

MSB Board
Class XII Mathematics & Statistics
Board Paper – 2015 Solution

SECTION – I

1. (A)

i. (d)

Given that

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We can write

$$A = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here 2 is the scalar multiple.

Therefore, $A = 2 \times I$, where $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\text{Thus, } A^6 = \left\{ 2 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}^6$$

$$= 2^6 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^6$$

$$= 2^6 \times I^6, \text{ where } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 2^6 \times I, \text{ since } I^6 = I$$

$$= 2^5 \times 2 \times I$$

$$= 2^5 \times A \quad [\because A = 2I]$$

$$= 32A$$

ii. (c)

The principal solution of $\cos^{-1}\left(-\frac{1}{2}\right)$ = An angle in $[0, \pi]$ whose cosine is $-\frac{1}{2}$.

$$\Rightarrow \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

iii. (a)

Consider the general equation in second degree,

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

The above equation will represent a pair of straight lines if,

$$a'f'^2 + b'g'^2 + c'h'^2 = 2f'g'h' + a'b'c' \dots (1)$$

Here, the given equation is, $hxy + gx + fy + c = 0$

Thus, comparing the coefficients, we have,

$$a' = 0, b' = 0, c' = 0, h' = \frac{h}{2}, g' = \frac{g}{2}, f' = \frac{f}{2}, c' = c$$

Substituting the above values in the condition (1),

we have,

$$(0)\left(\frac{f}{2}\right)^2 + (0)\left(\frac{g}{2}\right)^2 + c\left(\frac{h}{2}\right)^2 = 2 \times \frac{f}{2} \times \frac{g}{2} \times \frac{h}{2} + (0) \times (0) \times (0)$$

$$\Rightarrow c\left(\frac{h}{2}\right)^2 = 2 \times \frac{f}{2} \times \frac{g}{2} \times \frac{h}{2}$$

$$\Rightarrow \frac{ch^2}{4} = \frac{fg}{4}$$

$$\Rightarrow ch^2 = fgh$$

$$\Rightarrow ch = fg \quad [\because h \neq 0]$$

(B)

- i. The given statement- 'If two triangles are congruent, then their areas are equal.'

Converse of the above statement:

If the areas of the two triangles are equal, then the triangles are congruent.

Contrapositive of the given statement:

If the two triangles are not congruent, then their areas are not equal.

- ii. Consider the given equation of the straight lines

$$x^2 + kxy - 3y^2 = 0$$

Sum of the slopes is given by the formula = $\frac{-2H}{B}$

Comparing the given equation with the

standard equation,

$$Ax^2 + 2Hxy + By^2 = 0, \text{ we have,}$$

$$H = \frac{k}{2}, A = 1, B = -3$$

$$\text{Thus, sum of the slopes} = \frac{-2\left(\frac{k}{2}\right)}{-3} = \frac{k}{3}$$

$$\text{Product of the slopes} = \frac{A}{B} = \frac{1}{-3}$$

Also given that, sum of the slopes is twice their product.

$$\Rightarrow \frac{k}{3} = 2\left(\frac{1}{-3}\right)$$

$$\Rightarrow k = -2$$

iii. Given planes are:

$$\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) = 3 \text{ and } \vec{r} \cdot (\hat{i} + 2\hat{j} + \hat{k}) = 1$$

The angle between two planes with direction ratios,

(a_1, b_1, c_1) and (a_2, b_2, c_2) is

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \cos \theta = \frac{2 \times 1 + 1 \times 2 - 1 \times 1}{\sqrt{2^2 + 1^2 + (-1)^2} \sqrt{1^2 + 2^2 + 1^2}}$$

$$\Rightarrow \cos \theta = \frac{3}{\sqrt{6} \sqrt{6}}$$

$$\Rightarrow \cos \theta = \frac{3}{6}$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \cos \theta = \cos \frac{\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

iv. Given equations of the line are:

$$3x - 1 = 6y + 2 = 1 - z$$

Rewriting the above equation, we have,

$$\begin{aligned} 3\left(x - \frac{1}{3}\right) &= 6\left(y + \frac{2}{6}\right) = -(z - 1) \\ \Rightarrow \frac{x - \frac{1}{3}}{\frac{1}{3}} &= \frac{y + \frac{1}{3}}{\frac{1}{6}} = \frac{(z - 1)}{-1} \dots (1) \end{aligned}$$

Now consider the general equation of the line:

$$\frac{x-a}{\ell} = \frac{y-b}{m} = \frac{z-c}{n} \dots (2)$$

where, ℓ, m and n are the direction ratios of the line

and the point (a, b, c) lies on the line.

Compare the equation (1), with the general equation (2),

we have, $\ell = \frac{1}{3}$, $m = \frac{1}{6}$ and $n = -1$.

Also, $a = \frac{1}{3}$, $b = -\frac{1}{3}$ and $c = 1$

This shows that the given line passes through $\left(\frac{1}{3}, -\frac{1}{3}, 1\right)$

Therefore, the given line passes through the point having

position vector $\vec{a} = \frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}$ and is parallel to the

vector $\vec{b} = \frac{1}{3}\hat{i} + \frac{1}{6}\hat{j} - \hat{k}$

So its vector equation is

$$\vec{r} = \left(\frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}\right) + \lambda \left(\frac{1}{3}\hat{i} + \frac{1}{6}\hat{j} - \hat{k}\right)$$

v. Given that $\vec{a} = \hat{i} + 2\hat{j}$, $\vec{b} = -2\hat{i} + \hat{j}$, $\vec{c} = 4\hat{i} + 3\hat{j}$

We need to find x and y such that $\vec{c} = x\vec{a} + y\vec{b}$

Substituting the values of a, b and c, in $\vec{c} = x\vec{a} + y\vec{b}$, we have,

$$4\hat{i} + 3\hat{j} = x(\hat{i} + 2\hat{j}) + y(-2\hat{i} + \hat{j})$$

$$\Rightarrow 4\hat{i} + 3\hat{j} = (x - 2y)\hat{i} + (2x + y)\hat{j}$$

Comparing the coefficients of \hat{i} and \hat{j} on both the sides, we have,

$$x - 2y = 4$$

and

$$2x + y = 3$$

Solving the above simultaneous equations, we have,

$$x = 2 \text{ and } y = -1$$

2. (A)

- i. Given that A,B,C and D are $(1, 1, 1)$, $(2, 1, 3)$, $(3, 2, 2)$ and $(3, 3, 4)$ respectively.

