MSB Board
Class XII Mathematics & Statistics
Board Paper 2016 Solution

SECTION – I

1. (A)
(i) (d)

$$\sim\left[P\wedge(q\rightarrow r)\right] =\sim\left[(P)\right]\vee\left[\sim(q\rightarrow r)\right] \text{.................By De Morgan's law}$$

$$=\sim\left[(P)\right]\vee\left[\sim(\sim q\vee r)\right] \text{..............By Conditional Law}$$

$$=\sim\left[(P)\right]\vee\left[(q\wedge\sim r)\right] \text{..................By De Morgan's law}$$

$$\therefore\sim\left[P\wedge(q\rightarrow r)\right]=\sim P\vee(q\wedge\sim r)$$

(ii) (c)

$$\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$$

$$\sin^{-1}(1-x)=\frac{\pi}{2}+2\sin^{-1}x$$

$$(1-x)=\sin\left(\frac{\pi}{2}+2\sin^{-1}x\right)$$

$$(1-x)=\cos\left(2\sin^{-1}x\right)$$

$$(1-x)=\cos\left(\cos^{-1}\left(1-2x^2\right)\right)$$

$$(1-x)=1-2x^2$$

$$2x^2-x=0$$

$$x(2x-1)=0$$

$x=0$ or $2x - 1 = 0$

$x = 0$ or $x = \frac{1}{2}$

For $x = \frac{1}{2}$

$$\sin^{-1}(1-x)-2\sin^{-1}x=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)=-\sin^{-1}\left(\frac{1}{2}\right)=-\frac{\pi}{6}$$

So $x = \frac{1}{2}$ is not solution of the given equation.

For $x = 0$

$$\sin^{-1}(1-x)-2\sin^{-1}x=\sin^{-1}(1)-2\sin^{-1}(0)=\frac{\pi}{2}-0=\frac{\pi}{2}$$

So $x = 0$ is a valid solution of the given equation.

(iii) (a)

Equation of the coordinate axes are x = 0 and y = 0.

$\therefore$ The equations of the lines passing through (2, 3) and parallel to coordinate axes are, x = 2 and y = 3.

i.e. $x-2=0$ and $y-3=0$

The joint equation is given as

$(x-2)(y-3)=0$

$xy-3x-2y+6=0$

(B)

(i)

$$A=\begin{bmatrix}1 & 2 & 3\\ 1 & -2 & -3\end{bmatrix}, B=\begin{bmatrix}1 & -1\\ 1 & 2\\ 1 & -2\end{bmatrix}$$

$$AB=\begin{bmatrix}1 & 2 & 3\\ 1 & -2 & -3\end{bmatrix}\times\begin{bmatrix}1 & -1\\ 1 & 2\\ 1 & -2\end{bmatrix}$$

$$=\begin{bmatrix}6 & -3\\ -4 & 1\end{bmatrix}$$

$$(AB)^{-1}(AB)=I$$

$$(AB)^{-1}\begin{bmatrix}6 & -3\\ -4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Using $R_1\to\frac{1}{6}R_1$

$$\therefore (AB)^{-1}\begin{bmatrix}1 & \frac{-1}{2}\\ -4 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{6} & 0\\ 0 & 1\end{bmatrix}$$

Using $R_2\to R_2+4R_1$

$$\therefore (AB)^{-1}\begin{bmatrix}1 & \frac{-1}{2}\\ 0 & -1\end{bmatrix}=\begin{bmatrix}\frac{1}{6} & 0\\ \frac{2}{3} & 1\end{bmatrix}$$

Using $R_2\to(-1)R_2$

$$\therefore (AB)^{-1}\begin{bmatrix}1 & \frac{-1}{2}\\ 0 & 1\end{bmatrix}=\begin{bmatrix}\frac{1}{6} & 0\\ -\frac{2}{3} & -1\end{bmatrix}$$

Using $R_1 \to R_1 + \left(\frac{1}{2}\right)R_2$

$$\therefore (AB)^{-1}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-\frac{1}{6} & \frac{-1}{2}\\ -\frac{2}{3} & -1\end{bmatrix}$$

$$\therefore (AB)^{-1} I = \begin{bmatrix}-\frac{1}{6} & \frac{-1}{2}\\ -\frac{2}{3} & -1\end{bmatrix}$$

$$\therefore (AB)^{-1} = \begin{bmatrix}-\frac{1}{6} & \frac{-1}{2}\\ -\frac{2}{3} & -1\end{bmatrix}$$

(ii) We know that the vector equation of a plane passing through a point $A(\bar{a})$ and normal to $\bar{n}$ is $\bar{r}\bullet\bar{n} = \bar{a}\bullet\bar{n}$.

