rate of decay we have,

$\frac{dm}{dt} = -km$ **$(k > 0)$**

$\frac{dm}{m} = -k\ dt$

$\therefore$ $\int \frac{dm}{m} = \int -k\ dt$

$\therefore$ $\log m = -kt + \log c$

$\therefore$ $m = ce^{-kt}$

Now when $t = 0$, $m = m_0$

$m_0 = ce^0$

$\therefore$ $c = m_0$

$\therefore \quad m = m_0 e^{-kt}$ **(i)**

At $t = 2000$ years, $m = \frac{m_0}{2}$

So, $\frac{m_0}{2} = m_0 e^{-k\,2000}$

$\therefore \quad \frac{1}{2} = e^{-k\,2000}$

$\therefore \quad -k2000 = -\log 2$

$\therefore \quad k = \frac{\log 2}{2000}$

Now at some time t, m will be $\frac{m_0}{10}$,

From equation (i),

$\therefore \quad \frac{m_0}{10} = m_0 e^{-kt}$

$\therefore \quad -kt = \log \frac{1}{10}$

$\therefore \quad -kt = -\log 10$

$\therefore \quad kt = \log 10$

$\therefore \quad t = \frac{1}{k} \log 10 = \frac{2000}{\log_e 2} \cdot \log 10 \simeq 6644$ years

(5) Newton's Law of Cooling :

The rate of change of temperature of a body is proportional to the difference between the temperature of the body itself and that of the surroundings.

Let S be the constant temperature of surroundings. Let T be the temperature of the body at any time t. Then,

$$\frac{dT}{dt} \propto (T - S)$$

$\therefore \quad \frac{dT}{dt} = -k(T - S)$ **($k > 0$ is a constant)**

$\therefore \quad \frac{1}{T - S}\, dT = -kdt$

Integrating both the sides,

$$\log |T - S| = -kt + \log c$$

$\therefore \quad \log \left| \frac{T - S}{c} \right| = -kt$

$$T - S = ce^{-kt}$$

Example 30 : The temperature of a body in a room is 80° F. After five minutes the temperature of the body becomes 60° F. After another 5 minutes the temperature becomes 50° F. What is the temperature of surroundings ?

Solution : Let T be the temperature of the body at any time t.

Let S be the constant temperature of the surroundings. (i.e. room temperature)

Then by Newton's law of cooling.

$$\frac{dT}{dt} \propto (T - S)$$

$\therefore \quad \frac{dT}{dt} = -k(T - S)$ **($k > 0$ is a constant as temperature decreases in time interval)**

$\therefore \quad \frac{dT}{T - S} = -kT$

$\therefore \quad \int \frac{dT}{T - S} = \int -k \, dt$

$\therefore \quad \log (T - S) = -kt + c$ **(i)**

Now at $t = 0$, $T = 80^{\circ}$ F

$\therefore \quad \log (80 - S) = c$

From equation (i), we get

$\log (T - S) = -kt + \log (80 - S)$

Also at $t = 5$, $T = 60^{\circ}$ F

$\therefore \quad \log (60 - S) = -5k + \log (80 - S)$ **(ii)**

Also at $t = 10$, $T = 50^{\circ}$ F

$\therefore \quad \log (50 - S) = -10k + \log (80 - S)$ **(iii)**

From equations (ii) and (iii), we get

$\therefore \quad \frac{1}{5} \log \left(\frac{60 - S}{80 - S}\right) = -k = \frac{1}{10} \log \left(\frac{50 - S}{80 - S}\right)$

$\therefore \quad 2 \log \left(\frac{60 - S}{80 - S}\right) = \log \left(\frac{50 - S}{80 - S}\right)$

$\therefore \quad \left(\frac{60 - S}{80 - S}\right)^2 = \left(\frac{50 - S}{80 - S}\right)$

$\therefore \quad (60 - S)^2 = (80 - S)(50 - S)$

$\therefore \quad 3600 - 120S + S^2 = 4000 - 130S + S^2$

$\therefore \quad 10S = 400$

$\therefore \quad S = 40^{\circ}$ F

Hence, temperature of the room is 40° F.

Example 31 : Saptesh has a fixed deposit of ₹ 10,000 in a bank. Principal amount increases continuously at the rate of 7 % per year. In how many years will it get doubled ?

Solution : Let P be the amount at any time t.

According to the given conditions,

$\frac{dP}{dt} = \frac{7P}{100}$

$\therefore \quad \frac{dp}{P} = \frac{7}{100} dt$ **(Variables Separable form)**

Integrating both the sides,

$\int \frac{dp}{P} = \int \frac{7}{100} dt$

$\therefore \quad \log P = \frac{7}{100}t + \log c$

$\therefore \quad P = ce^{\frac{7t}{100}}$

At $t = 0$, P = ₹ 10000

$10000 = ce^{0}$

$\therefore \quad c = 10000$

$\therefore \quad P = 10000\, e^{\frac{7t}{100}}$ **(i)**

Let t be the time to double the investment.

After time t, P $= 2 \times$ principal

$= 2 \times 10000$

= ₹ 20000

From equation (i),

$\therefore \quad 20000 = 10000\, e^{\frac{7t}{100}}$

$\therefore \quad 2 = e^{\frac{7t}{100}}$

$\therefore \quad \log_e 2 = \frac{7}{100}t$

$\therefore \quad t = \frac{100}{7}\log_e 2$ which is approximately 9.9 years.

Exercise 5.6

1. If the X intercept of the tangent to a curve at any point is four times its y-coordinate, then find the equation of the curve.

2. In an experiment of culture of bacteria in a laboratory, the rate of increase of bacteria is proportional to the number of bacteria present at that time. If in one hour the number of bacteria gets doubled, then

(1) What is the number of bacteria at the end of 4 hours ?

(2) If the number of bacteria is 24,000 at the end of 3 hours. Find the number of bacteria in the beginning.

3. A curve passes through (3, −4). Slope of tangent at any point (x, y) is $\frac{2y}{x}$. Find the equation of the curve.

4. The increase in the principal amount kept at the compound interest in a bank is proportional to the product of the principal amount and annual rate of interest.

(1) Annual rate of interest in a bank is 5 %. How many years will it take to double the principal amount ?

(2) At what annual rate of interest, the principal amount will double in 10 years ?

5. Rate of decay of a radioactive body is proportional to its mass present at that time. After a decay of one year the mass of the body is 100 grams and after two years it is 80 grams. Find the initial mass of the body.

6. If the length of the subnormal of a curve is constant and if it passes through the origin, then find its equation.

7. Find the equation of the curve passing through the point (1, 2), given that at any point (x, y) on the curve, if the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point.

Exercise 5

1. Verify that the function $y = cx + \frac{a}{c}$ is the general solution of the differential equation,

$y = x\left(\frac{dy}{dx}\right) + a\left(\frac{dx}{dy}\right)$ (c is an arbitrary constant).

2. Show that the solution of the differential equation $\frac{dy}{dx} = 1 + xy^2 + x + y^2$, $y(0) = 0$ is

$y = tan\left(x + \frac{x^2}{2}\right)$.

3. Show that $y = e^{-x} + ax + b$ is a solution of the differential equation $e^x \frac{d^2y}{dx^2} - 1 = 0$.

4. Verify that the function $y = ae^{2x} + be^{-x}$ is a solution of the differential equation

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.

5. Find the differential equation for the family of the curves represented by $y^2 = a(b + x)(b - x)$.

(a, b arbitrary constant)

6. **Solve :**

(1) $\frac{dy}{dx} = cos\,(x + y) + sin\,(x + y)$

(2) $\frac{dy}{dx} + \frac{4xy}{x^2+1} = \frac{1}{(x^2+1)^3}$

(3) $2ye^{\frac{x}{y}}\,dx + (y - 2xe^{\frac{x}{y}})\,dy = 0$

(4) $xy\frac{dy}{dx} = x^2 - y^2$

(5) $(x^2 - y^2)\,dx + 2xy\,dy = 0 \qquad y(1) = 1$

(6) $cos^2x\,\frac{dy}{dx} + y = tan\,x$

7. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

(1) The order of a differential equation whose general solution is $y = \text{A}sinx + \text{B}cosx$ is ...... (A, B are arbitrary constants.) ☐

(a) 4 (b) 2 (c) 0 (d) 3

(2) The order and degree of $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + y = 0$ are ...... respectively. ☐

(a) 3, 2 (b) 2, 3 (c) 3, not defined (d) 2, 3

(3) $y' + y = \frac{5}{y'}$ has degree ....

(a) 1 (b) 2 (c) not defined (d) −1

(4) The differential equation $\frac{dy}{dx} = -\frac{x+y}{1+x^2}$ is ....

(a) of variable separable form (b) homogeneous

(c) linear (d) of second order

(5) $f(x, y) = \frac{x^3 - y^3}{x + y}$ is a homogeneous function of degree ....

(a) 1 (b) 2 (c) 3 (d) not defined

(6) An integrating factor of differential equation $\frac{dy}{dx} = \frac{1}{x + y + 2}$ is ....

(a) e^x (b) $e^{x + y + 2}$ (c) e^{-y} (d) $\log |x + y + 2|$

(7) The differential equation of the family of rectangular hyperbolas is ....

(a) $y_2 = 0$ (b) $xy + y_2 = 0$ (c) $yy_1 = x$ (d) $xy_1 + y = 0$

(8) The order and the degree of the differential equation $\frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} + xy = sinx$, are ...... respectively.

(a) 1, 1 (b) 2, 1 (c) 3, 2 (d) 2, not defined

(9) Which of the following function is a solution of the differential equation

$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$?

(a) $y = 4x$ (b) $y = 4$ (c) $y = 2x^2 + 4$ (d) $y = 2x - 4$

(10) Solution of the differential equation $x\frac{dy}{dx} + y = 0$ is ....

(a) $e^{xy} = c$ (b) $y = cx$ (c) $x = cy$ (d) $e^x y = c$

(11) The solution of the differential equation $\frac{dy}{dx} + \frac{2y}{x} = 0$ with $y(1) = 1$ is given by ......

(a) $y = \frac{1}{x}$ (b) $y = \frac{1}{x^2}$ (c) $x = \frac{1}{y^2}$ (d) $x^2 = \frac{1}{y^2}$

(12) The number of arbitrary constants in the general solution of differential equation of second order is ......

(a) 1 (b) 0 (c) 2 (d) 4

(13) The number of arbitrary constants in the particular solution of a differential equation of fourth order is ......

(a) 4 (b) 2 (c) 1 (d) 0

(14) The differential equation $\frac{dy}{dx} = e^{x + y}$ has solution ....

(a) $e^x + e^{-y} = c$ (b) $e^x + e^y = c$ (c) $e^{-x} + e^y = c$ (d) $e^{-x} + e^{-y} = c$

(15) The degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = x\left(\frac{d^2y}{dx^2}\right)$ is .... ☐

(a) 3 (b) 2 (c) 6 (d) 1

(16) The solution of the differential equation $2x\frac{dy}{dx} - y = 0$; $y(1) = 2$ represents .... ☐

(a) straight line (b) parabola (c) circle (d) ellipse

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Summary

We have studied the following points in this chapter :

1. An equation involving independent variable (x), dependent variable (y) and derivatives of the dependent variable *w.r.t.* independent variable is known as a differential equation.

2. Order of the highest order derivative occuring in the given differential equation is called the order of the differential equation.

3. If the differential equation is in a polynomial form in derivatives, then the highest power of the highest order derivative occurring in the differential equation is called the degree of the equation.

4. Solution of a differential equation of order n is a function which satisfies the differential equation. The solution which contain n arbitrary constants is called the general solution and the solution free from all arbitrary constants is called a particular solution.

5. Variables separable method is used to solve the differential equation in which variables can be separated completely.

6. If a two variable function $f(x, y)$ can be written as $f(x, y) = x^n \phi\left(\frac{y}{x}\right)$ form, then the function $f(x, y)$ is called homogeneous function having degree n.

7. P(x) and Q(x) are functions of variable x, then the differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ is called linear differential equation.

8. Applications of differential equations.

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VECTOR ALGEBRA 6

> **Mathematics knows no races or geographic boundaries;**
> **for mathematics, the cultural world is one country.**
> **– Jules Henri**

6.1 Introduction

In everyday conversation, when we talk of a quantity, we generally discuss a scalar quantity which has only magnitude. If we say that we drove through a distance of 50 *km*, we talk about the distance travelled. Here we do not bother in which direction we have travelled. 50 *km* is a scalar quantity. Now, if we drive towards our home, then simply to say driving 50 *km* is not enough, but we have to say that we should drive 50 *km* South to reach our home. This information provides not just magnitude but also the direction of the quantity. This quantity is a vector quantity.

The latin word **vector** means 'Carrier'. Vector 'carries' magnitude as the distance between two points (i.e. distance between initial point and terminal point) and also the direction from the first point to the last point (i.e. from initial point to terminal point). Most of the basic algebraic operations like addition, subtraction, multiplication and division are reflected equally well in vector-operations as addition, subtraction and multiplication by a scalar. Vector addition also follows the algebraic properties of R like commutativity, associativity.

Vector is a very important concept in the study of Physics. Many physical quantities like velocity, acceleration, force acting on an object etc. are described by vectors. Many physical quantities do not represent distance but are still represented by vectors and so it helps a lot to understand the concepts of Physics.

Generally, gravity, electrostatic force, magnetic force, electromagnetic force or mechanical force are studied in physics. Physicists had found by scientific experiments that these forces in general conditions act in a linear (vector) way and their resultant forces are also the result of the addition of vectors, e.g. **Coulomb's law of electrostatics**. So vector space and its algebraic operations etc are developed to study these forces.

Vectors are denoted by small arrow (→) or bar (–) sign above the letter or bold letters in print form. In Mathematics, Physics and Engineering, we frequently come across scalar quantities such as length, distance, speed, time, mass etc and also vector quantities like, displacement, velocity, acceleration, force, weight etc.

We have already studied in std. XI about vector space R^2 as well as R^3 and some operations on vectors like addition of vectors, multiplication of a vector by a scalar and their properties, magnitude of a vector, a unit vector etc. These concepts are needed for further study. So in this chapter, we shall summarise them and consolidate by solving some examples.

6.2 Vector as an Element of a Vector Space

$R^2 = \{(x, y) \mid x \in R, y \in R\}$

$R^3 = \{(x, y, z) \mid x \in R, y \in R, z \in R\}$

The sets R^2 and R^3 under operations of addition and multiplication by a scalar given on page 192 are called vector spaces over R.

The elements of R^2 and R^3 as vector space are denoted by $\bar{x}$, $\bar{y}$, $\bar{z}$ etc. $\bar{x}$, $\bar{y}$, $\bar{z}$ are called vectors. Elements of R are called scalars.

Equality of Vectors :

$(x_1, y_1, z_1) = (x_2, y_2, z_2) \Leftrightarrow x_1 = x_2, y_1 = y_2$ and $z_1 = z_2$.

Addition of Vectors :

$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Multiplication of a Vector by a Scalar :

$k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1), \quad \forall k \in R$

Properties of Addition of Elements of R^3 and Multiplication by a Scalar

(1) **Closure property :** $\forall \overline{x}, \overline{y} \in R^3, \overline{x} + \overline{y} \in R^3$

(2) **Commutative law of addition :** $\overline{x} + \overline{y} = \overline{y} + \overline{x}; \quad \forall \overline{x}, \overline{y} \in R^3$

(3) **Associative law of addition :** $(\overline{x} + \overline{y}) + \overline{z} = \overline{x} + (\overline{y} + \overline{z}); \quad \forall \overline{x}, \overline{y}, \overline{z} \in R^3$

(4) **Existence of additive identity :** There exists a vector $\overline{0} \in R^3$ such that $\overline{x} + \overline{0} = \overline{0} + \overline{x} = \overline{x}, \forall \overline{x} \in R^3$, $\overline{0}$ is called zero vector or null-vector. $\overline{0} = (0, 0, 0)$

(5) **Existence of additive inverse :** For every $\overline{x} \in R^3$, there exists a vector, $-\overline{x} \in R^3$ such that $\overline{x} + (-\overline{x}) = (-\overline{x}) + \overline{x} = \overline{0}$. This vector $-\overline{x}$ is called additive inverse vector of $\overline{x}$ or negation of $\overline{x}$.

(6) $\forall k \in R$ and $\overline{x} \in R^3$, $k\overline{x} \in R^3$.

(7) $\forall k \in R$, $k(\overline{x} + \overline{y}) = k\overline{x} + k\overline{y}; \quad \forall \overline{x}, \overline{y} \in R^3$

(8) $\forall k, l \in R$, $(k + l)\overline{x} = k\overline{x} + l\overline{x}; \quad \forall \overline{x} \in R^3$

(9) $\forall l, k \in R$, $(kl)\overline{x} = k(l\overline{x}); \ \forall \overline{x} \in R^3$

(10) $1\overline{x} = \overline{x}, \forall \overline{x} \in R^3$

The above rules are also true for the elements of R^2.

Some Basic Concepts

Magnitude of a Vector : If $\overline{x} = (x_1, x_2, x_3)$, then magnitude of $\overline{x}$, denoted by $|\overline{x}|$ is defined as $|\overline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. If $\overline{x} = (x_1, x_2)$, then $|\overline{x}| = \sqrt{x_1^2 + x_2^2}$.

For example, if $\overline{x} = (1, 2, -2)$, then $|\overline{x}| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = 3$.

Some obvious results : ($\overline{x} \in R^2$ or R^3)

(1) $|\overline{x}| \geq 0$

(2) $|\overline{x}| = 0 \Leftrightarrow \overline{x} = \overline{0}$

(3) $|k\overline{x}| = |k|\,|\overline{x}|, k \in R$

Unit Vector : If $|\overline{x}| = 1$, then $\overline{x}$ is called a unit vector. A unit vector is denoted by $\hat{x}$.

For example, if $\overline{x} = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, then $|\overline{x}| = 1$ and hence $\overline{x}$ is a unit vector.

$\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$ are unit vectors in the positive direction of X-axis, Y-axis and Z-axis respectively.

6.3 Direction of vectors

Let $\overline{x}$ and $\overline{y}$ be non-zero vectors of R^2 or R^3 and $k \in R$.

If (i) $\overline{x} = k\overline{y}$, $k > 0$, then $\overline{x}$ and $\overline{y}$ are vectors having same direction.

(ii) $\overline{x} = k\overline{y}$, $k < 0$, then $\overline{x}$ and $\overline{y}$ are vectors having opposite directions.

(iii) $\overline{x} \neq k\overline{y}$, for any $k \in R$, then $\overline{x}$ and $\overline{y}$ are vectors having different directions.

If directions of non-zero vectors $\overline{x}$ and $\overline{y}$ are same or opposite, they are called collinear vectors.

$\therefore$ If $\overline{x} = k\overline{y}$ then and only then $\overline{x}$ and $\overline{y}$ are collinear. ($\overline{x} \neq \overline{0}, \overline{y} \neq \overline{0}$)

Notation : Let $\overline{x} = (x_1, x_2, x_3)$. Direction of $\overline{x}$ is denoted by $<x_1, x_2, x_3>$ and direction opposite, to the direction of $\overline{x}$ is denoted by $-<x_1, x_2, x_3>$.

It follows from the definition that,

(i) $<x_1, x_2, x_3> = <kx_1, kx_2, kx_3>$, if $k > 0$.

(ii) $-<x_1, x_2, x_3> = <kx_1, kx_2, kx_3>$, if $k < 0$.

We also denote direction of $\overline{x}$ as (kx_1, kx_2, kx_3), $k \in R - \{0\}$

We accept the following theorems without proving them.

Theorem 6.1 : Non-zero vectors $\overline{x}$ and $\overline{y}$ are equal if and only if $|\overline{x}| = |\overline{y}|$ and $\overline{x}$ and $\overline{y}$ have the same direction.

Theorem 6.2 : If $\overline{x} \neq \overline{0}$ then there is a unique unit vector in the direction of $\overline{x}$.

Unit Vector in the Direction of a Given Vector : If $\overline{x}$ is any non-zero vector, then $\frac{1}{|\overline{x}|}\overline{x}$ is a unit vector in the direction of $\overline{x}$ and it is denoted by $\hat{x}$.

$\overline{y} = \frac{k\overline{x}}{|\overline{x}|}$, $k > 0$ has same direction as $\overline{x}$ and has magnitude k.

$\overline{y} = \frac{-k\overline{x}}{|\overline{x}|}$, $k > 0$ is in direction opposite to the direction of $\overline{x}$ and has magnitude k.

Example 1 : Find the vector of magnitude 10 in the direction opposite to the direction of $\overline{x} = (3, 0, -4)$.

Solution : $|\overline{x}| = \sqrt{9+0+16} = 5$

$\therefore$ The vector of magnitude 10 in the direction opposite to the direction of $\overline{x}$ is

$\frac{-10}{|\overline{x}|}\overline{x} = \frac{-10}{5}(3, 0, -4) = (-6, 0, 8)$.

Right Hand Thumb Rule : Let O be a fixed point in space and take three mutually perpendicular lines through O. These are taken as X-axis, Y-axis and Z-axis. Normally, X-axis and Y-axis are so arranged that they are in a horizontal plane. Z-axis is perpendicular to both X-axis and Y-axis. **The positive directions of these axes follow the Right Hand Thumb rule**, that is, if you curl the fingers of your right hand around the Z-axis in the direction of counter clockwise $\frac{\pi}{2}$ rotation from the positive X-axis to the positive Y-axis, then your thumb points in the positive direction of positive Z-axis.

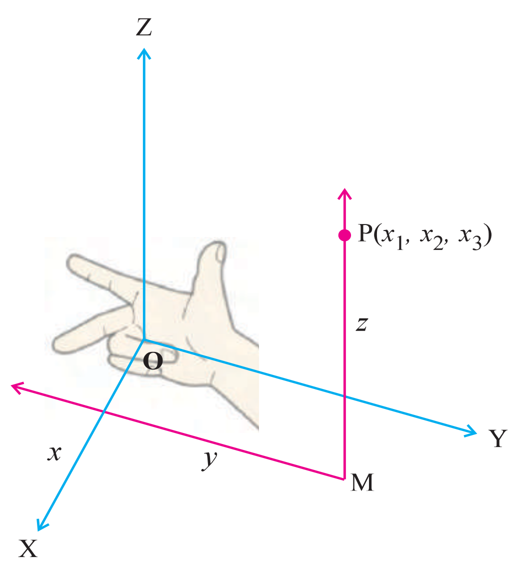


Figure 6.1

6.4 Position Vector

Let $\overline{x} = (x_1, x_2, x_3) \in R^3$ be a vector and a point P in space having coordinates (x_1, x_2, x_3). The directed line-segment $\overline{OP}$ with initial point O and terminal point P is called the position vector of the point P and it is denoted as $\overrightarrow{OP}$. Thus the position vector of P is $\overline{x} = (x_1, x_2, x_3)$, i.e. $\overrightarrow{OP} = (x_1, x_2, x_3)$. If the position vector of a point is $\overline{x}$, then $\overrightarrow{OP} = \overline{x}$ is the the geometrical representation of the vector.

If $A(x_1, x_2, x_3)$ and $B(y_1, y_2, y_3)$ are two distinct points in R^3, the vector joining the points A and B with initial point A is $\overrightarrow{AB}$.

Theorem 6.3 : (1) Every vector of R^2 can be uniquely expressed as linear combination of $\hat{i}$ and $\hat{j}$.

Proof : Suppose $\overline{x} = (x_1, x_2) \in R^2$.

Then $\overline{x} = (x_1, x_2) = (x_1, 0) + (0, x_2)$

$= x_1(1, 0) + x_2(0, 1)$

$= x_1\hat{i} + x_2\hat{j}$

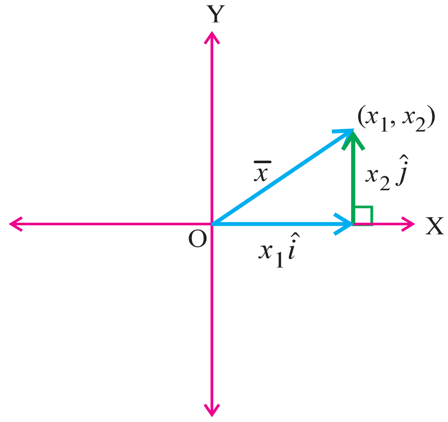


Figure 6.2

Thus, $\overline{x}$ is **a linear combination** of $\hat{i}$ and $\hat{j}$. Now, suppose $\overline{x}$ can be expressed as a linear combination of $\hat{i}$ and $\hat{j}$ as $\overline{x} = p\hat{i} + q\hat{j}$ also.

Then $(x_1, x_2) = \overline{x} = p\hat{i} + q\hat{j}$

$= p(1, 0) + q(0, 1)$

$= (p, 0) + (0, q)$

$= (p, q)$

$\therefore \quad x_1 = p$ and $x_2 = q$

$p\hat{i} + q\hat{j}$ and $x_1\hat{i} + x_2\hat{j}$ are same.

Thus $\overline{x} = x_1\hat{i} + x_2\hat{j}$ is a unique linear combination of $\hat{i}$ and $\hat{j}$.

(2) Every vector in R^3 can be uniquely expressed as a linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Proof : Suppose $\overline{x} = (x_1, x_2, x_3) \in R^3$.

Then $\overline{x} = (x_1, x_2, x_3) = (x_1, 0, 0) + (0, x_2, 0) + (0, 0, x_3)$

$= x_1(1, 0, 0) + x_2(0, 1, 0) + x_3(0, 0, 1)$

$= x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$

If $\overline{x} = p\hat{i} + q\hat{j} + r\hat{k}$, then we can prove $x_1 = p$, $x_2 = q$ and $x_3 = r$ as before.

Thus, $\overline{x} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ is unique linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Geometric Representation :

Let $\overrightarrow{OP} = (x_1, x_2, x_3)$.

Let L be the foot of perpendicular from P to XY plane (figure 6.3). So $L(x_1, x_2, 0)$.

Then $\overrightarrow{LP} = \overrightarrow{OC} = x_3\hat{k}$. Similarly, M and N are the feet of perpendiculars from P to YZ and ZX plane respectively. So $M(0, x_2, x_3)$ and $N(x_1, 0, x_3)$

and so $\overrightarrow{MP} = \overrightarrow{OA} = x_1\hat{i}$ and $\overrightarrow{NP} = \overrightarrow{OB} = x_2\hat{j}$.

$\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ are bound vectors corresponding to free vectors $\overrightarrow{MP}$, $\overrightarrow{NP}$, $\overrightarrow{LP}$ respectively.

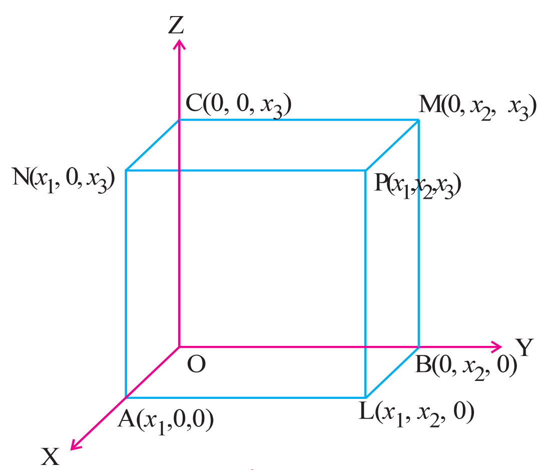


Figure 6.3

[The coordinates of A, B and C are A(x_1, 0, 0), B(0, x_2, 0) and C(0, 0, x_3).]

Now, $\overrightarrow{OL} = \overrightarrow{OA} + \overrightarrow{AL} = \overrightarrow{OA} + \overrightarrow{OB} = x_1\hat{i} + x_2\hat{j}$ $\qquad (\overrightarrow{OB} = \overrightarrow{AL})$

[The coordinates of L are (x_1, x_2, 0). Similarly coordinates of M and N are (0, x_2, x_3) and (x_1, 0, x_3) respectively.]

$\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$. The form $\overrightarrow{OP} = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$ of a vector is also called **component form**. Here x_1, x_2 and x_3 are the scalar components of $\overrightarrow{OP}$, while $x_1\hat{i}$, $x_2\hat{j}$ and $x_3\hat{k}$ are the **vector components** of $\overrightarrow{OP}$.

Note : (1) Distance of P(x_1, x_2, x_3) from XY plane is PL = $|x_3|$. Similarly, distance of P from YZ plane = PM = $|x_1|$ and distance from ZX plane = PN = $|x_2|$.

(2) Distance of P(x_1, x_2, x_3) from X-axis = AP = $\sqrt{x_2^2 + x_3^2}$. Similarly distance from Y-axis = BP = $\sqrt{x_3^2 + x_1^2}$ and distance from Z-axis = CP = $\sqrt{x_1^2 + x_2^2}$.

(3) Distance of P(x_1, x_2, x_3) from origin = OP = $\sqrt{x_1^2 + x_2^2 + x_3^2}$.

6.5 Triangle Law of Vector Addition

A particle is displaced from A to B and the displacement is represented by $\overrightarrow{AB}$ and the displacement from B to C is represented by $\overrightarrow{BC}$ as shown in figure 6.4. The displacement of the particle from A to C is given by the vector $\overrightarrow{AC}$. The result $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ is called the Triangle Law of Vector Addition.

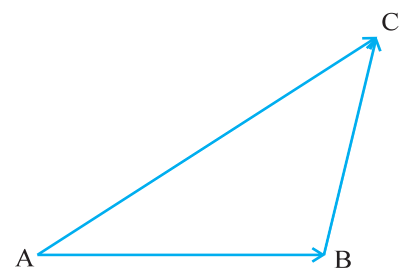


Figure 6.4

Let A, B, C have position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively.

$$\overrightarrow{AB} + \overrightarrow{BC} = (\bar{b} - \bar{a}) + (\bar{c} - \bar{b})$$
$$= \bar{c} - \bar{a} = \overrightarrow{AC}$$

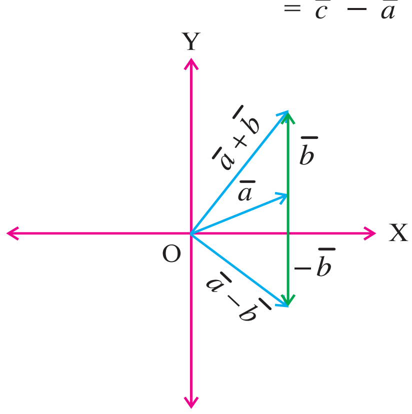


Figure 6.5

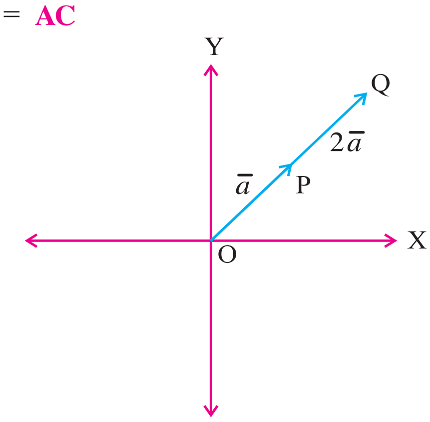


Figure 6.6

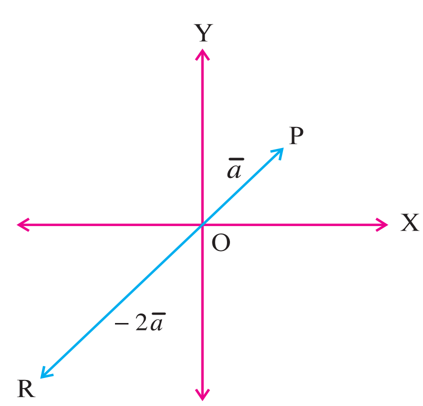


Figure 6.7

If $\bar{a}$ and $\bar{b}$ are two non-zero vectors, then the operations of addition and subtraction of vectors $\bar{a}$ and $\bar{b}$ in R^2 are shown in figure 6.5. Figures 6.6 and 6.7 illustrate scalar multiplication of vector in R^2. Here $\overrightarrow{OP} = \bar{a}$, $\overrightarrow{OQ} = 2\bar{a}$ and $\overrightarrow{OR} = -2\bar{a}$.

Parallelogram Law for Vector Addition :

Let $\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$ be two distinct vectors. We construct parallelogram OACB (figure 6.8). The vector along the diagonal from their common initial point $\overrightarrow{OC}$ represents the sum of vectors $\bar{a}$ and $\bar{b}$. Thus $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. This law is known as the parallelogram law for vector addition.

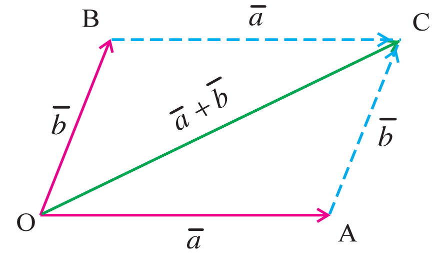


Figure 6.8

Note : $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$ $\quad (\overrightarrow{OB} = \overrightarrow{AC})$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

Properties of Vector Addition (Geometrically) :

Property 1 : For any two vectors $\bar{x}$ and $\bar{y}$, $\bar{x} + \bar{y} = \bar{y} + \bar{x}$ (Commutative property)

Let $\overrightarrow{AB} = \bar{x}$ and $\overrightarrow{AD} = \bar{y}$. We complete the parallelogram ABCD.

$\therefore$ Obviously, $\overrightarrow{BC} = \bar{y}$ and $\overrightarrow{DC} = \bar{x}$ **(By theorem 6.1)**

Now, applying triangle law for ΔABC,

we get $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \bar{x} + \bar{y}$.

Similarly, for ΔADC, $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$

$\therefore \bar{y} + \bar{x} = \overrightarrow{AC}$.

Thus, $\bar{x} + \bar{y} = \bar{y} + \bar{x}$.

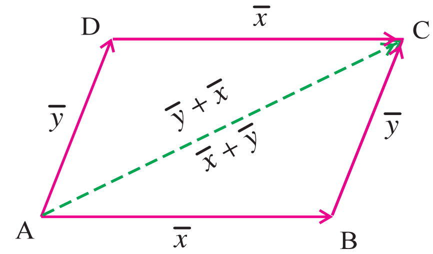


Figure 6.9

Property 2 : For vectors $\bar{x}$, $\bar{y}$, $\bar{z}$, $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$ (Associative property)

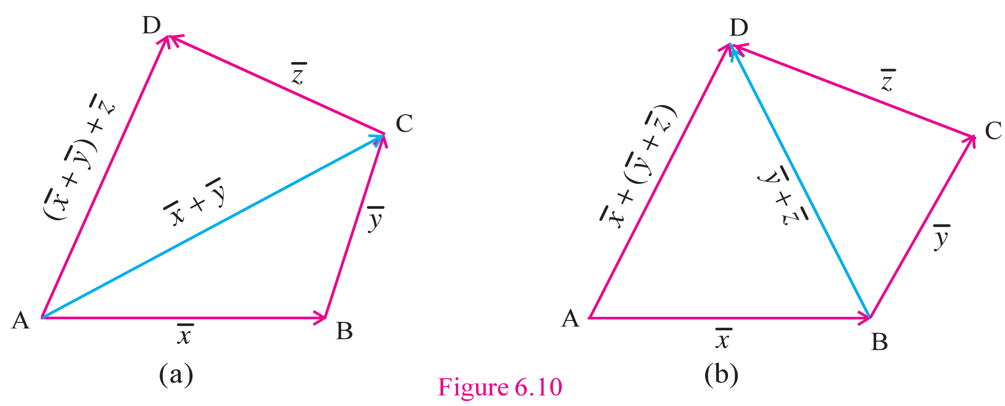


Figure 6.10

Let $\overrightarrow{AB} = \bar{x}$, $\overrightarrow{BC} = \bar{y}$, $\overrightarrow{CD} = \bar{z}$. Using triangle law of addition,

From figure 6.10(a)

From ΔABC,

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\therefore \bar{x} + \bar{y} = \overrightarrow{AC}$.

From ΔACD,

$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$\therefore (\bar{x} + \bar{y}) + \bar{z} = \overrightarrow{AD}$.

From figure 6.10(b)

From ΔBCD,

$\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

$\therefore \bar{y} + \bar{z} = \overrightarrow{BD}$.

From ΔABD,

$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\therefore \bar{x} + (\bar{y} + \bar{z}) = \overrightarrow{AD}$.

Thus, $(\bar{x} + \bar{y}) + \bar{z} = \bar{x} + (\bar{y} + \bar{z})$.

Example 2 : Find the vector having initial point (3, 2, −1) and terminal point (4, −2, 0) and its magnitude.

Solution : A(3, 2, −1) is the initial point and B(4, −2, 0) is the terminal point of $\overrightarrow{AB}$.

$\therefore \quad \overrightarrow{AB}$ = Position vector of B − Position vector of A

= (4, −2, 0) − (3, 2, −1)

= (1, −4, 1)

Magnitude of $\overrightarrow{AB} = |\overrightarrow{AB}| = \sqrt{(1)^2 + (-4)^2 + (1)^2}$

$\therefore \quad AB = \sqrt{18}$

$= 3\sqrt{2}$

Exercise 6.1

1. Find the magnitude of the following vectors :

(1) $(2, 3, \sqrt{3})$ **(2)** $3\hat{i} - 4\hat{k}$ **(3)** $\hat{i} + \hat{j} - 4\hat{k}$

2. Find the unit vector in the direction of $2\hat{i} - 2\hat{j} + \hat{k}$.

3. Find the vector of magnitude $2\sqrt{17}$ in the direction of (3, −2, −2).

4. Find the vector of magnitude 20 in the direction opposite to the direction of vector $-3\hat{i} + 2\sqrt{3}\hat{j} - 2\hat{k}$.

5. For vectors $\bar{x} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\bar{y} = 2\hat{i} + \hat{j}$, find the unit vector in the direction of $\bar{x} + 2\bar{y}$.

6. Find the scalar and vector components of the vector with initial point (−2, 1, 0) and terminal point (1, −5, 7).

7. If the position vector of a point P is (4, 5, −3), then find the distance of P, **(i)** from ZX plane **(ii)** from Y-axis and **(iii)** from the origin.

*

6.6 Inner Product of Vectors in R^2 and R^3

If $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ are vectors in R^2, their inner product is defined as $x_1y_1 + x_2y_2$ and is denoted by $\bar{x} \cdot \bar{y}$.

Similarly, for $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$ in R^3, $\bar{x} \cdot \bar{y} = x_1y_1 + x_2y_2 + x_3y_3$.

Here, $\bar{x}$ and $\bar{y}$ are vectors, but $\bar{x} \cdot \bar{y}$ is not a vector, it is a real number. Thus inner product of two vectors is a scalar, so the inner product is also called **Scalar Product**. This operation is known as **Scalar Multiplication.** Since notation for inner product is a dot (.) between the two vectors, so inner product is also called **Dot Product of Vectors**.

Note : Difference between scalar product and product by a scalar.

Scalar product is performed between two vectors and the result is a scalar quantity and product by a scalar with a vector is a vector quantity.

If $\bar{x} = (2, 3, -1)$ and $\bar{y} = (-1, 4, -2)$, then scalar product of $\bar{x}$ and $\bar{y}$ is

$\bar{x} \cdot \bar{y} = -2 + 12 + 2 = 12$ is a scalar quantity.

While product of $\bar{x} = (2, 3, -1)$ with a scalar, say 2 is $2\bar{x} = 2(2, 3, -1) = (4, 6, -2)$ is a vector quantity.

Properties of Inner Product :

Suppose $\overline{x} = (x_1, x_2, x_3)$, $\overline{y} = (y_1, y_2, y_3)$ and $\overline{z} = (z_1, z_2, z_3)$ are vectors in R^3 and $k \in R$.

(1) $\overline{x} \cdot \overline{x} \geq 0$ **and** $\overline{x} \cdot \overline{x} = 0 \Leftrightarrow \overline{x} = \overline{0}$.

$\overline{x} \cdot \overline{x} = (x_1, x_2, x_3) \cdot (x_1, x_2, x_3)$

$= x_1^2 + x_2^2 + x_3^2 \geq 0$ **(Property of R)**

$\overline{x} \cdot \overline{x} = 0 \Leftrightarrow x_1 = x_2 = x_3 = 0 \Leftrightarrow \overline{x} = \overline{0}$

(2) $\overline{x} \cdot \overline{x} = |\overline{x}|^2$ as $\overline{x} \cdot \overline{x} = x_1^2 + x_2^2 + x_3^2 = |\overline{x}|^2$

(3) $\overline{x} \cdot \overline{y} = \overline{y} \cdot \overline{x}$

(4) $\overline{x} \cdot (k\overline{y}) = (k\overline{x}) \cdot \overline{y} = k(\overline{x} \cdot \overline{y})$

(5) $\overline{x} \cdot (\overline{y} + \overline{z}) = \overline{x} \cdot \overline{y} + \overline{x} \cdot \overline{z}$

$\overline{x} \cdot (\overline{y} + \overline{z}) = (x_1, x_2, x_3) \cdot (y_1 + z_1, y_2 + z_2, y_3 + z_3)$

$= x_1(y_1 + z_1) + x_2(y_2 + z_2) + x_3(y_3 + z_3)$

$= x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2 + x_3y_3 + x_3z_3$ **(Distributive law in R)**

$= (x_1y_1 + x_2y_2 + x_3y_3) + (x_1z_1 + x_2z_2 + x_3z_3)$

$= \overline{x} \cdot \overline{y} + \overline{x} \cdot \overline{z}$

These properties are also valid for the vectors in R^2.