We need to find the volume of the parallelopiped with AB, AC and AD as the concurrent edges.

The volume of the parallelopiped whose

$$\text{coterminus edges are } \vec{a}, \vec{b} \text{ and } \vec{c} \text{ is } [\vec{a} \quad \vec{b} \quad \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

Given that A,B,C and D are $(1, 1, 1)$, $(2, 1, 3)$, $(3, 2, 2)$ and $(3, 3, 4)$

$$\overrightarrow{AB} = (2-1)\hat{i} + (1-1)\hat{j} + (3-1)\hat{k}$$

$$= \hat{i} + 2\hat{k}$$

$$\overrightarrow{AC} = (3-1)\hat{i} + (2-1)\hat{j} + (2-1)\hat{k}$$

$$= 2\hat{i} + \hat{j} + \hat{k}$$

$$\overrightarrow{AD} = (3-1)\hat{i} + (3-1)\hat{j} + (4-1)\hat{k}$$

$$= 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$[\vec{a} \quad \vec{b} \quad \vec{c}] = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= 1(3-2) - 0 + 2(4-2)$$

$$= 1 + 4$$

$$= 5 \text{ cubic units}$$

ii. Consider the statement pattern: $\sim(\sim p \wedge \sim q) \vee q$

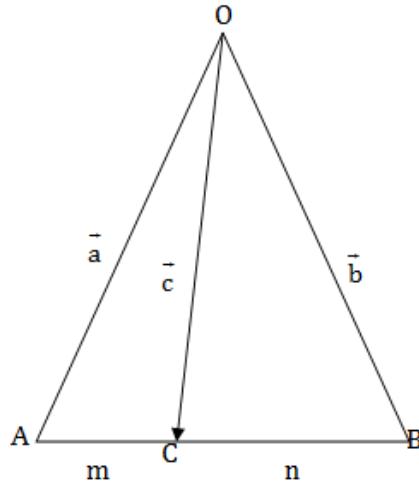
Thus the truth table of the given logical statement: $\sim(\sim p \wedge \sim q) \vee q$

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim(\sim p \wedge \sim q)$	$\sim(\sim p \wedge \sim q) \vee q$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

iii. Given that $C(\vec{c})$ divides the segment joining the points $A(\vec{a})$ and $B(\vec{b})$ internally in the ratio $m : n$

$$\text{We need to prove that } \vec{c} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

Consider the following figure.



$$\text{Since } \frac{\text{length}(AC)}{\text{length}(BC)} = \frac{m}{n}$$

$$\Rightarrow n \times \text{length}(AC) = m \times \text{length}(BC)$$

$$\Rightarrow n\vec{AC} = m\vec{CB}$$

$$\Rightarrow n(\vec{OC} - \vec{OA}) = m(\vec{OB} - \vec{OC})$$

$$\Rightarrow n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c})$$

$$\Rightarrow n\vec{c} - n\vec{a} = m\vec{b} - m\vec{c}$$

$$\Rightarrow n\vec{c} + m\vec{c} = m\vec{b} + n\vec{a}$$

$$\Rightarrow (n + m)\vec{c} = m\vec{b} + n\vec{a}$$

$$\Rightarrow \vec{c} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

Hence proved.

(B)

- i. Given direction ratios are:

$$-2, 1, -1 \text{ and } -3, -4, 1$$

Let a, b and c be the direction ratios of the line perpendicular to the given lines.

Thus, we have,

$$-2a + b - c = 0$$

$$-3a - 4b + c = 0$$

Cross multiplying, we get,

$$\frac{a}{1 \times 1 - (-4) \times (-1)} = \frac{b}{(-3) \times (-1) - (-2) \times 1} = \frac{c}{(-2) \times (-4) - (-3) \times 1}$$

$$\Rightarrow \frac{a}{1-4} = \frac{b}{3+2} = \frac{c}{8+3}$$

$$\Rightarrow \frac{a}{-3} = \frac{b}{5} = \frac{c}{11}$$

Let us find $\sqrt{a^2 + b^2 + c^2}$:

$$\begin{aligned}\sqrt{a^2 + b^2 + c^2} &= \sqrt{(-3)^2 + 5^2 + 11^2} \\ &= \sqrt{9 + 25 + 121} = \sqrt{155}\end{aligned}$$

Thus, the direction ratios of the required line are, $-3, 5, 11$

The direction cosines are: $\frac{-3}{\sqrt{155}}, \frac{5}{\sqrt{155}}, \frac{11}{\sqrt{155}}$

- ii. Given that a^2, b^2, c^2 are in arithmetic progression.

We need to prove that $\cot A, \cot B$ and $\cot C$ are in arithmetic progression.

a^2, b^2, c^2 are in A.P.

$\Rightarrow -2a^2, -2b^2, -2c^2$ are in A.P.