Here $\bar{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{n} = 4\hat{i} + 3\hat{j} + 2\hat{k}$

$\therefore$ The vector equation of the required plane is

$\bar{r}\bullet\bar{n} = \bar{a}\bullet\bar{n}$

$\bar{r}\bullet\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) = \left(3\hat{i} - 2\hat{j} + \hat{k}\right)\bullet\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right)$

$\bar{r}\bullet\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) = 12 - 6 + 2$

$\bar{r}\bullet\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) = 8$

$\therefore$ The vector equation of the required plane is $\bar{r}\bullet\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) = 8$.

(iii) R is the point which divides the line segment joining the points PQ internally in the ratio 2:1.

$$\bar{r} = \frac{2(\bar{q}) + 1(\bar{p})}{2+1}$$

$$= \frac{2\left(\hat{i} + 4\hat{j} - 2\hat{k}\right) + 1\left(\hat{i} - 2\hat{j} + \hat{k}\right)}{3}$$

$$= \frac{3\hat{i} + 6\hat{j} - 3\hat{k}}{3}$$

$$\therefore \bar{r} = \hat{i} + 2\hat{j} - \hat{k}$$

The position vector of point R is $\hat{i} + 2\hat{j} - \hat{k}$.

(iv) Let m_1 be the slope of $2x + y = 0$.

$\therefore m_1 = -2$

$6x^2 + kxy + y^2 = 0$

$\therefore a = 6,\ h = \frac{k}{2},\ b = 1$

$\therefore m_1 + m_2 = -\frac{2h}{b} = -k$

$\therefore -2 + m_2 = -k$

$\therefore m_2 = -k + 2$

Now, $m_1 m_2 = \frac{a}{b}$

$\therefore (-2)(-k + 2) = 6$

$2k - 4 = 6$

$k = 5$

The value of k is 5.

(v) Given equations of the line are:

Let $\bar{a}$ and $\bar{b}$ be vectors in the direction of lines $\frac{x - 1}{-3} = \frac{y - 2}{2k} = \frac{z - 3}{2}$ and $\frac{x - 1}{3k} = \frac{y - 5}{1} = \frac{z - 6}{-5}$ respectively.

$\therefore \bar{a} = -3\hat{i} + 2k\hat{j} + 2\hat{k}$ and $\bar{b} = 3k\hat{i} + \hat{j} - 5\hat{k}$

$\bar{a} \bullet \bar{b} = -9k + 2k - 10 = -7k - 10$

Given lines are at right angle

$\therefore \theta = 90^\circ$

$\cos\theta = \frac{\bar{a} \bullet \bar{b}}{|\bar{a}||\bar{b}|}$

$0 = \frac{\bar{a} \bullet \bar{b}}{|\bar{a}||\bar{b}|}$

$\bar{a} \bullet \bar{b} = 0$

$-7k - 10 = 0$

$k = -\frac{10}{7}$

The value of k is $-\frac{10}{7}$.

2. (A)

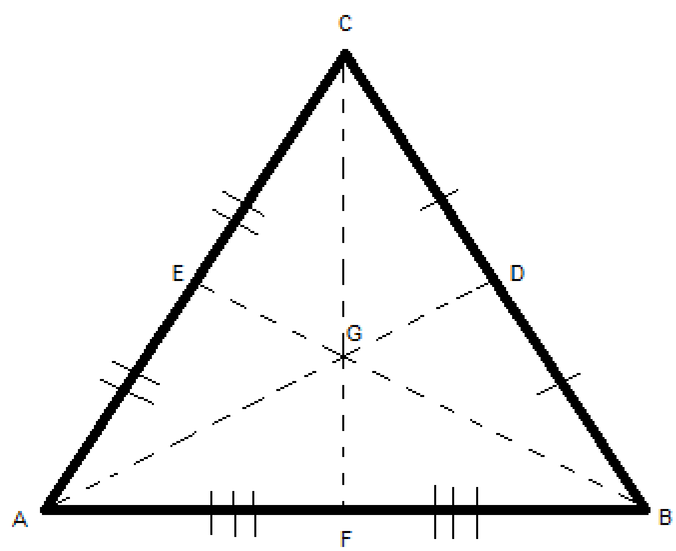
(i) Consider the statement pattern: $\left[(p \rightarrow q) \wedge q\right] \rightarrow p$

Thus the truth table of the given logical statement: $\left[(p \rightarrow q) \wedge q\right] \rightarrow p$

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$\left[(p \rightarrow q) \wedge q\right] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

From the above truth table we can say that given logical statement: $\left[(p \rightarrow q) \wedge q\right] \rightarrow p$ is contingency.