Example 3 : Find $\overline{x} \cdot \overline{y}$, where $\overline{x} = (1, 2, -1)$, $\overline{y} = (-3, 4, -2)$.

Solution : $\overline{x} \cdot \overline{y} = (1, 2, -1) \cdot (-3, 4, -2)$

$= -3 + 8 + 2$

$= 7$

Example 4 : If $\overline{x} = 5\hat{i} + 4\hat{j} - 3\hat{k}$ and $\overline{y} = 2\hat{i} - \hat{j} + 2\hat{k}$, then find $(\overline{x} + 2\overline{y}) \cdot (2\overline{x} - \overline{y})$.

Solution : $\overline{x} + 2\overline{y} = (5\hat{i} + 4\hat{j} - 3\hat{k}) + 2(2\hat{i} - \hat{j} + 2\hat{k})$

$= 5\hat{i} + 4\hat{j} - 3\hat{k} + 4\hat{i} - 2\hat{j} + 4\hat{k}$

$= 9\hat{i} + 2\hat{j} + \hat{k}$

or $\overline{x} + 2\overline{y} = (5, 4, -3) + 2(2, -1, 2) = (5, 4, -3) + (4, -2, 4) = (9, 2, 1)$

$2\overline{x} - \overline{y} = 2(5\hat{i} + 4\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + 2\hat{k})$

$= 10\hat{i} + 8\hat{j} - 6\hat{k} - 2\hat{i} + \hat{j} - 2\hat{k}$

$= 8\hat{i} + 9\hat{j} - 8\hat{k}$

or $2\overline{x} - \overline{y} = 2(5, 4, -3) - (2, -1, 2) = (10, 8, -6) + (-2, 1, -2) = (8, 9, -8)$

Now, $(\overline{x} + 2\overline{y})\cdot(2\overline{x} - \overline{y}) = (9\hat{i} + 2\hat{j} + \hat{k})\cdot(8\hat{i} + 9\hat{j} - 8\hat{k})$

$= (9, 2, 1)\cdot(8, 9, -8)$

$= 72 + 18 - 8$

$= 82$

Outer Product of Vectors in R^3 :

If $\overline{x} = (x_1, x_2, x_3)$ and $\overline{y} = (y_1, y_2, y_3)$ are vectors in R^3, their outer product is denoted by $\overline{x} \times \overline{y}$ and defined as

$$\overline{x} \times \overline{y} = (x_1, x_2, x_3) \times (y_1, y_2, y_3)$$

$$= \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)$$

$$= (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)$$

Here, $\overline{x}$ and $\overline{y}$ are vectors and their outer product $\overline{x} \times \overline{y}$ is also a vector. So outer product is also called **Vector Product**. The operation of obtaining outer product is known as **Vector Multiplication**. Since the notation for outer product is a cross (×) between the two vectors, outer product is also called **Cross Product**.

Properties of Outer Product :

(1) $\overline{x} \times \overline{y} = -\overline{y} \times \overline{x}$ **(Interchange of rows in a determinant)**

(2) $\overline{x} \times \overline{x} = \overline{0}$ **(Two identical rows in a determinant)**

(3) $\overline{x} \times (k\overline{y}) = (k\overline{x}) \times \overline{y} = k(\overline{x} \times \overline{y})$

(4) $\overline{x} \times (\overline{y} + \overline{z}) = \overline{x} \times \overline{y} + \overline{x} \times \overline{z}$

(5) $\overline{x} \times \overline{0} = \overline{0} \times \overline{x} = \overline{0}$

Difference Between Inner and Outer Product of Vectors :

(1) Inner product is a scalar quantity, while outer product is a vector quantity.

(2) Inner product is defined in R^2 as well as in R^3, while outer product is not defined in R^2.

(3) Inner product is commutative, while outer product is not commutative.

Note : $\overline{x}\cdot\overline{x} = |\overline{x}|^2$, but $\overline{x} \times \overline{x} = \overline{0}$.

Example 5 : Find $\overline{x} \times \overline{y}$, where $\overline{x} = (1, 3, -2)$ and $\overline{y} = (-2, 1, 5)$

Solution : $\overline{x} \times \overline{y} = \left(\begin{vmatrix} 3 & -2 \\ 1 & 5 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ -2 & 5 \end{vmatrix}, \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} \right)$

$= (15 + 2, -(5 - 4), 1 + 6) = (17, -1, 7)$

Example 6 : If $\overline{x} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\overline{y} = 3\hat{i} - 2\hat{j} + \hat{k}$, find $|\overline{x} \times \overline{y}|$.

Solution : $\overline{x} = (2, 1, -3)$, $\overline{y} = (3, -2, 1)$

$$\overline{x} \times \overline{y} = \left(\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \right)$$

$= (1 - 6, -(2 + 9), -4 - 3) = (-5, -11, -7)$

$\therefore \quad |\overline{x} \times \overline{y}| = \sqrt{25 + 121 + 49} = \sqrt{195}$

Box Product and Vector Triple Product :

If $\bar{x}$, $\bar{y}$ and $\bar{z}$ are vectors in R^3, the product $\bar{x} \cdot (\bar{y} \times \bar{z})$ is called the box product of $\bar{x}$, $\bar{y}$ and $\bar{z}$, it is denoted by $[\bar{x} \quad \bar{y} \quad \bar{z}]$.

Let $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$ and $\bar{z} = (z_1, z_2, z_3)$. Then

$$\bar{x} \cdot (\bar{y} \times \bar{z}) = (x_1, x_2, x_3) \cdot (y_2z_3 - y_3z_2, -(y_1z_3 - y_3z_1), y_1z_2 - y_2z_1)$$

$$\therefore \quad [\bar{x} \quad \bar{y} \quad \bar{z}] = x_1(y_2z_3 - y_3z_2) - x_2(y_1z_3 - y_3z_1) + x_3(y_1z_2 - y_2z_1)$$

$$\therefore \quad [\bar{x} \quad \bar{y} \quad \bar{z}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Properties of Box Product :

(1) $[\bar{x} \quad \bar{y} \quad \bar{z}] = [\bar{y} \quad \bar{z} \quad \bar{x}] = [\bar{z} \quad \bar{x} \quad \bar{y}]$

Proof : $[\bar{x} \quad \bar{y} \quad \bar{z}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

$$= -\begin{vmatrix} y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \qquad (R_{12})$$

$$= \begin{vmatrix} y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \end{vmatrix} \qquad (R_{23})$$

$$= [\bar{y} \quad \bar{z} \quad \bar{x}]$$

Similarly, we can prove that $[\bar{x} \quad \bar{y} \quad \bar{z}] = [\bar{z} \quad \bar{x} \quad \bar{y}]$.

(2) $[\bar{x} \quad \bar{x} \quad \bar{y}] = 0$, $[\bar{x} \quad \bar{y} \quad \bar{x}] = 0$, $[\bar{x} \quad \bar{y} \quad \bar{y}] = 0$

(3) $[m\bar{x} \quad \bar{y} \quad \bar{z}] = m[\bar{x} \quad \bar{y} \quad \bar{z}]$; $[\bar{x} \quad m\bar{y} \quad \bar{z}] = m[\bar{x} \quad \bar{y} \quad \bar{z}]$; $[\bar{x} \quad \bar{y} \quad m\bar{z}] = m[\bar{x} \quad \bar{y} \quad \bar{z}]$; $m \in R$

(4) $[\bar{x} \quad \bar{y} \quad \bar{0}] = 0$

Note : **(1)** If the vectors are changed in cyclic order, the box product remains unchanged.

(2) Interchange of any two vectors in $[\bar{x} \quad \bar{y} \quad \bar{z}]$ results in mere interchange of two rows in the determinant. So the value of the box product will be additive inverse, i.e. $[\bar{x} \quad \bar{y} \quad \bar{z}] = -[\bar{y} \quad \bar{x} \quad \bar{z}]$.

The product $\bar{x} \times (\bar{y} \times \bar{z})$ is called the vector triple product.

It can be proved that $\bar{x} \times (\bar{y} \times \bar{z}) = (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}$.

Similarly $(\bar{x} \times \bar{y}) \times \bar{z} = (\bar{z} \cdot \bar{x})\bar{y} - (\bar{z} \cdot \bar{y})\bar{x}$.

We shall prove the following result :

$\bar{x} \times (\bar{y} \times \bar{z}) = (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}$

Proof : Let $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$, $\bar{z} = (z_1, z_2, z_3)$

Then $\bar{x} \times (\bar{y} \times \bar{z}) = (x_1, x_2, x_3) \times (y_2z_3 - y_3z_2, y_3z_1 - y_1z_3, y_1z_2 - y_2z_1)$

$= (p_1, p_2, p_3)$, say

Now, $p_1 = x_2(y_1z_2 - y_2z_1) - x_3(y_3z_1 - y_1z_3)$

$= y_1(x_2z_2 + x_3z_3) - z_1(x_2y_2 + x_3y_3)$

$= y_1(x_1z_1 + x_2z_2 + x_3z_3) - z_1(x_1y_1 + x_2y_2 + x_3y_3)$ **(Adding and subtracting $x_1y_1z_1$)**

$= y_1(\bar{x}\cdot\bar{z}) - z_1(\bar{x}\cdot\bar{y})$

Similarly $p_2 = y_2(\bar{x}\cdot\bar{z}) - z_2(\bar{x}\cdot\bar{y})$ and $p_3 = y_3(\bar{x}\cdot\bar{z}) - z_3(\bar{x}\cdot\bar{y})$

$\therefore \quad \bar{x}\times(\bar{y}\times\bar{z}) = ((\bar{x}\cdot\bar{z})y_1 - (\bar{x}\cdot\bar{y})z_1, (\bar{x}\cdot\bar{z})y_2 - (\bar{x}\cdot\bar{y})z_2, (\bar{x}\cdot\bar{z})y_3 - (\bar{x}\cdot\bar{y})z_3)$

$= (\bar{x}\cdot\bar{z})(y_1, y_2, y_3) - (\bar{x}\cdot\bar{y})(z_1, z_2, z_3)$

$= (\bar{x}\cdot\bar{z})\bar{y} - (\bar{x}\cdot\bar{y})\bar{z}$

Example 7 : Find $[\bar{x} \quad \bar{y} \quad \bar{z}]$, if $\bar{x} = (1, 2, 0)$, $\bar{y} = (3, -1, 2)$, $\bar{z} = (1, 1, 1)$.

Solution : $[\bar{x} \quad \bar{y} \quad \bar{z}] = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(-3) - 2(1) + 0$

$= -5$

Example 8 : Find $\bar{x}\times(\bar{y}\times\bar{z})$, if $\bar{x} = (1, 2, 3)$, $\bar{y} = (2, 3, 5)$, $\bar{z} = (1, -1, -1)$.

Solution : Method 1 : $\bar{x}\cdot\bar{z} = (1, 2, 3)\cdot(1, -1, -1) = 1 - 2 - 3 = -4$

$\bar{x}\cdot\bar{y} = (1, 2, 3)\cdot(2, 3, 5) = 2 + 6 + 15 = 23$

$\bar{x}\times(\bar{y}\times\bar{z}) = (\bar{x}\cdot\bar{z})\bar{y} - (\bar{x}\cdot\bar{y})\bar{z}$

$= -4(2, 3, 5) - 23(1, -1, -1)$

$= (-8, -12, -20) + (-23, 23, 23)$

$= (-31, 11, 3)$

Method 2 : $\bar{y} = (2, 3, 5)$ and

$\bar{z} = (1, -1, -1)$

$\therefore \quad \bar{y}\times\bar{z} = (-3 + 5, -(-2 - 5), -2 - 3) = (2, 7, -5)$

$\therefore \quad \bar{x} = (1, 2, 3)$ and

$\bar{y}\times\bar{z} = (2, 7, -5)$

$\bar{x}\times(\bar{y}\times\bar{z}) = (-10 - 21, -(-5 - 6), 7 - 4) = (-31, 11, 3)$

Example 9 : $\forall \bar{x}, \bar{y}, \bar{z} \in R^3$, prove that, $[(\bar{x} + \bar{y})\times(\bar{y} + \bar{z})]\cdot(\bar{x} + \bar{z}) = 2[\bar{x} \quad \bar{y} \quad \bar{z}]$

Solution : L.H.S. $= [(\bar{x} + \bar{y})\times(\bar{y} + \bar{z})]\cdot(\bar{x} + \bar{z})$

$= [\bar{x}\times\bar{y} + \bar{x}\times\bar{z} + \bar{y}\times\bar{y} + \bar{y}\times\bar{z}]\cdot(\bar{x} + \bar{z})$

$= [\bar{x}\times\bar{y} + \bar{x}\times\bar{z} + \bar{y}\times\bar{z}]\cdot(\bar{x} + \bar{z})$ **$(\bar{y}\times\bar{y} = \bar{0})$**

$= (\bar{x}\times\bar{y})\cdot\bar{x} + (\bar{x}\times\bar{y})\cdot\bar{z} + (\bar{x}\times\bar{z})\cdot\bar{x} + (\bar{x}\times\bar{z})\cdot\bar{z} + (\bar{y}\times\bar{z})\cdot\bar{x} + (\bar{y}\times\bar{z})\cdot\bar{z}$

$= [\bar{x} \quad \bar{y} \quad \bar{x}] + [\bar{x} \quad \bar{y} \quad \bar{z}] + [\bar{x} \quad \bar{z} \quad \bar{x}] + [\bar{x} \quad \bar{z} \quad \bar{z}] + [\bar{y} \quad \bar{z} \quad \bar{x}] + [\bar{y} \quad \bar{z} \quad \bar{z}]$

$= 0 + [\bar{x} \quad \bar{y} \quad \bar{z}] + 0 + 0 + [\bar{x} \quad \bar{y} \quad \bar{z}] + 0$ **$([\bar{y} \quad \bar{z} \quad \bar{x}] = [\bar{x} \quad \bar{y} \quad \bar{z}])$**

$= 2[\bar{x} \quad \bar{y} \quad \bar{z}]$ = R.H.S.

Exercise 6.2

Find the vector or scalar as required :

1. $(2, 3, 1) \cdot (2, -1, 4)$

2. $(1, -1, 2) \times (2, 3, 1)$

3. $(2, -1, -2) \times (4, 1, 8)$

4. $|(2, 1, 3) \times (0, -4, -4)|$

5. $|(3, -4, -1) \cdot (1, 2, -2)|$

6. $(1, 1, 2) \times [(1, 2, 1) \times (2, 1, 1)]$

7. $(1, 0, 1) \cdot [(1, 1, 0) \times (1, 0, -1)]$

8. $(2, 3, 4) \cdot [(1, 1, 1) \times (3, 4, 5)]$

9. $[(1, 5, 1) \times (2, -1, 2)] \times (4, 1, -3)$

10. $|[(2, 3, 4) \cdot (-4, 3, -2)]\ (1, -1, 2)|$

*

6.7 Lagrange's Identity

If $x_1, x_2, x_3, y_1, y_2, y_3 \in R$, then

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 + (x_1y_2 - x_2y_1)^2 + (x_1y_3 - x_3y_1)^2 + (x_2y_3 - x_3y_2)^2 = (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) \quad \text{(Verify !)}$$

This identity is known as Lagrange's identity.

If we take $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, then vector form of Lagrange's identity is

$$|\bar{x} \cdot \bar{y}|^2 + |\bar{x} \times \bar{y}|^2 = |\bar{x}|^2 |\bar{y}|^2.$$

because $\bar{x} \cdot \bar{y} = x_1y_1 + x_2y_2 + x_3y_3$, $\bar{x} \times \bar{y} = (x_2y_3 - x_3y_2, -(x_1y_3 - x_3y_1), x_1y_2 - x_2y_1)$

$|\bar{x}|^2 = x_1^2 + x_2^2 + x_3^2$ and $|\bar{y}|^2 = y_1^2 + y_2^2 + y_3^2$.

Example 10 : If $\bar{x}$ and $\bar{y}$ are unit vectors and $\bar{x} \cdot \bar{y} = 0$, then prove that $\bar{x} \times \bar{y}$ is a unit vector.

Solution : $\bar{x}$ and $\bar{y}$ are unit vectors.

$\therefore \quad |\bar{x}| = 1 = |\bar{y}|$

Using Lagrange's identity,

$|\bar{x} \times \bar{y}|^2 + |\bar{x} \cdot \bar{y}|^2 = |\bar{x}|^2 |\bar{y}|^2.$

$\therefore \quad |\bar{x} \times \bar{y}|^2 + 0 = (1)(1)$

$\therefore \quad |\bar{x} \times \bar{y}| = 1$

$\therefore \quad \bar{x} \times \bar{y}$ is a unit vector.

Cauchy-Schwartz Inequality :

For any two vectors $\bar{x}$ and $\bar{y}$ of R^2 or R^3, $|\bar{x} \cdot \bar{y}| \le |\bar{x}|\,|\bar{y}|$.

This inequality is known as Cauchy - Schwartz inequality.

In R^3, according to the Lagrange's identity,

$|\bar{x} \times \bar{y}|^2 + |\bar{x} \cdot \bar{y}|^2 = |\bar{x}|^2 |\bar{y}|^2.$

$\therefore \quad |\bar{x} \cdot \bar{y}|^2 \le |\bar{x}|^2 |\bar{y}|^2 \qquad (|\bar{x} \times \bar{y}|^2 \ge 0)$

$\therefore \quad |\bar{x} \cdot \bar{y}| \le |\bar{x}|\,|\bar{y}|$

For R^2, let $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$

So, $\bar{x} \cdot \bar{y} = x_1y_1 + x_2y_2$

Now, $(x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2 = (x_1^2 + x_2^2)\cdot(y_1^2 + y_2^2)$ **(Verify !)**

$\therefore \quad |x_1y_1 + x_2y_2|^2 \le (x_1^2 + x_2^2)\cdot(y_1^2 + y_2^2)$ **($(x_1y_2 - x_2y_1)^2 \ge 0$)**

$\therefore \quad |\overline{x}\cdot\overline{y}|^2 \le |\overline{x}|^2|\overline{y}|^2$ and hence $|\overline{x}\cdot\overline{y}| \le |\overline{x}||\overline{y}|$.

Second Proof : This is valid for R^2 and R^3.

If $\overline{x} = \overline{0}$ or $\overline{y} = \overline{0}$, then $\overline{x}\cdot\overline{y} = 0$ and $|\overline{x}||\overline{y}| = 0$

So $|\overline{x}\cdot\overline{y}| = |\overline{x}||\overline{y}|$

Let $\overline{x} \ne \overline{0}$ and $\overline{y} \ne \overline{0}$

Suppose $|\overline{x}| = 1$ and $|\overline{y}| = 1$.

Now, $(\overline{x} - \overline{y})\cdot(\overline{x} - \overline{y}) \ge 0$

$\therefore \quad \overline{x}\cdot\overline{x} - 2\overline{x}\cdot\overline{y} + \overline{y}\cdot\overline{y} \ge 0$

$\therefore \quad |\overline{x}|^2 - 2\overline{x}\cdot\overline{y} + |\overline{y}|^2 \ge 0$

$\therefore \quad 2 - 2\overline{x}\cdot\overline{y} \ge 0$ **($|\overline{x}| = |\overline{y}| = 1$)**

Hence, $\overline{x}\cdot\overline{y} \le 1$

Similarly, $(\overline{x} + \overline{y})\cdot(\overline{x} + \overline{y}) \ge 0$

$\therefore \quad |\overline{x}|^2 + 2\overline{x}\cdot\overline{y} + |\overline{y}|^2 \ge 0$

$\therefore \quad 2 + 2\overline{x}\cdot\overline{y} \ge 0$

$\therefore \quad -1 \le \overline{x}\cdot\overline{y}$

Thus, $-1 \le \overline{x}\cdot\overline{y} \le 1$

$\therefore \quad |\overline{x}\cdot\overline{y}| \le 1$

$\therefore \quad |\overline{x}\cdot\overline{y}| \le |\overline{x}||\overline{y}|$ **($|\overline{x}| = 1 = |\overline{y}|$) (i)**

Finally, let $\overline{x} \ne \overline{0}$ and $\overline{y} \ne \overline{0}$, so $|\overline{x}| \ne 0$, $|\overline{y}| \ne 0$

Let $\overline{u} = \dfrac{\overline{x}}{|\overline{x}|}$, $\overline{v} = \dfrac{\overline{y}}{|\overline{y}|}$. Then $|\overline{u}| = 1 = |\overline{v}|$

So by (i), $|\overline{u}\cdot\overline{v}| \le |\overline{u}||\overline{v}|$

$$\therefore \quad \left|\frac{\overline{x}}{|\overline{x}|}\cdot\frac{\overline{y}}{|\overline{y}|}\right| \le \left|\frac{\overline{x}}{|\overline{x}|}\right|\left|\frac{\overline{y}}{|\overline{y}|}\right| = \frac{|\overline{x}|}{|\overline{x}|}\frac{|\overline{y}|}{|\overline{y}|} = 1$$

$\therefore \quad |\overline{x}\cdot\overline{y}| \le |\overline{x}||\overline{y}|$

For non-zero vectors $\overline{x}$ and $\overline{y}$,

if $\overline{x}\cdot\overline{y} = |\overline{x}||\overline{y}|$, then

$$\begin{aligned} |t\overline{x} - \overline{y}|^2 &= (t\overline{x} - \overline{y})\cdot(t\overline{x} - \overline{y}) \\ &= t^2|\overline{x}|^2 - 2t\overline{x}\cdot\overline{y} + |\overline{y}|^2 \\ &= t^2|\overline{x}|^2 - 2t|\overline{x}||\overline{y}| + |\overline{y}|^2 \\ &= (t|\overline{x}| - |\overline{y}|)^2 \end{aligned}$$

($\overline{x}\cdot\overline{y} = |\overline{x}||\overline{y}|$)

Taking $t = \dfrac{|\overline{y}|}{|\overline{x}|}$ **($|\overline{x}| \ne 0$)**

$|t\overline{x} - \overline{y}|^2 = 0$

$\therefore \quad t\overline{x} = \overline{y}$

$\therefore \quad \overline{y} = t\overline{x}$ **($t > 0$)**

$\therefore$ $\bar{x}$ and $\bar{y}$ are in the same direction.

If $\bar{x} \cdot \bar{y} = -|\bar{x}||\bar{y}|$, then $(|t\bar{x} - \bar{y}|)^2 = (t|\bar{x}| + |\bar{y}|)^2$

Now taking $t = -\frac{|\bar{y}|}{|\bar{x}|}$, we get

$t\bar{x} - \bar{y} = \bar{0}$

$\therefore$ $\bar{y} = t\bar{x}$ **$(t < 0)$**

$\therefore$ $\bar{x}$ and $\bar{y}$ are in the opposite direction.

$\therefore$ In Cauchy-Schwartz inequality, if $|\bar{x} \cdot \bar{y}| = |\bar{x}||\bar{y}|$, for non-zero vectors $\bar{x}$ and $\bar{y}$, then $\bar{x}$ and $\bar{y}$ are in the same or in the opposite direction.

Triangle Inequality :

For vectors $\bar{x}$, $\bar{y}$ in R^2 as well as in R^3, $|\bar{x} + \bar{y}| \le |\bar{x}| + |\bar{y}|$.

Proof : $|\bar{x} + \bar{y}|^2 = (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y})$

$= \bar{x} \cdot \bar{x} + \bar{x} \cdot \bar{y} + \bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{y}$

$= |\bar{x}|^2 + 2\bar{x} \cdot \bar{y} + |\bar{y}|^2$ **$(\bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x})$**

$\le |\bar{x}|^2 + 2|\bar{x} \cdot \bar{y}| + |\bar{y}|^2$ **$(\forall a \in R, a \le |a|)$**

$\le |\bar{x}|^2 + 2|\bar{x}||\bar{y}| + |\bar{y}|^2$ **(Cauchy-Schwartz Inequality)**

$\le (|\bar{x}| + |\bar{y}|)^2$

$\therefore$ $|\bar{x} + \bar{y}| \le |\bar{x}| + |\bar{y}|$

Geometric Interpretation :

Let $P(\bar{x})$ and $Q(\bar{y})$ be two distinct points. In figure 6.11, □OPRQ is a parallelogram whose sides $\overline{OP}$ and $\overline{OQ}$ represent two vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ respectively. By the parallelogram law of vector addition,

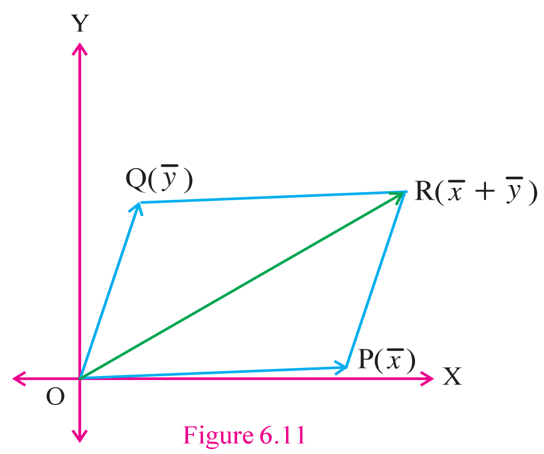


Figure 6.11

$\overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OR}$

In ΔOPR, OP + PR > OR

$\therefore$ OP + OQ > OR **(Opposite sides of a parallelogram are congruent)**

$\therefore$ $|\bar{x}| + |\bar{y}| > |\bar{x} + \bar{y}|$

If O, P, Q are collinear and O–P–Q or O–Q–P (See figure 6.12),

then OP + OQ = OR

$\therefore$ $|\bar{x}| + |\bar{y}| = |\bar{x} + \bar{y}|$

Also, if O–P–Q or O–Q–P is not the case and O, P, Q are collinear, then OP + OQ > OR.

Thus $|\bar{x}| + |\bar{y}| > |\bar{x} + \bar{y}|$

$\therefore$ $|\bar{x} + \bar{y}| \le |\bar{x}| + |\bar{y}|$

In all cases $|\bar{x} + \bar{y}| \le |\bar{x}| + |\bar{y}|$.

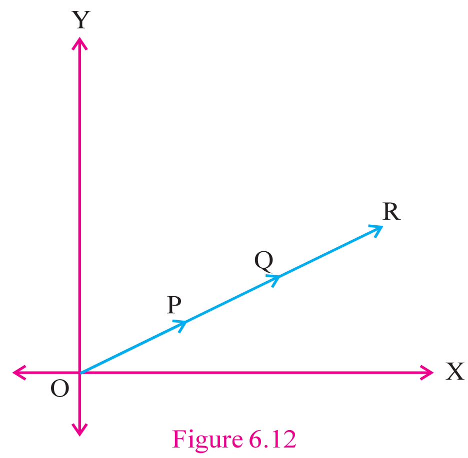


Figure 6.12

6.8 Collinear and Coplanar Vectors

We know that, if $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$ and if $\overline{x} = k\overline{y}$, $k \neq 0$ then $\overline{x}$ and $\overline{y}$ have same or opposite directions. If two vectors have same or opposite directions, then they are called collinear vectors. Free vectors equivalent to the same bound vector or a non-zero multiple of it are conventionally called parallel vectors. If the bound vectors are not collinear, their directions are different. Hence either two bound vectors are collinear or have different directions. They can not be parallel.

Theorem 6.4 : Non-zero vectors $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$ of R^2 are collinear if and only if $x_1y_2 - x_2y_1 = 0$.

Proof : $\overline{x}$ and $\overline{y}$ are collinear $\Rightarrow \overline{x} = k\overline{y}$, $k \in R - \{0\}$, $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$

$\Rightarrow (x_1, x_2) = k(y_1, y_2)$

$\therefore \quad x_1 = ky_1, x_2 = ky_2$

$\therefore \quad x_1y_2 - x_2y_1 = ky_1y_2 - ky_2y_1 = 0$

Conversely, let $x_1y_2 - x_2y_1 = 0$

$\therefore \quad x_1y_2 = x_2y_1$

Let $y_1 \neq 0$, $y_2 \neq 0$

Then $\frac{x_1}{y_1} = \frac{x_2}{y_2} = k$, say.

If $k = 0$, then $x_1 = 0$, $x_2 = 0$. So $\overline{x} = \overline{0}$, But $\overline{x} \neq \overline{0}$. So $k \neq 0$.

$\therefore \quad \overline{x} = (x_1, x_2) = (ky_1, ky_2) = k(y_1, y_2) = k\overline{y}$, $k \in R - \{0\}$

If $y_1 = 0$ or $y_2 = 0$, (both cannot be zero as $\overline{y} \neq \overline{0}$),

let for definiteness $y_2 = 0$, $y_1 \neq 0$

$\therefore \quad x_1y_2 = 0$

$\therefore \quad x_2y_1 = 0$ $\quad$ **($x_1y_2 = x_2y_1$)**

$\therefore \quad x_2 = 0$ as $y_1 \neq 0$

Let $\frac{x_1}{y_1} = k$

$\therefore \quad (x_1, x_2) = (ky_1, 0) = (ky_1, ky_2)$ $\quad$ **($y_2 = 0$)**

$= k(y_1, y_2)$

Again $k = 0 \Rightarrow x_1 = 0$, $x_2 = 0$. So $\overline{x} = \overline{0}$, But $\overline{x} \neq \overline{0}$.

$\therefore \quad \overline{x} = k\overline{y}$, $\quad k \in R - \{0\}$

$\therefore \quad$ If $x_1y_2 - x_2y_1 = 0$, then for $k \in R - \{0\}$, $\overline{x} = k\overline{y}$ and hence $\overline{x}$ and $\overline{y}$ are collinear.

(1) $|\overline{x} \cdot \overline{y}| = |\overline{x}||\overline{y}|$, if and only if $\overline{x} = k\overline{y}$, $k \in R - \{0\}$, $\overline{x} \neq \overline{0}$, $\overline{y} \neq \overline{0}$

Proof : Let $\overline{x} = k\overline{y}$, $k \in R - \{0\}$

$\therefore \quad |\overline{x} \cdot \overline{y}| = |(k\overline{y}) \cdot \overline{y}| = |k(\overline{y} \cdot \overline{y})|$

$= |k| \, |\overline{y} \cdot \overline{y}|$

$= |k|\ |\bar{y}|^2$

$= |k|\ |\bar{y}|\ |\bar{y}|$

$= |k\bar{y}|\ |\bar{y}|$

$= |\bar{x}||\bar{y}|$

Conversely, let $|\bar{x} \cdot \bar{y}| = |\bar{x}||\bar{y}|$.

Now, vector form of Lagrange's identity is

$|\bar{x} \times \bar{y}|^2 + |\bar{x} \cdot \bar{y}|^2 = |\bar{x}|^2 |\bar{y}|^2$

$\therefore\ |\bar{x} \times \bar{y}|^2 = 0$ **($|\bar{x} \cdot \bar{y}| = |\bar{x}||\bar{y}|$)**

$\therefore\ \bar{x} \times \bar{y} = \bar{0}$

We can prove that $\bar{x} = k\bar{y}$ $\quad k \in R - \{0\}$. **(See exercise 6)**

Thus $|\bar{x} \cdot \bar{y}| < |\bar{x}||\bar{y}|$ if and only if $\bar{x} \neq k\bar{y}$, for any $k \in R - \{0\}$, $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$

(2) $|\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}|$, if and only if $\bar{x} = k\bar{y}$, $k > 0$, $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$

i.e. $\bar{x}$ and $\bar{y}$ have the same direction.

Proof : Let $\bar{x} = k\bar{y}$, $k > 0$.

$\therefore\ |\bar{x} + \bar{y}| = |(k\bar{y}) + \bar{y}| = |(k+1)\bar{y}| = |k+1||\bar{y}|$

$= (k+1)|\bar{y}|$ **($k > 0$)**

$= k|\bar{y}| + |\bar{y}|$

$= |k||\bar{y}| + |\bar{y}|$ **($k > 0$)**

$= |k\bar{y}| + |\bar{y}|$

$= |\bar{x}| + |\bar{y}|$

Conversely, let $|\bar{x} + \bar{y}| = |\bar{x}| + |\bar{y}|$

$|\bar{x} + \bar{y}|^2 = (|\bar{x}| + |\bar{y}|)^2$

$\therefore\ (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y}) = |\bar{x}|^2 + 2|\bar{x}||\bar{y}| + |\bar{y}|^2$

$\therefore\ |\bar{x}|^2 + 2\bar{x} \cdot \bar{y} + |\bar{y}|^2 = |\bar{x}|^2 + 2|\bar{x}||\bar{y}| + |\bar{y}|^2$

$\therefore\ \bar{x} \cdot \bar{y} = |\bar{x}||\bar{y}|$

$\therefore$ From the equality in Cauchy-Schwartz inequality, $\bar{x} = k\bar{y}$, $k > 0$.

$\therefore\ \bar{x}$ and $\bar{y}$ are in the same direction.

Theorem 6.5 : Non-zero vectors $\bar{x}$ and $\bar{y}$ of R^3 are collinear if and only if $\bar{x} \times \bar{y} = \bar{0}$.

Proof : Since, $\bar{x}$ and $\bar{y}$ are collinear $\bar{x} = k\bar{y}$, $k \in R - \{0\}$, $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$

$\therefore\ \bar{x} \times \bar{y} = (k\bar{y} \times \bar{y}) = k(\bar{y} \times \bar{y}) = k\bar{0} = \bar{0}$

Conversely, let $\bar{x} \times \bar{y} = \bar{0}$.

$\therefore\ |\bar{x} \cdot \bar{y}| = |\bar{x}||\bar{y}|$ **(Lagrange's identity)**

$\therefore$ Cauchy Schwarz inequality gives $\overline{x} = k\overline{y}$, $k \in \mathrm{R} - \{0\}$ as $\overline{x} \neq \overline{0}$.

$\therefore$ $\overline{x}$, $\overline{y}$ are collinear.

Coplanar Vectors : Let $\overline{x}$, $\overline{y}$ and $\overline{z}$ be vectors of R^3. If we can find $\alpha, \beta, \gamma \in \mathrm{R}$ with at least one of them non-zero, such that $\alpha\overline{x} + \beta\overline{y} + \gamma\overline{z} = \overline{0}$, then $\overline{x}$, $\overline{y}$ and $\overline{z}$ are said to be coplanar vectors.

If $\overline{x}$, $\overline{y}$, $\overline{z}$ are not coplanar, they are called non-coplanar or linearly independent vectors. Thus if $\overline{x}$, $\overline{y}$ and $\overline{z}$ are non-coplanar vectors, then

$\alpha\overline{x} + \beta\overline{y} + \gamma\overline{z} = \overline{0} \Rightarrow \alpha = 0, \beta = 0$ and $\gamma = 0$.

Theorem 6.6 : Distinct non-zero vectors $\overline{x}$, $\overline{y}$, $\overline{z}$ of R^3 are coplanar if and only if $[\overline{x}\ \ \overline{y}\ \ \overline{z}] = 0$.

Proof : Suppose $\overline{x}$, $\overline{y}$, $\overline{z}$ are coplanar.

$\therefore$ We can find α, β, γ with at least one non-zero in R such that $\alpha\overline{x} + \beta\overline{y} + \gamma\overline{z} = \overline{0}$.

Let us assume that $\gamma \neq 0$

$$\therefore \quad \overline{z} = \left(\frac{-\alpha}{\gamma}\right)\overline{x} + \left(\frac{-\beta}{\gamma}\right)\overline{y}$$

$$\therefore \quad [\overline{x}\ \ \overline{y}\ \ \overline{z}] = (\overline{x} \times \overline{y}) \cdot \overline{z} = (\overline{x} \times \overline{y}) \cdot \left[\left(\frac{-\alpha}{\gamma}\right)\overline{x} + \left(\frac{-\beta}{\gamma}\right)\overline{y}\right]$$

$$= (\overline{x} \times \overline{y}) \cdot \left(\frac{-\alpha}{\gamma}\right)\overline{x} + (\overline{x} \times \overline{y}) \cdot \left(\frac{-\beta}{\gamma}\right)\overline{y}$$

$$= \left(\frac{-\alpha}{\gamma}\right)((\overline{x} \times \overline{y}) \cdot \overline{x}) + \left(\frac{-\beta}{\gamma}\right)((\overline{x} \times \overline{y}) \cdot \overline{y})$$

$$= 0 + 0 = 0$$

$$\therefore \quad [\overline{x}\ \ \overline{y}\ \ \overline{z}] = 0$$

Conversely, suppose $[\overline{x}\ \ \overline{y}\ \ \overline{z}] = 0$.

$\therefore$ $\overline{x} \cdot (\overline{y} \times \overline{z}) = 0$

If $\overline{y} \times \overline{z} = \overline{0}$, then $\overline{y}$ and $\overline{z}$ are collinear.

$\therefore$ $\overline{y} = k\overline{z}$, $k \neq 0$

$\therefore$ $0\overline{x} + 1\overline{y} - k\overline{z} = \overline{0}$

Comparing it with $\alpha\overline{x} + \beta\overline{y} + \gamma\overline{z} = \overline{0}$, $\alpha = 0$, $\beta = 1$ and $\gamma = -k \neq 0$

$\therefore$ $\overline{x}$, $\overline{y}$, $\overline{z}$ are coplanar.

Now suppose $\overline{y} \times \overline{z} \neq \overline{0}$.

$\therefore$ At least one of the numbers $y_1z_2 - y_2z_1$, $y_2z_3 - y_3z_2$ and $y_1z_3 - y_3z_1$ is non-zero.

Assume that $y_1z_2 - y_2z_1 \neq 0$

Now, we will prove $\overline{x} - \alpha\overline{y} - \beta\overline{z} = \overline{0}$ for some $\alpha, \beta \in \mathrm{R}$ **(i)**

Consider the equations $\alpha y_1 + \beta z_1 - x_1 = 0$ **(ii)**

$\alpha y_2 + \beta z_2 - x_2 = 0$ **(iii)**

and $\alpha y_3 + \beta z_3 - x_3 = 0$ **(iv)**

Since $y_1z_2 - y_2z_1 \neq 0$, we can solve (ii) and (iii) to find α and β and these α and β satisfy (iv) as $[\overline{x}\ \overline{y}\ \overline{z}] = 0$.

$\therefore$ We can find $\alpha, \beta \in R$ such that $\alpha\overline{y} + \beta\overline{z} = \overline{x}$.

Here $1\overline{x} - \alpha\overline{y} - \beta\overline{z} = \overline{0}$

Also $1 \neq 0$.

$\therefore$ $\overline{x} - \alpha\overline{y} - \beta\overline{z} = \overline{0}$ with at least one coefficient $1 \neq 0$.

Thus, $\overline{x}, \overline{y}, \overline{z}$ are coplanar.

Example 11 : Prove that $(-1, 0, -1)$, $(0, -1, 1)$ and $(-1, 1, 0)$ are non-coplanar and that every $\overline{x} \in R^3$ can be written as $\overline{x} = \alpha(-1, 0, -1) + \beta(0, -1, 1) + \gamma(-1, 1, 0)$ for some real numbers α, β and γ.

Solution : $\begin{vmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -1(-1) + 0 - 1(-1) = 2 \neq 0$

$\therefore$ Vectors $(-1, 0, -1)$, $(0, -1, 1)$ and $(-1, 1, 0)$ are non-coplanar.

Now, let $\overline{x} = \alpha(-1, 0, -1) + \beta(0, -1, 1) + \gamma(-1, 1, 0)$ for some $\alpha, \beta, \gamma \in R$,

where $\overline{x} = (x_1, x_2, x_3)$.

$\therefore$ $(x_1, x_2, x_3) = (-\alpha - \gamma, -\beta + \gamma, -\alpha + \beta)$

$-\alpha - \gamma = x_1, \quad -\beta + \gamma = x_2, \quad -\alpha + \beta = x_3$

Solving them, we get

$$\alpha = -\frac{x_1 + x_2 + x_3}{2},\ \beta = \frac{x_3 - x_1 - x_2}{2},\ \gamma = \frac{x_2 + x_3 - x_1}{2}$$

$$\therefore \quad \overline{x} = -\frac{x_1 + x_2 + x_3}{2}(-1, 0, -1) + \frac{x_3 - x_1 - x_2}{2}(0, -1, 1) + \frac{x_2 + x_3 - x_1}{2}(-1, 1, 0)$$

Example 12 : Give one example of vectors $\overline{x}$ and $\overline{y}$ such that $|\overline{x} \cdot \overline{y}| < |\overline{x}||\overline{y}|$.

Solution : Let, $\overline{x} = (1, -1, 2)$ and $\overline{y} = (2, 1, -2)$ **(choose $\overline{x} \neq k\overline{y}$)**

$\overline{x} \cdot \overline{y} = 2 - 1 - 4 = -3$

$\therefore$ $|\overline{x} \cdot \overline{y}| = 3$ **(i)**

$|\overline{x}||\overline{y}| = \sqrt{6} \cdot \sqrt{9}$

$= 3\sqrt{6}$ **(ii)**

From results (i) and (ii), we have $|\overline{x} \cdot \overline{y}| < |\overline{x}||\overline{y}|$, since $3 < 3\sqrt{6}$.

Example 13 : When is $|\overline{x} + \overline{y}| = |\overline{x}| + |\overline{y}|$? Verify your answer by giving one example of $\overline{x}$ and $\overline{y}$.

Solution : If $\overline{x}$ and $\overline{y}$ are in the same direction, then $|\overline{x} + \overline{y}| = |\overline{x}| + |\overline{y}|$.

Let $\overline{x} = (1, -1, 1)$ and $\overline{y} = (2, -2, 2)$

Here, $\overline{x} = \frac{1}{2}\overline{y}$; $\frac{1}{2} > 0$, so $\overline{x}$ and $\overline{y}$ are in the same direction.