$\Rightarrow (a^2 + b^2 + c^2) - 2a^2, (a^2 + b^2 + c^2) - 2b^2, (a^2 + b^2 + c^2) - 2c^2$ are in A.P.

$\Rightarrow (b^2 + c^2 - a^2), (c^2 + a^2 - b^2), (a^2 + b^2 - c^2)$ are in A.P.

$\Rightarrow \frac{(b^2 + c^2 - a^2)}{2abc}, \frac{(c^2 + a^2 - b^2)}{2abc}, \frac{(a^2 + b^2 - c^2)}{2abc}$ are in A.P.

$\Rightarrow \frac{1}{a} \frac{(b^2 + c^2 - a^2)}{2bc}, \frac{1}{b} \frac{(c^2 + a^2 - b^2)}{2ac}, \frac{1}{c} \frac{(a^2 + b^2 - c^2)}{2ab}$ are in A.P.

$\Rightarrow \frac{1}{a} \frac{(b^2 + c^2 - a^2)}{2bc}, \frac{1}{b} \frac{(c^2 + a^2 - b^2)}{2ac}, \frac{1}{c} \frac{(a^2 + b^2 - c^2)}{2ab}$ are in A.P.

$$\Rightarrow \frac{1}{a} \cos A, \frac{1}{b} \cos B, \frac{1}{c} \cos C \text{ are in A.P.}$$

$$\Rightarrow \frac{k}{a} \cos A, \frac{k}{b} \cos B, \frac{k}{c} \cos C \text{ are in A.P.}$$

$$\Rightarrow \frac{\cos A}{\sin A}, \frac{\cos B}{\sin B}, \frac{\cos C}{\sin C} \text{ are in A.P.}$$

$$\Rightarrow \cot A, \cot B, \cot C \text{ are in A.P.}$$

- iii. Given that the sum of three numbers, x, y and z is 6.

From the given statement, we have,

$$3(x+z) - y = 10$$

$$5(x+y) - 4z = 3$$

Thus, the system of equations are:

$$x+y+z=6$$

$$3x-y+3z=10$$

$$5x+5y-4z=3$$

Let us write the above equations in the matrix form as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 5 & 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ 3 \end{bmatrix}$$

$$\Rightarrow AX=B$$

$$\text{where, } A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 5 & 5 & -4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 10 \\ 3 \end{bmatrix}$$

$$\text{We know that } A^{-1} \text{ exists only if } \begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 5 & 5 & -4 \end{vmatrix} \neq 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 5 & 5 & -4 \end{vmatrix} = 1(4-15) - 1(-12-15) + 1(15+5) \neq 0$$

Thus, A^{-1} exists

We know that $AA^{-1} = I$

$$\text{Thus, } \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 3 \\ 5 & 5 & -4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - 3R_1$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & 0 \\ 5 & 5 & -4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - 5R_1$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & -9 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \rightarrow \frac{R_3}{-9}$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{-5}{-9} & 0 & \frac{-1}{9} \end{bmatrix}$$

Applying $R_2 \rightarrow \frac{R_2}{-4}$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & \frac{-1}{4} & 0 \\ \frac{5}{9} & 0 & \frac{-1}{9} \end{bmatrix}$$

Applying $R_1 \rightarrow R_1 - R_2$, we have

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{3}{4} & \frac{-1}{4} & 0 \\ \frac{5}{9} & 0 & \frac{-1}{9} \end{bmatrix}$$

Applying $R_1 \rightarrow R_1 - R_3$, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{-11}{36} & \frac{1}{4} & \frac{1}{9} \\ \frac{3}{4} & \frac{-1}{4} & 0 \\ \frac{5}{9} & 0 & \frac{-1}{9} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -\frac{11}{36} & \frac{1}{4} & \frac{1}{9} \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ \frac{5}{9} & 0 & -\frac{1}{9} \end{bmatrix}$$

$$AX=B$$

$$\Rightarrow A^{-1}AX=A^{-1}B$$

$$\Rightarrow IX=A^{-1}B$$

$$\Rightarrow X=A^{-1}B$$

$$\Rightarrow X = \begin{bmatrix} -\frac{11}{36} & \frac{1}{4} & \frac{1}{9} \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ \frac{5}{9} & 0 & -\frac{1}{9} \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Thus, the numbers are 1, 2 and 3.

3. (A)

i. Case (1)

Let m_1 and m_2 are the slopes of the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$,

$$\text{then } m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}.$$

\therefore If θ is the acute angle between the lines,

$$\text{then } \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|.$$

$$\text{Now, } (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$

$$(m_1 - m_2)^2 = \left(-\frac{2h}{b} \right)^2 - 4 \left(\frac{a}{b} \right)$$

$$(m_1 - m_2)^2 = \frac{4(h^2 - ab)}{b^2}$$

$$\therefore |m_1 - m_2| = \left| \frac{2\sqrt{h^2 - ab}}{b} \right|$$

$$\text{Similarly } 1 + m_1 m_2 = 1 + \frac{a}{b} = \frac{a+b}{b}$$

Substituting in $\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$, we get

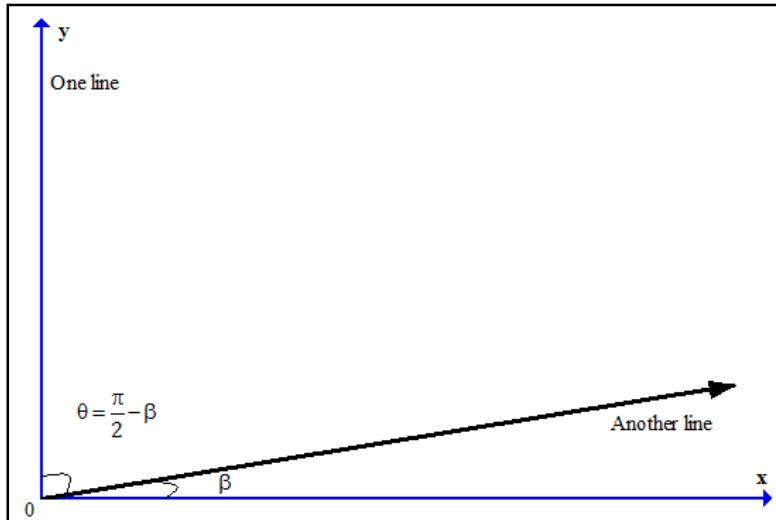
$$\tan \theta = \left| \frac{\frac{2\sqrt{h^2 - ab}}{b}}{\frac{a+b}{b}} \right|$$