(ii)



Let A, B and C be the vertices of a triangle.

Let D, E and F be the midpoints of the sides BC, AC and AB respectively.

Let $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}, \overline{OD} = \bar{d}, \overline{OE} = \bar{e}$ and $\overline{OF} = \bar{f}$ be position vectors of points A, B, C, D, E and F respectively.

Therefore, by midpoint formula,

$\bar{d} = \dfrac{\bar{b} + \bar{c}}{2}, \bar{e} = \dfrac{\bar{a} + \bar{b}}{2}$ and $\bar{f} = \dfrac{\bar{a} + \bar{b}}{2}$

$\therefore 2\bar{d} = \bar{b} + \bar{c}, 2\bar{e} = \bar{a} + \bar{c}$ and $2\bar{f} = \bar{a} + \bar{b}$

$\therefore 2\bar{d} + \bar{a} = \bar{a} + \bar{b} + \bar{c},\ 2\bar{e} + \bar{b} = \bar{a} + \bar{b} + \bar{c}$ and $2\bar{f} + \bar{c} = \bar{a} + \bar{b} + \bar{c}$

$$\therefore \frac{2\bar{d} + \bar{a}}{3} = \frac{2\bar{e} + \bar{b}}{3} = \frac{2\bar{f} + \bar{c}}{3} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$$

Let $\bar{g} = \dfrac{\bar{a} + \bar{b} + \bar{c}}{3}$.

$\therefore$ We have $\bar{g} = \dfrac{\bar{a} + \bar{b} + \bar{c}}{3} = \dfrac{(2)\bar{d} + (1)\bar{a}}{3} = \dfrac{(2)\bar{e} + (1)\bar{b}}{3} = \dfrac{(2)\bar{f} + (1)\bar{c}}{3}$

If G is the point whose position vector is $\bar{g}$, then from the above equation it is clear that the point G lies on the medians AD, BE, CF and it divides each of the medians AD, BE, CF internally in the ratio 2:1.

Therefore, three medians are concurrent.

(iii) We know that the shortest distance between the lines

$\bar{r} = \overline{a_1} + \lambda \overline{b_1}$ and $\bar{r} = \overline{a_2} + \mu \overline{b_2}$ is given as $d = \left| \dfrac{(\overline{a_2} - \overline{a_1}) \cdot (\overline{b_1} \times \overline{b_2})}{|\overline{b_1} \times \overline{b_2}|} \right|$

Given equation of lines are

$\bar{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\bar{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(\hat{i} + 4\hat{j} - 5\hat{k})$.

$\therefore \overline{a_2} - \overline{a_1} = (\hat{i} - \hat{j} + 2\hat{k}) - (4\hat{i} - \hat{j}) = -3\hat{i} + 2\hat{k}$

$$\overline{b_1} \times \overline{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{vmatrix}$$

$= \hat{i}(-10 + 12) - \hat{j}(-5 + 3) + \hat{k}(4 - 2)$

$= 2\hat{i} + 2\hat{j} + 2\hat{k}$

$(\overline{a_2} - \overline{a_1}) \cdot (\overline{b_1} \times \overline{b_2}) = (-3\hat{i} + 2\hat{k}) \cdot (2\hat{i} + 2\hat{j} + 2\hat{k}) = -3 \times 2 + 0 \times 2 + 2 \times 2 = -2$

$|\overline{b_1} \times \overline{b_2}| = \sqrt{4 + 4 + 4} = 2\sqrt{3}$

Shortest distance $= d = \left| \dfrac{-2}{2\sqrt{3}} \right| = \left| \dfrac{-1}{\sqrt{3}} \right| = \dfrac{1}{\sqrt{3}}$ units.