Now, $\overline{x} + \overline{y} = (3, -3, 3)$

$\therefore$ $|\overline{x} + \overline{y}| = 3|(1, -1, 1)| = 3\sqrt{3}$

$\therefore$ $|\overline{x} + \overline{y}| = 3\sqrt{3}$ **(i)**

$|\overline{x}| = \sqrt{3}$, $|\overline{y}| = 2\sqrt{3}$

$\therefore$ $|\overline{x}| + |\overline{y}| = \sqrt{3} + 2\sqrt{3} = 3\sqrt{3}$

Hence, $|\overline{x} + \overline{y}| = |\overline{x}| + |\overline{y}|$.

6.9 Angle Between Two Non-zero Vectors

If two non-zero vectors in R^3 are given, then the measure of the angle between their corresponding bound vectors is defined as the measure of the angle between the given vectors.

Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be the corresponding bound vectors of $\bar{a}$ and $\bar{b}$ respectively. The measure of the angle between $\bar{a}$ and $\bar{b}$ is the measure of the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

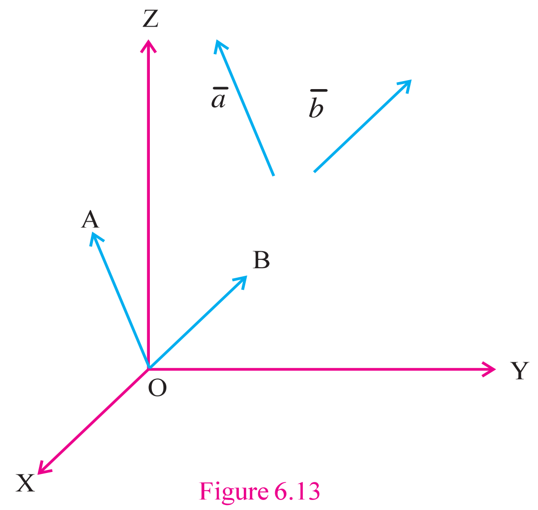


Figure 6.13

Let $\bar{x}$ and $\bar{y}$ be two non-zero vectors.

(1) **If $\bar{x} = k\bar{y}$, $k > 0$, then $\bar{x}$ and $\bar{y}$ have the same directions and so the measure of the angle between them is defined to be 0.**

(2) **If $\bar{x} = k\bar{y}$, $k < 0$, then $\bar{x}$ and $\bar{y}$ have opposite directions and so the measure of the angle between them is defined to be π.**

(3) Now, suppose that $\bar{x}$ and $\bar{y}$ have different directions. So by Cauchy-Schwartz inequality, $|\bar{x} \cdot \bar{y}| < |\bar{x}|\,|\bar{y}|$.

$\therefore \quad -|\bar{x}|\,|\bar{y}| < \bar{x} \cdot \bar{y} < |\bar{x}|\,|\bar{y}|$ $\qquad$ **($|x| < a \Leftrightarrow -a < x < a$)**

$\therefore \quad -1 < \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} < 1$

$\therefore$ **There is a unique $\alpha \in (0, \pi)$ such that,**

$$cos^{-1} \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = \alpha$$

The number α is defined to be the measure of the angle between $\bar{x}$ and $\bar{y}$. It is denoted by $\alpha = (\bar{x}, \hat{\ } \bar{y})$.

Thus $(\bar{x}, \hat{\ } \bar{y}) = cos^{-1} \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$ if $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$.

Also, if $|\bar{x} \cdot \bar{y}| = |\bar{x}|\,|\bar{y}|$, then $\bar{x} \cdot \bar{y} = |\bar{x}|\,|\bar{y}|$ or $\bar{x} \cdot \bar{y} = -|\bar{x}|\,|\bar{y}|$. The directions of $\bar{x}$ and $\bar{y}$ are same or opposite respectively. Hence respective measure of the angle between $\bar{x}$ and $\bar{y}$ is 0 or π.

Let us justify.

If $\bar{x}$ and $\bar{y}$ have same direction, then $\bar{x} = k\bar{y}$, $k > 0$.

Now $\dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = \dfrac{(k\bar{y}) \cdot \bar{y}}{|k\bar{y}||\bar{y}|} = \dfrac{k(\bar{y} \cdot \bar{y})}{|k|\,|\bar{y}|\,|\bar{y}|} = \dfrac{k|\bar{y}|^2}{k|\bar{y}|^2} = 1$ $\qquad$ **($k > 0$)**

$\therefore \quad cos^{-1} \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = cos^{-1} 1 = 0$

If $\bar{x}$ and $\bar{y}$ have opposite directions, then $\bar{x} = k\bar{y}$, $k < 0$.

Now $\dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = \dfrac{(k\bar{y}) \cdot \bar{y}}{|k\bar{y}||\bar{y}|} = \dfrac{k(\bar{y} \cdot \bar{y})}{|k|\,|\bar{y}|\,|\bar{y}|} = \dfrac{k|\bar{y}|^2}{-k|\bar{y}|^2} = -1$ $\qquad$ **($k < 0$)**

$\therefore \quad cos^{-1} \dfrac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = cos^{-1}(-1) = \pi$

Thus, for all non-zero vectors $\bar{x}$ and $\bar{y}$, there exists $\alpha \in [0, \pi]$ such that,

$$\alpha = (\bar{x}, \hat{\ } \bar{y}) = \cos^{-1} \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$$

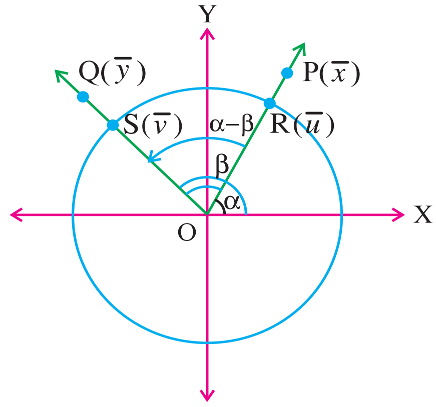


Figure 6.14

Geometrical Interpretation : Our definition of the measure of the angle between two vectors is quite consistent with our understanding of the measure of the angle in geometry.

Suppose, position vectors of P and Q are $\bar{x}$ and $\bar{y}$ respectively, where $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$.

Let $\frac{\bar{x}}{|\bar{x}|} = \bar{u}$ and $\frac{\bar{y}}{|\bar{y}|} = \bar{v}$ be unit vectors in the direction of $\bar{x}$ and $\bar{y}$ respectively.

$(\bar{x}, \hat{\ } \bar{y}) = (\bar{u}, \hat{\ } \bar{v})$

Suppose $\bar{u}$ and $\bar{v}$ are the position vectors of R and S respectively. R and S are the points on the unit circle, so for some α and β with $0 \leq \alpha, \beta < 2\pi$, we would have $\bar{u} = (\cos\alpha, \sin\alpha)$ and $\bar{v} = (\cos\beta, \sin\beta)$.

Now if the radian measure of the angle formed by the rays $\overrightarrow{OR}$ and $\overrightarrow{OS}$ is θ, then it is clear that, $\theta = \alpha - \beta$ or $\beta - \alpha$.

Now, $\cos(\bar{x}, \hat{\ } \bar{y}) = \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = \bar{u} \cdot \bar{v}$

$= (\cos\alpha, \sin\alpha) \cdot (\cos\beta, \sin\beta)$

$= \cos\alpha \cos\beta + \sin\alpha \sin\beta$

$= \cos(\alpha - \beta)$ or $\cos(\beta - \alpha)$

$= \cos\theta$ **$(0 < \theta < \pi, 0 < (\bar{x}, \hat{\ } \bar{y}) < \pi)$**

$\therefore \quad \theta = (\bar{x}, \hat{\ } \bar{y}) = \cos^{-1} \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$

Thus, the measure of angle θ formed by $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, as we understand from geometry is same as $(\bar{x}, \hat{\ } \bar{y})$.

Orthogonal Vectors : If $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$ and $(\bar{x}, \hat{\ } \bar{y}) = \frac{\pi}{2}$, then $\bar{x}$ and $\bar{y}$ are said to be orthogonal or perpendicular to each other. Perpendicularity of $\bar{x}$ and $\bar{y}$ denoted by $\bar{x} \perp \bar{y}$. We say $\bar{x}$ is perpendicular to $\bar{y}$.

Necessary and sufficient condition for two non-zero vectors to be perpendicular to each other :

Let $\bar{x}$ and $\bar{y}$ be two non-zero vectors.

$\bar{x} \perp \bar{y} \Leftrightarrow (\bar{x}, \hat{\ } \bar{y}) = \frac{\pi}{2}$

$\Leftrightarrow \cos(\bar{x}, \hat{\ } \bar{y}) = \cos\frac{\pi}{2}$

$\Leftrightarrow \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = 0$

$\Leftrightarrow \bar{x} \cdot \bar{y} = 0$

Thus $\bar{x}$ and $\bar{y}$ are orthogonal if and only if $\bar{x} \cdot \bar{y} = 0$.

Theorem 6.7 : If $\bar{x}, \bar{y} \in R^3$, $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$ and $(\bar{x} \wedge \bar{y}) = \alpha$, then

(1) $\bar{x} \cdot \bar{y} = |\bar{x}| |\bar{y}| \cos\alpha$

(2) $|\bar{x} \times \bar{y}| = |\bar{x}| |\bar{y}| \sin\alpha$

(3) $\bar{x} \perp (\bar{x} \times \bar{y})$, $\bar{y} \perp (\bar{x} \times \bar{y})$

Proof : (1) By definition of the measure of the angle between two vectors, $\alpha = \cos^{-1} \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$

$\therefore \quad \cos\alpha = \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$

$\therefore \quad \bar{x} \cdot \bar{y} = |\bar{x}| |\bar{y}| \cos\alpha$

(2) By Lagrange's identity,

$|\bar{x} \times \bar{y}|^2 + |\bar{x} \cdot \bar{y}|^2 = |\bar{x}|^2 |\bar{y}|^2$

$$\begin{aligned} \therefore \quad |\bar{x} \times \bar{y}|^2 &= |\bar{x}|^2 |\bar{y}|^2 - |\bar{x} \cdot \bar{y}|^2 \\ &= |\bar{x}|^2 |\bar{y}|^2 - |\bar{x}|^2 |\bar{y}|^2 \cos^2\alpha \\ &= |\bar{x}|^2 |\bar{y}|^2 (1 - \cos^2\alpha) \\ &= |\bar{x}|^2 |\bar{y}|^2 \sin^2\alpha \end{aligned}$$

$\therefore \quad |\bar{x} \times \bar{y}| = |\bar{x}| |\bar{y}| \sin\alpha$ **($\sin\alpha \geq 0$ as $0 \leq \alpha \leq \pi$)**

(3) Let $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$

$$\text{Now, } \bar{x} \cdot (\bar{x} \times \bar{y}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0$$

$\therefore \quad \bar{x} \perp (\bar{x} \times \bar{y})$

Similarly, $\bar{y} \cdot (\bar{x} \times \bar{y}) = 0$. So $\bar{y} \perp (\bar{x} \times \bar{y})$.

Thus, $(\bar{x} \times \bar{y})$ is a vector orthogonal to both $\bar{x}$ and $\bar{y}$. And so $\pm \frac{\bar{x} \times \bar{y}}{|\bar{x} \times \bar{y}|}$ are unit vectors orthogonal to both $\bar{x}$ and $\bar{y}$.

Geometrical Interpretation of $\bar{x} \times \bar{y}$:

When the positive X-axis is rotated in anticlockwise direction to the positive Y-axis, a right handed screw would advance in positive direction of Z-axis as shown in figure 6.15.

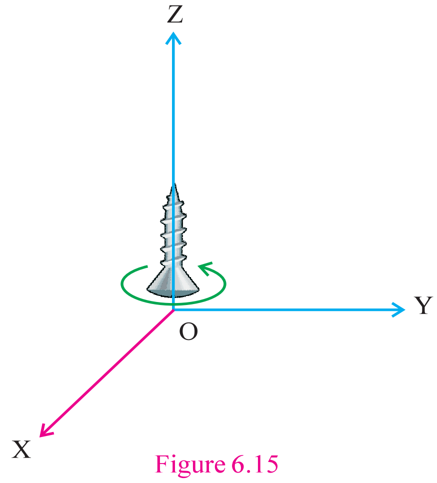


Figure 6.15

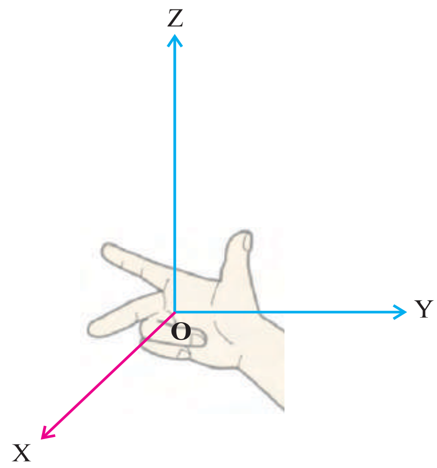


Figure 6.16

As $|\overline{x} \times \overline{y}| = |\overline{x}|\,|\overline{y}|\,sin\theta$, $\theta = (\overline{x}, \hat{\ } \overline{y})$

So, $\overline{x} \times \overline{y} = |\overline{x}|\,|\overline{y}|\,sin\theta\ \hat{n}$, where $\hat{n}$ is the unit vector in the direction of $\overline{x} \times \overline{y}$.

Direction of $\overline{x} \times \overline{y}$ can be determined by using right hand thumb rule i.e. if we keep fingers of our right hand in the direction of $\overline{x}$ and turning the fingers towards $\overline{y}$, then the direction shown by the thumb of the right hand is the direction of $\overline{x} \times \overline{y}$.

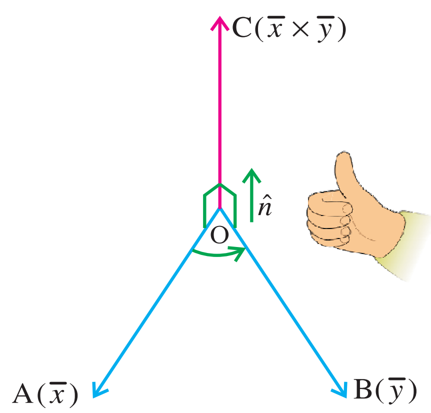


Figure 6.17

Example 14 : Find the measure of the angle between the vectors (1, −1, 2) and (2, −1, 1).

Solution : Let, $\overline{x} = (1, -1, 2)$ and $\overline{y} = (2, -1, 1)$

Now, $cos(\overline{x}, \hat{\ } \overline{y}) = \dfrac{\overline{x} \cdot \overline{y}}{|\overline{x}|\,|\overline{y}|}$

$$= \frac{(1, -1, 2) \cdot (2, -1, 1)}{\sqrt{1+1+4}\ \sqrt{4+1+1}} = \frac{2+1+2}{\sqrt{6}\sqrt{6}}$$

$$= \frac{5}{6}$$

$\therefore \quad (\overline{x}, \hat{\ } \overline{y}) = cos^{-1} \frac{5}{6}$

Example 15 : If the measure of the angle between the vectors $\sqrt{3}\,\hat{i} + \hat{j}$ and $a\hat{i} + \sqrt{3}\,\hat{j}$ is $\frac{\pi}{3}$, find a.

Solution : Let, $\overline{x} = \sqrt{3}\,\hat{i} + \hat{j} = (\sqrt{3}, 1)$ and $\overline{y} = a\hat{i} + \sqrt{3}\,\hat{j} = (a, \sqrt{3})$

It is given that $(\widehat{\bar{x}, \bar{y}}) = \frac{\pi}{3}$

$\therefore \quad cos(\widehat{\bar{x}, \bar{y}}) = cos\frac{\pi}{3}$

$\therefore \quad \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} = \frac{1}{2}$ **(i)**

Now $\bar{x} \cdot \bar{y} = (\sqrt{3}, 1) \cdot (a, \sqrt{3}) = \sqrt{3}a + \sqrt{3}$, $|\bar{x}| = \sqrt{3+1} = 2$, $|\bar{y}| = \sqrt{a^2+3}$

$\therefore \quad \frac{\sqrt{3}a + \sqrt{3}}{2\sqrt{a^2+3}} = \frac{1}{2}$ **(using (i))**

$\therefore \quad \sqrt{3}(a + 1) = \sqrt{a^2+3}$ **(ii)**

$\therefore \quad 3(a^2 + 2a + 1) = a^2 + 3$

$\therefore \quad 2a^2 + 6a = 0$

$\therefore \quad 2a(a + 3) = 0$

$\therefore \quad a = 0$ or $a = -3$

$a = -3$ does not satisfy (ii) as $\sqrt{3}(-2) \neq \sqrt{12} = 2\sqrt{3}$

For $a = 0$, $\sqrt{3}(a + 1) = \sqrt{3}$, $\sqrt{a^2+3} = \sqrt{3}$. Hence $a = 0$.

Example 16 : If $|\bar{x}| = |\bar{y}| = 1$ and $(\widehat{\bar{x}, \bar{y}}) = \theta$, then prove that $|\bar{x} - \bar{y}\,cos\theta| = sin\theta$

Solution : $|\bar{x} - \bar{y}\,cos\theta|^2 = |\bar{x}|^2 - 2\bar{x} \cdot \bar{y}\,cos\theta + |\bar{y}\,cos\theta|^2$

$= 1 - 2\,cos\theta \cdot cos\theta + |\bar{y}|^2 cos^2\theta$ **($|\bar{x}| = 1$)**

$\left(cos\theta = \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|} \Rightarrow cos\theta = \frac{\bar{x} \cdot \bar{y}}{1}\right)$

$= 1 - 2\,cos^2\theta + cos^2\theta$ **($|\bar{y}| = 1$)**

$= 1 - cos^2\theta$

$= sin^2\theta$

$\therefore \quad |\bar{x} - \bar{y}\,cos\theta| = sin\theta$ **($0 \leq \theta \leq \pi$)**

Example 17 : If $\bar{x} = \hat{i} + a\hat{j} + 3\hat{k}$ and $\bar{y} = 2\hat{i} - \hat{j} + 5\hat{k}$ are orthogonal, find a.

Solution : Here $\bar{x} = (1, a, 3)$, $\bar{y} = (2, -1, 5)$

$\bar{x} \perp \bar{y} \Leftrightarrow \bar{x} \cdot \bar{y} = 0$

$\Leftrightarrow 2 - a + 15 = 0$

$\Leftrightarrow a = 17$

$\therefore \quad a = 17$

Example 18 : Find unit vectors orthogonal to both (1, 2, 3) and (2, −1, 4).

Solution : $\bar{x} = (1, 2, 3)$,

$\bar{y} = (2, -1, 4)$

$\therefore \quad \bar{x} \times \bar{y} = (11, 2, -5)$ and $|\bar{x} \times \bar{y}| = \sqrt{121+4+25} = \sqrt{150} = 5\sqrt{6}$

$\therefore$ Unit vectors orthogonal to the given vectors are $\pm \frac{\bar{x} \times \bar{y}}{|\bar{x} \times \bar{y}|} = \pm\left(\frac{11}{5\sqrt{6}}, \frac{2}{5\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$

6.10 Projection of a Vector

If $\bar{a}$ and $\bar{b}$ are non-zero vectors and they are not orthogonal to each other, then the projection of $\bar{a}$ on $\bar{b}$ is defined as the vector $\left(\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}\right)\bar{b}$ and is denoted by $\text{Proj}_{\bar{b}}\,\bar{a}$.

Let $\overrightarrow{PR} = \bar{a}$ and $\overrightarrow{PQ} = \bar{b}$ have the same initial point P. Also S is the foot of perpendicular from R to $\overleftrightarrow{PQ}$. Then we assert that $\overrightarrow{PS} = \text{Proj}_{\bar{b}}\,\bar{a}$. (as shown in figure 6.18)

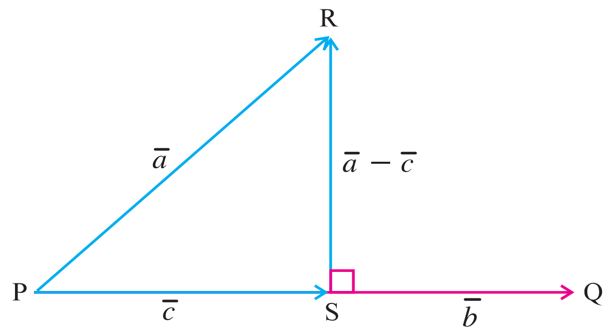


Figure 6.18

Let $\bar{c} = \overrightarrow{PS}$, $\bar{c} \neq \bar{0}$ **(Why ?)**

Then $\overrightarrow{SR} = \bar{a} - \bar{c}$ since $\overrightarrow{PS} + \overrightarrow{SR} = \overrightarrow{PR} = \bar{a}$

$\bar{c}$ and $\bar{b}$ are in the same or in the opposite directions.

$\therefore \quad \bar{c} = k\bar{b},\ k \in R - \{0\}$

$\therefore \quad \bar{c}\cdot\bar{b} = k\bar{b}\cdot\bar{b} = k|\bar{b}|^2$

$\therefore \quad k = \frac{\bar{c}\cdot\bar{b}}{|\bar{b}|^2}$

As $\overleftrightarrow{RS} \perp \overleftrightarrow{PS}$, $(\bar{a} - \bar{c}) \perp \bar{b}$

$\therefore \quad (\bar{a} - \bar{c})\cdot\bar{b} = 0$

$\therefore \quad \bar{a}\cdot\bar{b} = \bar{c}\cdot\bar{b}$

$\therefore \quad k = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}$, since $k = \frac{\bar{c}\cdot\bar{b}}{|\bar{b}|^2}$

$\therefore \quad \overrightarrow{PS} = \bar{c} = \left(\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}\right)\bar{b} = \text{Proj}_{\bar{b}}\,\bar{a}$.

Magnitude of projection vector is PS $= \frac{|\bar{a}\cdot\bar{b}|}{|\bar{b}|^2}\,|\bar{b}| = \frac{|\bar{a}\cdot\bar{b}|}{|\bar{b}|}$.

$\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$ is called the component of $\bar{a}$ along $\bar{b}$ and is denoted by $\text{Comp}_{\bar{b}}\,\bar{a}$.

Note : If two vectors of R^3 are given, then we can think as above by taking corresponding two bound vectors.

If $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are two vectors in R^3, then if we take equal vector as $\overrightarrow{AC}$ with initial point A, then we have the same result. Projection of $\overrightarrow{AB}$ on $\overrightarrow{PQ}$ is the vector $\overrightarrow{AC}$.

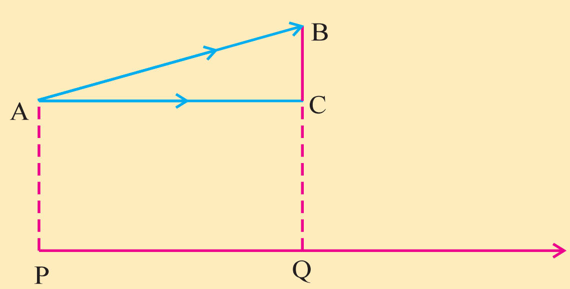


Figure 6.19

Area of a Triangle :

In ΔABC $\overrightarrow{AB} = \bar{c}$, $\overrightarrow{BC} = \bar{a}$, $\overrightarrow{CA} = \bar{b}$.

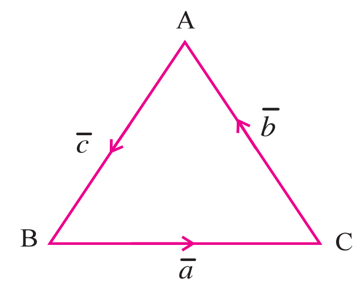


Figure 6.20

$$\text{Area of } \Delta ABC = \frac{1}{2} bc \, sinA$$

$$= \frac{1}{2} |\bar{b}| \, |\bar{c}| \, sinA$$

$$= \frac{1}{2} |\bar{b} \times \bar{c}|$$

$$(\bar{b} \, \hat{,} \, \bar{c}) = \pi - A \text{ and } sin(\pi - A) = sinA$$

Thus, **area of ΔABC = $\frac{1}{2} |\bar{b} \times \bar{c}| = \frac{1}{2} |\bar{a} \times \bar{b}| = \frac{1}{2} |\bar{c} \times \bar{a}|$**

Note : This formula can be applied in R^3 only.

Area of ΔABC is also given by

$$\Delta = \frac{1}{2} bc \sqrt{1 - cos^2 A}$$

$$= \frac{1}{2} |\bar{b}| \, |\bar{c}| \cdot \sqrt{1 - \left(\frac{\bar{b} \cdot \bar{c}}{|\bar{b}||\bar{c}|}\right)^2}$$

$$\therefore \quad \Delta = \frac{1}{2} \sqrt{|\bar{b}|^2 |\bar{c}|^2 - |\bar{b} \cdot \bar{c}|^2}$$

Note : This formula can be applied in R^2 as well as in R^3.

Area of a Parallelogram :

$\square$OACB is a parallelogram with

$\overrightarrow{OA} = \bar{a}$ and $\overrightarrow{OB} = \bar{b}$.

$\overline{BM} \perp \overline{OA}$.

$\therefore$ BM = OB$sin\alpha$ = $|\bar{b}| \, sin\alpha$

$\therefore$ Area of $\square^m$ OACB = OA $\cdot$ BM

$= |\bar{a}| \, |\bar{b}| \, sin\alpha$

$\therefore$ **Area of $\square^m$ OACB = $|\bar{a} \times \bar{b}|$**

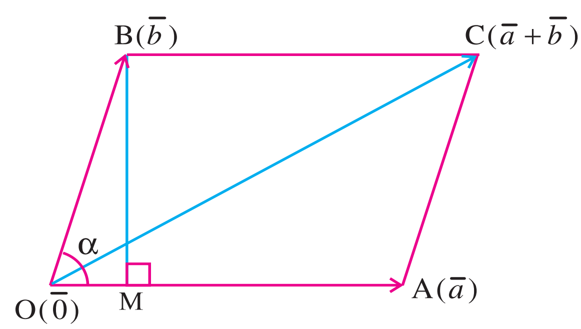


Figure 6.21

Note : **Area of $\square^m$ ABCD = $\frac{1}{2} |\bar{x} \times \bar{y}|$**

if $\overrightarrow{AC} = \bar{x}$, $\overrightarrow{BD} = \bar{y}$.

Let M be the point of intersection of the diagonals, then

$\overrightarrow{AM} = \frac{1}{2} \bar{x}$ and $\overrightarrow{BM} = \frac{1}{2} \bar{y}$

Area of $\square^m$ ABCD = 4(Area of ΔABM)

$= 4(\frac{1}{2} |\overrightarrow{AM} \times \overrightarrow{BM}|)$

Area of $\square^m$ ABCD = $2 \left| \frac{1}{2} \bar{x} \times \frac{1}{2} \bar{y} \right| = \frac{1}{2} |\bar{x} \times \bar{y}|$

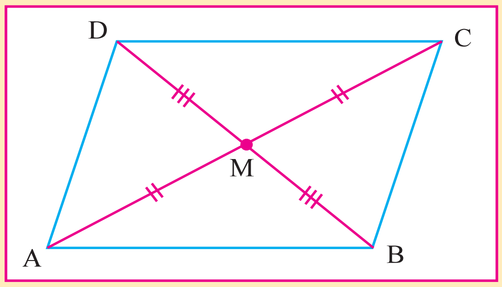


Figure 6.22

Example 19 : Find projection, component and magnitude of projection of $2\hat{i} + \hat{j} + \hat{k}$ on $-4\hat{i} - 2\hat{j} + 4\hat{k}$.

Solution : Here $\bar{a} = (2, 1, 1)$, $\bar{b} = (-4, -2, 4)$

$\therefore \quad \bar{a} \cdot \bar{b} = -8 - 2 + 4 = -6$ and $|b| = \sqrt{16+4+16} = 6$

$\therefore \quad \text{Proj}_{\bar{b}}\,\bar{a} = \left(\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}\right)\bar{b} = \frac{-6}{36}(-4, -2, 4) = \frac{1}{6}(4, 2, -4) = \frac{1}{3}(2, 1, -2)$

$\therefore \quad \text{Comp}_{\bar{b}}\,\bar{a} = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \frac{-6}{6} = -1$

Magnitude of $\text{Proj}_{\bar{b}}\,\bar{a} = \frac{|\bar{a}\cdot\bar{b}|}{|\bar{b}|} = \frac{|-6|}{6} = 1.$

Volume of a Parallelopiped :

A parallelopiped is a solid consisting of six faces which are parallelograms.

Suppose $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors in R^3,

$\therefore \quad (\bar{a} \times \bar{b}) \cdot \bar{c} \neq 0$

Let the position vector of O be $\bar{0}$.

$\overrightarrow{OA} = \bar{a}$, $\overrightarrow{OC} = \bar{b}$ represent vectors $\bar{a}$ and $\bar{b}$ respectively.

Here, $\square$ OABC is a parallelogram.

$\therefore$ Area of $\square^m$OABC $= |\bar{a} \times \bar{b}|$

Also $\bar{a} \times \bar{b}$ (i.e. $\overrightarrow{OM}$) is perpendicular to $\bar{a}$ and $\bar{b}$ both.

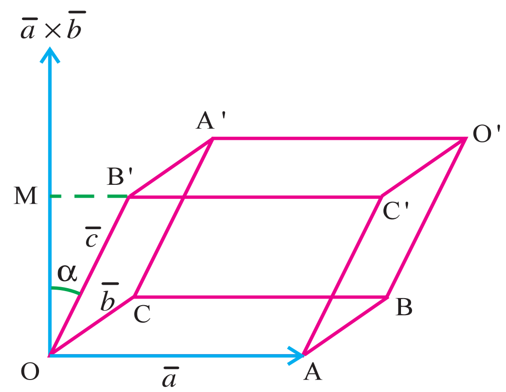


Figure 6.23

$\therefore$ Height of parallelopiped OABC $-$ B'C'O'A' $=$ Magnitude of projection of $\bar{c}$ on $\bar{a} \times \bar{b}$ (i.e. OM)

$$= \frac{|\bar{c}\cdot(\bar{a}\times\bar{b})|}{|\bar{a}\times\bar{b}|}$$

Volume of parallelopiped $=$ Area of base $\times$ height

$$= |\bar{a} \times \bar{b}| \frac{|\bar{c}\cdot(\bar{a}\times\bar{b})|}{|\bar{a}\times\bar{b}|}$$

$$= |\bar{c}\cdot(\bar{a} \times \bar{b})|$$

$\therefore$ **Volume of parallelopiped** $= |[\bar{c}\ \bar{a}\ \bar{b}]| = |[\bar{a}\ \bar{b}\ \bar{c}]|$

Note : Let us note that $\bar{a}$, $\bar{b}$, $\bar{c}$ are the vectors denoting three consecutive edges of the parallelopiped.

Example 20 : Find the volume of the parallelopiped three of whose edges are $\overrightarrow{OA} = (2, 1, 1)$, $\overrightarrow{OB} = (3, -1, 1)$, $\overrightarrow{OC} = (-1, 1, -1)$.

Solution : Here, $\bar{a} = (2, 1, 1)$, $\bar{b} = (3, -1, 1)$, $\bar{c} = (-1, 1, -1)$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 2 & 1 & 1 \\ 3 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = 2(0) - 1(-2) + 1(2) = 4$$

Volume of parallelopiped $= \left| [\bar{a}\ \bar{b}\ \bar{c}] \right| = |4| = 4$

6.11 Direction *cosines*, Direction Angles and Direction Ratios of a Vector

We know that $\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$ are unit vectors of R^3 in the positive directions of X-axis, Y-axis and Z-axis respectively. If $\bar{x} = (x_1, x_2, x_3)$ is a non-zero vector of R^3 and makes angles of measures α, β and γ with the positive directions of X-axis, Y-axis and Z-axis respectively, then α, β and γ are called the direction angles of $\bar{x}$ and $cos\alpha$, $cos\beta$, $cos\gamma$ are called the direction *cosines* of $\bar{x}$.

As α is the measure of the angle between $\bar{x}$ and $\hat{i}$, we have,

$$cos\alpha = \frac{\bar{x} \cdot \hat{i}}{|\bar{x}||\hat{i}|} = \frac{(x_1, x_2, x_3) \cdot (1, 0, 0)}{\sqrt{x_1^2 + x_2^2 + x_3^2} \cdot 1} = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$$

Similarly, $cos\beta = \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$ **and** $cos\gamma = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$.

If we take $l = cos\alpha$, $m = cos\beta$, $n = cos\gamma$

$$\text{then } (l, m, n) = (cos\alpha, cos\beta, cos\gamma) = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \right)$$

$$= \left(\frac{x_1}{|\bar{x}|}, \frac{x_2}{|\bar{x}|}, \frac{x_3}{|\bar{x}|} \right)$$

$$= \frac{1}{|\bar{x}|}(x_1, x_2, x_3) = \frac{\bar{x}}{|\bar{x}|} = \hat{x}$$

Now, $l^2 + m^2 + n^2 = cos^2\alpha + cos^2\beta + cos^2\gamma = \frac{x_1^2 + x_2^2 + x_3^2}{x_1^2 + x_2^2 + x_3^2} = 1$

Also $(cos\alpha, cos\beta, cos\gamma) = \frac{\bar{x}}{|\bar{x}|} = \hat{x}$

$\therefore$ $(cos\alpha, cos\beta, cos\gamma)$ is the unit vector in direction of $\bar{x}$ as $\frac{\bar{x}}{|\bar{x}|} = k\bar{x}$, where $k = \frac{1}{|\bar{x}|} > 0$.

If $\bar{x} = (x_1, x_2, x_3)$, $\bar{x} \neq \bar{0}$ and $m \neq 0$, then let $m\bar{x} = (mx_1, mx_2, mx_3)$. The components of $m\bar{x}$, namely, mx_1, mx_2 and mx_3 are called direction ratios (or direction numbers) of $\bar{x}$. Direction ratios of $k\bar{x}$ are $m(kx_1)$, $m(kx_2)$, $m(kx_3)$ $(m \neq 0, k \neq 0)$. Direction numbers of $\bar{x}$ and $m\bar{x}$ are same. For $m > 0$, $\bar{x}$, $m\bar{x}$ have same direction *cosines*. For $m < 0$, direction *cosines* of $\bar{x}$ and $m\bar{x}$ are additive inverses. Also, the direction angles of $\bar{x}$ are $\alpha = cos^{-1}\frac{x_1}{|\bar{x}|}$, $\beta = cos^{-1}\frac{x_2}{|\bar{x}|}$, $\gamma = cos^{-1}\frac{x_3}{|\bar{x}|}$.

$$\frac{m\bar{x}}{|m\bar{x}|} = \frac{m\bar{x}}{|m||\bar{x}|} = \frac{m\bar{x}}{m|\bar{x}|} = \frac{\bar{x}}{|\bar{x}|},\ m > 0$$

$\therefore$ If $m > 0$, direction *cosines* of $\bar{x}$ and $m\bar{x}$ are same.

And if $m < 0$, $|m| = -m$. Hence direction *cosines* of $\bar{x}$ and $m\bar{x}$ are additive inverses.

Example 21 : Find direction *cosines* and direction angles of $\sqrt{2}\hat{i} - \hat{j} + \hat{k}$.

Solution : Since $\bar{x} = (\sqrt{2}, -1, 1)$, $|\bar{x}| = \sqrt{2+1+1} = 2$

If α, β and γ are the direction angles of $\bar{x}$, then $cos\alpha = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, $cos\beta = -\frac{1}{2}$, $cos\gamma = \frac{1}{2}$

$\therefore \quad \alpha = \frac{\pi}{4}$, $\beta = \pi - cos^{-1}\frac{1}{2} = \frac{2\pi}{3}$ and $\gamma = \frac{\pi}{3}$

$\therefore$ Direction *cosines* of $\bar{x}$ are $\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}$ and direction angles are $\frac{\pi}{4}, \frac{2\pi}{3}$ and $\frac{\pi}{3}$.

Example 22 : If a vector $\bar{x}$ makes angles with measure $\frac{\pi}{3}, \frac{2\pi}{3}$ with X-axis and Y-axis respectively, then find the measure of the angle made by $\bar{x}$ with Z-axis.

Solution : Let $\bar{x}$ make angles with measures α, β and γ with X-axis, Y-axis and Z-axis respectively. Then $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$. Here $\alpha = \frac{\pi}{3}$, $\beta = \frac{2\pi}{3}$

$\therefore \quad cos^2\frac{\pi}{3} + cos^2\frac{2\pi}{3} + cos^2\gamma = 1$

$\therefore \quad \frac{1}{4} + \frac{1}{4} + cos^2\gamma = 1$

$\therefore \quad cos^2\gamma = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore \quad cos\gamma = \pm\frac{1}{\sqrt{2}}$

$\therefore \quad \gamma = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

Miscellaneous Examples

Example 23 : If $|\bar{x}| = 2$, $|\bar{y}| = 4$, $|\bar{z}| = 1$ and $\bar{x} + \bar{y} + \bar{z} = \bar{0}$, find $\bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{z} + \bar{z}\cdot\bar{x}$.

Solution : $|\bar{x} + \bar{y} + \bar{z}|^2 = |\bar{x}|^2 + |\bar{y}|^2 + |\bar{z}|^2 + 2\bar{x}\cdot\bar{y} + 2\bar{y}\cdot\bar{z} + 2\bar{z}\cdot\bar{x}$.

$\therefore \quad 0 = 4 + 16 + 1 + 2(\bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{z} + \bar{z}\cdot\bar{x})$

$\therefore \quad \bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{z} + \bar{z}\cdot\bar{x} = -\frac{21}{2}$.

Example 24 : If A(1, 1, 1), B(0, 2, 5), C(−3, 3, 2) and D(−1, 1, −6) are four points in R^3, find the measure of the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. What can you conclude about $\overrightarrow{AB}$ and $\overrightarrow{CD}$?

Solution : $\overrightarrow{AB} = (0, 2, 5) - (1, 1, 1) = (-1, 1, 4)$ and $\overrightarrow{CD} = (-1, 1, -6) - (-3, 3, 2) = (2, -2, -8)$

$|\overrightarrow{AB}| = \sqrt{1+1+16} = 3\sqrt{2}$ and $|\overrightarrow{CD}| = \sqrt{4+4+64} = 6\sqrt{2}$

$cos(\overrightarrow{AB} \wedge \overrightarrow{CD}) = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\,|\overrightarrow{CD}|} = \frac{-2-2-32}{3\sqrt{2}\times 6\sqrt{2}} = \frac{-36}{36} = -1$

$\therefore \quad (\overrightarrow{AB} \wedge \overrightarrow{CD}) = \pi$

As the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ has measure π, they are in opposite directions.

Also, $\overrightarrow{AB} \times \overrightarrow{CD} = \bar{0}$, so $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Note : $\overrightarrow{CD} = -2\overrightarrow{AB}$. Hence $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear and in opposite directions.

Example 25 : Express $\bar{x} = 3\hat{i} - \hat{j} + 2\hat{k}$ as a sum of two vectors $\bar{a}$ and $\bar{b}$ such that $\bar{a}$ is parallel to $\bar{y}$ and $\bar{b}$ is perpendicular to vector $\bar{y}$, where $\bar{y} = 2\hat{i} - \hat{k}$.

Solution : $\bar{a}$ is parallel to $\bar{y}$.

So $\bar{a} = m\bar{y}$, $m \in R - \{0\}$

$\therefore \quad \bar{a} = 2m\hat{i} - m\hat{k} = (2m, 0, -m)$

Now, $\bar{x} = \bar{a} + \bar{b}$

$\therefore \quad \bar{b} = \bar{x} - \bar{a} = (3, -1, 2) - (2m, 0, -m) = (3 - 2m, -1, 2 + m)$

Again, $\bar{b} \perp \bar{y}$.

$\therefore \quad \bar{b} \cdot \bar{y} = 0$

$\therefore \quad (3 - 2m, -1, 2 + m) \cdot (2, 0, -1) = 0$

$\therefore \quad 6 - 4m - 2 - m = 0$

$\therefore \quad m = \frac{4}{5}$

$\therefore \quad \bar{a} = \frac{8}{5}\hat{i} - \frac{4}{5}\hat{k}$ and $\bar{b} = \left(3 - 2\left(\frac{4}{5}\right)\right)\hat{i} - \hat{j} + \left(2 + \frac{4}{5}\right)\hat{k} = \frac{7}{5}\hat{i} - \hat{j} + \frac{14}{5}\hat{k}$

$\therefore$ For the above $\bar{a}$ and $\bar{b}$, we have $\bar{x} = \bar{a} + \bar{b}$.

Example 26 : Prove that $\bar{a} \times [\bar{a} \times (\bar{a} \times \bar{b})] = |\bar{a}|^2(\bar{b} \times \bar{a})$

Solution : $\bar{a} \times [\bar{a} \times (\bar{a} \times \bar{b})] = [\bar{a} \cdot (\bar{a} \times \bar{b})]\,\bar{a} - (\bar{a} \cdot \bar{a})(\bar{a} \times \bar{b})$

$= [\bar{a}\ \bar{a}\ \bar{b}]\,\bar{a} - |\bar{a}|^2(-(\bar{b} \times \bar{a}))$

$= \bar{0} + |\bar{a}|^2(\bar{b} \times \bar{a})$

$= |\bar{a}|^2(\bar{b} \times \bar{a})$

Example 27 : For non-zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$, if $\bar{a} \times \bar{b} = \bar{c}$, $\bar{b} \times \bar{c} = \bar{a}$, then prove that $|\bar{b}| = 1$.