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right|, \text{ if } a+b \neq 0.$$

Case (2)

If one of the lines is parallel to the y-axis then one of the slopes m_1, m_2 does not exist. As the line passes through the origin so one line parallel is the y-axis, its equation is $x=0$ and $b=0$.

The other line is $ax + 2hy = 0$, whose slope $\tan \beta = -\frac{a}{2h}$.



\therefore The acute angle θ between the pair of lines is $\frac{\pi}{2} - \beta$.

$$\therefore \tan \theta = \left| \tan \left(\frac{\pi}{2} - \beta \right) \right| = |\cot \beta| = \left| \frac{2h}{a} \right|$$

$$\text{Put } b=0 \text{ in } \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right|, \text{ we get } \tan \theta = \left| \frac{2h}{a} \right|.$$

$$\text{Hence } \tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a+b} \right| \text{ is valid in both the cases.}$$

ii. Let $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = u$, where u is any constant.

So for any point on this line has co-ordinates
in the form $(2u+1, 3u-1, 4u+1)$.

$$\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} = v$$

So for any point on this line has co-ordinates
in the form $(v+3, 2v+k, v)$.

Point of intersection of these two lines will have
co-ordinates of the form

$$(2u+1, 3u-1, 4u+1) \text{ and } (v+3, 2v+k, v).$$

Equating the x, y and z co-ordinates for both the forms
we get three equations

$$2u+1=v+3$$

$$\Rightarrow 2u-v=2 \dots\dots\dots(1)$$

$$3u-1=2v+k$$

$$3u-2v=k+1 \dots\dots\dots(2)$$

$$4u+1=v$$

$$4u-v=-1 \dots\dots\dots(3)$$

Subtracting equation (1) from equation (3) we get,

$$2u=-3$$

$$u=-\frac{3}{2}$$

Substitute value of u in equation (1) we get,

$$2\left(-\frac{3}{2}\right)-v=2$$

$$v=-5$$

Substitute value of u and v in equation (2) we get,

$$3\left(-\frac{3}{2}\right)-2(-5)=k+1$$

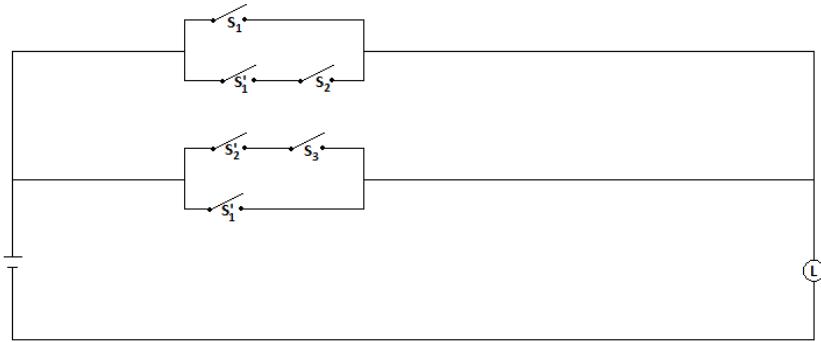
$$k=\frac{9}{2}$$

The value of k is $\frac{9}{2}$.

iii. Let p: the switch S_1 is closed.

q: the switch S_2 is closed.

r: the switch S_3 is closed.



(B)

$$\text{i. } \cos x - \sin x = 1$$

Taking square on both the sides we get,

$$(\cos x - \sin x)^2 = (1)^2$$

$$\therefore \cos^2 x + \sin^2 x - 2\sin x \cos x = 1$$

$$\therefore 1 - 2\sin x \cos x = 1$$

$$\therefore 2\sin x \cos x = 0$$

$$\Rightarrow \sin 2x = 0$$

$$\Rightarrow 2x = n\pi + (-1)^n \alpha$$

But here $\alpha = 0$

$$\Rightarrow 2x = n\pi$$

$$\Rightarrow x = \frac{n\pi}{2}$$

The general solution is $x = \frac{n\pi}{2}$.

ii. The equation of the planes parallel to the plane $x - 2y + 2z - 4 = 0$

are of the form $x - 2y + 2z + k = 0$.