(B)

(i) $\text{LHS} = (a - b)^2 \cos^2\left(\dfrac{C}{2}\right) + (a + b)^2 \sin^2\left(\dfrac{C}{2}\right)$

$= a^2\left[\cos^2\left(\dfrac{C}{2}\right) + \sin^2\left(\dfrac{C}{2}\right)\right] + b^2\left[\cos^2\left(\dfrac{C}{2}\right) + \sin^2\left(\dfrac{C}{2}\right)\right] - 2ab\left[\cos^2\left(\dfrac{C}{2}\right) - \sin^2\left(\dfrac{C}{2}\right)\right]$

$= a^2(1) + b^2(1) - 2ab[\cos C]$

..........$[\because \cos^2\theta + \sin^2\theta = 1 \text{ and } \cos^2\theta - \sin^2\theta = \cos 2\theta]$

$$= a^2 + b^2 - 2ab\left[\frac{a^2 + b^2 - c^2}{2ab}\right]$$

$$\ldots\ldots\ldots\ldots\left[\text{Cosine Rule } \cos C = \frac{a^2 + b^2 - c^2}{2ab}\right]$$

$$= a^2 + b^2 - a^2 - b^2 + c^2$$

$$= c^2$$

$= \text{RHS}$

Hence proved.

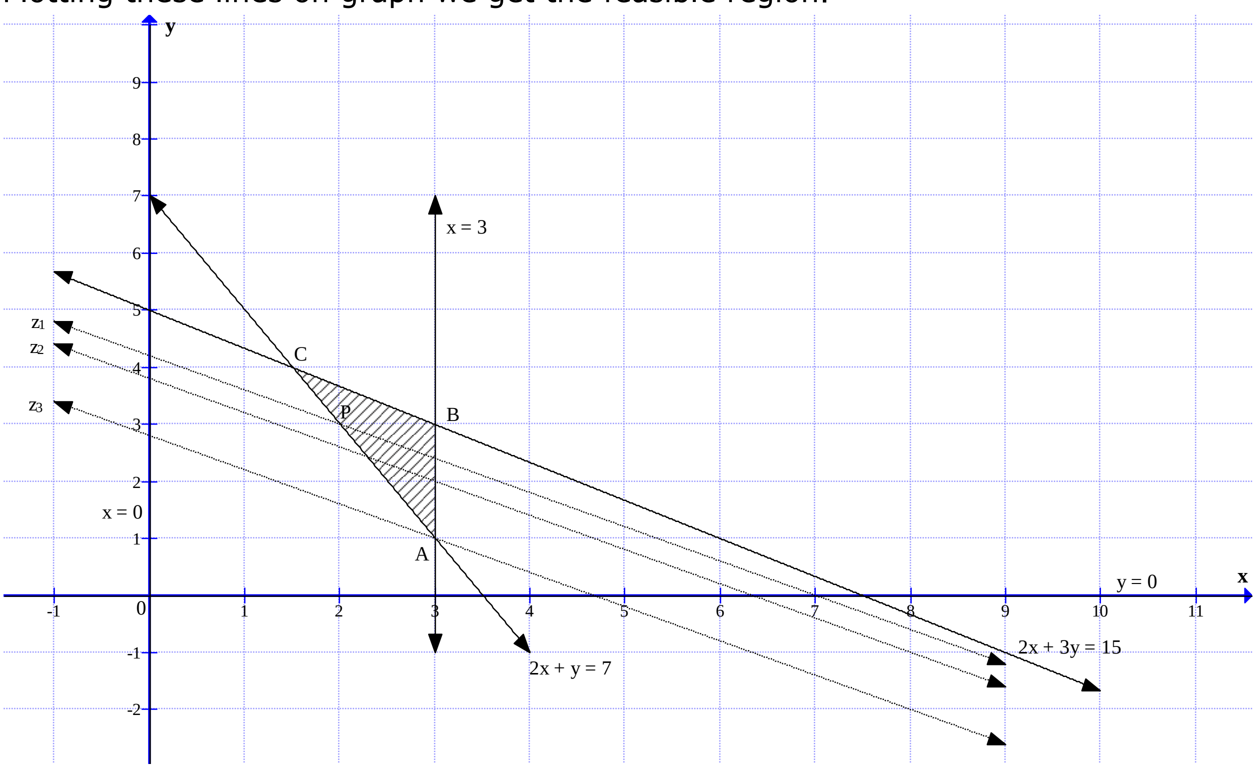
(ii) Consider equations obtained by converting all inequations representing the constraints.

$2x + y = 7$ i.e. $\frac{x}{3.5} + \frac{y}{7} = 1$

$2x + 3y = 15$ i.e. $\frac{x}{7.5} + \frac{y}{5} = 1$

$x = 3$, $x = 0$, $y = 0$

Plotting these lines on graph we get the feasible region.