Solution : $\bar{b} \times \bar{c} = \bar{a}$

$\therefore \quad (\bar{b} \times \bar{c}) \cdot \bar{b} = \bar{a} \cdot \bar{b}$

$\therefore \quad [\bar{b}\ \bar{c}\ \bar{b}] = \bar{a} \cdot \bar{b}$

$\therefore \quad \bar{a} \cdot \bar{b} = 0$ **(i)**

Now, $\bar{b} \times \bar{c} = \bar{a}$

$\therefore \quad \bar{b} \times (\bar{a} \times \bar{b}) = \bar{a}$ **($\bar{c} = \bar{a} \times \bar{b}$)**

$\therefore \quad (\bar{b} \cdot \bar{b})\bar{a} - (\bar{b} \cdot \bar{a})\bar{b} = \bar{a}$

$\therefore \quad |\bar{b}|^2\bar{a} = \bar{a}$ **(using (i))**

$\therefore \quad (|\bar{b}|^2 - 1)\bar{a} = \bar{0}$

$\therefore$ Since $\bar{a} \neq \bar{0}$, $|\bar{b}|^2 = 1$ **($\alpha\bar{x} = \bar{0} \Rightarrow \alpha = 0$ or $\bar{x} = \bar{0}$)**

$\therefore$ $|\bar{b}| = 1$

Example 28 : A(1, 1, 2), B(2, 3, 5), C(1, 3, 4) and D(0, 1, 1) are the vertices of a parallelogram ABCD. Find its area.

Solution : Method 1 : Adjacent sides of $\square^m$ABCD are

$\overrightarrow{AB} = (2, 3, 5) - (1, 1, 2) = (1, 2, 3)$ and

$\overrightarrow{BC} = (1, 3, 4) - (2, 3, 5) = (-1, 0, -1)$

$$\text{Area} = |\overrightarrow{AB} \times \overrightarrow{BC}| = |(-2 - 0, -(-1 + 3), 0 + 2)|$$
$$= |(-2, -2, 2)|$$
$$= \sqrt{4+4+4}$$
$$= 2\sqrt{3}$$

Method 2 : Vector along the diagonal $\overline{AC}$ is $\overrightarrow{AC} = (0, 2, 2)$ and

Vector along the diagonal $\overline{BD}$ is $\overrightarrow{BD} = (-2, -2, -4)$.

$$\therefore \overrightarrow{AC} \times \overrightarrow{BD} = (-8 + 4, -(0 + 4), 0 + 4)$$
$$= (-4, -4, 4)$$

$$\therefore \text{Area} = \frac{1}{2}|\overrightarrow{AC} \times \overrightarrow{BD}|$$
$$= \frac{1}{2}|(-4, -4, 4)|$$
$$= \frac{1}{2}\sqrt{16+16+16}$$
$$= 2\sqrt{3}$$

Example 29 : If α, β, γ are the direction angles of $\bar{x}$, prove that $sin^2\alpha + sin^2\beta + sin^2\gamma = 2$. Also find the value of $cos2\alpha + cos2\beta + cos2\gamma$.

Solution : α, β, γ are the direction angles of $\bar{x}$.

$\therefore$ $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$

$\therefore$ $1 - sin^2\alpha + 1 - sin^2\beta + 1 - sin^2\gamma = 1$

$\therefore$ $sin^2\alpha + sin^2\beta + sin^2\gamma = 2$

Again, $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$

$$\therefore \frac{1+cos2\alpha}{2} + \frac{1+cos2\beta}{2} + \frac{1+cos2\gamma}{2} = 1$$

$\therefore$ $3 + cos2\alpha + cos2\beta + cos2\gamma = 2$

$\therefore$ $cos2\alpha + cos2\beta + cos2\gamma = -1$

Example 30 : Find a unit vector in XY-plane perpendicular to $4\hat{i} - 3\hat{j} + 2\hat{k}$.

Solution : Let the required vector in XY-plane be $(a, b, 0)$ and it is perpendicular to $(4, -3, 2)$.

$\therefore \quad (a, b, 0) \cdot (4, -3, 2) = 0$

$\therefore \quad 4a - 3b = 0$

$\therefore \quad a = \frac{3b}{4}$

Now, $(a, b, 0)$ is a unit vector.

$\therefore \quad a^2 + b^2 = 1$

$\therefore \quad \frac{9b^2}{16} + b^2 = 1$

$\therefore \quad 25b^2 = 16$

$\therefore \quad b = \pm\frac{4}{5}, \ a = \pm\frac{3}{5}$

$\therefore \quad$ Required vector is $\pm\frac{1}{5}(3, 4, 0)$.

Example 31 : $\bar{a}$ is a unit vector and $\bar{b} = (3, 0, -4)$. The measure of the angle between them is $\frac{\pi}{6}$. If the diagonals of the parallelogram are $(3\bar{a} + \bar{b})$ and $(\bar{a} + 3\bar{b})$, then obtain the area of the parallelogram.

Solution : Area of parallelogram $= \frac{1}{2}\,|(3\bar{a} + \bar{b}) \times (\bar{a} + 3\bar{b})|$

$= \frac{1}{2}\,|3(\bar{a} \times \bar{a}) + \bar{b} \times \bar{a} + 9(\bar{a} \times \bar{b}) + 3(\bar{b} \times \bar{b})|$

$= \frac{1}{2}\,|-(\bar{a} \times \bar{b}) + 9(\bar{a} \times \bar{b})| = 4\,|\bar{a} \times \bar{b}|$

Now, $|\bar{a} \times \bar{b}| = |\bar{a}|\,|\bar{b}|\ sin(\bar{a}, \hat{\ } \bar{b})$

$= (1)\,(\sqrt{9+16})\,\left(sin\frac{\pi}{6}\right)$

$= (5)\left(\frac{1}{2}\right)$

$= \frac{5}{2}$

$\therefore \quad$ Area $= 4 \times \frac{5}{2} = 10$

Exercise 6

1. If $\bar{x} = (-1, 2, 3)$, $\bar{y} = (2, -1, 3)$ and $\bar{z} = (3, 2, 1)$, show that $\bar{x} \times (\bar{y} \times \bar{z}) \neq (\bar{x} \times \bar{y}) \times \bar{z}$.

2. Prove that $[\bar{x} + \bar{y} \quad \bar{y} + \bar{z} \quad \bar{z} + \bar{x}] = 2\,[\bar{x} \quad \bar{y} \quad \bar{z}]$.

3. Does $\bar{x} \cdot \bar{y} = \bar{x} \cdot \bar{z}$ imply $\bar{y} = \bar{z}$? Why ?

4. Does $\bar{x} \times \bar{y} = \bar{x} \times \bar{z}$ imply $\bar{y} = \bar{z}$? Why ?

5. If $\bar{x} \cdot \bar{y} = \bar{x} \cdot \bar{z}$ and $\bar{x} \times \bar{y} = \bar{x} \times \bar{z}$ and $\bar{x} \neq \bar{0}$, then prove that $\bar{y} = \bar{z}$.

6. Find a, b, c if $a(1, 3, 2) + b(1, -5, 6) + c(2, 1, -2) = (4, 10, -8)$.

7. If $m\bar{a} = n\bar{b}$, $m, n \in$ N, then prove that $\bar{a} \cdot \bar{b} = |\bar{a}|\,|\bar{b}|$. If $m, n \in$ Z $-$ {0}, what can be said ?

8. Prove that $\bar{x} \times (\bar{y} \times \bar{z}) + \bar{y} \times (\bar{z} \times \bar{x}) + \bar{z} \times (\bar{x} \times \bar{y}) = \bar{0}$.

9. Find direction angles and direction *cosines* of the following vectors :

(1) (1, 0, −1) (2) $\hat{j} + \hat{k}$ (3) $5\hat{i} + 12\hat{j} + 84\hat{k}$.

10. If $(\widehat{\bar{x}, \bar{y}}) = \alpha$, then prove that $\sin\frac{\alpha}{2} = \frac{1}{2}|\bar{x} - \bar{y}|$, where $\bar{x}$ and $\bar{y}$ are unit vectors.

11. Find unit vectors in R^2 orthogonal to (5, −12).

12. If $\bar{x}, \bar{y}, \bar{z}$ are non-coplanar, then prove that $\bar{x} + \bar{y}, \bar{y} + \bar{z}$ and $\bar{z} + \bar{x}$ are non-coplanar.

13. Prove that $(\bar{a} - \text{Proj}_{\bar{b}}\bar{a})$ is orthogonal to $\bar{b}$.

14. Prove that (1, 2, 3) and (2, 1, 3) are not collinear.

15. Prove that (1, 2, 3), (2, 3, 5) and (5, 8, 13) are coplanar.

16. If the angle between (*a*, 2) and (*a*, −2) has measure $\frac{\pi}{3}$, find *a*.

17. Prove that $a\hat{i} + 3\hat{j} + 2\hat{k}$ cannot be orthogonal to $-a\hat{i} + \hat{j} - 2\hat{k}$.

18. Find $|\bar{a} \times \bar{b}|$, if $|\bar{a}| = 4$, $|\bar{b}| = 5$ and $(\bar{a} \cdot \bar{b}) = -6$.

19. If (*a*, 1, 1), (1, *b*, 1) and (1, 1, *c*) are coplanar, prove that $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$.

20. $\bar{a} \times \bar{b} = \bar{a} \times \bar{c}$, $\bar{a} \neq \bar{0}$, $\bar{b} \neq \bar{c}$, then show that $\bar{b} = \bar{c} + k\bar{a}$, $k \in R$

21. If $\bar{a}$ is orthogonal to both $\bar{b}$ and $\bar{c}$ and $\bar{a}, \bar{b}, \bar{c}$ are unit vectors and $(\widehat{\bar{b}, \bar{c}}) = \frac{\pi}{6}$, show that $\bar{a} = \pm 2(\bar{b} \times \bar{c})$.

22. Prove that $[(\bar{a} \times \bar{b}) \times (\bar{a} \times \bar{c})] \cdot \bar{d} = (\bar{a} \cdot \bar{d})[\bar{a}\ \ \bar{b}\ \ \bar{c}]$.

23. Prove by using vectors that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$.

24. Find the area of the triangle whose verticies are (4, −3, 1), (2, −4, 5), (1, −1, 0).

25. Find the projection of $4\hat{i} + \hat{j} + 3\hat{k}$ on $\hat{i} - \hat{j} + \hat{k}$ and its magnitude.

26. Find the projection of (*a, b, c*) on Y-axis and its magnitude.

27. If A(3, 2, −4), B(4, 3, −4), C(3, 3, 3) and D(4, 2, −3), find projection of $\overrightarrow{AD}$ on $\overrightarrow{AB} \times \overrightarrow{AC}$.

28. Use vectors to prove $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ for ΔABC.

29. Obtain *cosine* formula for a triangle by using vectors.

30. Express $2\hat{i} + 3\hat{j} + \hat{k}$ as a sum of two vectors out of which one vector is perpendicular to $2\hat{i} - 4\hat{j} + \hat{k}$ and another is parallel to $2\hat{i} - 4\hat{j} + \hat{k}$.

31. Find unit vector in R^3 which makes an angle of measure $\frac{\pi}{4}$ with $\hat{i}$ and perpendicular to $\hat{k}$.

32. If the sum of two unit vectors is a unit vector, show that the magnitude of their difference is $\sqrt{3}$.

33. If $\bar{a} = (1, 1, 1)$ and $\bar{c} = (0, 1, -1)$ are two given vectors, find $\bar{b}$ such that $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$.

34. Find the volume of parallelopiped whose edges are $\overrightarrow{OA} = (3, 1, 4)$, $\overrightarrow{OB} = (1, 2, 3)$, $\overrightarrow{OC} = (2, 1, 5)$.

35. Prove that if $\bar{x} \times \bar{y} = \bar{0}$, then $\bar{x} = k\bar{y}$, $k \in R - \{0\}$, $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$

36. **Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :**

(1) If $\bar{x} = (-2, 1, -2)$, then a unit vector in the direction of $\bar{x}$ is ...... ☐

(a) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ (c) $\left(-\frac{2}{9}, \frac{1}{9}, -\frac{2}{9}\right)$ (d) $\left(\frac{2}{9}, -\frac{1}{9}, \frac{2}{9}\right)$

(2) ...... is not a unit vector. ($\alpha \neq \frac{n\pi}{2}$, $n \in Z$)

(a) $(cos\alpha, sin\alpha)$ (b) $(-cos\alpha, -sin\alpha)$ (c) $(-cos2\alpha, sin2\alpha)$ (d) $(cos2\alpha, sin\alpha)$

(3) $\bar{x} \times \bar{y} = (7, 2, -3)$, then $\bar{y} \times \bar{x} =$ ......

(a) (7, 2, −3) (b) (−3, 2, 7) (c) (−7, −2, 3) (d) (3, −2, −7)

(4) $|\bar{x}| = |\bar{y}| = 1$, $\bar{x} \perp \bar{y}$, $|\bar{x} + \bar{y}| =$ ......

(a) $\sqrt{3}$ (b) $\sqrt{2}$ (c) 1 (d) 0

(5) If $\bar{x} = 3\bar{y}$, then $\bar{x} \times \bar{y} =$ ......

(a) $3|\bar{y}|^2$ (b) $3|\bar{x}|^2$ (c) $\bar{0}$ (d) $\frac{1}{3}|\bar{y}|^2$

(6) $\bar{x} = (2, 3)$, $\bar{y} = (5, -2)$ are ...... vectors.

(a) collinear (b) non-collinear (c) same directional (d) of opposite direction

(7) If $\bar{x} = (a, 4, 2a)$ and $\bar{y} = (2a, -1, a)$ are perpendicular to each other, then $a =$ ....

(a) 2 (b) 1 (c) 4 (d) any real number

(8) $(a, 1, -2)$, $(1, 1, 3)$, $(8, 5, 0)$ are coplanar then $a =$ ....

(a) −5 (b) 5 (c) −2 (d) 2

(9) If $\bar{x} = (3, 1, 0)$, $\bar{y} = (2, 2, 3)$, $\bar{z} = (-1, 2, 1)$ and $\bar{x} \perp (\bar{y} + k\bar{z})$, then $k =$ ......

(a) 8 (b) 4 (c) $\frac{1}{8}$ (d) $\frac{1}{4}$

(10) If $\bar{x} = (1, 2, 4)$, $\bar{y} = (-1, -2, k)$, $k \neq -4$, then $|\bar{x} \cdot \bar{y}|$ ...... $|\bar{x}||\bar{y}|$.

(a) < (b) > (c) = (d) ≥

(11) $\bar{x} = (-1, 4, -2)$, $\bar{y} = (-4, 16, -8)$, then $|\bar{x} + \bar{y}|$ ...... $|\bar{x}| + |\bar{y}|$.

(a) = (b) > (c) ≥ (d) ≤

(12) (3, 6, −9) and ...... have same direction ratios.

(a) (1, 2, 3) (b) $(\pi, 2\pi, 3\pi)$ (c) (−1, −2, 3) (d) (1, 2, 0)

(13) If $\bar{a} = (-3, 1, 0)$ and $\bar{b} = (1, -1, -1)$, then $\text{Comp}_{\bar{a}}\bar{b} =$ ....

(a) $\frac{4}{\sqrt{10}}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\frac{-4}{\sqrt{10}}$ (d) $-\frac{\sqrt{3}}{4}$

(14) The area of the parallelogram whose diagonals are $\hat{j} + \hat{k}$ and $\hat{i} + \hat{k}$ is ....

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{3}{2}$ (c) 3 (d) $\sqrt{3}$

(15) Magnitude of the projection of (−1, 2, −1) on $\hat{i}$ is ....

(a) $\frac{1}{\sqrt{6}}$ (b) $-\frac{1}{\sqrt{6}}$ (c) 1 (d) −1

(16) $\bar{a}$ is a non-zero vector, then number of unit vectors collinear with $\bar{a}$ is ....

(a) 1 (b) 2 (c) 3 (d) infinitely many.

(17) The area of the parallelogram whose adjacent sides are $\hat{i} + \hat{k}$ and $\hat{i} + \hat{j}$ is ....

(a) 3 (b) $\sqrt{3}$ (c) $\frac{3}{2}$ (d) $\frac{\sqrt{3}}{2}$

(18) If $\bar{x}$ and $\bar{y}$ are non-collinear, non-zero vectors, then number of unit vectors orthogonal to both $\bar{x}$ and $\bar{y}$ is ....

(a) 2 (b) 4 (c) none (d) infinitely many.

(19) If θ is the measure of the angle between vectors $\bar{x}$ and $\bar{y}$ such that $\bar{x} \cdot \bar{y} \geq 0$, then ......

(a) $0 \leq \theta \leq \pi$ (b) $\frac{\pi}{2} \leq \theta \leq \pi$ (c) $0 \leq \theta \leq \frac{\pi}{2}$ (d) $0 < \theta < \frac{\pi}{2}$

(20) The unit vector in the direction of sum of the vectors $(1, 1, 1)$, $(2, -1, -1)$ and $(0, 2, 6)$ is ....

(a) $-\frac{1}{7}(3, 2, 6)$ (b) $\frac{1}{49}(3, 2, 6)$ (c) $\frac{1}{7}(3, -2, 6)$ (d) $\frac{1}{7}(3, 2, 6)$

(21) The expression ...... is meaningless.

(a) $\bar{a} \cdot (\bar{b} \times \bar{c})$ (b) $(\bar{a} \cdot \bar{b})\ \bar{c}$ (c) $\bar{a} \times (\bar{b} \cdot \bar{c})$ (d) $\bar{a} \times (\bar{b} \times \bar{c})$

(22) If $\bar{x} = \hat{i} - \hat{j} + \hat{k}$, $\bar{y} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\bar{z} = \hat{i} + a\hat{j} + b\hat{k}$ are coplanar and $|\bar{z}| = \sqrt{3}$, then ......

(a) $a = 1, b = -1$ (b) $a = 1, b = \pm 1$ (c) $a = -1, b = \pm 1$ (d) $a = \pm 1, b = 1$

(23) If A(3, −1), B(2, 3) and C(5, 1), then $m\angle A$ = ...

(a) $\cos^{-1}\frac{3}{\sqrt{34}}$ (b) $\pi - \cos^{-1}\frac{3}{\sqrt{34}}$ (c) $\sin^{-1}\frac{5}{\sqrt{34}}$ (d) $\frac{\pi}{2}$

(24) If $|\bar{x} \cdot \bar{y}| = \cos\alpha$, then $|\bar{x} \times \bar{y}|$ = ...

(a) $\pm \sin\alpha$ (b) $\sin\alpha$ (c) $-\sin\alpha$ (d) $\sin^2\alpha$

(25) If $\bar{x} \cdot \bar{y} = 0$, then $\bar{x} \times (\bar{x} \times \bar{y})$ = ..., where $|\bar{x}| = 1$.

(a) $\bar{x} \times \bar{y}$ (b) $\bar{x}$ (c) $-\bar{y}$ (d) $\bar{y} \times \bar{x}$

•

Summary

We have studied the following points in this chapter :

1. $R^2 = \{(x, y) \mid x \in R, y \in R\}$ and $R^3 = \{(x, y, z) \mid x \in R, y \in R, z \in R\}$ are vector spaces over R.

2. Properties of vector space were listed.

3. Magnitude of a Vector : If $\bar{x} = (x_1, x_2, x_3)$, then magnitude of $\bar{x}$ is $|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

$\hat{i} = (1, 0, 0)$, $\hat{j} = (0, 1, 0)$, $\hat{k} = (0, 0, 1)$ are unit vectors in the positive direction of X-axis, Y-axis and Z-axis respectively. If $\bar{x} = (x_1, x_2)$, then $|\bar{x}| = \sqrt{x_1^2 + x_2^2}$. In R^2, $\hat{i} = (1, 0)$, $\hat{j} = (0, 1)$.

4. Direction of vectors : Let $\bar{x} \neq \bar{0}$, $\bar{y} \neq \bar{0}$

If **(i)** $\bar{x} = k\bar{y}$, $k > 0$, then $\bar{x}$ and $\bar{y}$ are vectors having same direction.

(ii) $\bar{x} = k\bar{y}$, $k < 0$, then $\bar{x}$ and $\bar{y}$ are vectors having opposite directions.

(iii) $\bar{x} \neq k\bar{y}$, for any $k \in R$, then $\bar{x}$ and $\bar{y}$ are vectors having different directions.

5. Non-zero vectors $\bar{x}$ and $\bar{y}$ are equal if and only if $|\bar{x}| = |\bar{y}|$ and $\bar{x}$ and $\bar{y}$ have the same direction.

6. If $\bar{x} \neq \bar{0}$, then $\frac{1}{|\bar{x}|}\bar{x}$ is a unit vector in the direction of $\bar{x}$ and it is denoted by $\hat{x}$.

7. If $A(x_1, x_2, x_3)$ and $B(y_1, y_2, y_3)$ are two distinct points in R^3, then

$\overrightarrow{AB} = (y_1 - x_1, y_2 - x_2, y_3 - x_3)$

8. $P(x_1, x_2, x_3) \in R^3$, then

(i) Distance of P from XY-plane $= |x_3|$. from YZ-plane $= |x_1|$ and from ZX-plane $= |x_2|$.

(ii) Distance of P from X-axis $= \sqrt{x_2^2 + x_3^2}$.

(iii) Distance of P from origin $= \sqrt{x_1^2 + x_2^2 + x_3^2}$.

9. **Triangle law of vector addition :** If A, B and C are non-collinear points, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

10. **Inner Product :** If $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, then inner product of $\bar{x}$ and $\bar{y}$ is $\bar{x} \cdot \bar{y} = x_1y_1 + x_2y_2 + x_3y_3$. If $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$, then $\bar{x} \cdot \bar{y} = x_1y_1 + x_2y_2$.

Properties of inner product were studied.

11. **Outer Product :** If $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$, then outer product of $\bar{x}$ and $\bar{y}$ is $\bar{x} \times \bar{y} = \left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)$.

Properties of outer product were studied.

12. **Box Product :** If $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$ and $\bar{z} = (z_1, z_2, z_3)$, then box product of $\bar{x}$, $\bar{y}$ and $\bar{z}$ is

$$\bar{x} \cdot (\bar{y} \times \bar{z}) = [\bar{x} \quad \bar{y} \quad \bar{z}] = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

Properties of box product were studied.

13. **Vector Triple Product :** If $\bar{x}, \bar{y}, \bar{z} \in R^3$, then vector triple product of $\bar{x}$, $\bar{y}$ and $\bar{z}$ is $\bar{x} \times (\bar{y} \times \bar{z}) = (\bar{x} \cdot \bar{z})\bar{y} - (\bar{x} \cdot \bar{y})\bar{z}$.

14. **Lagrange's Identity :** $(\bar{x} \cdot \bar{y})^2 + |\bar{x} \times \bar{y}|^2 = |\bar{x}|^2 |\bar{y}|^2$

15. **Cauchy-Schwartz Inequality :** $|\bar{x} \cdot \bar{y}| \leq |\bar{x}| |\bar{y}|$

16. **Triangle Inequality :** $|\bar{x} + \bar{y}| \leq |\bar{x}| + |\bar{y}|$.

17. **Measure of the angle between two non-zero vecotrs :** $(\bar{x} \,\hat{,}\, \bar{y}) = cos^{-1} \frac{\bar{x} \cdot \bar{y}}{|\bar{x}||\bar{y}|}$

18. If $\bar{x} \cdot \bar{y} = 0 \Leftrightarrow \bar{x} \perp \bar{y}$

19. **Projection of a Vector :** If $\bar{a}$ and $\bar{b}$ are non-zero vectors and they are not orthogonal, then the projection of $\bar{a}$ on $\bar{b}$ is $\text{Proj}_{\bar{b}}\,\bar{a} = \left(\frac{\bar{a} \cdot \bar{b}}{|\bar{b}|^2} \right) \bar{b}$.

Component of $\bar{a}$ on $\bar{b}$ is $\text{Comp}_{\bar{b}}\,\bar{a} = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$.

Magnitude of $\text{Proj}_{\bar{b}}\,\bar{a} = \dfrac{|\bar{a}\cdot\bar{b}|}{|\bar{b}|}$.

20. Area of ΔABC : If $\bar{a} = \overrightarrow{BC}$, $\bar{b} = \overrightarrow{CA}$, $\bar{c} = \overrightarrow{AB}$, then

area of $\Delta ABC = \frac{1}{2}|\bar{b} \times \bar{c}|$

$= \frac{1}{2}\sqrt{|\bar{b}|^2 |\bar{c}|^2 - |\bar{b}\cdot\bar{c}|^2}$

21. Area of a Parallelogram : Area of $\square^m$ ABCD $= |\overrightarrow{AB} \times \overrightarrow{BC}|$

$= \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{BD}\right|$

22. Volume of a Parallelopiped : If $\bar{a}$, $\bar{b}$ and $\bar{c}$ are the edges of a parallelopiped, then volume of parallelopiped $= |[\bar{a}\ \bar{b}\ \bar{c}]|$.

23. Collinear Vectors : Non-zero vectors $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ are collinear if and only if $x_1y_2 - x_2y_1 = 0$.

Non-zero vectors $\bar{x}$ and $\bar{y}$ of R^3 are collinear if and only if $\bar{x} \times \bar{y} = \bar{0}$.

24. Coplanar Vectors : If $\bar{x}$, $\bar{y}$ and $\bar{z}$ are the vectors of R^3 and we can find $\alpha, \beta, \gamma \in R$ with at least one of them non-zero, such that $\alpha\bar{x} + \beta\bar{y} + \gamma\bar{z} = \bar{0}$, then $\bar{x}$, $\bar{y}$ and $\bar{z}$ are said to be coplanar vectors.

The vectors which are not coplanar are said to be non-coplanar or linearly independent vectors.

25. Distinct non-zero vectors $\bar{x}$, $\bar{y}$, $\bar{z}$ of R^3 are coplanar if and only if $[\bar{x}\ \bar{y}\ \bar{z}] = 0$.

26. Direction *cosines*, Direction Angles and Direction Ratios of a Vector : If $\bar{x} = (x_1, x_2, x_3)$ is a non-zero vector of R^3 and makes angles of measures α, β and γ with the positive directions of X-axis, Y-axis and Z-axis respectively, then α, β and γ are called the **direction angles** of $\bar{x}$ and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the **direction *cosines*** of $\bar{x}$.

Here, $\cos\alpha = \dfrac{\bar{x}\cdot\hat{i}}{|\bar{x}||\hat{i}|} = \dfrac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$, $\cos\beta = \dfrac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$ and $\cos\gamma = \dfrac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}$.

For $m \neq 0$, mx_1, mx_2, mx_3 are called direction ratios of $\bar{x}$.

7 THREE DIMENSIONAL GEOMETRY

To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it.

– Pierre de Fermat

7.1 Introduction

We have studied plane geometry in standard IX and X and studied the same concepts in the light of coordinate geometry in standard X and XI. Now in the semester II, we studied about the vector space which was explained with the concept of three dimensional coordinate system in R^3 and vectors in R^3. Now, question arises whether we can study a line, a plane, a square, a triangle, a sphere,... in R^3 ? The answer is yes. Vectors can help us to study such concepts. In this chapter, we shall study about the equations of a line and a plane in space.

Before we study lines in space, let us be clear about some differences in plane geometry and three dimensional geometry. Given two lines in a plane, there are three possibilities : (1) lines are parallel, (2) lines are coincident and (3) lines intersect in unique point. These can be very easily seen by drawing lines on a paper, but when we think of two lines in R^3, basically there are two possibilities : They are in the same plane or there is no plane containing these two lines. If they are in the same plane, they are called coplanar and for them, there are three possibilities as discussed above. If two lines are not in the same plane, they are called non-coplanar or skew.

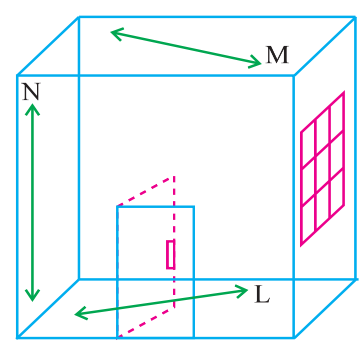


Figure 7.1

In figure 7.1, we see that line L is in the plane of floor and line M is in the plane of ceiling. These lines L and M are in different parallel planes and there is no plane containing them. Hence these lines are skew lines or non-coplanar lines. Such a possibility cannot be observed in plane geometry. Observing carefully one can imagine that $L \perp N$ and $M \perp N$ but L and N as well as M and N are not intersecting each other. This is not observed in the plane geometry.

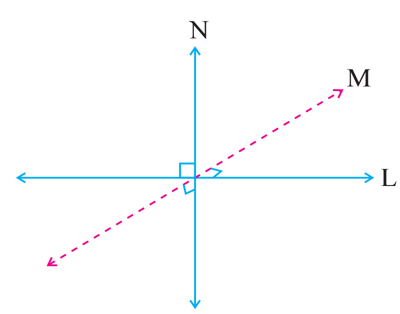


Figure 7.2

Figure 7.2 is a picture of three mutually perpendicular lines in space. This is not possible in plane geometry.

7.2 Direction of a line

We know about the direction of a vector. If A and B are two distinct points of a line L in R^3, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have opposite directions. If direction of $\overrightarrow{AB}$ is $\overline{l}$, then direction of $\overrightarrow{BA}$ is $-\overline{l}$. Both $\pm\overline{l}$ are called directions of $\overleftrightarrow{AB}$. (i.e. line L)

Thus, when we talk about $\overline{l}$ as the direction of a line L, we mean to say that direction of any non-zero vector on L can be $\overline{l}$ or $-\overline{l}$.

Note : **(1)** Lines in space will be denoted by letters L, M, N,...

(2) A line in space can uniquely be determined if

(i) it passes through a given point and has the direction of a non-zero vector $\bar{l}$. (or $-\bar{l}$) written briefly as 'direction $\bar{l}$'.

(ii) it passes through two distinct points.

7.3 Equation of a line passing through A($\bar{a}$) and having the same direction as a non-zero vector $\bar{l}$

Let L be the line passing through A($\bar{a}$) and having direction $\bar{l}$.

Let P($\bar{r}$) be any point on the line L and P $\neq$ A.

$\therefore$ Direction of $\overrightarrow{AP}$ is $\bar{l}$ or $-\bar{l}$.

$\therefore$ $\overrightarrow{AP} = k\bar{l}$, $k \in R - \{0\}$. **($k \neq 0$ as P $\neq$ A)**

$\therefore$ $\bar{r} - \bar{a} = k\bar{l}$

$\therefore$ $\bar{r} = \bar{a} + k\bar{l}$

Also, if $k = 0$, then $\bar{r} = \bar{a}$

i.e. P = A and A is also on L.

$\therefore$ For every point P($\bar{r}$) on L,

$\bar{r} = \bar{a} + k\bar{l}$, $k \in R$.

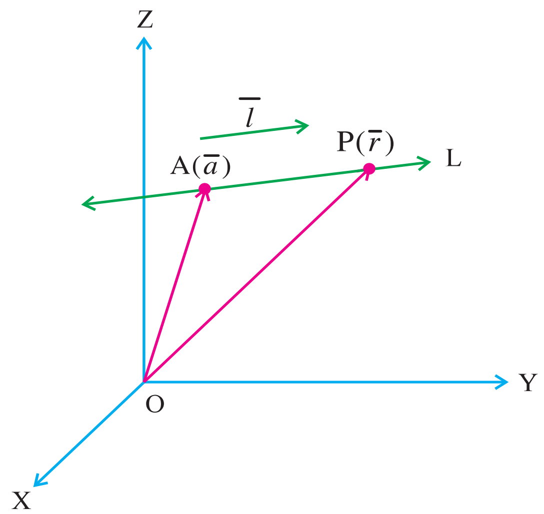


Figure 7.3

Conversely, if P($\bar{r}$) is any point in space such that $\bar{r} = \bar{a} + k\bar{l}$ for some $k \in R$, then (i) if $k = 0$ then $\bar{r} = \bar{a}$ or P = A.

and (ii) if $k \neq 0$, then $\bar{r} \neq \bar{a}$ and $\bar{r} - \bar{a} = k\bar{l}$, where $k \neq 0$

$\therefore$ $\overrightarrow{AP} = k\bar{l}$

$\therefore$ $\overrightarrow{AP}$ has the same direction as $\bar{l}$ or direction opposite to that of $\bar{l}$. But A $\in$ L (given). So P $\in$ L.

Thus, P($\bar{r}$) $\in$ L $\Leftrightarrow$ $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$

$\therefore$ **The vector equation of line L is $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$**

Vector equation of a line gives the position vector of any point on the line.

The equation does not depend upon the choice of $\bar{a}$. If $\bar{b} \in$ L, let $\bar{b} = \bar{a} + k_1\bar{l}$

Then $\bar{b} + k\bar{l} = \bar{a} + k_1\bar{l} + k\bar{l}$

$= \bar{a} + (k_1 + k)\bar{l}$

$= \bar{a} + t\bar{l}$, $t \in R$

$\therefore$ $\{\bar{b} + k\bar{l} \mid k \in R\} = \{\bar{a} + k\bar{l} \mid k \in R\}$

Parametric Equations of a Line :

Suppose a line L has direction $\bar{l} = (l_1, l_2, l_3)$ and passes through $\bar{a} = (x_1, y_1, z_1)$. Let P($\bar{r}$) $\in$ L.

Suppose $\bar{r} = (x, y, z)$. Also $\bar{a} = (x_1, y_1, z_1)$ and $\bar{l} = (l_1, l_2, l_3)$.

$\therefore \quad \overline{r} = \overline{a} + k\overline{l}, \quad k \in R$

$\therefore \quad (x, y, z) = (x_1, y_1, z_1) + k(l_1, l_2, l_3), \quad k \in R$

$\therefore \quad (x - x_1, y - y_1, z - z_1) = (kl_1, kl_2, kl_3)$

$\therefore \quad x - x_1 = kl_1, \quad y - y_1 = kl_2, z - z_1 = kl_3$ **(i)**

$$\therefore \quad \left.\begin{aligned} x &= x_1 + kl_1 \\ y &= y_1 + kl_2 \\ z &= z_1 + kl_3 \end{aligned}\right\} \quad k \in R$$

These equations are called the parametric equations of line L passing through (x_1, y_1, z_1) and having direction (l_1, l_2, l_3) and k is the parameter.

Cartesian Equation (Symmetric Form) :

If we eliminate the parameter k from above equations, we get

$\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{l_2} = \dfrac{z - z_1}{l_3}$ **$(= k)$ provided $l_1 \neq 0, l_2 \neq 0, l_3 \neq 0$** **(using (i)) (ii)**

This is called the symmetric form of the Cartesian equations of line L.

If $l_1 = 0$ and $l_2 \neq 0$, $l_3 \neq 0$, then (i) gives

$$x = x_1, \ \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$$

[Here actually $x - x_1 = kl_1$ and as $l_1 = 0$, so $x - x_1 = 0$, i.e. $x = x_1$.]

This can also be written as $\dfrac{x - x_1}{0} = \dfrac{y - y_1}{l_2} = \dfrac{z - z_1}{l_3}$ $(= k)$

[Here, $\dfrac{x - x_1}{0}$ does not mean that denominator is zero. This is only a symbolic form.]

It simply means $x = x_1 + 0k$, $y = y_1 + kl_2$, $z = z_1 + kl_3$

$\therefore \quad x = x_1, y = y_1 + kl_2, z = z_1 + kl_3$.

Similarly, we can write the equation if any of l_1, l_2, l_3 is zero (of course not for $l_1 = l_2 = l_3 = 0$).

If $l_1 = l_2 = 0$ in equation (i) then $x = x_1$, $y = y_1$ and z is arbitrary.

This can be written symbolically as $\dfrac{x - x_1}{0} = \dfrac{y - y_1}{0} = \dfrac{z - z_1}{l_3} = k$ $(l_3 \neq 0$ as $\overline{l} \neq \overline{0})$

Agian 0 in denominator does not mean division by zero. It simply means $x - x_1 = 0$ or $x = x_1$ and $y = y_1$.

Note : If l_1, l_2, l_3 are direction *cosines* of a line L passing through $A(x_1, y_1, z_1)$, then the equation of L is $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{l_2} = \dfrac{z - z_1}{l_3}$, where $l_1^2 + l_2^2 + l_3^2 = 1$.

Example 1 : Find the equation of the line passing through A(2, 1, −4) and having direction (1, −1, 2), in the vector form and also in the symmetric form.

Solution : Here, $\overline{a} = (2, 1, -4)$ and $\overline{l} = (1, -1, 2)$.

$\therefore$ The vector equation of the line L, $\overline{r} = \overline{a} + k\overline{l}$, $k \in R$ gives,

$\bar{r} = (2, 1, -4) + k(1, -1, 2), \quad k \in R$

This is the vector equation of the line.

Symmetric Form : Symmetric form of the equation of line is $\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$

$\therefore \quad \frac{x-2}{1} = \frac{y-1}{-1} = \frac{z+4}{2}$ is the equation of the line in symmetric form.

7.4 Equation of a line passing through two distinct points

Suppose a line L passes through $A(\bar{a})$ and $B(\bar{b})$, $A \neq B$.

Let $P(\bar{r})$ be a point on $\overleftrightarrow{AB}$ and $P \neq A$.

$P(\bar{r}) \in \overleftrightarrow{AB} \Leftrightarrow$ directions of $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are same or opposite.

$\Leftrightarrow \overrightarrow{AP} = k\overrightarrow{AB}, \; k \in R - \{0\}$

($k \neq 0$ as $P \neq A$)

$\Leftrightarrow \bar{r} - \bar{a} = k(\bar{b} - \bar{a})$

$\Leftrightarrow \bar{r} = \bar{a} + k(\bar{b} - \bar{a})$

$\Leftrightarrow \bar{r} = (1 - k)\bar{a} + k\bar{b}, \; k \in R - \{0\}$

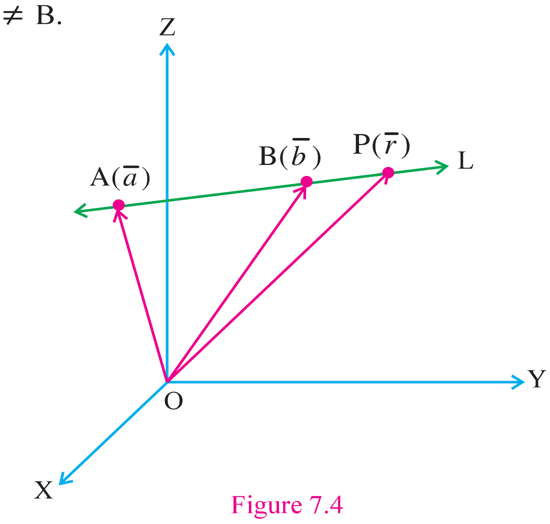


Figure 7.4

Also, $k = 0 \Leftrightarrow \bar{r} = \bar{a}$ and $A(\bar{a}) \in \overleftrightarrow{AB}$

$\therefore$ **The vector equation of $\overleftrightarrow{AB}$ is $\bar{r} = (1 - k)\bar{a} + k\bar{b}, \; k \in R$**

or $\bar{r} = \bar{a} + k(\bar{b} - \bar{a}), \; k \in R$

Taking $k = 1 - t$, $\bar{r} = (1 - (1 - t))\bar{a} + (1 - t)\bar{b}, \; t \in R$

$= t\bar{a} + (1 - t)\bar{b} = \bar{b} + t(\bar{a} - \bar{b})$.

[**Compare :** In R^2, $x = tx_2 + (1 - t)x_1$, $y = ty_2 + (1 - t)y_1$]

Thus roles of $\bar{a}$ and $\bar{b}$ can be interchanged or you can choose any pair of distinct points of L, to get its equation.

Parametric Form :

Suppose $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{r} = (x, y, z)$.

$\therefore \quad \bar{r} = \bar{a} + k(\bar{b} - \bar{a}), \; k \in R$ gives,

$(x, y, z) = (x_1, y_1, z_1) + k(x_2 - x_1, y_2 - y_1, z_2 - z_1), \quad k \in R$

$\therefore \quad x - x_1 = k(x_2 - x_1), \; y - y_1 = k(y_2 - y_1), \; z - z_1 = k(z_2 - z_1)$ **(i)**

$$\therefore \quad \left.\begin{aligned} x &= x_1 + k(x_2 - x_1) \\ y &= y_1 + k(y_2 - y_1) \\ z &= z_1 + k(z_2 - z_1) \end{aligned}\right\} \quad k \in R$$

are the parametric equations of $\overleftrightarrow{AB}$, k is a parameter.

Symmetric Form :

Eliminating parameter k from above equations, we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

[**Compare :** In R^2 $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$]

This is the symmetric form of the Cartesian equation of $\overleftrightarrow{AB}$.

Here, also if $x_1 = x_2$, then we get

$$\frac{x - x_1}{0} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

which can be understood as $x = x_1$, $\frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$.

[Here denominator of $x - x_1$ is not zero, it only means $x = x_1$. The form is only symbolic.]

Example 2 : Write vector form of the line $\frac{3 - x}{3} = \frac{2y - 3}{5} = \frac{z}{2}$.

Solution : Line is $\frac{x - 3}{-3} = \frac{y - \frac{3}{2}}{\frac{5}{2}} = \frac{z - 0}{2}$.

Here, $\bar{a} = \left(3, \frac{3}{2}, 0\right)$ and $\bar{l} = \left\langle -3, \frac{5}{2}, 2 \right\rangle = \langle -6, 5, 4 \rangle$

$\therefore$ The vector form of the equation of the line is $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$

$\therefore$ $\bar{r} = \left(3, \frac{3}{2}, 0\right) + k(-6, 5, 4)$, $k \in R$

Example 3 : Convert the equation of the line $\bar{r} = (5, -2, 4) + k(0, -4, 3)$, $k \in R$ in the Cartesian form.