The distance of a plane $ax + by + cz + \lambda$ from a

point (x_1, y_1, z_1) is given by

$$d = \frac{|ax_1 + by_1 + cz_1 + \lambda|}{\sqrt{a^2 + b^2 + c^2}}$$

It is given the plane $x - 2y + 2z + k = 0$ is at an unit

distance from the point $(1, 2, 3)$.

$$\therefore d = \frac{|1 - 2(2) + 2(3) + k|}{\sqrt{1^2 + (-2)^2 + (2)^2}}$$

$$\therefore 1 = \left| \frac{k+3}{3} \right|$$

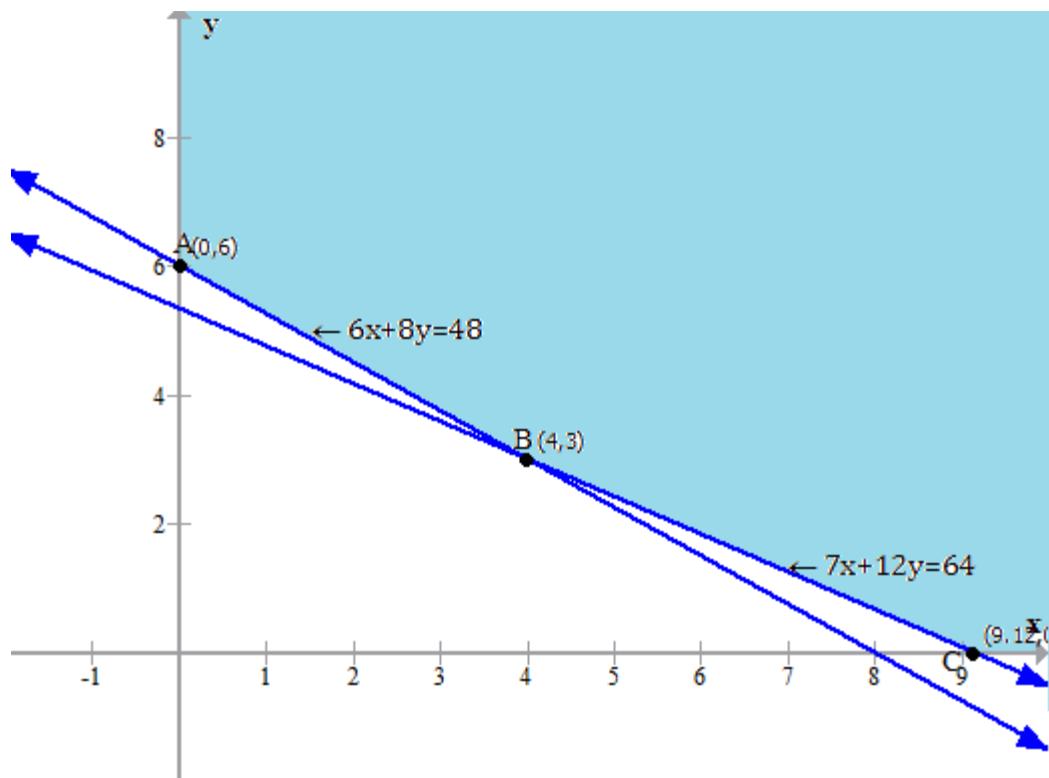
$$\therefore |k+3| = 3$$

$$\therefore k = 0 \text{ or } k = -6$$

The equation of the planes parallel to the plane $x - 2y + 2z - 4 = 0$
are of $x - 2y + 2z = 0$ and $x - 2y + 2z = 6$

- iii. Let x and y be two different types of food.
Thus, our objective function is minimise the cost
 $Z = 6x + 10y$, subject to the constraints,
 $6x + 8y \geq 48$
 $7x + 12y \geq 64$

Plotting the above lines in a graph, we have,



Thus, the region above ABC is unbounded.

Let us check the value of the function at the corner points A, B and C

Corner point	Value of $Z = 6x + 10y$
(0, 6)	$Z = 0 + 10 \times 6 = 60$
(4, 3)	$Z = 6 \times 4 + 10 \times 3 = 54$
$\left(\frac{64}{7}, 0\right)$	$Z = 6 \times \frac{64}{7} + 10 \times 0 = 54.85$

Minimum of the function is at (4, 3)

Minimum cost of the optimum diet is Rs. 54

SECTION - II

4. (A)

i. (a)

X = x	-2	-1	0	1	2	3
P(X)	0.1	0.1	0.2	0.2	0.3	0.1

$$\begin{aligned}
 E(X) &= \sum x_i P(x_i) \\
 &= (-2) \times 0.1 + (-1) \times 0.1 + 0 \times 0.2 + 1 \times 0.2 + 2 \times 0.3 + 3 \times 0.1 \\
 &= -0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3 \\
 &= 0.8
 \end{aligned}$$

ii. (c)

$$\int_0^\alpha 3x^2 dx = 8$$

$$\Rightarrow \left[\frac{3x^3}{3} \right]_0^\alpha = 8$$

$$\Rightarrow [x^3]_0^\alpha = 8$$

$$\Rightarrow \alpha^3 = 8$$

$$\therefore \alpha = 2$$

iii. (a)

Differentiating w.r.t.x,

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = \left(-\frac{c}{x^2} \right)^2$$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = \frac{c^2}{x^4}$$

$$\Rightarrow x^4 \cdot \left(\frac{dy}{dx} \right)^2 = c^2$$

$$\Rightarrow x^4 \cdot \left(\frac{dy}{dx} \right)^2 = y - \frac{c}{x} \quad [\text{From(i)}]$$

$$\Rightarrow x^4 \cdot \left(\frac{dy}{dx} \right)^2 = y + x \frac{dy}{dx}$$

$$\Rightarrow x^4 \cdot \left(\frac{dy}{dx} \right)^2 - x \frac{dy}{dx} = y$$

(B)

i.

$$\int e^x \left[\frac{\sqrt{1-x^2} \sin^{-1} x + 1}{\sqrt{1-x^2}} \right] dx$$

$$= \int e^x \left[\sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \right] dx$$

We know that $\int e^x [f(x) + f'(x)] dx = e^x \cdot f(x) + C$

$$= e^x \cdot \sin^{-1} x + C$$

$$\text{ii. } y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots \dots \infty}}}$$

$$\text{Let } y = \sqrt{\sin x + y}$$

$$y^2 = \sin x + y \quad [\text{Squaring both sides}]$$

Differentiating w.r.t.x,

$$\Rightarrow 2y \cdot \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y-1}$$

iii.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} dx$$

Solving the integral without limits,

$$\int \frac{1}{1+\cos x} dx$$

$$= \int \frac{1}{2\cos^2 \frac{x}{2}} dx$$

$$= \frac{1}{2} \int \sec^2 \frac{x}{2} dx$$

$$= \frac{1}{2} \left[\frac{\tan \frac{x}{2}}{\frac{1}{2}} \right] + C$$

$$= \tan \frac{x}{2} + C$$

Substituting the limits, we get

$$= \left[\tan \frac{x}{2} \right]_0^{\frac{\pi}{2}}$$

$$= \left[\tan \frac{\pi}{4} - \tan 0 \right]$$

$$= 1$$

iv. $y = e^{ax}$

$$y = e^{ax} \dots\dots\dots (i)$$

$$\Rightarrow \log y = ax \dots\dots\dots (ii)$$

$$\frac{dy}{dx} = ae^{ax}$$

$$\Rightarrow \frac{dy}{dx} = ay$$

$$\Rightarrow x \frac{dy}{dx} = axy \text{ [Multiplying both sides by } x \text{]}$$

$$\Rightarrow x \frac{dy}{dx} = y \log y \text{ [From (ii)]}$$

v. Let X be the random variable.

let 'p' be the success and 'q' be the failure

$$p = \frac{1}{2}, q = \frac{1}{2}$$

P(Coin shows 3 heads)

$$= P(X=3) = {}^5C_3 p^3 q^2$$

$$= 10 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$$= 10 \frac{1}{32}$$

$$= \frac{5}{16}$$

5. (A)

$$\text{i. } \int \sec^3 x dx$$

$$= \int \sec x \cdot \sec^2 x dx$$

$$= \int \sqrt{1 + \tan^2 x} \cdot \sec^2 x dx$$

Let $\tan x = z$

$$\sec^2 x dx = dz$$

$$= \int \sqrt{1+z^2} dz$$

$$= \frac{z}{2} \sqrt{1+z^2} + \frac{1}{2} \log |z + \sqrt{1+z^2}| + C$$

$$= \frac{\tan x}{2} \sqrt{1+\tan^2 x} + \frac{1}{2} \log |\tan x + \sqrt{1+\tan^2 x}| + C$$

$$= \frac{\tan x \cdot \sec x}{2} + \frac{1}{2} \log |\tan x + \sec x| + C$$

$$\text{ii. } y = (\tan^{-1} x)^2$$

Differentiating w.r.t. x, we get

$$\Rightarrow \frac{dy}{dx} = \frac{2 \tan^{-1} x}{(1+x^2)}$$

$$\Rightarrow (1+x^2) \frac{dy}{dx} = 2 \tan^{-1} x$$

$$\Rightarrow (1+x^2) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} = \frac{2}{(1+x^2)}$$

$$\Rightarrow (1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} = 2$$

$$\therefore (1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2) \frac{dy}{dx} - 2 = 0$$

iii. $f(x) = \left[\tan\left(\frac{\pi}{4} + x\right) \right]^{\frac{1}{x}}$, for $x \neq 0$
 $= k$, for $x = 0$

The function would be continuous if $\lim_{x \rightarrow 0} f(x) = f(0)$

$$\lim_{x \rightarrow 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left[\tan\left(\frac{\pi}{4} + x\right) \right]^{\frac{1}{x}} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \left[\frac{1 + \tan x}{1 - \tan x} \right]^{\frac{1}{x}} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \left[1 + \frac{1 + \tan x}{1 - \tan x} - 1 \right]^{\frac{1}{x}} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \left[1 + \frac{1 + \tan x - 1 + \tan x}{1 - \tan x} \right]^{\frac{1}{x}} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \left[1 + \frac{2 \tan x}{1 - \tan x} \right]^{\frac{1}{x}} = k$$

$$\Rightarrow \lim_{x \rightarrow 0} \left[1 + \frac{2 \tan x}{1 - \tan x} \right]^{\frac{1}{2 \tan x} \times \frac{2 \tan x}{x(1 - \tan x)}} = k$$

$$\Rightarrow e^{\lim_{x \rightarrow 0} \frac{2 \tan x}{x(1 - \tan x)}} = k \quad \left\{ \because \lim_{x \rightarrow 0} [1 + x]^{\frac{1}{x}} = e \right\}$$

$$\Rightarrow e^{2 \lim_{x \rightarrow 0} \frac{\tan x}{x} \times \lim_{x \rightarrow 0} \frac{1}{(1 - \tan x)}} = k \quad \left\{ \because \lim_{x \rightarrow 0} \left[\frac{\tan x}{x} \right] = 1 \right\}$$

$$\Rightarrow e^{2 \times 1 \times \frac{1}{(1-0)}} = k$$

$$\therefore k = e^2$$

(B)

i.

$$y = x - \frac{4}{x}$$

Differentiating w.r.t. x,

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{4}{x^2}$$

$$\Rightarrow \left| \frac{dy}{dx} \right|_{(x_1, y_1)} = 1 + \frac{4}{x_1^2} = m$$

$$\text{Slope of the tangent, } m = 1 + \frac{4}{x_1^2}$$

Also Slope of the line $y = 2x, m_1 = 2$

m is parallel to m_1

$$1 + \frac{4}{x_1^2} = 2$$

$$\Rightarrow \frac{4}{x_1^2} = 1$$

$$\Rightarrow x_1^2 = 4$$

$\therefore x_1 = 2, x_2 = -2$ [Two points of point of contact of tangent]

$$y_1 = 4, y_2 = -4$$

Co-ordinates of the point of contact are (2, 4) and (-2, -4).