Solution : Here, $\bar{a} = (5, -2, 4) = (x_1, y_1, z_1)$ and $\bar{l} = (0, -4, 3) = (l_1, l_2, l_3)$

Cartesian form of the equation of line is $\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$

$\therefore$ $x - 5 = 0$, $\frac{y + 2}{-4} = \frac{z - 4}{3}$ **($l_1 = 0$)**

Example 4 : Find the equation of the line passing through the points (2, 2, -3) and (1, 3, 5).

Solution : The equation of the line pasing through $\bar{a}$ and $\bar{b}$ is $\bar{r} = \bar{a} + k(\bar{b} - \bar{a})$, $k \in R$

Here $\bar{a} = (2, 2, -3)$ and $\bar{b} = (1, 3, 5)$, $\bar{b} - \bar{a} = (-1, 1, 8)$.

$\therefore$ $\bar{r} = (2, 2, -3) + k(-1, 1, 8)$, $k \in R$

Cartesian form of the equation of the line L is $\frac{x - 2}{-1} = \frac{y - 2}{1} = \frac{z + 3}{8}$.

7.5 Collinear Points

Let $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ be distinct points in R^3.

A, B, C are collinear $\Leftrightarrow C \in \overleftrightarrow{AB}$

$\Leftrightarrow \bar{c} = \bar{a} + k(\bar{b} - \bar{a})$, for some $k \in R$,

($\overleftrightarrow{AB}$ has equation $\bar{r} = \bar{a} + k(\bar{b} - \bar{a})$, $k \in R$)

$\Leftrightarrow \bar{c} - \bar{a} = k(\bar{b} - \bar{a})$

$\therefore$ A, B, C are collinear $\Leftrightarrow (\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$

Thus, $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$ is necessary and sufficient condition for $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ to be collinear.

There is a theorem also stating the necessary and sufficient condition for collinearity. This theorem is stated below and we accept it without proof.

Theorem 7.1 : If A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are three distinct points in space, then a necessary and sufficient condition for A, B, C to be collinear is that there exist three non-zero real numbers *l, m, n* such that $l + m + n = 0$ and $l\bar{a} + m\bar{b} + n\bar{c} = \bar{0}$.

We obtain a necessary condition for collinearity of three points.

A, B, C are collinear $\Rightarrow (\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$

$\Rightarrow (\bar{c} \times \bar{b}) - (\bar{a} \times \bar{b}) - (\bar{c} \times \bar{a}) + (\bar{a} \times \bar{a}) = \bar{0}$

Also $\bar{a} \times \bar{a} = \bar{0}$ and $\bar{c} \times \bar{b} = -\bar{b} \times \bar{c}$

$\Rightarrow (\bar{a} \times \bar{b}) + (\bar{b} \times \bar{c}) + (\bar{c} \times \bar{a}) = \bar{0}$

$\Rightarrow (\bar{a} \times \bar{b}) \cdot \bar{c} + (\bar{b} \times \bar{c}) \cdot \bar{c} + (\bar{c} \times \bar{a}) \cdot \bar{c} = 0$

$\Rightarrow [\bar{a}\ \bar{b}\ \bar{c}] = 0$

$[\bar{a}\ \bar{b}\ \bar{c}] = 0$ is a necessary condition for A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) to be collinear. However as a following examples show that it is not a sufficient condition.

We also note that $[\bar{a}\ \bar{b}\ \bar{c}] \neq 0 \Rightarrow$ A, B, C are non-collinear as contrapositive of above statement, but $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ does not guarantee any conclusion. Following examples will clear this.

For example : Consider A(1, 2, 0), B(−4, 1, 9) and C(2, 4, 0).

Let $\bar{a} = (1, 2, 0)$, $\bar{b} = (-4, 1, 9)$ and $\bar{c} = (2, 4, 0)$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} 1 & 2 & 0 \\ -4 & 1 & 9 \\ 2 & 4 & 0 \end{vmatrix} = 1(-36) - 2(-18) + 0 = 0$$

Now, $\bar{c} - \bar{a} = (1, 2, 0)$

$\bar{b} - \bar{a} = (-5, -1, 9)$

$(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = (18, -9, 9) \neq \bar{0}$

$\therefore$ A, B, C are non-collinear, though $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

We shall take one simple example, let $\bar{a} = (0, 0, 0)$, $\bar{b} = (1, 2, 3)$, $\bar{c} = (2, 3, 4)$.

Then $[\bar{a}\ \bar{b}\ \bar{c}] = 0$

But $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{c} \times \bar{b} \neq \bar{0}$

$\therefore$ $\bar{a}$, $\bar{b}$, $\bar{c}$ are not collinear.

Example 5 : Prove that (−1, 2, 5), (−2, 4, 2) and (1, −2, 11) are collinear.

Solution : Method 1 : $\bar{a} = (-1, 2, 5)$, $\bar{b} = (-2, 4, 2)$, $\bar{c} = (1, -2, 11)$

$\therefore$ $\bar{c} - \bar{a} = (2, -4, 6)$ and

$\bar{b} - \bar{a} = (-1, 2, -3)$

$\therefore$ $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = (0, 0, 0) = \bar{0}$

$\therefore$ The given points are collinear.

Method 2 : First of all, we shall find the equation of the line passing through two points $A(\bar{a}) = (-1, 2, 5)$ and $B(\bar{b}) = (-2, 4, 2)$.

Equation of $\overleftrightarrow{AB}$ is $\bar{r} = \bar{a} + k(\bar{b} - \bar{a}),\ k \in R$

$\therefore \quad \bar{r} = (-1, 2, 5) + k(-1, 2, -3),\ k \in R$

Now we shall prove that the third point $C(\bar{c}) = (1, -2, 11)$ is on this line.

Let, if possible $\bar{r} = \bar{c} = (1, -2, 11)$ lie on $\overleftrightarrow{AB}$.

We must have $(1, -2, 11) = (-1 - k, 2 + 2k, 5 - 3k)$ for some $k \in R$.

$\therefore$ We must have $1 = -1 - k,\ -2 = 2 + 2k,\ 11 = 5 - 3k$ for some $k \in R$.

$\therefore$ $k = -2$ satisfies all the three equations. So $C(\bar{c})$ lie on $\overleftrightarrow{AB}$.

$\therefore$ A, B, C are collinear.

7.6 The Measure of the Angle Between Two Lines in Space

Suppose $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ are two lines in space.

(i) **If $\bar{l} = \bar{m}$ or $\bar{l} = -\bar{m}$ then $\bar{l} \times \bar{m} = \bar{0}$. Then the measure of the angle between the lines is defined to be zero. Since direction of lines are same, they are coincident or parallel.**

(ii) **If $\bar{l} \perp \bar{m}$ i.e. $\bar{l} \cdot \bar{m} = 0$, then the lines are mutually perpendicular. Then the measure of angle between the lines is defined to be $\frac{\pi}{2}$.**

(iii) **If $\bar{l} \neq \pm\bar{m}$ and $\bar{l} \cdot \bar{m} \neq 0$ i.e. lines are neither perpendicular nor parallel or coincident. We define the measure of the acute angle between $\bar{l}$ and $\bar{m}$ as the measure of the angle between the lines.**

If α is the measure of angle between the lines, then

$$\cos\alpha = \frac{|\bar{l} \cdot \bar{m}|}{|\bar{l}||\bar{m}|},\ 0 < \alpha < \frac{\pi}{2}$$

which also holds good for $\alpha = 0$ and $\frac{\pi}{2}$

Note : For $\alpha = 0$, $|\bar{l} \cdot \bar{m}| = |\bar{l}||\bar{m}|$.

$\therefore \quad \bar{l} \times \bar{m} = \bar{0}$

Thus, $\cos\alpha = \dfrac{|\bar{l} \cdot \bar{m}|}{|\bar{l}||\bar{m}|},\ 0 \le \alpha \le \dfrac{\pi}{2}$

7.7 Condition for intersection of two distinct lines

Theorem 7.2 : If two distinct lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ intersect in a point, then $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Proof : Suppose two distinct lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ intersect at $C(\bar{c})$.

$\therefore \quad \bar{c} = \bar{a} + k_1\bar{l} = \bar{b} + k_2\bar{m}$, for some $k_1, k_2 \in R$

$\therefore \quad \bar{a} - \bar{b} = k_2\bar{m} - k_1\bar{l}$

$\therefore \quad (\bar{a} - \bar{b})\cdot(\bar{l} \times \bar{m}) = (k_2\bar{m} - k_1\bar{l})\cdot(\bar{l} \times \bar{m}) = k_2\bar{m}\cdot(\bar{l} \times \bar{m}) - k_1\bar{l}\cdot(\bar{l} \times \bar{m})$

$= 0 - 0 = 0$

$\therefore \quad (\bar{a} - \bar{b})\cdot(\bar{l} \times \bar{m}) = 0$

$\therefore$ If the lines intersect in a point, then $(\bar{a} - \bar{b})\cdot(\bar{l} \times \bar{m}) = 0$

Note : This condition is necessary but it is not sufficient. Why ?

If $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{l} = (l_1, l_2, l_3)$, $\bar{m} = (m_1, m_2, m_3)$, then the condition $(\bar{a} - \bar{b})\cdot(\bar{l} \times \bar{m}) = 0$ is transformed into

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

This is the Cartesian form of the condition, when two lines intersect.

Example 6 : Find the measure of the angle between the lines $\frac{x-2}{2} = \frac{y-1}{2} = \frac{z+3}{1}$ and $\frac{x+2}{4} = \frac{y-4}{1} = \frac{z-3}{8}$.

Solution : Line L has equations $\frac{x-2}{2} = \frac{y-1}{2} = \frac{z+3}{1}$ and M has equations $\frac{x+2}{4} = \frac{y-4}{1} = \frac{z-3}{8}$.

$\therefore \quad \bar{l} = (2, 2, 1)$ and $\bar{m} = (4, 1, 8)$

If α is measure of the angle between the given lines, then $\left(0 \le \alpha \le \frac{\pi}{2}\right)$

$$cos\alpha = \frac{|\bar{l}\cdot\bar{m}|}{|\bar{l}||\bar{m}|} = \frac{|8+2+8|}{\sqrt{9}\cdot\sqrt{81}} = \frac{18}{3\cdot 9} = \frac{2}{3}$$

$\therefore \quad \alpha = cos^{-1}\frac{2}{3}$

Example 7 : If the lines $\frac{x-5}{7} = \frac{y-5}{k} = \frac{z-2}{1}$ and $\frac{x}{1} = \frac{y-3}{2} = \frac{z+1}{3}$ are perpendicular to each other, find k.

Solution : Here, $\bar{l} = (7, k, 1)$ and $\bar{m} = (1, 2, 3)$

As the lines are perpendicular, $\bar{l}\cdot\bar{m} = 0$

$\therefore \quad 7 + 2k + 3 = 0$

$\therefore \quad 2k = -10$

$\therefore \quad k = -5$

Example 8 : Find the Cartesian equation of the line which passes through the point $(2, -4, 5)$ and is parallel to the line $\bar{r} = (-3, 4, 8) + k(3, 5, 6)$, $k \in R$.

Solution : Here lines are parallel, so the direction of both the lines should be same.

$\therefore$ Direction of required line is $\bar{l} = (3, 5, 6) = (l_1, l_2, l_3)$ and it passes through the point $(2, -4, 5) = (x_1, y_1, z_1)$.

$(2, -4, 5)$ does not lie on $\bar{r} = (-3, 4, 8) + k(3, 5, 6)$, $k \in R$

as $(2, -4, 5) = (-3, 4, 8) + k(3, 5, 6)$ for some $k \in R$

$\Rightarrow (5, -8, -3) = k(3, 5, 6)$

But $5 = 3k$, $-8 = 5k$, $-3 = 6k$ is not true for any $k \in R$.

$\therefore$ The equation of the line parallel to the given line and passing through (x_1, y_1, z_1) is

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$$

$\therefore$ $\frac{x - 2}{3} = \frac{y + 4}{5} = \frac{z - 5}{6}$ is the equation of the line passing through $(2, -4, 5)$ and parallel to given line.

Condition for coplanar and non-coplanar lines :

Theorem 7.3 : A necessary condition for lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$, to be coplanar is that $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Proof : If the two distinct lines L and M are coplanar, then either they intersect or they are parallel.

If they intersect, then by theorem 7.2, $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

If they are parallel, then $\bar{l} \times \bar{m} = \bar{0}$. So $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Thus, if the lines are coplanar, then $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Is this condition sufficient also ?

Non-coplanar or skew lines : If there is no plane that contains both the lines L and M, then L and M are called non-coplanar or skew lines.

From theorem 7.3, it is clear that $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) \neq 0 \Rightarrow$ lines $\bar{r} = \bar{a} + k\bar{l}$ and $\bar{r} = \bar{b} + k\bar{m}$ are skew lines.

Example 9 : Examine whether the lines L : $\frac{x - 3}{4} = \frac{y + 2}{-1} = \frac{z + 1}{-1}$ and M : $\frac{x}{2} = \frac{z + 3}{3}$, $y = -1$ are coplanar or not.

Solution : M can be taken as $\frac{x}{2} = \frac{y + 1}{0} = \frac{z + 3}{3}$

Here $\bar{a} = (3, -2, -1)$, $\bar{l} = (4, -1, -1)$ and $\bar{b} = (0, -1, -3)$, $\bar{m} = (2, 0, 3)$

$\therefore$ $\bar{a} - \bar{b} = (3, -1, 2)$

$$(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = \begin{vmatrix} 3 & -1 & 2 \\ 4 & -1 & -1 \\ 2 & 0 & 3 \end{vmatrix}$$

$$= 3(-3) + 1(14) + 2(2)$$

$$= -9 + 14 + 4 = 9$$

Hence L and M are non-coplanar or skew.

7.8 Perpendicular distance of a point from a line

Suppose $\bar{r} = \bar{a} + k\bar{l}$ is the equation of a line L passing through $A(\bar{a})$ and having direction $\bar{l}$ and $P(\bar{p})$ is any point in R^3.

If $P \in L$, then perpendicular distance between P and L is zero.

If $P \notin L$, P and L determine unique plane π.

Let M be the foot of perpendicular in the plane π from P to line L and $(\bar{l}, \widehat{\overrightarrow{AP}}) = \alpha$, let $M \neq A$.

where $0 < \alpha < \frac{\pi}{2}$.

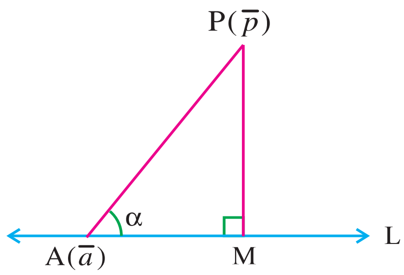


Figure 7.5

$\therefore$ PM = Perpendicular distance from P to L.

$= \text{AP } sin\alpha$

$= \frac{|\overrightarrow{AP}||\bar{l}| sin\alpha}{|\bar{l}|}$ $(\bar{l} \neq \bar{0})$

$= \frac{|\overrightarrow{AP} \times \bar{l}|}{|\bar{l}|}$ $(\alpha = (\overrightarrow{AP}, \hat{\bar{l}}))$

$= \frac{|(\bar{p} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$

Thus, PM $= \frac{|(\bar{p} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$ or $|(\bar{p} - \bar{a}) \times \hat{l}|$ $\left(\hat{l} = \frac{\bar{l}}{|\bar{l}|}\right)$

Second proof :

$\text{AM} = |\text{Proj}_{\bar{l}} \overrightarrow{AP}| = \frac{|\overrightarrow{AP} \cdot \bar{l}|}{|\bar{l}|}$

Now, $\text{PM}^2 = \text{AP}^2 - \text{AM}^2$

$= \text{AP}^2 - \frac{|\overrightarrow{AP} \cdot \bar{l}|^2}{|\bar{l}|^2}$

$= \frac{|\overrightarrow{AP}|^2 |\bar{l}|^2 - |\overrightarrow{AP} \cdot \bar{l}|^2}{|\bar{l}|^2}$

$\therefore$ $\text{PM}^2 = \frac{|\overrightarrow{AP} \times \bar{l}|^2}{|\bar{l}|^2}$ **(Lagrange's identity)**

$\therefore$ **PM** $= \frac{|\overrightarrow{AP} \times \bar{l}|}{|\bar{l}|} = \frac{|(\bar{p} - \bar{a}) \times \bar{l}|}{|\bar{l}|} = |(\bar{p} - \bar{a}) \times \hat{l}|$

Note : If P lies on perpendicular to A, both the proofs fail, but the result is true.

Example 10 : Find the perpendicular distance of the point (1, 2, –4) from the line $\frac{x-3}{2} = \frac{y-3}{3} = \frac{z+5}{6}$.

Solution : Here, point P(1, 2, –4) and A($\bar{a}$) = (3, 3, –5), $\bar{l}$ = (2, 3, 6)

$\overrightarrow{AP} = (1 - 3, 2 - 3, -4 + 5) = (-2, -1, 1)$ and

$\bar{l} = (2, 3, 6)$

$\overrightarrow{AP} \times \bar{l} = (-9, 14, -4)$

$|\bar{l}| = \sqrt{4 + 9 + 36} = 7$

Perpendicular distance of P from the given line $= \frac{|\overrightarrow{AP} \times \bar{l}|}{|\bar{l}|} = \frac{|(-9, 14, -4)|}{7}$

$= \frac{\sqrt{81 + 196 + 16}}{7} = \frac{\sqrt{293}}{7}$

Perpendicular distance between two parallel lines :

Let L : $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and M : $\bar{r} = \bar{b} + k\bar{l}$, $k \in R$ be two parallel lines in R^3.

Since L || M, they determine unique plane.

The distance between L and M is the perpendicular distance between A($\bar{a}$) and M (or between B($\bar{b}$) and L).

$\therefore$ **Distance between L and M is**

$$\frac{|\overrightarrow{AB} \times \bar{l}|}{|\bar{l}|} = \frac{|(\bar{b} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$$

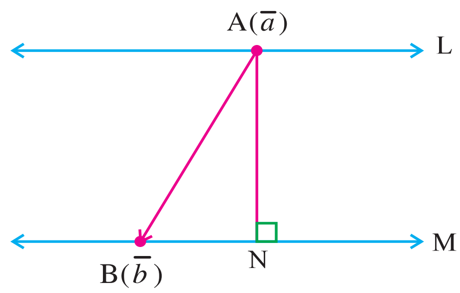


Figure 7.6

Example 11 : Find the distance between the lines L : $\frac{x-4}{3} = \frac{y+1}{-2} = \frac{z-2}{6}$ and

M : $\bar{r} = (2, 3, -1) + k(-3, 2, -6)$, $k \in R$

Solution : Here, $\bar{a} = (4, -1, 2)$; $\bar{l} = (3, -2, 6)$, $\bar{b} = (2, 3, -1)$; $\bar{m} = (-3, 2, -6)$

If possible, let A($\bar{a}$) $\in$ M.

Then $(4, -1, 2) = (2, 3, -1) + k(-3, 2, -6)$ for some $k \in R$

$\therefore$ $(2, -4, 3) = k(-3, 2, -6)$ for some $k \in R$

$\therefore$ $2 = -3k$, $-4 = 2k$, $3 = -6k$

This is not possible for any $k \in R$ as first equation gives $k = -\frac{2}{3}$ and this k does not satisfy other two equations.

$\therefore$ A($\bar{a}$) $\notin$ M

Also $\bar{l} = -\bar{m}$

$\therefore$ $\bar{l} \times \bar{m} = -\bar{m} \times \bar{m} = \bar{0}$

Now $\bar{l} \times \bar{m} = \bar{0}$ and A($\bar{a}$) $\notin$ M

$\therefore$ Given lines are parallel.

$\bar{a} - \bar{b} = (2, -4, 3)$ and

$\bar{l} = (3, -2, 6)$

$\therefore$ $(\bar{a} - \bar{b}) \times \bar{l} = (-18, -3, 8)$, $|\bar{l}| = \sqrt{9+4+36} = 7$

Perpendicular distance between given lines $= \frac{|(\bar{a} - \bar{b}) \times \bar{l}|}{|\bar{l}|}$

$$= \frac{\sqrt{324+9+64}}{7} = \frac{\sqrt{397}}{7}$$

Perpendicular distance between two skew lines :

Let L : $\overline{r} = \overline{a} + k\overline{l}$, $k \in R$ and M : $\overline{r} = \overline{b} + k\overline{m}$, $k \in R$ be skew lines of R^3.

As L and M are skew lines, $(\overline{a} - \overline{b}) \cdot (\overline{l} \times \overline{m}) \neq 0$ **(Theorem 7.3)**

We shall assume that for skew lines L and M, there exist points $P \in L$ and $Q \in M$ such that $\overleftrightarrow{PQ} \perp L$ and $\overleftrightarrow{PQ} \perp M$.

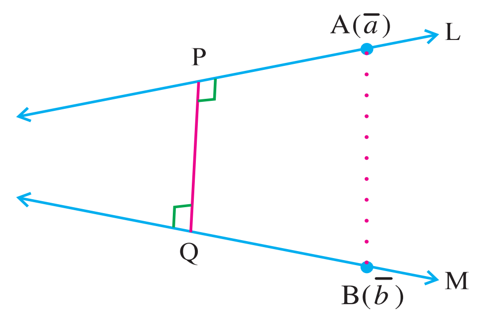


Figure 7.7

$\therefore$ $\overrightarrow{PQ} \cdot \overline{l} = 0$, $\overrightarrow{PQ} \cdot \overline{m} = 0$

$\therefore$ Direction *cosines* of $\overrightarrow{PQ}$ and $\overline{l} \times \overline{m}$ are same.

Now, $\overrightarrow{PQ}$ = projection of $\overrightarrow{AB}$ on $\overrightarrow{PQ}$.

$$\therefore \quad \overrightarrow{PQ} = \left[\frac{|(\overline{b} - \overline{a}) \cdot (\overline{l} \times \overline{m})|}{|\overline{l} \times \overline{m}|}\right]\left[\frac{\overline{l} \times \overline{m}}{|\overline{l} \times \overline{m}|}\right]$$

$$\therefore \quad PQ = \frac{|(\overline{b} - \overline{a}) \cdot (\overline{l} \times \overline{m})|}{|\overline{l} \times \overline{m}|}$$

$$\text{Also, } PQ = \frac{|\overline{b} - \overline{a}||\overline{l} \times \overline{m}|\,cos\,\alpha}{|\overline{l} \times \overline{m}|}$$

$= |\overline{b} - \overline{a}|\,|cos\alpha|$ where $\alpha = ((\overline{b} - \overline{a}) \widehat{,} (\overline{l} \times \overline{m}))$

$\therefore$ $PQ \leq |\overline{b} - \overline{a}|$ **($|cos\alpha| \leq 1$)**

$\therefore$ Distance PQ is less than or equal to the distance between any pair of points on L and M.

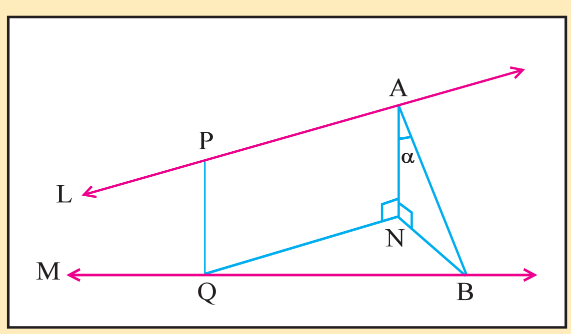
$\therefore$ PQ is the shortest distance between L and M.

Thus, $PQ = \dfrac{|(\overline{b} - \overline{a}) \cdot (\overline{l} \times \overline{m})|}{|\overline{l} \times \overline{m}|}$ is the perpendicular distance or the shortest distance between L and M.

$\overleftrightarrow{PQ}$ and L are intersecting lines, so there is a plane π containing them. □ PANQ is a rectangle in the plane π.

$\overleftrightarrow{AN}$ and $\overleftrightarrow{PQ}$ are parallel lines.

If the measure of the angle between $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$ is α, then the measure of the angle between $\overleftrightarrow{AB}$ and $\overleftrightarrow{AN}$ is α. Now, in the plane containing $\overleftrightarrow{AN}$ and $\overleftrightarrow{AB}$,



AN = AB*cos*α, because in ΔANB,

$m\angle ANB = \frac{\pi}{2}$

($\because$ $\overline{AN} \perp \overline{QN}$ and $\overline{AN} \perp \overline{QB}$, so $\overline{AN}$ is perpendicular to the plane containing $\overline{QN}$ and $\overline{QB}$.)

$\therefore \quad PQ = AN = |AB\cos\alpha|$

$$= \frac{|\overrightarrow{AB}\cdot \bar{l}\times\bar{m}|}{|\bar{l}\times\bar{m}|} = \frac{|(\bar{b}-\bar{a})\cdot(\bar{l}\times\bar{m})|}{|\bar{l}\times\bar{m}|}$$

Example 12 : Find the shortest distance between the lines $\bar{r} = (1, 1, 0) + k(2, -1, 1)$, $k \in R$ and $\bar{r} = (2, 1, -1) + k(3, -5, 2)$, $k \in R$.

Solution : Here, $\bar{a} = (1, 1, 0)$; $\bar{l} = (2, -1, 1)$ and $\bar{b} = (2, 1, -1)$; $\bar{m} = (3, -5, 2)$

$\bar{b} - \bar{a} = (1, 0, -1)$

$$(\bar{b} - \bar{a})\cdot(\bar{l}\times\bar{m}) = \begin{vmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= 1(3) - 1(-7) = 10 \neq 0$$

$\therefore$ Given lines are skew lines.

$\bar{l} = (2, -1, 1)$,

$\bar{m} = (3, -5, 2)$

$\therefore \quad \bar{l}\times\bar{m} = (3, -1, -7)$

$\therefore \quad |\bar{l}\times\bar{m}| = \sqrt{9+1+49} = \sqrt{59}$, $(\bar{b} - \bar{a})\cdot(\bar{l}\times\bar{m}) = 3 + 0 + 7 = 10$

$\therefore$ The shortest distance between given lines $= \dfrac{|(\bar{b}-\bar{a})\cdot(\bar{l}\times\bar{m})|}{|\bar{l}\times\bar{m}|} = \dfrac{10}{\sqrt{59}}$

7.9 To determine the nature of pair of lines of R^3

Let $L : \bar{r} = \bar{a} + k\bar{l}$, $k \in R$

$M : \bar{r} = \bar{b} + k\bar{m}$, $k \in R$ be two lines

If $\bar{l}\times\bar{m} = \bar{0}$, then L and M are parallel or coincident.

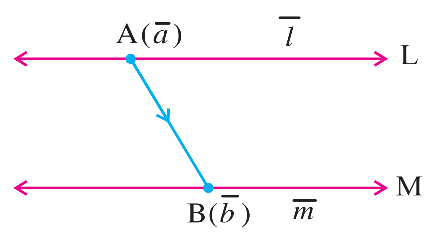


Figure 7.8

Suppose L || M

Here, $\overrightarrow{AB}$ and $\bar{l}$ are non-collinear vectors.

$\therefore \quad \overrightarrow{AB}\times\bar{l} = (\bar{b} - \bar{a})\times\bar{l} \neq \bar{0}$

Conversely if $\overrightarrow{AB}\times\bar{l} = (\bar{b} - \bar{a})\times\bar{l} \neq \bar{0}$, then $\overrightarrow{AB}$ and $\bar{l}$ are non-collinear.

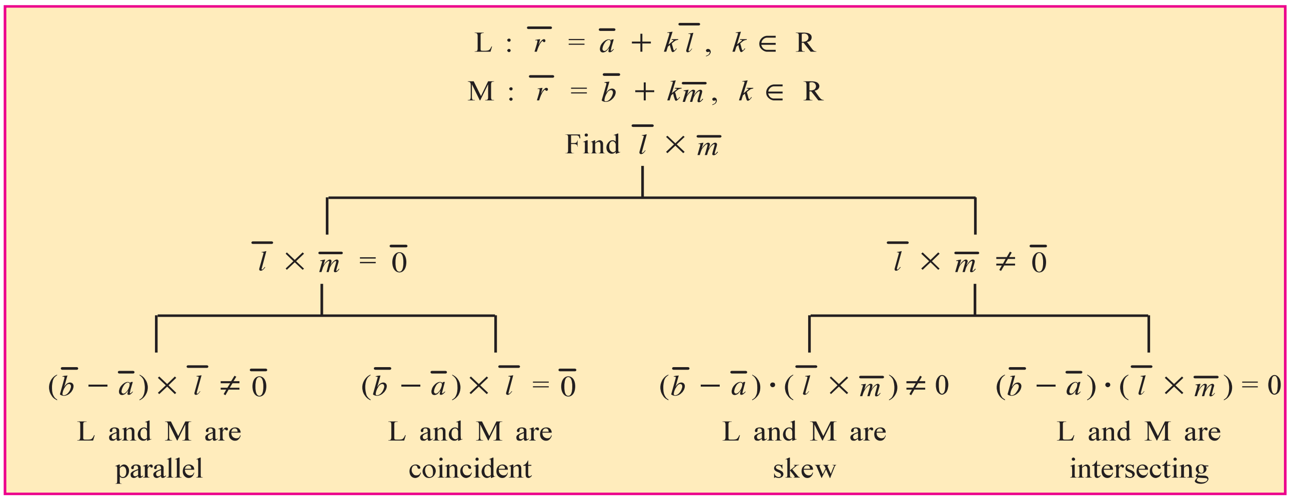
$\therefore$ L || M, if L and M have same directions and $(\bar{b} - \bar{a})\times\bar{l} \neq \bar{0}$.

But, if $(\bar{b} - \bar{a})\times\bar{l} = \bar{0}$, then L is not parallel to M, so L and M are coincident.

Hence if $\bar{l}\times\bar{m} = \bar{0}$, $(\bar{b} - \bar{a})\times\bar{l} = \bar{0}$, lines are coincident.

If $\bar{l}\times\bar{m} = \bar{0}$, $(\bar{b} - \bar{a})\times\bar{l} \neq \bar{0}$, lines are parallel.

If two lines of R^3 are given, then we want to determine whether they are parallel or intersecting or coincident or skew. We can decide by the following flow-chart, based on the entire previous discussion.



Example 13 : Identify the nature (i.e. skew, parallel, coincident and intersecting) of the following lines :

(1) $\overline{r} = (2, -5, 1) + k(3, 2, 6)$, $k \in R$ and $\frac{x-7}{1} = \frac{y}{2} = \frac{z+6}{2}$

(2) $\frac{2x-4}{1} = \frac{3-y}{3} = \frac{z}{1}$ and $\overline{r} = (1, 1, -1) + k(1, -6, 2)$, $k \in R$

(3) $\overline{r} = (1, -2, -3) + k(-1, 1, -2)$, $k \in R$ and $\overline{r} = (4, -2, -1) + k(1, 2, -2)$, $k \in R$

(4) $\overline{r} = (3 + t)\hat{i} + (1 - t)\hat{j} + (-2 - 2t)\hat{k}$, $t \in R$ and $x = 4 + k$, $y = -k$, $z = -4 - 2k$, $k \in R$

Solution : (1) Here, $\overline{a} = (2, -5, 1)$, $\overline{l} = (3, 2, 6)$

$\overline{b} = (7, 0, -6)$; $\overline{m} = (1, 2, 2)$

$\overline{b} - \overline{a} = (5, 5, -7)$

$\overline{l} \times \overline{m} = (-8, 0, 4) \neq \overline{0}$ and $(\overline{b} - \overline{a}) \cdot (\overline{l} \times \overline{m}) = (5, 5, -7) \cdot (-8, 0, 4)$

$= -40 - 28 = -68 \neq 0$

$\therefore$ The given lines are skew lines.

(2) The equation of the first line is $\frac{x-2}{\frac{1}{2}} = \frac{y-3}{-3} = \frac{z}{1}$

$\therefore$ $\overline{a} = (2, 3, 0)$; $\overline{l} = <\frac{1}{2}, -3, 1> = <1, -6, 2>$

$\overline{b} = (1, 1, -1)$; $\overline{m} = (1, -6, 2)$

$(\overline{b} - \overline{a}) = (-1, -2, -1)$

Now $\overline{l} \times \overline{m} = (0, 0, 0) = \overline{0}$ and $(\overline{b} - \overline{a}) \times \overline{m} = (-1, -2, -1) \times (1, -6, 2) = (-10, 1, 8) \neq \overline{0}$

$\therefore$ Lines are parallel.

(3) $\overline{a} = (1, -2, -3)$; $\overline{l} = (-1, 1, -2)$

$\overline{b} = (4, -2, -1)$; $\overline{m} = (1, 2, -2)$

$(\overline{b} - \overline{a}) = (3, 0, 2)$

Now $\overline{l} \times \overline{m} = (2, -4, -3) \neq \overline{0}$ and $(\overline{b} - \overline{a}) \cdot (\overline{l} \times \overline{m}) = (3, 0, 2) \cdot (2, -4, -3)$

$= 6 + 0 - 6 = 0$

$\therefore$ The lines are intersecting.

(4) $\overline{a} = (3, 1, -2)$; $\overline{l} = (1, -1, -2)$

$\overline{b} = (4, 0, -4)$; $\overline{m} = (1, -1, -2)$

$(\overline{b} - \overline{a}) = (1, -1, -2)$

Now $\overline{l} \times \overline{m} = (0, 0, 0) = \overline{0}$ and $(\overline{b} - \overline{a}) \times \overline{l} = (1, -1, -2) \times (1, -1, -2) = \overline{0}$

$\therefore$ The lines are coincident.

Exercise 7.1

1. Find the vector and Cartesian equation of the line passing through (2, −1, 3) and having direction $2\hat{i} - 3\hat{j} + 4\hat{k}$.

2. Find the equation of the line passing through the points (2, 3, −9) and (4, 3, −5) in symmetric and in vector form.

3. Are the points (0, 1, 1), (0, 4, 4) and (2, 0, 1) collinear ? Why ?

4. Find the direction *cosine*s of the line $x = 4z + 3$, $y = 2 - 3z$.

5. Find the vector and Cartesian equation of the line passing through (1, −2, 1) and perpendicular to the lines $x + 3 = 2y = -12z$ and $\frac{x}{2} = \frac{y+6}{2} = \frac{3z-9}{1}$.

6. Prove that the lines L : $\frac{x+2}{3} = \frac{y-2}{-1}$, $z + 1 = 0$ and M : $\{(4 + 2k, 0, -1 + 3k) \mid k \in R\}$ intersect each other. Also find the point of their intersection.

7. Find the measure of the angle between the lines $\overline{r} = (1, 2, 1) + k(2, 3, -1)$, $k \in R$ and $\frac{x-1}{4} = \frac{y-2}{3}$, $z = 3$.

8. Show that the line through the points (2, 1, −1) and (−2, 3, 4) is perpendicular to the line through the points (9, 7, 8) and (11, 6, 10).

9. Identify whether the following lines are parallel, intersecting, skew or coincident :

(1) $\overline{r} = (1, 2, -3) + k(3, -2, 1)$, $k \in R$ and $\frac{x-1}{2} = \frac{3-y}{2} = \frac{z-5}{-1}$.

(2) $\frac{x-5}{-2} = \frac{y-3}{-2} = \frac{z+2}{4}$ and $\frac{x-2}{1} = \frac{3-y}{-1} = \frac{z+2}{-2}$.

(3) $x = \frac{y-1}{1} = \frac{z+1}{3}$ and $\{(2, 1 + 3k, 2 + k) \mid k \in R\}$.

(4) $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-1}{2}$ and $x = 1 + 2t$, $y = t$, $z = 4 + 5t$, $t \in R$.

(5) $\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z-1}{3}$ and $\frac{x-1}{-2} = \frac{y+2}{4} = \frac{z-2}{-6}$.

10. Show that $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$ and $\frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$ are skew lines. Find the shortest distance between them.

11. Find the perpendicular distance of (−5, 3, 4) from the line $\frac{x+2}{-4} = \frac{y-6}{5} = \frac{z-5}{3}$.

12. Find the perpendicular distance between the lines $x = 3 - 2k$, $y = k$, $z = 3 - k$, $k \in R$ and $x = 2k - 3$, $y = 2 - k$, $z = 7 + k$, $k \in R$.

*

7.10 Plane

Let us recall the postulates of plane we studied in earlier class.

(1) Three distinct non-collinear points determine unique plane.

(2) There is a unique plane containing two parallel lines.

(3) There is a unique plane containing two intersecting lines.

Plane passing through three distinct non-collinear points :

Suppose $A(\overline{a})$, $B(\overline{b})$, $C(\overline{c})$ are three distinct non-collinear points of R^3.

∴ A, B, C determine unique plane π.

Let $P(\overline{r})$ be any point of the plane π and let P ≠ A.

$\therefore\quad \overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ are coplanar.

$\therefore\quad \overrightarrow{AP}$ is a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\therefore\quad \overrightarrow{AP} = m\overrightarrow{AB} + n\overrightarrow{AC}$, where $m, n \in R$ and $m^2 + n^2 \neq 0$.

$\therefore\quad \overline{r} - \overline{a} = m(\overline{b} - \overline{a}) + n(\overline{c} - \overline{a})$

If $\overline{r} = \overline{a}$ and $A(\overline{a}) \in \pi$, then $m = n = 0$.

$\therefore\quad \boldsymbol{\overline{r} = \overline{a} + m(\overline{b} - \overline{a}) + n(\overline{c} - \overline{a}), \ m, n \in R}$ **(i)**

Conversely, if $P(\overline{r})$ satisfies

$$\overline{r} - \overline{a} = m(\overline{b} - \overline{a}) + n(\overline{c} - \overline{a}),\ m, n \in R,\ m^2 + n^2 \neq 0$$

then $\overrightarrow{AP} = m(\overrightarrow{AB}) + n(\overrightarrow{AC})$

i.e. $\overrightarrow{AP}$ is in the plane of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$\therefore\quad$ A is in π. So $P \in \pi$.

If $m = n = 0$, then $\overline{r} = \overline{a}$ i.e. $P = A \in \pi$.

Thus, $P(\overline{r}) \in \pi$ if and only if $\overline{r}$ satisfies (i).

$\therefore\quad$ The plane π determined by $A(\overline{a})$, $B(\overline{b})$, $C(\overline{c})$ has equation

$$\overline{r} = \overline{a} + m(\overline{b} - \overline{a}) + n(\overline{c} - \overline{a}) \quad m, n \in R$$

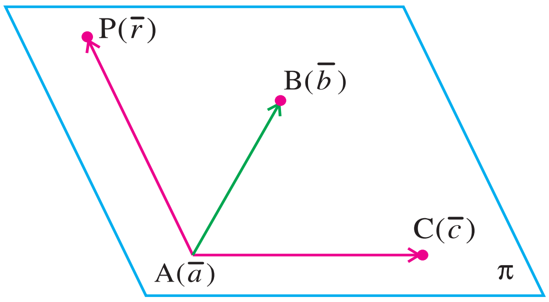


Figure 7.9

Now, $\overline{r} = (1 - m - n)\overline{a} + m\overline{b} + n\overline{c}$ if $P(\overline{r}) \in \pi$.

Let $1 - m - n = l$ i.e. $l + m + n = 1$

$\therefore\quad \boldsymbol{\overline{r} = l\overline{a} + m\overline{b} + n\overline{c}}$**, where** $\boldsymbol{l, m, n \in R}$ **and** $\boldsymbol{l + m + n = 1}$**.**

This is the vector equation of the plane containing three distinct non-collinear points $A(\overline{a})$, $B(\overline{b})$ and $C(\overline{c})$.

Parametric equations of a plane :

Let $P(x, y, z)$, be any point of the plane passing through non-collinear points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

$\therefore\quad \overline{r} = l\overline{a} + m\overline{b} + n\overline{c}$,

$\therefore\quad (x, y, z) = l(x_1, y_1, z_1) + m(x_2, y_2, z_2) + n(x_3, y_3, z_3)$

$\therefore\quad \boldsymbol{x = lx_1 + mx_2 + nx_3}$

$\boldsymbol{y = ly_1 + my_2 + ny_3}$

$\boldsymbol{z = lz_1 + mz_2 + nz_3}$ **where** $\boldsymbol{l, m, n \in R}$ **and** $\boldsymbol{l + m + n = 1}$

are the parametric equations of the plane through A, B, C and l, m, n are the parameters.

Other forms :

If $A(\overline{a})$, $B(\overline{b})$, $C(\overline{c})$ are three non-collinear distinct points, they determine a unique plane π.

$P(\overline{r}) \in \pi \Leftrightarrow \overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ are coplanar **(P ≠ A)**

$\Leftrightarrow (\overline{r} - \overline{a}), (\overline{b} - \overline{a}), (\overline{c} - \overline{a})$ are coplanar **(P ≠ A)**

$\Leftrightarrow (\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})] = 0$ **(ii)**

Also, if $\overline{r} = \overline{a}$, then $\overline{r} - \overline{a} = \overline{0}$.

$\therefore\quad (\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})] = 0, \quad \forall P(\overline{r}) \in \pi$

Thus, the vector equation of the plane through distinct non-collinear points A, B, C is $\boldsymbol{(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})] = 0}$

Cartesian form (Scalar form) :

Let $\overline{r} = (x, y, z)$, $\overline{a} = (x_1, y_1, z_1)$, $\overline{b} = (x_2, y_2, z_2)$, $\overline{c} = (x_3, y_3, z_3)$

$\therefore$ **The equation $(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times (\overline{c} - \overline{a})] = 0$ becomes,**

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

This is the Cartesian equation or scalar form of the equation of the plane passing through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Condition for four distinct points of R^3 to be coplanar :

Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ be points of R^3.