ii. Let $I = \int \sqrt{x^2 - a^2} dx$

$$\Rightarrow I = \int \sqrt{x^2 - a^2} \cdot 1 dx$$

$$\Rightarrow I = \sqrt{x^2 - a^2} \int dx - \int \left[\frac{d}{dx} (\sqrt{x^2 - a^2}) \int dx \right] dx$$

$$\Rightarrow I = x\sqrt{x^2 - a^2} - \int \left[\frac{2x}{2\sqrt{x^2 - a^2}} x \right] dx$$

$$\Rightarrow I = x\sqrt{x^2 - a^2} - \int \left[\frac{x^2}{\sqrt{x^2 - a^2}} \right] dx$$

$$\Rightarrow I = x\sqrt{x^2 - a^2} - \int \left[\frac{x^2 - a^2 + a^2}{\sqrt{x^2 - a^2}} \right] dx$$

$$\Rightarrow I = x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2}{\sqrt{x^2 - a^2}} dx + a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$\Rightarrow I = x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2} dx + a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}$$

$$\begin{aligned}
&\Rightarrow I = x\sqrt{x^2 - a^2} - I + a^2 \int \frac{dx}{\sqrt{x^2 - a^2}} \\
&\Rightarrow 2I = x\sqrt{x^2 - a^2} + a^2 \log|x + \sqrt{x^2 - a^2}| + C' \\
&\Rightarrow I = \frac{x\sqrt{x^2 - a^2}}{2} + \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + \frac{C'}{2} \\
&\therefore I = \frac{x\sqrt{x^2 - a^2}}{2} + \frac{a^2}{2} \log|x + \sqrt{x^2 - a^2}| + C
\end{aligned}$$

iii. Let $I = \int_0^\pi \frac{x \sin x}{1 + \sin x} dx$

$$\begin{aligned}
&= \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{1 + \sin(\pi - x)} dx \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a - x) dx \right] \\
&= \int_0^\pi \frac{(\pi - x) \sin x}{1 + \sin x} dx \\
&= \int_0^\pi \frac{\pi \sin x}{1 + \sin x} dx - \int_0^\pi \frac{x \sin x}{1 + \sin x} dx \\
&= \int_0^\pi \frac{\pi \sin x}{1 + \sin x} dx - I \\
&\therefore I = \int_0^\pi \frac{\pi \sin x}{1 + \sin x} dx - I \\
&\Rightarrow 2I = \int_0^\pi \frac{\pi \sin x (1 - \sin x)}{(1 + \sin x)(1 - \sin x)} dx \\
&\Rightarrow 2I = \int_0^\pi \frac{\pi \sin x (1 - \sin x)}{(1 - \sin^2 x)} dx \\
&\Rightarrow \frac{2I}{\pi} = \int_0^\pi \frac{\sin x (1 - \sin x)}{\cos^2 x} dx \\
&\Rightarrow \frac{2I}{\pi} = \int_0^\pi \frac{\sin x - \sin^2 x}{\cos^2 x} dx \\
&\Rightarrow \frac{2I}{\pi} = \int_0^\pi \frac{\sin x}{\cos^2 x} dx - \int_0^\pi \frac{\sin^2 x}{\cos^2 x} dx
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{2I}{\pi} = \int_0^{\pi} \sec x \cdot \tan x \, dx - \int_0^{\pi} \tan^2 x \, dx \\
&\Rightarrow \frac{2I}{\pi} = [\sec x]_0^{\pi} - \int_0^{\pi} (\sec^2 x - 1) \, dx \\
&\Rightarrow \frac{2I}{\pi} = [\sec \pi - \sec 0] - \int_0^{\pi} \sec^2 x \, dx + \int_0^{\pi} 1 \, dx \\
&\Rightarrow \frac{2I}{\pi} = [-1 - 1] - [\tan x]_0^{\pi} + [x]_0^{\pi} \\
&\Rightarrow \frac{2I}{\pi} = [-1 - 1] - [\tan \pi - \tan 0] + [\pi] \\
&\Rightarrow \frac{2I}{\pi} = -2 - [0] + [\pi] \\
&\therefore I = \frac{(\pi - 2)\pi}{2}
\end{aligned}$$

6. (A)

i. $f(x) = -2 \sin x$, for $-\pi \leq x \leq -\frac{\pi}{2}$
 $= a \sin x + b$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$
 $= \cos x$, for $\frac{\pi}{2} \leq x \leq \pi$

$f(x)$ is continuous for $x = -\frac{\pi}{2}$

RHL,

$$= \lim_{x \rightarrow -\frac{\pi}{2}} a \sin x + b$$

$$= a \sin\left(-\frac{\pi}{2}\right) + b$$

$$= -a + b$$

$$f\left(-\frac{\pi}{2}\right) = -2 \sin\left(-\frac{\pi}{2}\right)$$

$$\therefore -a + b = 2 \dots \text{(i)} \quad \left[\because f(x) \text{ is continuous for } x = -\frac{\pi}{2} \right]$$

$f(x)$ is continuous for $x = \frac{\pi}{2}$

LHL,

$$= \lim_{\substack{x \rightarrow \frac{\pi}{2}}} a \sin x + b$$

$$= a \sin\left(\frac{\pi}{2}\right) + b$$

$$= a + b$$

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\therefore a + b = 0 \dots\dots\dots(ii)$$