A, B, C, D are coplanar $\Leftrightarrow$ D lies on the plane determined by A, B, C

$$\Leftrightarrow D(x_4, y_4, z_4) \text{ satisfies } \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Thus $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = 0$$

Example 14 : Find the equation of the plane passing through $A(-6, 0, 7)$, $B(1, 2, 2)$ and $C(3, -5, -4)$, if possible.

Solution : Let us examine if A, B, C are collinear or not.

$$\begin{vmatrix} -6 & 0 & 7 \\ 1 & 2 & 2 \\ 3 & -5 & -4 \end{vmatrix} = -6(2) + 7(-11) = -89 \neq 0$$

$\therefore$ A, B, C are non-collinear.

$\therefore$ There is a unique plane passing through A, B, C.

Cartesian equation of the plane passing through A, B, C is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x + 6 & y - 0 & z - 7 \\ 1 + 6 & 2 - 0 & 2 - 7 \\ 3 + 6 & -5 - 0 & -4 - 7 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x + 6 & y & z - 7 \\ 7 & 2 & -5 \\ 9 & -5 & -11 \end{vmatrix} = 0$$

$\therefore \quad (x + 6)(-47) - y(-32) + (z - 7)(-53) = 0$

$\therefore \quad -47x - 282 + 32y - 53z + 371 = 0$

$\therefore \quad -47x + 32y - 53z + 89 = 0$

$\therefore \quad 47x - 32y + 53z - 89 = 0$ is the equation of the plane passing through A, B and C.

Example 15 : Does a unique plane pass through A(4, −2, −1), B(5, 0, −3) and C(3, −4, 1)? If so, find its equation.

Solution : Let us examine if A, B, C are collinear or not.

$$\begin{vmatrix} 4 & -2 & -1 \\ 5 & 0 & -3 \\ 3 & -4 & 1 \end{vmatrix} = 4(-12) + 2(14) - 1(-20)$$

$$= -48 + 28 + 20 = 0$$

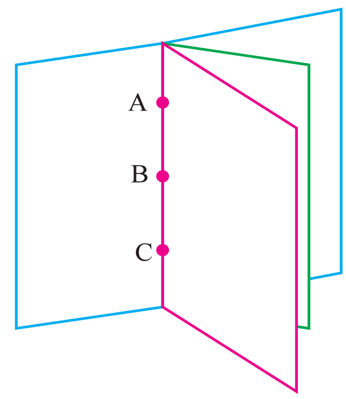


Figure 7.10

This is not enough to confirm that given points are collinear. So let us verify using the condition whether

$(\overline{c} - \overline{a}) \times (\overline{b} - \overline{a}) = \overline{0}$ is true or not.

$\therefore \quad \overline{c} - \overline{a} = (-1, -2, 2)$

$\overline{b} - \overline{a} = (1, 2, -2) = -(\overline{c} - \overline{a})$

$\therefore \quad (\overline{c} - \overline{a}) \times (\overline{b} - \overline{a}) = \overline{0}$

$\therefore$ A, B, C are collinear.

$\therefore$ A, B, C do not determine a unique plane.

Example 16 : Show that the points A(1, 0, 2), B(−1, 2, 0), C(2, 3, 11) and D(1, −3, −4) are coplanar.

Solution :
$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \end{vmatrix} = \begin{vmatrix} -2 & 2 & -2 \\ 1 & 3 & 9 \\ 0 & -3 & -6 \end{vmatrix} = -2(9) - 2(-6) - 2(-3)$$

$$= -18 + 12 + 6 = 0$$

$\therefore$ A, B, C, D are coplanar points.

7.11 Intercepts of a Plane

If a plane π intersects three coordinate axes at points A(a, 0, 0), B(0, b, 0) and C(0, 0, c), then a, b, c are called the X-intercept, the Y-intercept and the Z-intercept of the plane π respectively.

If the plane π does not intersect X-axis, then X-intercept of the plane π is said to be undefined and similarly for intersection of the plane with Y-axis or Z-axis also.

Equation of a plane making intercepts *a, b, c* on the coordinate axes :

Suppose intercepts made by a the plane π on X-axis, Y-axis and Z-axis are respectively a, b and c. (where $a \neq 0$, $b \neq 0$, $c \neq 0$).

$\therefore$ A(a, 0, 0), B(0, b, 0) and C(0, 0, c) are points on the plane π.

Obviously, A, B, C are non-collinear. **(Why ?)**

$\therefore$ Parametric equations of the plane π through A, B, C are

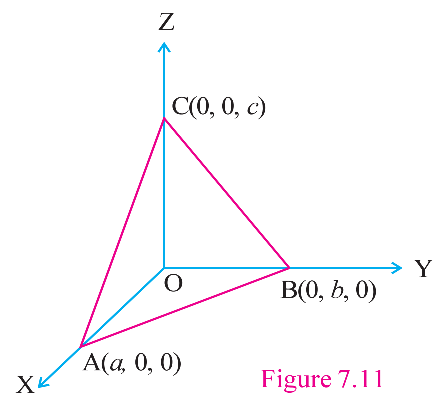


Figure 7.11

$$\therefore \quad \left.\begin{aligned} x &= lx_1 + mx_2 + nx_3 = la \\ y &= ly_1 + my_2 + ny_3 = mb \\ z &= lz_1 + mz_2 + nz_3 = nc \end{aligned}\right\} \text{ where } l, m, n \in R, \ l + m + n = 1$$

$$\therefore \quad l = \frac{x}{a}, \ m = \frac{y}{b}, \ n = \frac{z}{c}$$

Since $l + m + n = 1$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ **is the equation of the plane having intercepts** ***a, b*** **and** ***c.*** **($abc \neq 0$)**

Another Method :

Using cartesian form of the equation of the plane passing through A(a, 0, 0), B(0, b, 0), C(0, 0, c), we get $\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$ as the equation of the plane through A, B and C.

$$\therefore \quad \begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

$\therefore \quad (x - a)bc - y(-ac) + z(ab) = 0$

$\therefore \quad bcx - abc + acy + abz = 0$

$\therefore \quad bcx + acy + abz = abc$

$\therefore \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is the equation of the plane having intercepts a, b, c. **($abc \neq 0$)**

Example 17 : Find the equation of the plane making X-intercept 4, Y-intercept -6 and Z-intercept 3.

Solution : Here $a = 4$, $b = -6$, $c = 3$ is given.

$\therefore$ The equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\therefore \quad \frac{x}{4} + \frac{y}{-6} + \frac{z}{3} = 1$

$\therefore \quad 3x - 2y + 4z = 12$ is the equation of the plane having X-intercept 4, Y-intercept -6 and Z-intercept 3.

Example 18 : Find the intercepts made by the plane $2x - 3y + 5z = 15$ on the coordinate axes.

Solution : The equation of the given plane is $2x - 3y + 5z = 15$

$\therefore \quad \frac{x}{\frac{15}{2}} + \frac{y}{-5} + \frac{z}{3} = 1$ **(dividing both the sides by 15)**

$\therefore$ Comparing with the equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, X-intercept $= \frac{15}{2}$, Y-intercept $= -5$, Z-intercept $= 3$.

Example 19 : Find the intercepts made by the plane $3y + 2z = 12$ on the coordinate axes.

Solution : The equation of plane is $3y + 2z = 12$

$\therefore \quad \frac{y}{4} + \frac{z}{6} = 1$

$\therefore$ Comparing with $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, X-intercept is undefined, Y-intercept $= 4$ and Z-intercept $= 6$.

Another Method :

The equation of the plane is $3y + 2z = 12$.

It intersects X-axis where $y = 0 = z$.

$\therefore$ But then $0 + 0 = 12$

This is not true.

$\therefore$ $3y + 2z = 12$ does not intersect X-axis.

$\therefore$ It has no X-intercept.

To find Y-intercept, let $x = 0 = z$.

$\therefore$ $3y = 12$

$\therefore$ $y = 4$

$\therefore$ Y-intercept is 4.

To find Z-intercept, let $x = y = 0$.

$\therefore$ $2z = 12$

$\therefore$ $z = 6$

$\therefore$ Z-intercept is 6.

7.12 Normal to a plane

There exists a line which is perpendicular to every line in the plane. It is called a normal to the plane. Usually, normal is denoted by $\bar{n}$ or $\bar{n}_1$, $\bar{n}_2$, $\bar{n}_3$,...

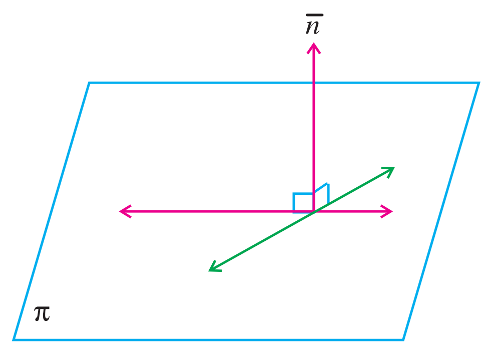


Figure 7.12

Vector equation of the plane passing through $A(\bar{a})$ and having normal $\bar{n}$:

Let the plane passing through $A(\bar{a})$ and having normal $\bar{n}$ be π.

Let $P(\bar{r})$ be any point in the plane π.

$\therefore$ $P(\bar{r}) \in \pi$, $P \neq A \Rightarrow \overrightarrow{AP} \in \pi$

$\Rightarrow \overrightarrow{AP} \perp \bar{n}$

$\Rightarrow \overrightarrow{AP} \cdot \bar{n} = 0$

$\Rightarrow (\bar{r} - \bar{a}) \cdot \bar{n} = 0$

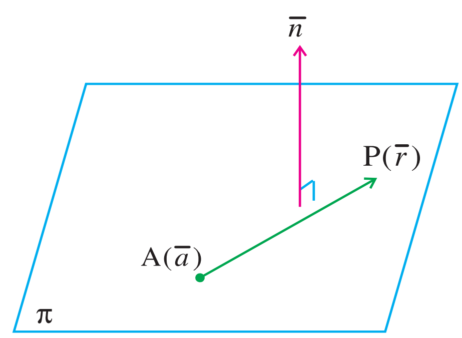


Figure 7.13

If $P = A$, then $\bar{r} = \bar{a}$ so $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$ holds good.

$\therefore$ $\forall P(\bar{r}) \in \pi$, $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$

Conversely, if $P(\bar{r})$ is any point in space such that $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$, then $\overrightarrow{AP} \perp \bar{n}$.

As $A \in \pi$, $P \in \pi$.

Thus, $P(\bar{r}) \in \pi \Leftrightarrow (\bar{r} - \bar{a}) \cdot \bar{n} = 0$

$\Leftrightarrow \bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$

$\therefore$ $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ is the vector equation of the plane passing through $A(\bar{a})$ and having normal $\bar{n}$.

Let $\bar{a} \cdot \bar{n} = d$

$\therefore$ $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ becomes $\bar{r} \cdot \bar{n} = d$

Cartesian form :

Let $\bar{r} = (x, y, z)$, $\bar{n} = (a, b, c)$ and $\bar{a} = (x_1, y_1, z_1)$

$\therefore$ $\bar{r} \cdot \bar{n} = d$ becomes $(x, y, z) \cdot (a, b, c) = d$ where $d = \bar{a} \cdot \bar{n} = ax_1 + by_1 + cz_1$.

$\therefore$ **$ax + by + cz = d$, $a^2 + b^2 + c^2 \neq 0$ as $\bar{n} \neq \bar{0}$ is the equation of the plane having normal $\bar{n} = (a, b, c)$**

> **Note :** The ordered triplet formed by the coefficient of x, y, z in the equation of a plane represents the normal $\bar{n}$ of the plane.

Example 20 : Find the equation of the plane passing through $(4, 5, -1)$ having normal $3\hat{i} - \hat{j} + \hat{k}$.

Solution : Here $\bar{a} = (4, 5, -1)$, $\bar{n} = (3, -1, 1)$

$\therefore$ The equation of the plane $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ gives $(x, y, z) \cdot (3, -1, 1) = (4, 5, -1) \cdot (3, -1, 1)$

$\therefore$ $3x - y + z = 12 - 5 - 1 = 6$

$\therefore$ $3x - y + z = 6$ is the equation of the plane passing through $(4, 5, -1)$ and having normal $3\hat{i} - \hat{j} + \hat{k}$.

Example 21 : Find the normal and the vector equation of the plane $2x - z + 1 = 0$.

Solution : Cartesian equation of plane is $2x - z + 1 = 0$.

$\therefore$ Normal $\bar{n} = (2, 0, -1)$ **(see note)**

$\therefore$ Vector equation $\bar{r} \cdot \bar{n} = d$ is $2x - z + 1 = (2, 0, -1) \cdot (x, y, z) + 1 = 0$

$\therefore$ The vector equation is $\bar{r} \cdot (2, 0, -1) + 1 = 0$

7.13 Equation of the plane using normal through the origin

Let $N(\bar{n})$ be the foot of perpendicular from origin to the plane π.

Let $ON = p$

$\therefore$ $|\bar{n}| = p$.

Let α, β, γ be the direction angles of $\bar{n}$.

$\therefore$ Direction *cosines* of $\bar{n}$ are $\cos\alpha, \cos\beta, \cos\gamma$.

$\therefore$ Unit vector in the direction of $\bar{n}$ (i.e. $\hat{n}$) is

$$\hat{n} = \frac{\bar{n}}{|\bar{n}|} = \frac{\bar{n}}{p} = (\cos\alpha, \cos\beta, \cos\gamma)$$

$\therefore$ $\bar{n} = (p\cos\alpha, p\cos\beta, p\cos\gamma)$

Let $P(\bar{r})$ be any point of the plane π.

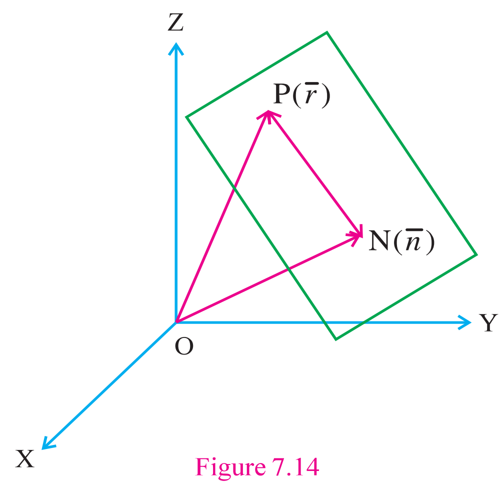


Figure 7.14

Here $\bar{a} = \bar{n} = (p\cos\alpha, p\cos\beta, p\cos\gamma)$

The equation of the plane $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ becomes

$(x, y, z) \cdot (p\cos\alpha, p\cos\beta, p\cos\gamma) = p^2$ **(as $\bar{a} \cdot \bar{n} = \bar{n} \cdot \bar{n} = |\bar{n}|^2 = p^2$)**

$\therefore$ $x\cos\alpha + y\cos\beta + z\cos\gamma = p$ is the equation of a plane having α, β, γ as the direction angles of the normal and p, the length of the normal.

> **Note :** If the equation of the plane is $ax + by + cz = d$, then to convert such an equation into the form of $x\cos\alpha + y\cos\beta + z\cos\gamma = p$, we divide the given equation by $|\bar{n}|$. That is
>
> $$\frac{a}{|\bar{n}|}x + \frac{b}{|\bar{n}|}y + \frac{c}{|\bar{n}|}z = \frac{d}{|\bar{n}|}$$

If $d > 0$, then let $\overline{n} = (a, b, c)$ so that $\frac{d}{|\overline{n}|} = p$ is positive.

$\therefore \quad \frac{\overline{n}}{|\overline{n}|} = \left(\frac{a}{|\overline{n}|}, \frac{b}{|\overline{n}|}, \frac{c}{|\overline{n}|}\right) = \hat{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ and $\frac{d}{|\overline{n}|} = p$

If $d < 0$, then let $\overline{n} = (-a, -b, -c)$ so that $\frac{-d}{|\overline{n}|} = p$ is positive.

$\therefore \quad -ax - by - cz = -d$

$\therefore \quad \frac{\overline{n}}{|\overline{n}|} = \left(\frac{-a}{|\overline{n}|}, \frac{-b}{|\overline{n}|}, \frac{-c}{|\overline{n}|}\right) = (\cos\alpha, \cos\beta, \cos\gamma)$ and $\frac{-d}{|\overline{n}|} = p$.

Example 22 : Find the direction *cosines* and the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 14 = 0$.

Solution : The plane π has the equation $2x - 3y + 6z = -14$ (given) **(i)**

We shall represent the equation in the form $\frac{a}{|\overline{n}|}x + \frac{b}{|\overline{n}|}y + \frac{c}{|\overline{n}|}z = \frac{d}{|\overline{n}|}$.

Here $d = -14 < 0$.

The equation can be written as $-2x + 3y - 6z = 14$, so that $d > 0$.

Let $\overline{n} = (-2, 3, -6)$, $|\overline{n}| = \sqrt{4+9+36} = 7$.

$\therefore \quad p = \frac{-d}{|\overline{n}|} = \frac{14}{7} = 2$, $(\cos\alpha, \cos\beta, \cos\gamma) = \left(\frac{-2}{7}, \frac{3}{7}, \frac{-6}{7}\right)$

Thus, the length of perpendicular from origin is 2 and direction *cosines* of the normal are $\frac{-2}{7}, \frac{3}{7}, \frac{-6}{7}$.

Intersection of a Line and a plane :

Let the equation $\overline{r} = \overline{a} + k\overline{l}$, $k \in R$ represent a line and the equation $\overline{r} \cdot \overline{n} = d$ represents a plane. $(\overline{n} \neq \overline{0})$

Consider the intersection of the line $\overline{r} = \overline{a} + k\overline{l}$ and the plane $\overline{r} \cdot \overline{n} = d$. $(\overline{l} \neq \overline{0}, \overline{n} \neq \overline{0})$

Suppose $\overline{l} = (l_1, l_2, l_3)$, $\overline{n} = (a, b, c)$, $\overline{a} = (x_1, y_1, z_1)$.

If the point $\overline{r}_1 = \overline{a} + k_1\overline{l}$ for some $k_1 \in R$ of the line is also on the plane, then

$(\overline{a} + k_1\overline{l}) \cdot \overline{n} = d$.

$\therefore \quad k_1(\overline{l} \cdot \overline{n}) = d - \overline{a} \cdot \overline{n}$ **(i)**

Now,

(1) If $\overline{l} \cdot \overline{n} = 0$ and $d - \overline{a} \cdot \overline{n} \neq 0$, then (i) is impossible.

$\therefore$ If $\overline{l} \cdot \overline{n} = 0$ and $ax_1 + by_1 + cz_1 \neq d$, then the line and the plane do not intersect. We say that the line is parallel to the plane.

(2) If $\overline{l} \cdot \overline{n} = 0$ and also $d - \overline{a} \cdot \overline{n} = 0$, then (i) is satisfied for every $k_1 \in R$.

In this case, every point of the line is in the plane.

Thus, if $ax_1 + by_1 + cz_1 = d$ and $\overline{l} \cdot \overline{n} = 0$ then the line lies in the plane.

(3) If $\overline{l} \cdot \overline{n} \neq 0$, then we get a unique value of k_1 by $k_1 = \frac{d - \overline{a} \cdot \overline{n}}{\overline{l} \cdot \overline{n}}$. So in this case, exactly one point of the line is on the plane. i.e. the line intersects the plane in exactly one point.

7.14 Measure of the Angle between two planes

The measure of the angle of between two planes is defined to be the measure of the angle between their normals.

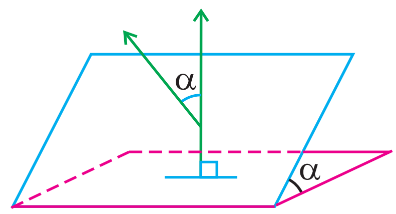


Figure 7.15

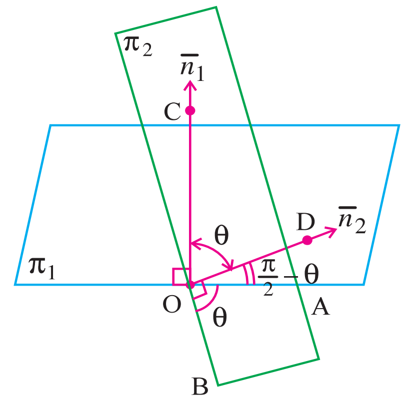


Figure 7.16

Since we take angle between two lines (normals) to be the acute angle between the lines, the angle between the planes is an acute angle.

Figure 7.16 shows that the measure of the angle between two normals $\overline{n}_1$ and $\overline{n}_2$ is θ, i.e. $(\overline{n}_1, \hat{\ } \overline{n}_2) = \theta = m\angle COD$,

but $m\angle COA = \frac{\pi}{2}$, so $m\angle DOA = \frac{\pi}{2} - \theta$.

Again, $\overline{n}_2$ is a normal of π_2, so $m\angle BOD = \frac{\pi}{2}$

$\therefore$ $m\angle AOB = \theta$, the angle between two planes.

Let $\pi_1 : \overline{r} \cdot \overline{n}_1 = d_1$ and

$\pi_2 : \overline{r} \cdot \overline{n}_2 = d_2$ be the equations of given planes.

(1) $\pi_1 \perp \pi_2 \Leftrightarrow \overline{n}_1 \perp \overline{n}_2 \Leftrightarrow \overline{n}_1 \cdot \overline{n}_2 = 0$

$\therefore$ **The measure of the angle between the planes π_1 and π_2 is $\frac{\pi}{2} \Leftrightarrow \overline{n}_1 \cdot \overline{n}_2 = 0$.**

(2) For distinct planes π_1 and π_2 we define π_1 is parallel to π_2 if they do not intersect. In this case

$\overline{n}_1 = \overline{n}_2 = \overline{n}$.

$\therefore$ **$\pi_1 \parallel \pi_2 \Leftrightarrow \overline{n}_1 \times \overline{n}_2 = \overline{0}$**

$\therefore$ **The measure of the angle between π_1 and π_2 is zero $\Leftrightarrow \overline{n}_1 \times \overline{n}_2 = \overline{0}$.**

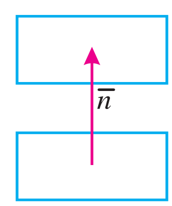


Figure 7.17

(3) Let θ be the measure of the angle between the planes π_1 and π_2, so that $0 < \theta < \frac{\pi}{2}$.

$\therefore$ $cos\theta = \frac{|\overline{n}_1 \cdot \overline{n}_2|}{|\overline{n}_1||\overline{n}_2|}$

$\therefore$ $\theta = cos^{-1} \frac{|\overline{n}_1 \cdot \overline{n}_2|}{|\overline{n}_1||\overline{n}_2|}$

which also holds true for $\theta = 0$ and $\frac{\pi}{2}$. **(Verify !)**

If $\pi_1 : a_1x + b_1y + c_1z = d_1$ and $\pi_2 : a_2x + b_2y + c_2z = d_2$ are given planes, then $\overline{n}_1 = (a_1, b_1, c_1)$ and $\overline{n}_2 = (a_2, b_2, c_2)$.

$$\theta = cos^{-1} \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

Example 23 : Find the measure of the angle between the planes $2x - y + z + 6 = 0$ and $x + y + 2z - 3 = 0$.

Solution : $\pi_1 : 2x - y + z + 6 = 0$. So $\overline{n}_1 = (2, -1, 1)$

$\pi_2 : x + y + 2z - 3 = 0$. So $\overline{n}_2 = (1, 1, 2)$

Now, $\overline{n}_1 \cdot \overline{n}_2 = 2(1) + (-1)1 + 1(2) = 3$

$$|\overline{n}_1| = \sqrt{4+1+1} = \sqrt{6}, \quad |\overline{n}_2| = \sqrt{1+1+4} = \sqrt{6}$$

$$\therefore \quad \theta = cos^{-1} \frac{|\overline{n}_1 \cdot \overline{n}_2|}{|\overline{n}_1||\overline{n}_2|} = cos^{-1} \frac{|3|}{\sqrt{6}\sqrt{6}} = cos^{-1} \frac{1}{2} = \frac{\pi}{3}$$

$\therefore$ The measure of the angle between given planes is $\frac{\pi}{3}$.

7.15 Equation of the plane passing through two parallel lines

Let $\overline{r} = \overline{a} + k\overline{l}$, $k \in$ R and

$\overline{r} = \overline{b} + k\overline{l}$, $k \in$ R be two parallel lines.

$\therefore$ They determine unique plane π.

Also, $\overline{b} \notin \{\overline{r} \mid \overline{r} = \overline{a} + k\overline{l}, k \in \text{R}\}$

$\therefore$ $\overline{b} \neq \overline{a} + k\overline{l}$, for any $k \in$ R

$\therefore$ $\overline{b} - \overline{a} \neq k\overline{l}$, for any $k \in$ R

$\therefore$ $(\overline{b} - \overline{a}) \times \overline{l} \neq \overline{0}$

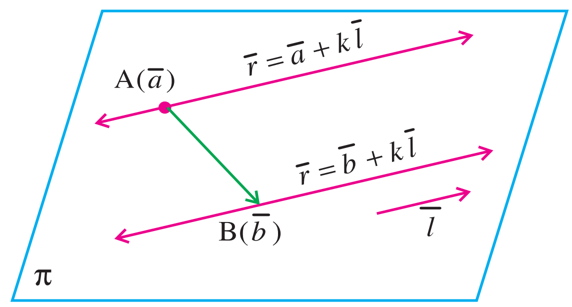


Figure 7.18

So let $\overline{n} = (\overline{b} - \overline{a}) \times \overline{l}$. Then $\overline{n} \neq \overline{0}$

We assert that the equation of the required plane π is $(\overline{r} - \overline{a}) \cdot \overline{n} = 0$.

i.e. $(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}] = 0$

Now, we shall show that this plane π contains both of the given lines.

For $\overline{r} = \overline{a} + k\overline{l}$

$$(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}] = (k\overline{l}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}] = 0$$

$\therefore$ Every point of line $\overline{r} = \overline{a} + k\overline{l}$ is in the plane $(\overline{r} - \overline{a}) \cdot \overline{n} = 0$.

$\therefore$ The plane π contains the line $\overline{r} = \overline{a} + k\overline{l}$, $k \in$ R.

Similarly, for $\overline{r} = \overline{b} + k\overline{l}$

$$(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}] = (\overline{b} + k\overline{l} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}]$$

$$= (\overline{b} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}] + k\overline{l} \cdot [(\overline{b} - \overline{a}) \times \overline{l}]$$

$$= 0$$

$\therefore$ The line $\overline{r} = \overline{b} + k\overline{l}$ is a subset of the plane $(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}] = 0$.

Hence, **$(\overline{r} - \overline{a}) \cdot [(\overline{b} - \overline{a}) \times \overline{l}] = 0$ is the equation of plane containing given parallel lines.**

Cartesian form :

Let $\overline{a} = (x_1, y_1, z_1)$, $\overline{b} = (x_2, y_2, z_2)$ and $\overline{l} = (l_1, l_2, l_3)$.

The Cartesian form of the equation of the plane containing two parallel lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & l_2 & l_3 \end{vmatrix} = 0$$

Example 24 : Show that lines L : $\frac{x-3}{3} = \frac{y-3}{-4} = \frac{z-5}{2}$ and M : $\frac{x}{6} = \frac{y-5}{-8} = \frac{z-2}{4}$ are parallel and find the equation of the plane containing them.

Solution : Here, $\bar{l} = (3, -4, 2)$, $\bar{m} = (6, -8, 4)$. So, $\bar{l} \times \bar{m} = \bar{0}$.

$\therefore$ L = M or L $\parallel$ M

Also, for (3, 3, 5) and $\frac{3}{6} = \frac{3-5}{-8} = \frac{5-2}{4}$ is not true. So $(3, 3, 5) \notin$ M.

$\therefore$ $(3, 3, 5) \in$ L, $(3, 3, 5) \notin$ M

$\therefore$ L $\neq$ M

Hence L $\parallel$ M

Now, $\bar{a} = (3, 3, 5)$, $\bar{b} = (0, 5, 2)$ and $\bar{l} = (3, -4, 2)$.

$\therefore$ The equation of the plane containing L and M is $\begin{vmatrix} x-3 & y-3 & z-5 \\ 0-3 & 5-3 & 2-5 \\ 3 & -4 & 2 \end{vmatrix} = 0$

$$\therefore \begin{vmatrix} x-3 & y-3 & z-5 \\ -3 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix} = 0$$

$\therefore$ $(x-3)(-8) - (y-3)(3) + (z-5)(6) = 0$

$\therefore$ $-8x + 24 - 3y + 9 + 6z - 30 = 0$

$\therefore$ $8x + 3y - 6z = 3$ is the equation of the plane passing through given parallel lines.

7.16 Equation of the plane containing two intersecting lines

Let $\bar{r} = \bar{a} + k\bar{l}$, $k \in$ R and

$\bar{r} = \bar{b} + k\bar{m}$, $k \in$ R be two intersecting lines.

$\therefore$ They determine unique plane π.

Also, $\bar{l} \times \bar{m} \neq \bar{0}$ and $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

(Why ?)

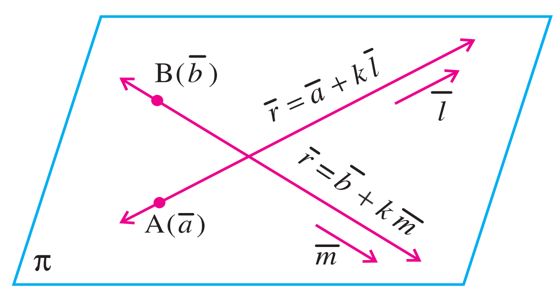


Figure 7.19

Taking $\bar{n} = \bar{l} \times \bar{m}$, we have $\bar{n} \neq \bar{0}$.

$(\bar{r} - \bar{a}) \cdot \bar{n} = 0$ represents a plane π.

i.e. $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$ is the equation of a plane π. **(as $\bar{n} \neq \bar{0}$)**

Now, we shall show that plane π contains given lines.

For $\bar{r} = \bar{a} + k\bar{l}$,

$$(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = (k\bar{l}) \cdot (\bar{l} \times \bar{m}) = 0$$

$\therefore$ Every point of $\bar{r} = \bar{a} + k\bar{l}$ is in the plane $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$.

Similarly, for $\bar{r} = \bar{b} + k\bar{m}$,

$(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = (\bar{b} + k\bar{m} - \bar{a}) \cdot (\bar{l} \times \bar{m})$

$= (\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) + (k\bar{m}) \cdot (\bar{l} \times \bar{m})$ $\quad ((\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0)$

$= 0$

$\therefore$ Every point of $\bar{r} = \bar{b} + k\bar{m}$ is in the plane $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$.

Hence, $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$ is the equation of a plane containing given intersecting lines.

Cartesian form :

Let $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$, $\bar{l} = (l_1, l_2, l_3)$ and $\bar{m} = (m_1, m_2, m_3)$.

The Cartesian form of the equation of the plane containing two intersecting lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

Note : (1) In the formula $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$, we can also use $\bar{b}$ in place of $\bar{a}$ i.e. $(\bar{r} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$ is also the equation of plane containing two intersecting lines.

(2) To get the equation of the plane we need three non-collinear points. So A($\bar{a}$) and B($\bar{b}$) are two given points of the plane. The third point C can be any point of the given lines (which can be obtained by taking $k \in R - \{0\}$ in any of the given equations.)

Example 25 : Prove that L : $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and M : $\frac{x-4}{5} = \frac{y-1}{2} = z$ are coplanar and find the equation of the plane containing them.

Solution : Here, $\bar{a} = (1, 2, 3)$, $\bar{l} = (2, 3, 4)$ and

$\bar{b} = (4, 1, 0)$, $\bar{m} = (5, 2, 1)$.

$\bar{l} \times \bar{m} = (-5, 18, -11) \neq \bar{0}$ and $\bar{b} - \bar{a} = (3, -1, -3)$

$(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = (3, -1, -3) \cdot (-5, 18, -11) = -15 - 18 + 33 = 0$

$\therefore$ L and M are intersecting lines and so coplanar.

The equation of the plane containing L and M is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} x - 1 & y - 2 & z - 3 \\ 2 & 3 & 4 \\ 5 & 2 & 1 \end{vmatrix} = 0$$

$\therefore$ $(x - 1)(-5) - (y - 2)(-18) + (z - 3)(-11) = 0$

$\therefore$ $-5x + 5 + 18y - 36 - 11z + 33 = 0$

$\therefore$ $5x - 18y + 11z - 2 = 0$ is the equation of the required plane.

Another Method : A(1, 2, 3), B(4, 1, 0) are given.

Taking $k = 1$ in the equation $\bar{r} = (1, 2, 3) + k(2, 3, 4)$, $k \in R$ of line L, we get C(3, 5, 7) as a point on line L.

Obviously, A, B, C are not collinear as $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 3 & 5 & 7 \end{vmatrix} = 7 - 56 + 51 \neq 0$.

So the equation of the required plane through A, B, C is $\begin{vmatrix} x-1 & y-2 & z-3 \\ 4-1 & 1-2 & 0-3 \\ 3-1 & 5-2 & 7-3 \end{vmatrix} = 0$

$\therefore \quad \begin{vmatrix} x-1 & y-2 & z-3 \\ 3 & -1 & -3 \\ 2 & 3 & 4 \end{vmatrix} = 0$

$\therefore \quad (x - 1)(5) - (y - 2)(18) + (z - 3)(11) = 0$

$\therefore \quad 5x - 5 - 18y + 36 + 11z - 33 = 0$

$\therefore \quad 5x - 18y + 11z - 2 = 0$

Note : Similar approach can also be taken for finding the equation of the plane containing two parallel lines.

7.17 Perpendicular distance from a point outside a plane to the plane

Let $\pi : \bar{r} \cdot \bar{n} = d$ be the equation of a given plane and $P(\bar{p})$ be a given point, $P \notin \pi$.

If $M(\bar{m})$ is the foot of the perpendicular from $P(\bar{p})$ to the plane π, then we need to find the distance PM.

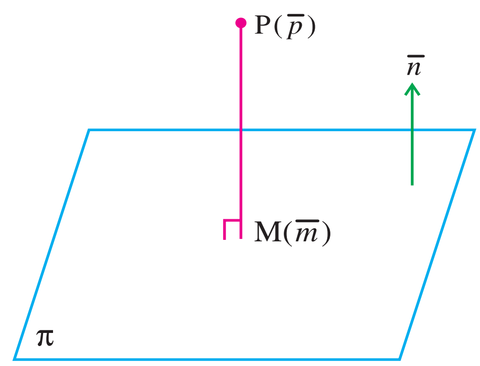


Figure 7.20

$\therefore$ Direction of $\overrightarrow{MP}$ and $\bar{n}$ are same.

$\therefore$ The equation of $\overleftrightarrow{MP}$ is $\bar{r} = \bar{p} + k\bar{n}$, $k \in R$

As $M(\bar{m}) \in \overleftrightarrow{MP}$ so $\bar{m} = \bar{p} + k_1\bar{n}$,

for some $k_1 \in R - \{0\}$

Also, $M(\bar{m}) \in \pi$. So $\bar{m} \cdot \bar{n} = d$

$\therefore \quad (\bar{p} + k_1\bar{n}) \cdot \bar{n} = d$

$\therefore \quad k_1 |\bar{n}|^2 = d - \bar{p} \cdot \bar{n}$

$\therefore \quad k_1 = \dfrac{d - \bar{p} \cdot \bar{n}}{|\bar{n}|^2}$ $\qquad (\bar{n} \neq \bar{0})$ **(i)**

Now, $PM = |\overrightarrow{PM}| = |\bar{m} - \bar{p}|$ $\qquad (\because \bar{m} = \bar{p} + k_1\bar{n})$

$= |k_1\bar{n}| = |k_1| \, |\bar{n}|$

$\therefore \quad$ **PM** $= \dfrac{|d - \bar{p} \cdot \bar{n}|}{|\bar{n}|^2} \times |\bar{n}| = \dfrac{|\bar{p} \cdot \bar{n} - d|}{|\bar{n}|}$

Cartesian form :

Let $P(x_1, y_1, z_1)$ be the given point and $ax + by + cz = d$ be the given plane.

$\therefore \quad \bar{p} = (x_1, y_1, z_1), \quad \bar{n} = (a, b, c)$

$\therefore$ Perpendicular distance from P to $\pi = \dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$

Also, if the equation of the plane is taken as $ax + by + cz + d = 0$, the perpendicular distance

$$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \quad \textbf{(replacing } d \textbf{ by } -d \textbf{ in } \overline{r} \cdot \overline{n} = d\textbf{)}$$

Note : (1) The foot of perpendicular from the point $P(\overline{p})$ to the plane $\overline{r} \cdot \overline{n} = d$ is $M(\overline{m})$ where $\overline{m} = \overline{p} + k_1\overline{n}$, $k_1 = \dfrac{d - \overline{p} \cdot \overline{n}}{|\overline{n}|^2}$.

(2) Compare with $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$, the perpendicular distance of (x_1, y_1) from $ax + by + c = 0$.

Example 26 : Find the perpendicular distance from point $(-1, 2, -2)$ to the plane $3x - 4y + 2z + 44 = 0$.

Solution : $\overline{p} = (-1, 2, -2)$ and $\pi : 3x - 4y + 2z = -44$ are given. So $d = -44$.

$\therefore$ Perpendicular distance from P to plane $\pi = \dfrac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$

$$= \frac{|3(-1) - 4(2) + 2(-2) + 44|}{\sqrt{3^2 + (-4)^2 + 2^2}} = \frac{29}{\sqrt{29}} = \sqrt{29}$$

Distance between two parallel planes :

Suppose $\pi_1 : \overline{r} \cdot \overline{n} = d_1$ and $\pi_2 : \overline{r} \cdot \overline{n} = d_2$ are two parallel planes.

The perpendicular distance of any point $A(\overline{a})$ in π_1 to the plane π_2 is the distance between two parallel planes.

$A(\overline{a}) \in \pi$. Hence $\overline{a} \cdot \overline{n} = d_1$

$\therefore$ Perpendicular distance of $A(\overline{a})$ from

$\overline{r} \cdot \overline{n} = d_2$ is $\dfrac{|\overline{a} \cdot \overline{n} - d_2|}{|\overline{n}|} = \dfrac{|d_1 - d_2|}{|\overline{n}|}$

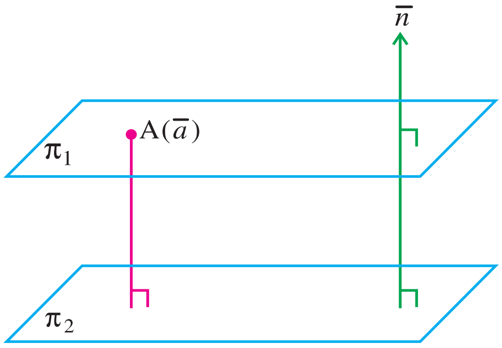


Figure 7.21

Example 27 : Find the distance between the planes $2x - 2y - z + 4 = 0$ and $4y + 2z - 4x + 1 = 0$.

Solution : $\left.\begin{aligned} \pi_1 &: 2x - 2y - z + 4 = 0 \\ \pi_2 &: 4y + 2z - 4x + 1 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \pi_1 &: 4x - 4y - 2z = -8 \\ \pi_2 &: 4x - 4y - 2z = 1 \end{aligned}$

$\therefore$ $\overline{n} = (4, -4, -2)$, $d_1 = -8$, $d_2 = 1$

$\therefore$ Perpendicular distance between the given planes $= \dfrac{|d_1 - d_2|}{|\overline{n}|}$

$$= \frac{|-8 - 1|}{\sqrt{4^2 + (-4)^2 + (-2)^2}}$$

$$= \frac{9}{6} = \frac{3}{2}$$

By using above formula, we can obtain the formula for the shortest distance between two skew lines.

Let $\overline{r} = \overline{a} + k\overline{l}$, $k \in R$ and $\overline{r} = \overline{b} + k\overline{m}$, $k \in R$ be two skew lines. So $(\overline{a} - \overline{b}) . (\overline{l} \times \overline{m}) \neq 0$

First of all, let P $(\bar{a}+k_2\bar{l})$ for some $k_2 \in R$ be any point on L and Q $(\bar{b}+k_1\bar{m})$ for some $k_1 \in R$ be any point on M.

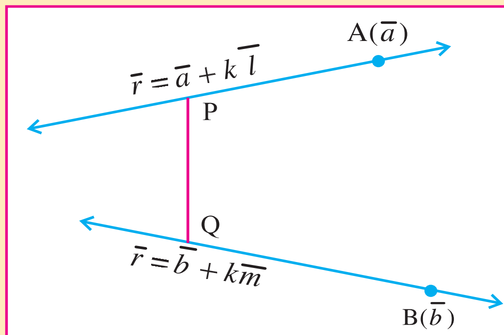


Figure 7.22

$\therefore \overrightarrow{PQ} = \bar{b} - \bar{a} + k_1\bar{m} - k_2\bar{l}$

Now, if $\overleftrightarrow{PQ}$ is perpendicular to both L and M, then

$(\bar{b} - \bar{a} + k_1\bar{m} - k_2\bar{l}) \cdot \bar{l} = 0$

and $(\bar{b} - \bar{a} + k_1\bar{m} - k_2\bar{l}) \cdot \bar{m} = 0$

$\therefore (\bar{l} \cdot \bar{m})\, k_1 - |\bar{l}|^2\, k_2 = (\bar{a} - \bar{b}) \cdot \bar{l}$

$|\bar{m}|^2\, k_1 - (\bar{l} \cdot \bar{m})\, k_2 = (\bar{a} - \bar{b}) \cdot \bar{m}$

As, lines are skew lines, so

$$(\bar{l} \cdot \bar{m})(\bar{l} \cdot \bar{m}) - |\bar{l}|^2 |\bar{m}|^2 = |\bar{l} \cdot \bar{m}|^2 - |\bar{l}|^2 |\bar{m}|^2$$

$$= -|\bar{l} \times \bar{m}|^2 \neq 0$$

$\therefore$ There exist unique $k_1 \in R$ and $k_2 \in R$, such that $\overleftrightarrow{PQ} \perp L$ and $\overleftrightarrow{PQ} \perp M$

But directions of L and M are $\bar{l}$ and $\bar{m}$ respectively.