Solving (i) and (ii),

$a = -1$ and $b = 1$

$$\text{ii. } \log_{10} \left(\frac{x^3 - y^3}{x^3 + y^3} \right) = 2$$

$$\Rightarrow \left(\frac{x^3 - y^3}{x^3 + y^3} \right) = 10^2 \text{ [Removing logarithm from both sides]}$$

$$\Rightarrow \left(\frac{x^3 - y^3}{x^3 + y^3} \right) = 100$$

$$\Rightarrow x^3 - y^3 = 100x^3 + 100y^3$$

$$\Rightarrow -101v^3 = 99x^3$$

Differentiating w.r.t. x on both sides

$$\Rightarrow -101 \times 3y^2 \cdot \frac{dy}{dx} = 99 \times 3x^2$$

$$\therefore \frac{dy}{dx} = -\frac{99x^2}{101y^2}$$

$$\text{iii. } P(x) = \left(\frac{4}{x}\right) \left(\frac{5}{9}\right)^x \left(\frac{4}{9}\right)^{4-x}$$

The probability distribution table of the function is

X_i	0	1	2	3	4
$P(X_i)$	∞	$\frac{1280}{6561}$	$\frac{1600}{6561 \times 2}$	$\frac{2000}{6561 \times 3}$	$\frac{625}{6561}$
$P(X_i)X_i^2$	0	$\frac{1280}{6561}$	$\frac{3200}{6561}$	$\frac{6000}{6561}$	$\frac{10000}{6561}$

$$\begin{aligned}
E(X) &= x_1 \times P(x_1) + x_2 \times P(x_2) + x_3 \times P(x_3) + x_4 \times P(x_4) + x_5 \times P(x_5) \\
&= 0 \times \infty + 1 \times \frac{4}{1} \times \frac{5}{9} \times \left(\frac{4}{9}\right)^3 + 2 \times \frac{4}{2} \times \left(\frac{5}{9}\right)^2 \times \left(\frac{4}{9}\right)^2 + 3 \times \frac{4}{3} \times \left(\frac{5}{9}\right)^3 \times \left(\frac{4}{9}\right)^1 + 4 \times \frac{4}{4} \times \left(\frac{5}{9}\right)^4 \times \left(\frac{4}{9}\right)^0 \\
&= 0 + \frac{1280}{6561} + \frac{1600}{6561} + \frac{2000}{6561} + \frac{2500}{6561} \\
&= \frac{7380}{6561} = 1.12
\end{aligned}$$

$$\begin{aligned}
\text{Var}(x) &= \sum P(X_i) X_i^2 - [E(X)]^2 \\
&= \frac{1280}{6561} + \frac{3200}{6561} + \frac{6000}{6561} + \frac{10000}{6561} - 1.12 \\
&= \frac{20480}{6561} - 1.12 \\
&= 3.12 - 1.12 \\
&= 2
\end{aligned}$$

(B)

i. $f(x) = 2x^3 - 21x^2 + 36x - 20$

$$f'(x) = 6x^2 - 42x + 36$$

For finding critical points, we take $f'(x)=0$

$$\therefore 6x^2 - 42x + 36 = 0$$

$$\Rightarrow x^2 - 7x + 6 = 0$$

$$\Rightarrow (x-6)(x-1) = 0$$

For finding the maxima and minima, find $f''(x)$

$$f'(x) = 12x - 42$$

For $x = 6$,

$$f''(6) = 30 > 0$$

Minima

For $x = 1$,

$$f''(x) = -30 < 0$$

Maxima

Maximum values of $f(x)$ for $x = 1$

$$f(1) = -3$$

Minimum values of $f(x)$ for $x = 6$,

$$f(6) = -128$$

\therefore the maximum values of the function is -3 and the minimum value of the function is -128.

$$\text{ii. } (x^2 + y^2).dx = 2xy dy \dots\dots(i)$$

The equation is a homogeneous equation

Let $y = vx$,

Differentiating w.r.t. x, we get,

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x^2 + y^2)}{2xy} \quad [\text{From(i)}]$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{(x^2 + v^2 x^2)}{2x.vx}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{(1 + v^2)}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{(1 + v^2)}{2v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + v^2 - 2v^2}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 - v^2}{2v}$$

$$\Rightarrow \frac{2v}{1 - v^2} \cdot dv = \frac{1}{x} dx$$

$$\Rightarrow \int \frac{2v}{1 - v^2} dv = \int \frac{1}{x} dx$$

$$\Rightarrow - \int \frac{-2v}{1 - v^2} dv = \log x$$

$$\Rightarrow -\log|1 - v^2| = \log x + \log C$$

$$\Rightarrow \log x - \log|1 - v^2| = \log(C)$$

$$\Rightarrow \frac{x}{1 - v^2} = C$$

$$\Rightarrow \frac{x}{1 - \frac{y^2}{x^2}} = C$$

$$\Rightarrow \frac{x^3}{x^2 - y^2} = C, \text{ where } C \text{ is a constant.}$$

iii. c.d.f. of the continuous random variable is given by

$$F(x) = \int_{-1}^x \frac{y^2}{3} dx$$

$$= \left[\frac{y^3}{9} \right]_{-1}^x$$

$$= \frac{(x^3 + 1)}{9}, x \in \mathbb{R}$$

$$\text{Consider } P(X < 1) = F(1) = \frac{(1^3 + 1)}{9} = \frac{2}{9}$$

$$P(X \leq -2) = 0$$

$$P(X > 0) = 1 - P(X \leq 0)$$

$$= 1 - F(0)$$

$$= 1 - \left(\frac{0}{9} + \frac{1}{9} \right)$$

$$= \frac{8}{9}$$

$$P(1 < X < 2) = F(2) - F(1)$$

$$= 1 - \left(\frac{1}{9} + \frac{1}{9} \right)$$

$$= \frac{7}{9}$$