$\therefore$ Direction of $\overleftrightarrow{PQ}$ is $\bar{l} \times \bar{m}$.

The plane $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$ passes through L. Since $(\bar{a} + k\bar{l} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$

Similarly $(\bar{r} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$ passes through M.

.Direction of $\overleftrightarrow{PQ}$ is $\bar{l} \times \bar{m}$ and it is perpendicular to both the planes.

$$\therefore PQ = \frac{|d_1 - d_2|}{|\bar{l} \times \bar{m}|}$$

$$= \frac{|\bar{a}.(\bar{l} \times \bar{m}) - \bar{b}.(\bar{l} \times \bar{m})|}{|\bar{l} \times \bar{m}|}$$

$$= \frac{|(\bar{a} - \bar{b})(\bar{l} \times \bar{m})|}{|\bar{l} \times \bar{m}|}$$

7.18 Angle between a line and a plane

Suppose $\bar{r} = \bar{a} + k\bar{l}$ is the equation of a given line and $\bar{r} \cdot \bar{n} = d$ is the equation of a given plane. Suppose the line intersects the plane at P and is not perpendicular to the plane. M is the foot of the perpendicular from A($\bar{a}$) on the plane. Then $\angle$APM is called the angle between the given line and the given plane.

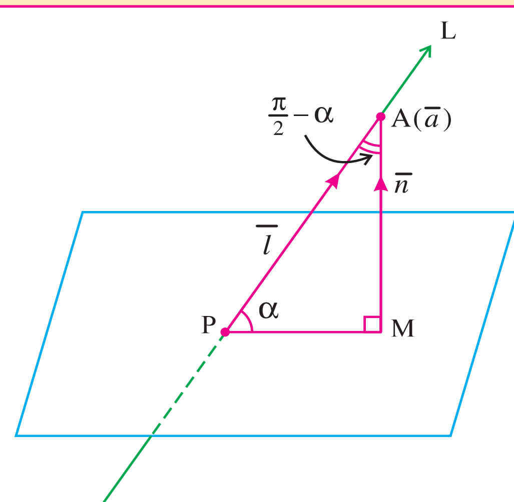


Figure 7.23

Let $m\angle APM = \alpha,\ 0 < \alpha < \frac{\pi}{2}$

$$\therefore \frac{\pi}{2} - \alpha = (\bar{l}\ \hat{,}\ \bar{n})$$

$$\therefore \cos\left(\frac{\pi}{2} - \alpha\right) = \frac{|\bar{l} \cdot \bar{n}|}{|\bar{l}||\bar{n}|}$$

$\therefore \quad sin\alpha = \dfrac{|\bar{l} \cdot \bar{n}|}{|\bar{l}||\bar{n}|}$

$\therefore \quad \alpha = sin^{-1} \dfrac{|\bar{l} \cdot \bar{n}|}{|\bar{l}||\bar{n}|}$ **is the measure of the angle between the line and the plane.**

Example 28 : Find the measure of the angle between the line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z+1}{1}$ and the plane $\bar{r} \cdot (-2, 2, -1) = 1$.

Solution : Here $\bar{l} = (2, 2, 1)$, $\bar{n} = (-2, 2, -1)$

$\bar{l} \cdot \bar{n} = 2(-2) + 2(2) + 1(-1) = -1$

$|\bar{l}| = \sqrt{2^2 + 2^2 + 1^2} = 3, \quad |\bar{n}| = \sqrt{(-2)^2 + 2^2 + (-1)^2} = 3$

$\therefore$ The measure of the angle between the given line and the plane $= sin^{-1} \dfrac{|\bar{l} \cdot \bar{n}|}{|\bar{l}||\bar{n}|}$

$= sin^{-1} \dfrac{|-1|}{3(3)} = sin^{-1} \frac{1}{9}$

7.19 Intersection of two planes

Let $\pi_1 : \bar{r} \cdot \bar{n}_1 = d_1$ and $\pi_2 : \bar{r} \cdot \bar{n}_2 = d_2$ be two intersecting planes.

$\therefore \quad \bar{n}_1 \times \bar{n}_2 \neq \bar{0}$

Let $\bar{n} = \bar{n}_1 \times \bar{n}_2$.

Suppose $A(\bar{a})$ is a point of intersection of π_1 and π_2.

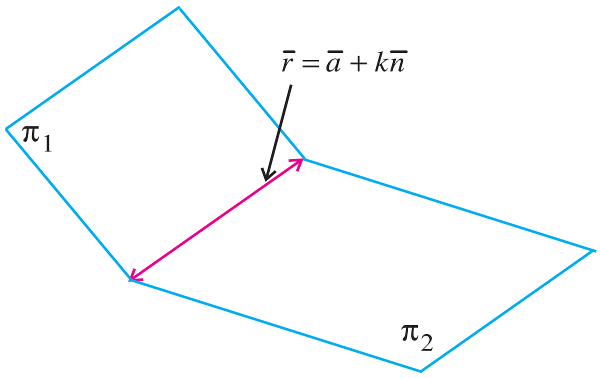


Figure 7.24

$\therefore \quad A(\bar{a}) \in \pi_1$ and $A(\bar{a}) \in \pi_2$.

$\therefore \quad \bar{a} \cdot \bar{n}_1 = d_1$ and $\bar{a} \cdot \bar{n}_2 = d_2$

$\therefore$ The equations of π_1 and π_2 are

$\bar{r} \cdot \bar{n}_1 = d_1 = \bar{a} \cdot \bar{n}_1$

$\therefore \quad (\bar{r} - \bar{a}) \cdot \bar{n}_1 = 0$

Similarly $(\bar{r} - \bar{a}) \cdot \bar{n}_2 = 0$

$\therefore$ If $P(\bar{r})$ is on both π_1 and π_2, then $(\bar{r} - \bar{a}) \perp \bar{n}_1$ and $(\bar{r} - \bar{a}) \perp \bar{n}_2$, $P \neq A$.

$\therefore \quad \bar{r} - \bar{a} = k(\bar{n}_1 \times \bar{n}_2), \ k \in R - \{0\}$

$\therefore \quad \bar{r} - \bar{a} = k\bar{n}, \ k \in R - \{0\}$ $\qquad (\bar{n} = \bar{n}_1 \times \bar{n}_2)$

If $k = 0$, then $P = A$. So $\bar{r} = \bar{a} + k\bar{n}, \ k \in R$.

Thus, if $P(\bar{r}) \in \pi_1 \cap \pi_2$, then $\bar{r} = \bar{a} + k\bar{n}, \ k \in R$.

This is the equation of a line.

$\therefore$ Every point of $\pi_1 \cap \pi_2$ is on the line $\bar{r} = \bar{a} + k\bar{n}, \ k \in R$.

Conversely, if $P(\bar{r})$ is on the line $\bar{r} = \bar{a} + k\bar{n}, \ k \in R$, then

$(\bar{r} - \bar{a}) \cdot \bar{n}_1 = k\bar{n} \cdot \bar{n}_1 = k(\bar{n}_1 \times \bar{n}_2) \cdot \bar{n}_1 = 0$

and $(\bar{r} - \bar{a}) \cdot \bar{n}_2 = k\bar{n} \cdot \bar{n}_2 = k(\bar{n}_1 \times \bar{n}_2) \cdot \bar{n}_2 = 0$

Thus, $P(\bar{r}) \in \pi_1 \cap \pi_2$.

Hence, $\pi_1 \cap \pi_2$ is the line given by the equation $\overline{r} = \overline{a} + k\overline{n}$, $k \in R$ where $\overline{n} = \overline{n}_1 \times \overline{n}_2$.

Thus two planes $\overline{r} \cdot \overline{n}_1 = d_1$ and $\overline{r} \cdot \overline{n}_2 = d_2$ intersect in the line $\overline{r} = \overline{a} + k(\overline{n}_1 \times \overline{n}_2)$ $k \in R$ provided $\overline{n}_1 \times \overline{n}_2 \neq \overline{0}$.

Equation of a plane passing through the intersection of two planes :

Suppose $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are two intersecting planes.

The equation of any plane passing through their line of intersection is

$$l(a_1x + b_1y + c_1z + d_1) + m(a_2x + b_2y + c_2z + d_2) = 0, \ l^2 + m^2 \neq 0$$

Conversely, any plane whose equation can be expressed in the form,

$l(a_1x + b_1y + c_1z + d_1) + m(a_2x + b_2y + c_2z + d_2) = 0$, $l^2 + m^2 \neq 0$ will certainly contain the line of intersection of the two given planes.

We shall assume both these statements without proof.

Here $l^2 + m^2 \neq 0$ means atleast one of l, m is non-zero.

If $l = 0$, then $m \neq 0$ and hence the required plane is $a_2x + b_2y + c_2z + d_2 = 0$.

If $l \neq 0$, then the required plane is not $a_2x + b_2y + c_2z + d_2 = 0$.

$\therefore$ $l(a_1x + b_1y + c_1z + d_1) + m(a_2x + b_2y + c_2z + d_2) = 0$ becomes

$$a_1x + b_1y + c_1z + d_1 + \frac{m}{l}(a_2x + b_2y + c_2z + d_2) = 0$$

Let $\frac{m}{l} = \lambda$

If $a_2x + b_2y + c_2z + d_2 = 0$ is not the required plane, then

the equation of the required plane is

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0, \ \lambda \in R$$

Example 29 : Find the equation of the plane passing through the intersection of the planes $2x + 3y + z - 1 = 0$ and $x + y - z - 7 = 0$ and also passing through the point (1, 2, 3). Also obtain the equation of the line of intersection of these planes.

Solution : For (1, 2, 3), $x + y - z - 7 = 1 + 2 - 3 - 7 = -7 \neq 0$

$\therefore$ (1, 2, 3) is not in the plane $x + y - z - 7 = 0$.

$\therefore$ $x + y - z - 7 = 0$ is not the required plane.

Suppose the required plane has equation $2x + 3y + z - 1 + \lambda(x + y - z - 7) = 0$ **(i)**

It passes through (1, 2, 3)

$\therefore$ $2 + 6 + 3 - 1 + \lambda(1 + 2 - 3 - 7) = 0$

$\therefore$ $-7\lambda = -10$

$\therefore$ $\lambda = \frac{10}{7}$. Substitute $\lambda = \frac{10}{7}$ in (i).

$$2x + 3y + z - 1 + \frac{10}{7}(x + y - z - 7) = 0$$

$\therefore$ $14x + 21y + 7z - 7 + 10x + 10y - 10z - 70 = 0$

$\therefore$ $24x + 31y - 3z - 77 = 0$

The direction of the line of intersection is $\overline{n} = \overline{n}_1 \times \overline{n}_2 = (2, 3, 1) \times (1, 1, -1) = (-4, 3, -1)$.

Let us take $z = 0$ in both the equations of planes.

$\therefore$ We get $2x + 3y = 1$ and $x + y = 7$.

Solving these equations we get $x = 20$, $y = -13$.

$\therefore$ A point of intersection is A(20, −13, 0)

$\therefore$ The equation of the required line $\overline{r} = \overline{a} + k\overline{n}$, $k \in$ R gives,

$\overline{r} = (20, -13, 0) + k(-4, 3, -1)$, $k \in$ R

Note : To find a common point of two planes, we can take any one of x, y, z as known number so that the other two can be uniquely determined.

Exercise 7.2

1. Find the unit normal to the plane $4x - 2y + z - 7 = 0$.

2. If possible, find the vector and Cartesian equation of the plane passing through (1, 1, −1), (2, −1, −3) and (3, 0, 1).

3. Find the equation of the plane parallel to $2x - 3y - 5z + 1 = 0$ and passing through (1, 2, −3).

4. Find the equation of the plane passing through (5, −1, 2) and perpendicular to the line which passes through (−2, 1, 1) and (0, 5, 1). Also find the intercepts made by this plane on the co-ordinate axes.

5. Find the equation of the plane passing through (2, 0, 1) and containing the line

$\overline{r} = (1, 4, -1) + k(2, -3, 3)$, $k \in$ R.

6. Show that the points (2, 7, 3), (−10, −10, 2), (−3, 3, 2) and (0, −2, 4) are coplanar. Also find the equation of the plane passing through them.

7. Obtain the equation of the plane which passes through (3, 4, −5) and (1, 2, 3) and parallel to Z-axis.

8. Find the measure of the angle between the planes $2x + y - z - 1 = 0$ and $x - y - 2z + 7 = 0$.

9. Find the measure of the angle between the line $\frac{x-2}{2} = \frac{y-2}{-3} = \frac{z-1}{2}$ and the plane $2x + y - 3z + 4 = 0$.

10. Find the perpendicular distance to the plane $3x + 2y - 5z - 13 = 0$ from the point (5, 3, 4).

11. Find the perpendicular distance between the planes $12x - 6y + 4z - 21 = 0$ and $6x - 3y + 2z - 1 = 0$.

12. Find the equation of the plane passing through A(1, 3, 5) and perpendicular to $\overline{AP}$, where P is (3, −2, 1)

13. Find the equation of the plane passing through the point (1, 1, −1) and containing the line

$\overline{r} = (2, -4, -6) + k(1, 8, -3)$, $k \in$ R.

14. Find the equation of the plane passing through the intersecting lines $\frac{x+1}{1} = \frac{3-y}{1} = \frac{z+5}{2}$ and $\frac{x+1}{3} = \frac{y-3}{1} = \frac{z+5}{2}$.

*

Miscellaneous Examples

Example 30 : If a line makes angles of measures α, β, γ, δ with the four diagonals of a cube, prove that $cos2\alpha + cos2\beta + cos2\gamma + cos2\delta = -\frac{4}{3}$.

Solution : Assume that each side of the cube is of unit length. Then the vertices can be taken as shown in the figure 7.25.

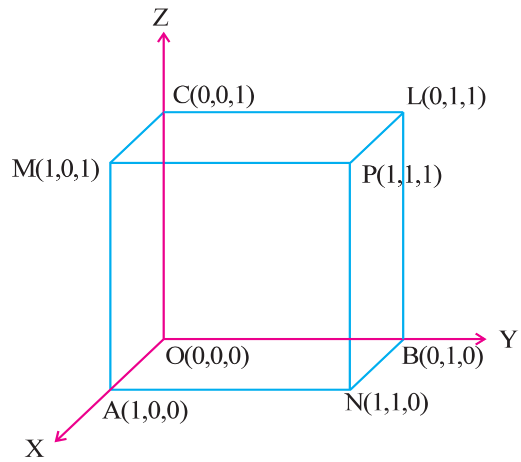


Figure 7.25

The four diagonals of the cube are $\overrightarrow{OP} = (1, 1, 1)$, $\overrightarrow{AL} = (-1, 1, 1)$, $\overrightarrow{BM} = (1, -1, 1)$, $\overrightarrow{CN} = (1, 1, -1)$.

Suppose the line has direction *cosines* l, m, n. So $l^2 + m^2 + n^2 = 1$.

If α, β, γ and δ are the measure of the angles made by the line with the diagonals $\overrightarrow{OP}$, $\overrightarrow{AL}$, $\overrightarrow{BM}$ and $\overrightarrow{CN}$ respectively, then

$$cos\alpha = \frac{|l + m + n|}{\sqrt{3}},\ cos\beta = \frac{|-l + m + n|}{\sqrt{3}},\ cos\gamma = \frac{|l - m + n|}{\sqrt{3}} \text{ and } cos\delta = \frac{|l + m - n|}{\sqrt{3}}.$$

Now, $cos2\alpha + cos2\beta + cos2\gamma + cos2\delta = 2cos^2\alpha - 1 + 2cos^2\beta - 1 + 2cos^2\gamma - 1 + 2cos^2\delta - 1$

$$= 2(cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta) - 4$$

$$= \frac{2}{3}[(l + m + n)^2 + (-l + m + n)^2 + (l - m + n)^2 + (l + m - n)^2] - 4$$

$$= \frac{2}{3}[4(l^2 + m^2 + n^2)] - 4$$

$$= \frac{8}{3} - 4 \qquad (l^2 + m^2 + n^2 = 1)$$

$$= -\frac{4}{3}$$

Image of a point in the line (plane) : If M is the foot of perpendicular from A to a line (plane) and B is the point such that M is the mid-point of $\overline{AB}$, then B is called the image of A in the line (plane).

Example 31 : Find the image of A(1, 2, 3) in the line L : $\frac{x - 6}{3} = \frac{y - 7}{2} = \frac{7 - z}{2}$.

Solution : The line has equation $\frac{x - 6}{3} = \frac{y - 7}{2} = \frac{z - 7}{-2}$.

Here $\bar{a} = (6, 7, 7)$, $\bar{l} = (3, 2, -2)$. Let M be the foot of perpendicular from A(1, 2, 3) to L.

M $\in$ L. So M is $(6 + 3k, 7 + 2k, 7 - 2k)$ for some $k \in$ R.

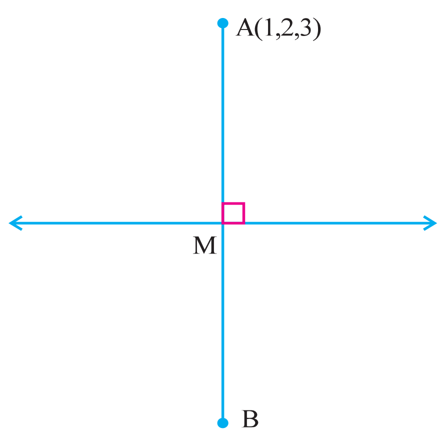


Figure 7.26

$$\overrightarrow{AM} = (6 + 3k, 7 + 2k, 7 - 2k) - (1, 2, 3)$$
$$= (5 + 3k, 5 + 2k, 4 - 2k)$$

$\overrightarrow{AM} \perp L$

$\therefore \overrightarrow{AM} \cdot \bar{l} = 0$

$\therefore (5 + 3k, 5 + 2k, 4 - 2k) \cdot (3, 2, -2) = 0$

$\therefore 15 + 9k + 10 + 4k - 8 + 4k = 0$

$\therefore$ $17k + 17 = 0$

$\therefore$ $k = -1$

$\therefore$ The foot of perpendicular is $M(6 + 3k, 7 + 2k, 7 - 2k) = M(3, 5, 9)$.

If $B(x, y, z)$ is the image of A in the given line, then M is the mid-point of $\overline{AB}$.

$\therefore$ $(3, 5, 9) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z+3}{2}\right)$

$\therefore$ $x = 5, y = 8, z = 15$

$\therefore$ The image of A is $B(5, 8, 15)$.

Example 32 : The direction numbers l, m, n of two lines satisfy $l + m + n = 0$ and $l^2 - m^2 + n^2 = 0$. Find the measure of the angle between the lines.

Solution : Here $l + m + n = 0$

$\therefore$ $m = -l - n$

Also $l^2 - m^2 + n^2 = 0$

$\therefore$ $l^2 - (-l - n)^2 + n^2 = 0$

$\therefore$ $l^2 - l^2 - 2ln - n^2 + n^2 = 0$

$\therefore$ $ln = 0$

$\therefore$ $l = 0$ or $n = 0$

As l, m, n are the direction numbers, $(l, m, n) \neq (0, 0, 0)$

If $l = 0$, then $n = -m$

$\therefore$ Direction numbers are $(0, m, -m)$

If $n = 0$, then $l = -m$

$\therefore$ Direction numbers are $(-m, m, 0)$

If α is the measure of the angle between the two lines, then

$$\cos\alpha = \frac{|(0, m, -m) \cdot (-m, m, 0)|}{\sqrt{2m^2} \cdot \sqrt{2m^2}}$$

$$= \frac{|m^2|}{2|m^2|} = \frac{1}{2}$$

$\therefore$ $\alpha = \frac{\pi}{3}$

Example 33 : Find the point of intersection of the line $\frac{x-4}{2} = \frac{y-5}{2} = \frac{z-3}{1}$ and the plane $x + y + z - 2 = 0$. Also find the distance between this point and the point $(8, 9, 5)$.

Solution : Here $\overline{a} = (4, 5, 3)$, $\overline{l} = (2, 2, 1)$.

Let P be the point of intersection. So P is on the given line.

∴ P is $(5 + 2k, 3 + k, 4 + 2k)$ for some $k \in$ R, P is also on the plane $x + y + z - 2 = 0$.

∴ $4 + 2k + 5 + 2k + 3 + k - 2 = 0$

∴ $5k = -10$

∴ $k = -2$

∴ The point of intersection is $(4 + 2(-2), 5 + 2(-2), 3 + (-2)) = (0, 1, 1)$

The distance between P(0, 1, 1) and Q(8, 9, 5) is given by

$$PQ = \sqrt{(8-0)^2 + (9-1)^2 + (5-1)^2} = \sqrt{64+64+16} = \sqrt{144} = 12$$

Example 34 : Find the equation of the plane passing through $(2, 2, -2)$ and $(-2, -2, 2)$ and perpendicular to the plane $2x - 3y + z - 7 = 0$.

Solution : Let the equation of the required plane be $ax + by + cz + d = 0$.

If $\overline{n}$ is normal to this plane, then $\overline{n} = (a, b, c)$.

Since this plane is perpendicular to $2x - 3y + z - 7 = 0$.

∴ $\overline{n} \cdot (2, -3, 1) = 0$ **(i)**

Also A(2, 2, −2) and B(−2, −2, 2) lie in the plane.

∴ $\overrightarrow{AB}$ lies in the plane. $\overrightarrow{AB} = (-4, -4, 4)$

∴ $\overline{n} \cdot (-4, -4, 4) = 0$

∴ $\overline{n} \cdot (-1, -1, 1) = 0$ **(ii)**

From (i) and (ii), $\overline{n} = (2, -3, 1) \times (-1, -1, 1)$

∴ $\overline{n} = (-2, -3, -5)$ or $\overline{n} = (2, 3, 5)$

Since the plane passes through $(2, 2, -2)$ its equation is given by $\overline{r} \cdot \overline{n} = \overline{a} \cdot \overline{n}$.

∴ $\overline{r} \cdot (2, 3\ 5) = (2, 2, -2) \cdot (2, 3\ 5)$

∴ $2x + 3y + 5z = 4 + 6 - 10 = 0$

∴ The equation of the required plane is $2x + 3y + 5z = 0$.

Example 35 : Find the foot of the perpendicular from P(9, 6, −2) to the plane passing through the point A(4, 5, 2), B(2, 3, −1) and C(6, −1, −1). Also find the perpendicular distance from P to this plane.

Solution : The equation of the plane is

$$\begin{vmatrix} x-4 & y-5 & z-2 \\ 2-4 & 3-5 & -1-2 \\ 6-4 & -1-5 & -1-2 \end{vmatrix} = 0$$

∴ $$\begin{vmatrix} x-4 & y-5 & z-2 \\ -2 & -2 & -3 \\ 2 & -6 & -3 \end{vmatrix} = 0$$

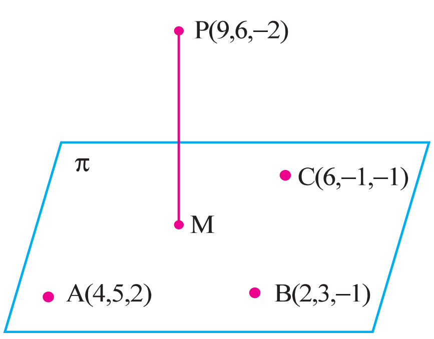


Figure 7.27

∴ $(x - 4)(-12) - (y - 5)(12) + (z - 2)(16) = 0$

∴ $3(x - 4) + 3(y - 5) - 4(z - 2) = 0$

∴ $3x + 3y - 4z - 19 = 0$ is the equation of plane through A, B and C.

Let M be the foot of perpendicular from the P(9, 6, −2) to the plane $\pi : 3x + 3y - 4z - 19 = 0$.

Here, $\bar{n} = (3, 3, -4)$

Equation of $\overleftrightarrow{PM}$ is $\bar{r} = \bar{p} + k\bar{n},\ k \in R$

$\therefore \quad \bar{r} = (9, 6, -2) + k(3, 3, -4),\ k \in R$

$\therefore \quad$ M is $(9 + 3k, 6 + 3k, -2 - 4k)$ for some $k \in R$

Now, $M \in \pi$

$\therefore \quad 3(9 + 3k) + 3(6 + 3k), -4(-2 - 4k) - 19 = 0$

$\therefore \quad 27 + 9k + 18 + 9k + 8 + 16k - 19 = 0$

$\therefore \quad 34k = -34$

$\therefore \quad k = -1$

$\therefore \quad$ The foot of the perpendicular is $M(9 + 3(-1), 6 + 3(-1), -2 - 4(-1))$

$\therefore \quad$ M is (6, 3, 2)

Perpendicular distance $PM = \sqrt{(9-6)^2 + (6-3)^2 + (-2-2)^2}$

$= \sqrt{9 + 9 + 16}$

$= \sqrt{34}$

Example 36 : Show that (i) The line $\bar{r} = (1, 2, -3) + k(4, -3, 2),\ k \in R$ is parallel to the plane $3x + 2y - 3z = 5$. (ii) The plane $2x - 3y + 4z = 0$ contains the line $\bar{r} = (1, -2, -2) + k(1, 2, 1)$, $k \in R$

Solution : (i) Here, the equation of the line L is $\bar{r} = (1, 2, -3) + k(4, -3, 2),\ k \in R$ and the plane π has equation $3x + 2y - 3z = 5$.

$\therefore \quad A(\bar{a}) = (1, 2, -3),\ \bar{l} = (4, -3, 2)$ and $\bar{n} = (3, 2, -3)$

Now, $\bar{l} \cdot \bar{n} = 4(3) - 3(2) + 2(-3) = 12 - 6 - 6 = 0$

$\therefore \quad \bar{l} \perp \bar{n}$. So L is parallel to π or π contains L.

Also $\bar{a} \cdot \bar{n} = (1, 2, -3) \cdot (3, 2, -3) = 3 + 4 + 9 = 16 \neq 0$

$\therefore \quad$ The line is parallel to the plane.

(ii) Here, the equation of the line L is $\bar{r} = (1, -2, -2) + k(1, 2, 1),\ k \in R$ and the equation of the plane π is $2x - 3y + 4z = 0$.

$\therefore \quad A(\bar{a}) = (1, -2, -2),\ \bar{l} = (1, 2, 1)$ and $\bar{n} = (2, -3, 4)$

Now, $\bar{l} \cdot \bar{n} = 1(2) + 2(-3) + 1(4) = 2 - 6 + 4 = 0$

$\therefore \quad \bar{l} \perp \bar{n}$. So L is parallel to π or π contains L.

$\bar{a} \cdot \bar{n} = (1, -2, -2) \cdot (2, -3, 4) = 2 + 6 - 8 = 0$

$\therefore \quad$ The plane π contains the line L.

Exercise 7

1. Find the foot of perpendicular from P(1, 0, 3) to the line passing through the points A(4, 7, 1) and B(5, 9, −1). Also find the equation of perpendicular line $\overleftrightarrow{AB}$ through P and perpendicular distance from P to $\overleftrightarrow{AB}$.

2. Find the measure of the angle between two lines, if their direction *cosines l, m, n* satisfy $l + m + n = 0$ and $m^2 + n^2 = l^2$.

3. Prove that the lines $x = 2, \frac{y-1}{3} = \frac{z-2}{1}$ and $x = \frac{y-1}{1} = \frac{z+1}{3}$ are skew. Find the shortest distance between them.

4. Find the point of intersection of the lines $\frac{x+3}{2} = \frac{5-y}{1} = \frac{1-z}{1}$ and $\frac{x+3}{2} = \frac{y-5}{3} = \frac{z-1}{1}$. Also find the measure of the angle between them.

5. Find the equation of the line passing through (1, 2, 3) and perpendicular to both the lines $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{-1}$ and $\frac{x-5}{-3} = \frac{y+8}{1} = \frac{z-5}{5}$.

6. Find the equation of the line equally inclined to the co-ordinate axes and passing through (3, −2, −4).

7. Find the point of intersection of the line $\frac{x-1}{2} = \frac{2-y}{3} = \frac{z+3}{4}$ and the plane $2x + 4y - z = 1$. Also find the measure of the angle between them.

8. Find the equation of the plane parallel to X-axis and whose Y and Z-intercepts are 2 and 3 respectively.

9. Find the image of (1, 5, 1) in the plane $x - 2y + z + 5 = 0$.

10. Find the foot of perpendicular from (0, 2, −2) to the plane $2x - 3y + 4z - 44 = 0$, the equation of this perpendicular and the perpendicular distance between the point and the plane.

11. Find the equation of the plane through the line of intersection of the planes $2x + 3y - z - 4 = 0$ and $x + y + z - 2 = 0$ and through the point (1, 2, 2). Also find the equation of the line of intersection of these planes.

12. If the centroid of the triangle formed by the intersection of a plane with the coordinate axes is (2, 1, −1), find the equation of this plane.

13. Prove that the lines $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z+4}{4}$ and $\frac{x-7}{5} = \frac{y+6}{1} = \frac{z+8}{2}$ intersect each other. Find the equation of the plane containing them.

14. Find the equation of the plane whose intercepts are equal to half of the intercepts of the plane $3x + 4y - 6z = 12$.

15. Find the equation of the perpendicular bisector plane of the line-segment joining the points (1, 2, −3) and (−3, 6, 4).

16. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The equation of the line passing through origin with direction angles $\frac{2\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3}$ is ...... ☐

(a) $x = \frac{y}{-\sqrt{2}} = z$ (b) $\frac{x}{-1} = \frac{y}{-\sqrt{2}} = z$ (c) $x = \frac{y}{-\sqrt{2}} = -z$ (d) $x = \frac{y}{\sqrt{2}} = z$

(2) Line passing through (3, 4, 5) and (4, 5, 6) has direction *cosines* ...... ☐

(a) 1, 1, 1 (b) $\sqrt{3}, \sqrt{3}, \sqrt{3}$ (c) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (d) 7, 9, 11

(3) Lines $\frac{x}{2} = \frac{y}{1} = \frac{z}{3}$ and $\frac{x-2}{2} = \frac{y+1}{1} = \frac{3-z}{-3}$ are ...... lines. ☐

(a) parallel (b) perpendicular

(c) coincident (d) intersecting in an acute angle

(4) Line through origin and parallel to Y-axis is ...... ☐

(a) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ (b) $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$ (c) $\frac{x}{1} = \frac{y}{0} = \frac{z}{1}$ (d) $\frac{x}{1} = \frac{y}{1} = \frac{z}{0}$

(5) The measure of the angle between the lines $x = k + 1$, $y = 2k - 1$, $z = 2k + 3$, $k \in$ R and $\frac{x-1}{2} = \frac{y+1}{1} = \frac{z-1}{-2}$ is ...... ☐

(a) $sin^{-1} \frac{4}{3}$ (b) $cos^{-1} \frac{4}{9}$ (c) $sin^{-1} \frac{\sqrt{5}}{3}$ (d) $\frac{\pi}{2}$

(6) A normal to the plane $x = 2$ is... ☐

(a) (0, 1, 1) (b) (2, 0, 2) (c) (1, 0, 0) (d) (0, 1, 0)

(7) Direction of the line perpendicular to the plane $3x - 4y + 7z = 2$ and passing through (−1, 2, 4) is ..... . ☐

(a) (3, 4, 7) (b) (4, −6, 3) (c) (−3, 4, −7) (d) (−1, 2, 4)

(8) Perpendicular distance of origin from the plane $\overline{r} \cdot (12, -4, 3) = 65$ is ...... ☐

(a) 65 (b) 5 (c) −5 (d) $\frac{5}{13}$

(9) Plane $2x + 3y + 6z - 15 = 0$ makes angle of measure ...... with X-axis. ☐

(a) $cos^{-1} \frac{3\sqrt{5}}{7}$ (b) $sin^{-1} \frac{3}{7}$ (c) $sin^{-1} \frac{2}{\sqrt{7}}$ (d) $tan^{-1} \frac{2}{7}$

(10) Perpendicular distance between the planes $2x - y + 2z = 1$ and $4x - 2y + 4z = 1$ is .... ☐

(a) $\frac{1}{3}$ (b) 3 (c) $\frac{1}{6}$ (d) 6

(11) The plane passing through the points (1, 1, 1), (1, −1, 1) and (−1, 3, −5) will pass through (2, k , 4) for .... ☐

(a) no value of k (b) two values of k

(c) any value of k (d) unique k

(12) If the foot of the perpendicular from the origin to a plane is (a, b, c), the equation of the plane is .... ☐

(a) $ax + by + cz = a + b + c$ (b) $ax + by + cz = abc$

(c) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (d) $ax + by + cz = a^2 + b^2 + c^2$

(13) Equation of the line L passing through A(–2, 2, 3) and perpendicular to $\overleftrightarrow{AB}$ is ...... where coordinates of B are (13, –3, 13). ☐

(a) $\frac{x-2}{3} = \frac{y+2}{13} = \frac{z+3}{2}$ (b) $\frac{x+2}{3} = \frac{y-2}{13} = \frac{z-3}{2}$

(c) $\frac{x+2}{15} = \frac{y-2}{-5} = \frac{z-3}{10}$ (d) $\frac{x-2}{15} = \frac{y+2}{-5} = \frac{z+3}{10}$

(14) If $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$, then $k =$ .... ☐

(a) 7 (b) 6 (c) –7 (d) any value of k

(15) Perpendicular distance of (2, –3, 6) from $3x - 6y + 2z + 10 = 0$ is .... ☐

(a) $\frac{13}{7}$ (b) $\frac{46}{7}$ (c) 7 (d) $\frac{10}{7}$

(16) Line passing through (2, –3, 1) and (3, –4, –5) intersects ZX-plane at .... ☐

(a) (–1, 0, 13) (b) (–1, 0, 19) (c) $\left(\frac{13}{6}, 0, \frac{-19}{6}\right)$ (d) (0, –1, 13)

(17) If lines $\overline{r} = (2, -3, 7) + k(2, a, 5)$, $k \in R$ and $\overline{r} = (1, 2, 3) + k(3, -a, a)$, $k \in R$ are perpendicular to each other, then a .... ☐

(a) 2 (b) –6 (c) 1 (d) –1

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Summary

We have studied the following points in this chapter :

1. Vector equation of the line pasing through A($\overline{a}$) and having the direction of a non-zero vector $\overline{l}$ is $\overline{r} = \overline{a} + k\overline{l}$, $k \in R$

Parametric equations :

$$\left.\begin{aligned} x &= x_1 + kl_1 \\ y &= y_1 + kl_2 \\ z &= z_1 + kl_3 \end{aligned}\right\} \quad k \in R$$

Cartesian equations (symmetric form) : $\frac{x - x_1}{l_1} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$

If $l_1 = 0$ and $l_2 \neq 0$ and $l_3 \neq 0$, then equation is $x = x_1$, $\frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$,

we can write it as $\frac{x - x_1}{0} = \frac{y - y_1}{l_2} = \frac{z - z_1}{l_3}$

2. Equation of a line passing through two distinct points A($\bar{a}$) and B($\bar{b}$) :

Vector equation : $\bar{r} = \bar{a} + k(\bar{b} - \bar{a}),\ k \in R$

Parametric equations :

$$\left.\begin{aligned} x &= x_1 + k(x_2 - x_1) \\ y &= y_1 + k(y_2 - y_1) \\ z &= z_1 + k(z_2 - z_1) \end{aligned}\right\} \quad k \in R$$

Symmetric Form :

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

3. Collinear points : A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are collinear if and only if $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$.

4. If A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) are collinear, $[\bar{a}\ \bar{b}\ \bar{c}] = 0$. But $[\bar{a}\ \bar{b}\ \bar{c}] = 0$ does not assure that points are collinear.

5. The measure of the angle between two lines : $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ are two distinct lines. If α is the measure of angle between the lines,

then $\cos\alpha = \dfrac{|\bar{l} \cdot \bar{m}|}{|\bar{l}||\bar{m}|}$; $0 \le \alpha \le \frac{\pi}{2}$

Lines are perpendicular if and only if $\bar{l} \cdot \bar{m} = 0$.

6. If two distinct lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ intersect in a point, then $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$, $\bar{l} \times \bar{m} \ne \bar{0}$.

It can also be stated as $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$,

where $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{l} = (l_1, l_2, l_3)$ and $\bar{m} = (m_1, m_2, m_3)$.

7. Non-coplanar or skew lines : For two distinct lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$, if $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) \ne 0$, then they are non-coplanar or skew lines.

8. Perpendicular distance of a point P($\bar{p}$) from a line $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ is $\dfrac{|(\bar{p} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$.

9. Perpendicular distance between two parallel lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{l},\ k \in R$ is $\dfrac{|(\bar{b} - \bar{a}) \times \bar{l}|}{|\bar{l}|}$.

10. Perpendicular (shortest) distance between two skew lines $\bar{r} = \bar{a} + k\bar{l},\ k \in R$ and $\bar{r} = \bar{b} + k\bar{m},\ k \in R$ is $\dfrac{|(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m})|}{|\bar{l} \times \bar{m}|}$.

11. Vector equation of the plane passing through three distinct non-collinear points A($\bar{a}$), B($\bar{b}$) and C($\bar{c}$) is $\bar{r} = l\bar{a} + m\bar{b} + n\bar{c}$, where $l, m, n \in R$ and $l + m + n = 1$.

Parametric Form :

$x = lx_1 + mx_2 + nx_3$

$y = ly_1 + my_2 + ny_3$

$z = lz_1 + mz_2 + nz_3$ where $l, m, n \in R$ and $l + m + n = 1$ and the points are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.

Cartesian Form : $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$

12. Four distinct points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $C(x_4, y_4, z_4)$ are coplanar if and only if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix} = 0.$

13. Equation of the plane making intercepts a, b and c on X-axis, Y-axis and Z-axis respectively is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ $(abc \neq 0)$.

14. Equation of the plane passing through $A(\bar{a})$ and having normal $\bar{n}$:

Vector equation : $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$

Cartesian form : If $\bar{r} = (x, y, z)$, $\bar{n} = (a, b, c)$, then the equation is $ax + by + cz = d$, $(d = \bar{a} \cdot \bar{n})$

15. Equation of the plane using normal through the origin : Let $N(\bar{n})$ be the foot of perpendicular from the origin and $|\bar{n}| = p$. Then the equation of the plane is $x\cos\alpha + y\cos\beta + z\cos\gamma = p$ where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction *cosines* of $\bar{n}$.

16. Measure of the angle between the planes $\bar{r} \cdot \bar{n}_1 = d_1$ and $\bar{r} \cdot \bar{n}_2 = d_2$: If θ is the measure of the angle between them, then $\cos\theta = \frac{|\bar{n}_1 \cdot \bar{n}_2|}{|\bar{n}_1||\bar{n}_2|}$; $0 \leq \theta \leq \frac{\pi}{2}$.

Planes are perpendicular if and only if $\bar{n}_1 \cdot \bar{n}_2 = 0$.

17. Equation of the plane passing through two parallel lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{l}$, $k \in R$ is $(\bar{r} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times \bar{l}] = 0$.

Cartesian form :

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & l_2 & l_3 \end{vmatrix} = 0$$

where $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$ and $\bar{l} = (l_1, l_2, l_3)$.

18. Equation of the plane passing through two intersecting lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ is $(\bar{r} - \bar{a}) \cdot (\bar{l} \times \bar{m}) = 0$.

Cartesian form :

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0, \text{ where } \bar{a} = (x_1, y_1, z_1),\ \bar{l} = (l_1, l_2, l_3) \text{ and } \bar{m} = (m_1, m_2, m_3).$$

19. Perpendicular distance from a point P($\overline{p}$) to the plane $\overline{r} \cdot \overline{n} = d$ is $\frac{|\overline{p} \cdot \overline{n} - d|}{|\overline{n}|}$.

Cartesian form :

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

where equation of the plane is $ax + by + cz = d$ and point P is (x_1, y_1, z_1).

20. Perpendicular distance between two parallel planes $\pi_1 : \overline{r} \cdot \overline{n} = d_1$ and $\pi_2 : \overline{r} \cdot \overline{n} = d_2$ is $\frac{|d_1 - d_2|}{|\overline{n}|}$.

21. If the measure of the angle between the line $\overline{r} = \overline{a} + k\overline{l}$, $k \in$ R and the plane $\overline{r} \cdot \overline{n} = d$ is α, then $\alpha = sin^{-1} \frac{|\overline{l} \cdot \overline{n}|}{|\overline{l}||\overline{n}|}$; $0 < \alpha < \frac{\pi}{2}$.

22. Intersection of two planes $\pi_1 : \overline{r} \cdot \overline{n}_1 = d_1$ and $\pi_2 : \overline{r} \cdot \overline{n}_2 = d_2$ is a line given by the equation $\overline{r} = \overline{a} + k\overline{n}$, $k \in$ R where $\overline{n} = \overline{n}_1 \times \overline{n}_2$.

23. Equation of a plane passing through the intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0$.

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Mahavira

Mahavira was a 9th-century Indian mathematician from Gulbarga who asserted that the square root of a negative number did not exist. He gave the sum of a series whose terms are squares of an arithmetical progression and empirical rules for area and perimeter of an ellipse. He was patronised by the great Rashtrakuta king Amoghavarsha. Mahavira was the author of Ganit Saar Sangraha. He separated Astrology from Mathematics. He expounded on the same subjects on which Aryabhata and Brahmagupta contended, but he expressed them more clearly. He is highly respected among Indian Mathematicians, because of his establishment of terminology for concepts such as equilateral, and isosceles triangle; rhombus; circle and semicircle. Mahavira's eminence spread in all South India and his books proved inspirational to other Mathematicians in Southern India.

ANSWERS

Exercise 1.1

1. $15\ cm^3/sec$ **2.** $\frac{2}{3}\pi rh$ **3.** $\frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}$ **4.** $4\ cm^2/sec$ **5.** $3\ cm^2/sec$

6. **(1)** $27\pi\ cm^3/sec$ **(2)** $36\pi\ cm^2/sec$ **7.** $80\pi\ cm^2/sec$

8. **(1)** $1\ cm^2/sec$ **(2)** $1\ cm/sec$ **(3)** $0.5\ cm/sec$ **9.** $4\ cm/sec$ **10.** $\frac{1}{8\pi}\ cm/sec$ **11.** ₹ 21.42

12. ₹ 615 **13.** $2\ m/min$ **14.** $0.1\ cm/sec$ **15.** $0.25\ m^2/sec$ **16.** $\frac{3}{20}\sqrt{\frac{3}{7}}\ m/sec$

17. $12\pi\ cm^2/sec$ **18.** $-36\ units/sec$ **19.** $(1, 1), (-1, -1)$ **20.** $(1, 2)$

Exercise 1.2

7. **(1)** Increasing on R **(2)** Decreasing on R **(3)** Increasing on $(1, \infty)$, Decreasing on $(-\infty, 1)$

(4) Increasing on $\left(-\infty, \frac{3}{2}\right)$, Decreasing on $\left(\frac{3}{2}, \infty\right)$ **(5)** Increasing on R

(6) Decreasing on $(-\infty, -1)$ and $(0, 2)$, Increasing on $(-1, 0)$ and $(2, \infty)$

(7) Increasing on $\left(0, \frac{\pi}{4}\right)$ and Decreasing on $\left(\frac{\pi}{4}, \pi\right)$

(8) Decreasing on $(-\infty, -2)$ and $(-1, \infty)$, Increasing on $(-2, -1)$

(9) Strictly increasing on $(1, 3), (3, \infty)$; Strictly decreasing on $(-\infty, -1), (-1, 1)$

(10) Decreasing **(11)** Increasing **(12)** Decreasing

11. Decreasing in $(-\infty, -2)$ and $(1, 3)$; Increasing in $(-2, 1)$ and in $(3, \infty)$

12. Increasing in $\left(2k\pi, (4k+1)\frac{\pi}{2}\right)$ and $\left((4k+3)\frac{\pi}{2}, (2k+2)\pi\right)$, $k \in Z$

Decreasing in $\left((4k+1)\frac{\pi}{2}, (2k+1)\pi\right)$ and $\left((2k+1)\pi, (4k+3)\frac{\pi}{2}\right)$, $k \in Z$

14. Decreasing in $\left(0, \frac{\pi}{4}\right)$, Increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

15. $a < -2$ **16.** $a \in \left[0, \frac{1}{3}\right)$

21. **(1)** Increasing in $(-\infty, -2)$ and $(6, \infty)$; Decreasing in $(-2, 6)$

(2) Increasing in $(1, \infty)$, Decreasing in $(-\infty, 1)$

(3) Increasing in $\left(-\infty, \frac{4}{3}\right)$, $(2, \infty)$ and Decreasing in $\left(\frac{4}{3}, 2\right)$

(4) Increasing in $(-\infty, 1)$ and $(3, \infty)$ Decreasing in $(1, 3)$

(5) Increasing on R^+ **(6)** Increasing on R^+

(7) Increasing in $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, Decreasing in $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$

(8) Increasing in $((2k-1)\pi, 2k\pi)$, Decreasing in $(2k\pi, (2k+1)\pi)$, $k \in Z$

(9) Increasing in $(0, \infty)$, Decreasing in $(-\infty, 0)$

(10) Decreasing in $(-\infty, -2)$, Increasing in $(-2, \infty)$

(11) Increasing in $(-1, \infty)$, Decreasing in $(-\infty, -1)$

(12) Increasing in $(-\infty, -2)$ and $(0, \infty)$, Decreasing in $(-2, 0)$

(13) Increasing in $(0, e^2)$, Decreasing in (e^2, ∞)

(14) Increasing in $\left(\frac{1}{e}, \infty\right)$, Decreasing in $\left(0, \frac{1}{e}\right)$

Exercise 1.3

1. $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ **2.** $yy_1 = 2a(x + x_1)$ **3.** 17 **4.** -1 **5.** $y = 4x + 1$ **6.** $2x + 4y = 9$

11. (1) $y = 1$ (2) $\left(2n\pi + \frac{\pi}{2}, 1\right)$, $n \in Z - \{0\}$ **12.** $x + y = \sqrt{2}$

13. (1) $y = 0$ at $(0, 0)$ (2) $y = 2x$ at $(1, 2)$ and $(-1, -2)$ **14.** $a = 2$, $b = -7$

15. $a = 5$, $b = -4$ **16.** $x\cos\frac{\theta}{2} - y\sin\frac{\theta}{2} = a\,\theta\cos\frac{\theta}{2} - 2a\sin\frac{\theta}{2}$ **18.** $(-1, -2)$

19. $a = 2$, $b = -1$ **20.** $x + y = 6$, horizontal at $(0, 0)$, $\left(2^{\frac{4}{3}}, 2^{\frac{5}{3}}\right)$, vertical at $(0, 0)$, $\left(2^{\frac{5}{3}}, 2^{\frac{4}{3}}\right)$

22. (1) $5x + 4y + 16 = 0$ (2) $x - \sqrt{2}y + 9 = 0$ (3) $x - y = 0$

(4) $9x - 2y - 5 = 0$ (5) $9x + 13y - 40 = 0$

23. $(1, 1)$, $(-1, -1)$ **24.** (1) $\tan^{-1}\frac{4}{3}$ (2) $\tan^{-1} 2$ at $(2, 1)$ and $(2, -1)$

25. $x + 2y = (4k + 1)\frac{\pi}{2}$, $k \in Z$ **26.** $x + y = 3$, $x + y + 1 = 0$ **28.** $a = -\frac{1}{2}$, $b = -\frac{3}{4}$, $c = 3$

Exercise 1.4

1. $\frac{73}{120}$ or 0.6083 **2.** 0.9999 **3.** $\frac{323}{108}$ or 2.9907 **4.** $\frac{1023}{256}$ or 3.9961 **5.** 19.975

6. 2.00125 **7.** $\frac{\sqrt{3}}{2} + \frac{\pi}{360}$ **8.** $\frac{\sqrt{3}}{2} + \frac{\pi}{360}$ **9.** $\frac{1}{\sqrt{3}} + \frac{\pi}{135}$ **10.** 4.6062

11. 1.0004343 **12.** 2.003125 **13.** $\frac{\pi}{2}\ cm^3$ **14.** $4\pi r^2\,\Delta r$ **15.** 0.5 %

16. $\frac{\sqrt{3}\pi x}{6}$ % **17.** 1.12 **18.** 4.05 **19.** 60 cm^3 **20.** $5.184\pi\ cm^2$

21. $\frac{1}{2} + \frac{\sqrt{3}\pi}{72}$

Exercise 1.5

1. Local minimum at $x = \frac{1}{3}$, $f\left(\frac{1}{3}\right) = \frac{122}{27}$; Local maximum at $x = 3$, $f(3) = 14$

2. Local minimum at $x = -\sqrt{3}$, $f(\sqrt{3}) = f(-\sqrt{3}) = -9$

Local minimum at $x = \sqrt{3}$

Local maximum at $x = 0$, $f(0) = 0$

3. No extreme value. f is increasing on R^+

4. Local minimum at $x = (2n + 1)\pi$, $f((2n + 1)\pi) = -2$

Local maximum at $x = 2n\pi$, $f(2n\pi) = 2$

5. Local and global minimum at $x = 0$, $f(0) = 0$

6. Local and global maximum at $x = 1$, $f(1) = \frac{1}{e}$

Global minimum at $x = 0$, $f(0) = 0$

7. Local and global maximum at $x = e$, $f(e) = \frac{1}{e}$

Global minimum at $x = 1$, $f(1) = 0$

8. Local and global maximum at $x = 0$, $f(0) = 4$

Global minimum at $x = \pm 4$, $f(\pm 4) = 0$

9. Global minimum $f(1) = \frac{1}{2}$; Global maximum $f(2) = \frac{2}{3}$; f is $\uparrow$. No local minimum or local maximum.

10. Local and global maximum at $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \sqrt{2}$

Local and global minimum at $x = \frac{5\pi}{4}$, $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$

11. Local and global maximum at $x = \frac{11\pi}{6}$, $f\left(\frac{11\pi}{6}\right) = \frac{1}{\sqrt{3}}$

Local and global minimum at $x = \frac{7\pi}{6}$, $f\left(\frac{7\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

12. Local maximum at $x = \frac{2}{3}$, $f\left(\frac{2}{3}\right) = \frac{2}{3^{\frac{3}{2}}}$

13. Local and global minimum at $x = 2$, $f(2) = 61$

Global maximum at $x = 0$, $f(0) = 125$

14. Local and global maximum at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 1$

Local and global minimum at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$, $f\left(\frac{3\pi}{4}\right) = f\left(\frac{7\pi}{4}\right) = -1$

15. Local and global minimum at $x = 2$, $f(2) = 75$

Global maximum at $x = 3$, $f(3) = 89$

16. Length $(l) = \frac{20}{\pi + 4}$ m, Breadth $(b) = \frac{10}{\pi + 4}$ m **18.** 8, 8 **19.** $x = 10$, $y = 25$

22. Minimum distance 10 for P(4, 3) and Maximum distance 20 for Q(−4, −3)

24. Length = Breadth = 2 m, Height = 1 m, Minimum surface = 12 m^2

25. $a = 0$, $b = -1$, $c = 2$ **26.** 25 cm^2

Exercise 1

1. $\frac{5}{2\sqrt{3\pi}}$ *cm/sec* **2.** 15 *m/sec* **3.** −3 *cm/min*

4. Increasing in $(-\infty, -2)$ and $(3, \infty)$, Decreasing in $(-2, 3)$

5. (1) $(1, 3), (3, \infty)$ (2) $(-\infty, -1), (-1, 1)$ **7.** Increasing in $(-2, \infty)$, Decreasing in $(-\infty, -2)$

8. Decreasing in $(-\infty, 0)$ and $(2, \infty)$, Increasing in $(0, 2)$

10. $\frac{x}{a} + \frac{y}{b} = 1$ **11.** $\frac{\pi}{2}$ at $(0, 0)$, $\tan^{-1}\frac{3}{4}$ at $(4a, 4a)$ **13.** $(-1, 2)$, $(1, -2)$

14. Local and global minimum at $x = \frac{\pi}{3}$, $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3}$

Local and global maximum at $x = \frac{5\pi}{3}$, $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3}$

15. Global minimum at $x = 0$, $f(0) = 0$

16. Local minimum at $x = 1$, $f(1) = 3$

17. Increasing in $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, Decreasing in $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ and $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$.

Local minimum $f\left(\frac{-\pi}{3}\right) = \frac{-4\pi}{3} + \sqrt{3}$, Local maximum $f\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$

18. Increasing in $\left(0, \frac{3}{4}\right)$, Decreasing in $\left(\frac{3}{4}, 1\right)$.

Local maximum at $x = \frac{3}{4}$, $f\left(\frac{3}{4}\right) = \frac{5}{4}$

19. Critical numbers 0, 4 and 6. Increasing in (0, 4) and Decreasing in (4, 6).

Local and global maximum at $x = 4$, $f(4) = 2^{\frac{5}{3}}$, Global minimum at $x = 0$ and 6, $f(0) = f(6) = 0$

20. Local and global minimum at $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$

Global maximum at $x = 0, \frac{\pi}{2}$, $f(0) = f\left(\frac{\pi}{2}\right) = 1$

27. (1) $\tan^{-1}\frac{3}{11}$ (2) $\tan^{-1}\frac{9}{2}$ (3) $\tan^{-1}\frac{1}{2}$ (4) $\tan^{-1}\left(\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}}+b^{\frac{2}{3}})}\right)$ at $(4a^{\frac{1}{3}}b^{\frac{2}{3}}, 4a^{\frac{2}{3}}b^{\frac{1}{3}})$ and $\frac{\pi}{2}$ at (0, 0)

(5) $\tan^{-1}\frac{9}{13}$ (6) $\tan^{-1}\frac{1}{2}$ at (1, 1) and (1, −1) and touch each other at (0, 0)

29. (1) (b) (2) (d) (3) (a) (4) (b) (5) (d) (6) (a) (7) (c) (8) (b) (9) (d) (10) (b)
(11) (a) (12) (c) (13) (b) (14) (d) (15) (d) (16) (b) (17) (b) (18) (a) (19) (c) (20) (d)
(21) (d) (22) (c) (23) (a) (24) (c) (25) (d) (26) (a) (27) (c) (28) (a) (29) (b) (30) (c)
(31) (b) (32) (b) (33) (b) (34) (d) (35) (d) (36) (c) (37) (a) (38) (a) (39) (b) (40) (c)
(41) (d) (42) (b) (43) (b) (44) (a) (45) (a) (46) (a) (47) (a) (48) (b) (49) (a) (50) (b)
(51) (b) (52) (a) (53) (b) (54) (a) (55) (b)

Exercise 2.1

1. $\frac{x^3}{3}\log x - \frac{1}{9}x^3 + c$

2. $\left(\frac{3+5x}{7}\right)\sin 7x + \frac{5}{49}\cos 7x + c$

3. $x\cos^{-1}x - \sqrt{1-x^2} + c$

4. $e^{3x}\left[\frac{x^2}{3} - \frac{2}{9}x + \frac{2}{27}\right] + c$

5. $\frac{x^3}{3}\tan^{-1}x - \frac{1}{6}[x^2 - \log(1+x^2)] + c$

6. $x\,\mathrm{cosec}^{-1}x + \log\left|x + \sqrt{x^2-1}\right| + c$

7. $\frac{x}{2}[\sin(\log x) - \cos(\log x)] + c$

8. $\frac{1}{2}\left[\sec x\tan x + \log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|\right] + c$

9. $-x\cot\frac{x}{2} + 2\log\left|\sin\frac{x}{2}\right| + c$

10. $-\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2 + c$

11. $2x\tan^{-1}x - \log(1+x^2) + c$

12. $-\frac{1}{2}(x\,\mathrm{cosec}^2x + \cot x) + c$

13. $\frac{3}{4}(x\sin x + \cos x) + \frac{1}{12}\left(x\sin 3x + \frac{1}{3}\cos 3x\right) + c$

14. $\frac{1}{n}(x^n\sin x^n + \cos x^n) + c$

15. $\left(x - \frac{x^3}{3}\right)\log x - x + \frac{x^3}{9} + c$

16. $-\frac{\log x}{x+1} + \log\left(\frac{x}{x+1}\right) + c$

17. $-\frac{\sin^{-1}x}{x} + \log\left|\frac{1-\sqrt{1-x^2}}{x}\right| + c$

18. $2(\sqrt{x} - \sqrt{1-x}\,\sin^{-1}\sqrt{x}) + c$

Exercise 2.2

1. $\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3} + c$

2. $\sqrt{2}\left[\frac{x}{2}\sqrt{x^2+5} + \frac{5}{2}\log\left|x + \sqrt{x^2+5}\right|\right] + c$

3. $\frac{x}{2}\sqrt{5x^2-3} - \frac{3}{2\sqrt{5}}\log\left|\sqrt{5}x + \sqrt{5x^2-3}\right| + c$

4. $\frac{4x+3}{8}\sqrt{4-3x-2x^2} + \frac{41}{16\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{\sqrt{41}}\right) + c$

5. $\frac{1}{2}\left[\frac{2x+1}{2}\sqrt{4x^2+4x-15} - 8\log\left|2x+1+\sqrt{4x^2+4x-15}\right|\right] + c$

6. $\frac{1}{3}\left[\frac{x^3}{2}\sqrt{8-x^6} + 4\sin^{-1}\frac{x^3}{2\sqrt{2}}\right] + c$

7. $\frac{\sin x}{2}\sqrt{4-\sin^2 x} + 2\sin^{-1}\left(\frac{\sin x}{2}\right) + c$

8. $e^x \log \sin x + c$

9. $-e^x \cot\frac{x}{2} + c$

10. $\frac{e^{2x}}{2}\tan x + c$

11. $e^x\left(\frac{x-2}{x+2}\right) + c$

12. $\frac{e^x}{\sqrt{x^2+1}} + c$

13. $\frac{e^x}{1+x^2} + c$

14. $-\frac{1}{3}(1 + x - x^2)^{\frac{3}{2}} + \frac{1}{8}(2x-1)\sqrt{1+x-x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$

15. $(x^2 + x + 1)^{\frac{3}{2}} - \frac{7(2x+1)}{8}\sqrt{x^2+x+1} - \frac{21}{16}\log\left|x + \frac{1}{2} + \sqrt{x^2+x+1}\right| + c$

16. $-\frac{2}{3}(2 + 3x - x^2)^{\frac{3}{2}} - \frac{2x-3}{2}\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$

17. $\frac{e^{2x}}{10}(\sin 4x - 2\cos 4x) + c$

18. $-e^{-\frac{x}{2}} + \frac{e^{-\frac{x}{2}}}{17}(-\cos 2x + 4\sin 2x) + c$

19. $\frac{3^x}{2\log 3} - \frac{3^x}{2(4+(\log 3)^2)}((\log 3)\cos 2x + 2\sin 2x) + c$

20. $\frac{e^{2x}}{8}(\sin 2x + \cos 2x) - \frac{e^{2x}}{20}(\cos 4x + 2\sin 4x) + c$

Exercise 2.3

1. $\log\left|\frac{x(x-1)^2}{(x+1)^2}\right| + c$

2. $\frac{5}{2}\log|x-1| - 8\log|x-2| + \frac{11}{2}\log|x-3| + c$

3. $\frac{x^2}{2} - x - 2\log|x-2| + \log|x-3| + c$

4. $\frac{1}{3\sqrt{2}}\ tan^{-1}(\sqrt{2}\,x) + \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + c$

5. $\frac{1}{3\sqrt{2}}\ tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{3\sqrt{2}}\ tan^{-1}(\sqrt{2}\,x) + c$

6. $\frac{5}{6}\log(x^2+5) - \frac{1}{3}\log(x^2+2) + c$

7. $-2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + c$

8. $-\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\ tan^{-1}\left(\frac{x}{3}\right) + c$

9. $x + 2\log|2e^x+1| - 3\log|3e^x+1| + c$

10. $\frac{1}{2}\log\left|\frac{tan\theta-3}{tan\theta-1}\right| + c$

11. $\frac{1}{2}\log|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\log(x^2+1) + c$

12. $-\frac{1}{8}\log|x+1| + \frac{1}{8}\log|x-1| - \frac{3}{4(x-1)} - \frac{1}{4(x-1)^2} + c$

13. $-\frac{1}{2}\log|1-cosx| - \frac{1}{6}\log|1+cosx| + \frac{2}{3}\log|1-2cosx| + c$

14. $\frac{1}{10}\log|1-cosx| - \frac{1}{2}\log|1+cosx| + \frac{2}{5}\log|3+2cosx| + c$

Exercise 2

1. $\frac{x^3}{3}\ sin^{-1}x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{\frac{3}{2}} + c$

2. $\frac{x}{2}\ cos^{-1}x - \frac{1}{2}\sqrt{1-x^2} + c$

3. $-x\ cot\frac{x}{2} + c$

4. $\frac{1}{2}\log\left|\frac{1+\sqrt{sinx}}{1-\sqrt{sinx}}\right| - tan^{-1}(\sqrt{sinx}) + c$

5. $x\log\left|x+\sqrt{x^2+a^2}\right| - \sqrt{x^2+a^2} + c$

6. $(x+a)\ tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + c$

7. $2\sqrt{x} - 2\sqrt{1-x}\ sin^{-1}\sqrt{x} + c$

8. $\frac{1}{2}\ e^x\ sec\,x + c$

9. $\frac{x}{\log x} + c$

10. $x\log(\log x) + c$

11. $-\frac{1}{3}(2ax-x^2)^{\frac{3}{2}} + \frac{a(x-a)}{2}\sqrt{2ax-x^2} + \frac{a^3}{2}\ sin^{-1}\left(\frac{x-a}{a}\right) + c$

12. $\frac{1}{3}(x^2+x)^{\frac{3}{2}} - \frac{11}{8}(2x+1)\sqrt{x^2+x} + \frac{11}{16}\log\left|x+\frac{1}{2}+\sqrt{x^2+x}\right| + c$

13. $\frac{1}{2}\log\left|\frac{sinx-1}{sinx+1}\right| - \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}\ sinx-1}{\sqrt{2}\ sinx+1}\right| + c$

14. $\frac{1}{6}\log|1-cosx| + \frac{1}{2}\log|1+cosx| - \frac{2}{3}\log|2cosx+1| + c$

15. $\frac{1}{8}\log\left|\frac{sinx-1}{sinx+1}\right| - \frac{1}{4\sqrt{2}}\log\left|\frac{\sqrt{2}\ sin\,x-1}{\sqrt{2}\ sin\,x+1}\right| + c$

16. $x\ tan^{-1}x + x\ tan^{-1}(1-x) + \frac{1}{2}\log|x^2-2x+2| + tan^{-1}(x-1) - \frac{1}{2}\log(x^2+1) + c$

17. $\frac{2}{\sqrt{\cos x}} - \frac{1}{2}\log\left|\frac{\sqrt{\cos x}+1}{\sqrt{\cos x}-1}\right| + \tan^{-1}(\sqrt{\cos x}) + c$

18. $\frac{1}{4}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{2(1+\sin x)} + c$

19. $\frac{1}{2}\log\left|\tan\frac{x}{2}\right| + \frac{1}{4}\sec^2\frac{x}{2} + \tan\frac{x}{2} + c$

20. (1) (a) (2) (b) (3) (c) (4) (a) (5) (c) (6) (c) (7) (a) (8) (d) (9) (b) (10) (b) (11) (a)

Exercise 3.1

1. 8
2. 10
3. $\frac{94}{3}$
4. $\frac{38}{3}$
5. $e - e^{-1}$
6. $\frac{1}{3}(e^2 - e^{-1})$
7. $6\log_3 e$
8. 3
9. $e^2 - 3$
10. $2\log_a e$
11. 26
12. $\sin b - \sin a$
13. 2
14. 1
15. 20

Exercise 3.2

1. $\frac{1}{3}\cdot 2^{\frac{5}{2}}$
2. $\left(1-\frac{\pi}{4}\right)$
3. $\frac{\pi}{4}$
4. $\frac{1}{2}\log 2$
5. $\sqrt{2}$
6. $\sqrt{2} - 1$
7. $\frac{\pi}{2}$
8. $\frac{1}{5}\log 6$
9. $\frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$
10. $2 - \frac{\pi}{2}$
11. $\frac{\pi}{3\sqrt{3}}$
12. $6 - 4\log 2$
13. $\frac{\pi}{6}$
14. $\tan^{-1}e - \frac{\pi}{4}$
15. $\frac{\pi}{4} - \frac{1}{2}\log 2$
16. $\frac{\pi}{3}$
17. $-\frac{\pi}{4}$
18. $\frac{1}{2} - \frac{\sqrt{3}\pi}{12}$
19. $\tan^{-1}\frac{1}{3}$
20. $\frac{1}{\sqrt{10}}\tan^{-1}\sqrt{\frac{2}{5}}$
21. $\frac{\pi}{2} - 1$
22. $\frac{\pi}{2} - 1$
23. $\frac{1}{2}\log\left(\frac{32}{27}\right)$
24. $\frac{1}{2}(\sqrt{2} - 1) + \frac{1}{2}\log(\sqrt{2} + 1)$
25. $\frac{1}{2}\log\frac{8}{5}$
26. $\frac{1}{4}\log 2 - \frac{\pi}{8} + \frac{1}{4}$
27. $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$
28. $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$
29. 4
30. 47
31. $e^4 + 5 - \frac{\pi}{2}$
32. $\frac{13}{10}$
33. 4
34. 0
35. 0
36. 2
37. $\frac{1}{2}$
38. $\frac{9}{2}$

Exercise 3.3

1. (1) 0 (2) 0 (3) 0 (4) 0 (5) $\frac{\pi}{2}$ (6) 2
2. (1) 0 (2) 0

Exercise 3

3. **(1)** $\frac{\pi^2}{4}$ **(2)** π **5.** -64 **8.** $\frac{1}{2}(1 - \log 2)$ **9.** $\frac{1}{2ab} \log \left|\frac{a+b}{a-b}\right|$

10. $\frac{1}{\sqrt{2}}\, tan^{-1}\frac{3}{2\sqrt{2}}$ **11.** 0 **12.** $\frac{\pi}{6\sqrt{3}} - \frac{1}{2} \log \frac{3}{2}$ **13.** $\frac{\pi}{4} - \frac{1}{2} \log 2$ **14.** $\frac{\pi}{8} \log 2$ **15.** $\frac{2}{3} + \log\left(\frac{2}{3}\right)$

16. $\frac{\pi^2}{4}$ **17.** $2(\sqrt{2} - 1)$ **18.** $\frac{38}{3}$ **19.** $\frac{15+e^8}{2}$

22. **(1)** (c) **(2)** (a) **(3)** (a) **(4)** (c) **(5)** (a) **(6)** (b) **(7)** (c) **(8)** (a) **(9)** (b) **(10)** (b)
(11) (a) **(12)** (a) **(13)** (b) **(14)** (b) **(15)** (d) **(16)** (b) **(17)** (d) **(18)** (a) **(19)** (b) **(20)** (c)

Exercise 4.1

1. $\frac{13}{3}$ **2.** 9 **3.** 3 **4.** $\frac{136}{3}$ **5.** $\frac{32}{3}$ **6.** 36 **7.** πa^2 **8.** $\frac{32}{3}$

Exercise 4.2

1. 27 **2.** $\frac{9}{2}$ **3.** $\frac{4}{\pi}$ **4.** $\frac{64}{3}$ **5.** $\frac{5}{6}$ **6.** $\frac{32}{3}$ **7.** $\frac{19}{6}$ **8.** $\frac{64}{3}$ **9.** 8 **10.** $\frac{15}{2}$

11. 4π **12.** $\frac{20}{3}(\sqrt{5} - 2)$

Exercise 4

1. $\frac{125}{6}$ **2.** $\frac{2}{3}$ **3.** $\frac{1}{6}$ **4.** $\frac{\pi}{4}$ **5.** $\frac{8}{3}$ **6.** $\frac{9}{8}$ **7.** $\frac{4}{3}(8 + 3\pi)$ **8.** $\frac{13}{3}$ **10.** $\frac{23}{6}$

11. $\frac{8\pi}{3} - 2\sqrt{3}$ **12.** $\frac{32}{3}$ **13.** 2 **14.** $\frac{5\pi}{4} - \frac{1}{2}$ **15.** $\frac{9}{2}$ **16.** $\frac{4}{3}$

17. **(1)** (c) **(2)** (d) **(3)** (c) **(4)** (b) **(5)** (c) **(6)** (c) **(7)** (b) **(8)** (d) **(9)** (c) **(10)** (b)
(11) (a) **(12)** (b) **(13)** (d) **(14)** (a) **(15)** (b) **(16)** (d) **(17)** (d) **(18)** (c) **(19)** (a) **(20)** (b)

Exercise 5.1

1.

Sr. No.	Order	Degree
1	2	1
2	1	4
3	2	Undefined
4	1	1
5	3	2
6	2	2
7	1	2
8	3	2
9	2	1
10	2	3

Exercise 5.2

1. $(x-y)^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=\left(x+y\frac{dy}{dx}\right)^2$

2. $\frac{d^2y}{dx^2}=0$ **3.** $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$ **4.** $(x^2-y^2)\frac{dy}{dx}=2xy$

9. (1) $\frac{d^2y}{dx^2}=0$ (2) $x\left(y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right)=y\frac{dy}{dx}$ (3) $2\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$

(4) $x^2y_2+xy_1-y=0$ (5) $x\frac{dy}{dx}=3y$ (6) $y_2=4(y_1-y)$ (7) $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=y\frac{dy}{dx}$

Exercise 5.3

1. (1) $2y^3+3y^2=3x^2+6\log|x|+c$

(2) $e^y=(y+1)(e^x+1)c$

(3) $\sin y = c\cos x$

(4) $\log|y|=\log|\sec x|-\tan x+c$

(5) $(e^y+1)\sin x=c$

(6) $\tan^{-1}y=x+\frac{x^3}{3}+c$

(7) $x=c\log y$

(8) $\frac{1}{y}=2x^2+1$

(9) $y=x^2+\log x$

(10) $8e^y=x(y+2)^2$

(11) $4e^x+\frac{1}{y^2}=8$

(12) $x\sec y=2$

(13) $y=(x+1)\log(x+1)-x+3$

(14) $y=\sin^{-1}a\cdot x+1$

(15) $y=\sec x$

(16) $y=\frac{e^x}{x+1}+c$

2. (1) $\tan(x+y)-\sec(x+y)=x+c$

(2) $c(x-y+2)=e^{2y-x}$

(3) $(x+y+2)c=e^{y+1}$

(4) $c(e^{x+y}+1)=e^y$

(5) $y-a\tan^{-1}\left(\frac{x+y}{a}\right)=c$

Exercise 5.4

1. (1) $(x-y)^2=cx\,e^{-\frac{y}{x}}$

(2) $\sec\frac{y}{x}=xyc$

(3) $x\tan\frac{y}{2x}=c$

(4) $e^{\frac{x}{y}}=y+c$

(5) $-\cos\frac{y}{x}=\log x+c$

(6) $2e^{\frac{x}{y}}=\log\frac{c}{y}$

(7) $\frac{\sqrt{2}y+x}{\sqrt{2}y-x}=cx^{2\sqrt{2}}$

(8) $ye^{\frac{x}{y}}+x=c$

(9) $e^{\frac{y}{x}}=xc$

(10) $y\left(\log\frac{y}{x}-1\right)=c$

(11) $-e^{-\frac{y}{x}}=\log xc$

(12) $yx^2=c(y+2x)$

(13) $\sin\frac{y}{x}=xc$

2. (1) $x^2(x^2 + 2y^2) = 3$ (2) $e^{-\frac{y}{x}} = \log x$

(3) $e^{\cos\frac{y}{x} - 1} = x$ (4) $xe^{\frac{y^2}{x^2}} = e$

(5) $x = e^{1 - \frac{2x}{y}}$ (6) $e^{\frac{y}{x+y}} = x$

Exercise 5.5

1. $y = \frac{1}{5}[2\sin x - \cos x] + ce^{-2x}$
2. $y = -e^{-x} + cx$
3. $\frac{y}{x} = \log x + c$
4. $\frac{y}{1+x^2} = x + c$
5. $y + x + 1 = ce^x$
6. $yx^2 = e^x(x^2 - 2x + 2) + c$
7. $y = -\frac{5}{4}e^{-3x} + ce^{-2x}$
8. $(1 + x^2)y = \frac{4x^3}{3} + c$
9. $xe^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$
10. $y\log x = -\frac{2}{x}(1 + \log x) + c$
11. $y = (\cot x + 1) + ce^{\cot x}$
12. $\frac{x}{y} = 2y + c$

Exercise 5.6

1. $y = ce^{-\frac{x}{4y}}$
2. 16 times, 3000
3. $x^2 = -\frac{9}{4}y$
4. 14 years, 6.9 %
5. $m_0 = 125\ g$
6. $y^2 = 2kx$, (k is arbitrary constant)
7. $y^2 - x^2 = 3$

Exercise 5

5. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx}$

6. (1) $1 + \tan\left(\frac{x+y}{2}\right) = ce^x$ (2) $y(x^2 + 1)^2 = \tan^{-1}x + c$

(3) $2e^{\frac{x}{y}} = \log\left|\frac{c}{y}\right|, y \neq 0$ (4) $x^2(x^2 - 2y^2) = c$

(5) $x^2 + y^2 = 2x$ (6) $y = \tan x - 1 + ce^{-\tan x}$

7. (1) (b) (2) (a) (3) (b) (4) (c) (5) (b) (6) (c) (7) (c) (8) (b) (9) (d) (10) (a) (11) (b) (12) (c) (13) (d) (14) (a) (15) (a) (16) (b)

Exercise 6.1

1. **(1)** 4 **(2)** 5 **(3)** $3\sqrt{2}$

2. $\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$

3. (6, −4, −4)

4. $12\hat{i} - 8\sqrt{3}\hat{j} + 8\hat{k}$

5. $\frac{7}{\sqrt{110}}\hat{i} + \frac{6}{\sqrt{110}}\hat{j} - \frac{5}{\sqrt{110}}\hat{k}$

6. Scalar components are 3, −6, 7; Vector components are $3\hat{i}, -6\hat{j}, 7\hat{k}$

7. **(i)** 5 **(ii)** 5 **(iii)** $5\sqrt{2}$

Exercise 6.2

1. 5 **2.** (−7, 3, 5) **3.** (−6, −24, 6) **4.** $8\sqrt{3}$

5. 3 **6.** (−5, 5, 0) **7.** −2 **8.** 0

9. (11, −11, 11) **10.** $7\sqrt{6}$

Exercise 6

6. $a = 1, b = -1, c = 2$

9. **(1)** $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}; \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$ **(2)** $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}; 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

(3) $\cos^{-1}\frac{1}{17}; \cos^{-1}\frac{12}{85}; \cos^{-1}\frac{84}{85}; \frac{1}{17}, \frac{12}{85}, \frac{84}{85}$

11. $\left(\frac{12}{13}, \frac{5}{13}\right)$ or $\left(-\frac{12}{13}, -\frac{5}{13}\right)$ **16.** $\pm 2\sqrt{3}$ **18.** $2\sqrt{91}$ **24.** $\frac{7\sqrt{6}}{2}$

25. (2, −2, 2), $2\sqrt{3}$ **26.** $b\hat{j}; |b|$ **27.** $\left(\frac{56}{99}, \frac{-56}{99}, \frac{8}{99}\right)$

30. $\left(\frac{8}{3}, \frac{5}{3}, \frac{4}{3}\right) + \left(\frac{-2}{3}, \frac{4}{3}, \frac{-1}{3}\right) = (2, 3, 1)$ **31.** $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right)$ **33.** $\left(\frac{5}{3}, \frac{2}{3}, \frac{2}{3}\right)$ **34.** 10

36. **(1)** (b) **(2)** (d) **(3)** (c) **(4)** (b) **(5)** (c) **(6)** (b) **(7)** (b) **(8)** (d) **(9)** (a) **(10)** (a)
(11) (a) **(12)** (c) **(13)** (c) **(14)** (a) **(15)** (c) **(16)** (b) **(17)** (b) **(18)** (a) **(19)** (c) **(20)** (d)
(21) (c) **(22)** (d) **(23)** (a) **(24)** (b) **(25)** (c)

Exercise 7.1

1. $\bar{r} = (2, -1, 3) + k(2, -3, 4), k \in R; \frac{x-2}{2} = \frac{y+1}{-3} = \frac{z-3}{4}$

2. $\frac{x-2}{2} = \frac{z+9}{4}, y - 3 = 0; \bar{r} = (2, 3, -9) + k(2, 0, 4), k \in R$

3. Non-collinear **4.** $\frac{4}{\sqrt{26}}, \frac{-3}{\sqrt{26}}, \frac{1}{\sqrt{26}}$

5. $\bar{r} = (1, -2, 1) + k\left(\frac{1}{3}, \frac{-1}{2}, 1\right), k \in R; \frac{3(x-1)}{1} = \frac{2(y+2)}{-1} = \frac{z-1}{1}$

6. (4, 0, −1) 7. $\cos^{-1}\frac{17}{5\sqrt{14}}$ 9. (1) Skew (2) Parallel (3) Skew (4) Intersecting (5) Parallel

10. $\frac{107}{\sqrt{1038}}$ 11. $\frac{\sqrt{457}}{5}$ 12. $\sqrt{\frac{118}{3}}$

Exercise 7.2

1. $\frac{1}{\sqrt{21}}(4, -2, 1)$ 2. $\bar{r} \cdot (2, 2, -1) = 5;\ 2x + 2y - z = 5$ 3. $2x - 3y - 5z = 11$

4. $x + 2y - 3 = 0;\ 3, \frac{3}{2}$, not defined 5. $6x - y - 5z = 7$ 6. $13x - 7y - 37z + 134 = 0$

7. $x - y + 1 = 0$ 8. $\frac{\pi}{3}$ 9. $\sin^{-1}\left(\frac{5}{\sqrt{238}}\right)$ 10. $\frac{12}{\sqrt{38}}$ 11. $\frac{19}{14}$

12. $2x - 5y - 4z + 33 = 0$ 13. $55x - 2y + 13z = 40$ 14. $x - y - z - 1 = 0$

Exercise 7

1. $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$, $\bar{r} = (1, 0, 3) + k(2, 7, 8),\ k \in R;\ \sqrt{13}$ 2. $\frac{\pi}{3}$ 3. $\frac{7}{\sqrt{74}}$ 4. $(-3, 5, 1), \frac{\pi}{2}$

5. $\frac{x-1}{11} = \frac{y-2}{-2} = \frac{z-3}{7}$ 6. $\frac{x-3}{1} = \frac{y+2}{1} = \frac{z+4}{1}$ 7. $(3, -1, 1);\ \sin^{-1}\frac{12}{\sqrt{609}}$

8. $\frac{y}{2} + \frac{z}{3} = 1$ 9. (2, 3, 2) 10. $(4, -4, 6);\ \frac{x}{2} = \frac{y-2}{-3} = \frac{z+2}{4};\ 2\sqrt{29}$

11. $4x + 7y - 5z - 8 = 0;\ \frac{x-2}{4} = \frac{y}{-3} = \frac{z}{-1}$ 12. $x + 2y - 2z = 6$ 13. $2x + 16y - 13z - 22 = 0$

14. $3x + 4y - 6z = 6$ 15. $8x - 8y - 14z = -47$

16. (1) (c) (2) (c) (3) (a) (4) (b) (5) (d) (6) (c) (7) (c) (8) (b) (9) (a) (10) (c) (11) (c) (12) (d) (13) (b) (14) (a) (15) (b) (16) (b) (17) (d)

TERMINOLOGY

(In Gujarati)

Approximate Value	આસન્ન મૂલ્ય
Box Product	પેટીગુણન
Coincident	સંપાતી
Collinear Vectors	સમરેખ સદિશો
Component	ઘટક
Coplanar Vectors	સમતલીય સદિશો
Coplanar	સમતલીય
Definite Integration	નિયત સંકલન
Degree	પરિમાણ
Dependent Variable	અવલંબી ચલ
Differential Equation	વિકલ સમીકરણ
Direction Angles	દિક્‌ખૂણા
Direction Cosines	દિક્‌કોસાઈન
Direction of Line	રેખાની દિશા
Direction Ratios	દિક્‌ગુણોત્તર (દિક્‌ સંખ્યાઓ)
Error	ત્રુટિ
Free Vector	મુક્ત સદિશ
Global	વૈશ્વિક
Having same Direction	સમદિશ
Homogeneous	સમપરિમાણ
Improper Rational Function	અનુચિત સંમેય વિધેય
Independent Variable	સ્વતંત્ર ચલ
Initial Condition	પ્રારંભિક શરત
Inner Product	અંતઃ ગુણન
Integrating Factor (I.F.)	સંકલ્યકારક અવયવ
Integration by Parts	ખંડશઃ સંકલન
Linear Combination	સુરેખ સંયોજન
Linear Differential Equation	સુરેખ વિકલ સમીકરણ
Lower Limit	અધઃસીમા
Monotonic	એકસૂત્રી

Normal	અભિલંબ
Opposite Direction	વિરુદ્ધ દિશા
Order	કક્ષા
Outer Product of Vectors	સદિશોનું બહિર્ગુણન
Parallelopiped	સમાંતર ફલક
Particular Solution	વિશિષ્ટ ઉકેલ
Perpendicular Bisector Plane	લંબદ્વિભાજક સમતલ
Projection Vector	પ્રક્ષેપ સદિશ
Proper Rational Function	ઉચિત સંમેય વિધેય
Rate	દર
Scalar Product	અદિશ ગુણાકાર
Singular Solution	અસામાન્ય ઉકેલ
Skew Lines	વિષમતલીય રેખાઓ
Strictly Decreasing Function	ચુસ્ત ઘટતું વિધેય
Strictly Increasing Function	ચુસ્ત વધતું વિધેય
Subnormal	અવાભિલંબ
Subtangent	અવસ્પર્શક
Symmetric Form	સંમિત સ્વરૂપ
Tangent	સ્પર્શક
Triangle Inequality	ત્રિકોણીય અસમતા
Upper Limit	ઉર્ધ્વસીમા
Variable Separable	વિયોજનીય ચલ
Vector Product	સદિશ ગુણાકાર
Vector Triple Produc t	સદિશનું ત્રિગુણન
Vector	સદિશ