From the graph we can see that ABC is the feasible region.

Take any one point on the feasible region say $P(2, 3)$.
Draw initial isocost line z_1 passing through the point $(2, 3)$.
$\therefore z_1 = 4(2) + 5(3) = 8 + 15 = 23$
$\therefore$ Initial isocost line is $4x + 5y = 23$.

Since the objective function is of minimization type, from the graph we can see that the line z_3 contains only one point A(3, 1) of the feasible region ABC.

Minimum value of $z = 4(3) + 5(1) = 12 + 5 = 17$
$\therefore$ z is minimum when x = 3 and y = 1.

(iii) Let ₹'x', ₹'y' and ₹'z' be the cost of one dozen pencils, one dozen pens and one dozen erasers.
Thus, the system of equations are:
$4x + 3y + 2z = 60$
$2x + 4y + 6z = 90$
$6x + 2y + 3z = 70$
Let us write the above equations in the matrix form as:

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix} \textit{i.e.} AX = B$$

Using $R_2 \to R_2 - \frac{1}{2}R_1$ and $R_3 \to R_3 - \frac{3}{2}R_1$

$$\begin{bmatrix} 4 & 3 & 2 \\ 0 & \frac{5}{2} & 5 \\ 0 & -\frac{5}{2} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \\ -20 \end{bmatrix}$$

Using $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 4 & 3 & 2 \\ 0 & \frac{5}{2} & 5 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \\ 40 \end{bmatrix}$$

As matrix A is reduced to its upper triangular form we can write

$4x + 3y + 2z = 60$..........(i)

$\frac{5}{2}y + 5z = 60$..........(ii)

$5z = 40$

$z = 8$.....(iii)

Substituting (iii) in (ii) we get,

$\frac{5}{2}y + 5(8) = 60$

$y = 8$.....(iv)

Substituting (iii) and (iv) in (i) we get,

$4x + 3(8) + 2(8) = 60$

$x = 5$

∴ Cost of one dozen pencils, one dozen pens and one dozen erasers is ₹5, ₹8 and ₹8 respectively.

3.

(A)

(i)

Volume of tetrahedron whose conterminus edges are $\bar{a}, \bar{b}$ and $\bar{c}$ is $\frac{1}{6}\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

Here $\bar{a} = 7\hat{i}+\hat{k}$; $\bar{b} = 2\hat{i}+5\hat{j}-3\hat{k}$; $\bar{c} = 4\hat{i}+3\hat{j}+\hat{k}$.

$$\text{Volume of tetrahedron} = \frac{1}{6}\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

$$= \frac{1}{6}\begin{vmatrix} 7 & 0 & 1 \\ 2 & 5 & -3 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{6}\left[7(5+9) - 0(2+12) + 1(6-20)\right]$$

$$= \frac{1}{6}\left[98 - 0 - 14\right]$$

$$= \frac{1}{6}\left[84\right]$$

$$= 14$$

Hence volume of tetrahedron is 14 cubic units.

(ii)

$$\sim(p \vee q) \vee (\sim p \wedge q)$$

$\equiv \sim(p \vee q) \vee \sim(p \vee \sim q)$ by De Morgan's Law

$\equiv \sim[(p \vee q) \wedge (p \vee \sim q)]$ by De Morgan's Law

$\equiv \sim\{[(p \vee q) \wedge p] \vee [(p \vee q) \wedge \sim q]\}$ by Distributive Law

$\equiv \sim\{[P] \vee [(p \vee q) \wedge \sim q]\}$ by Absorption Law

$\equiv \sim\{[P] \vee [(p \wedge \sim q) \vee (q \wedge \sim q)]\}$ by Distributive Law

$\equiv \sim\{[P] \vee [(p \wedge \sim q) \vee F]\}$ by Complement Law

$\equiv \sim\{[P] \vee [(p \wedge \sim q)]\}$ by Identity Law

$\equiv \sim P \wedge (\sim P \vee q)$ by De Morgan's Law

$\equiv \sim P$ by Absorption Law

(iii)

Consider a homogeneous equation of degree two in x and y

$ax^2 + 2hxy + by^2 = 0$.....................(i)

In this equation at least one of the coefficients a, b or h is non zero.

We consider two cases.

Case I: If b = 0 then equation becomes

$ax^2 + 2hxy = 0$

$x(ax + 2hy) = 0$

This is the joint equation of lines x = 0 and $(ax + 2hy) = 0$

These lines pass through the origin.

Case II: If b $\neq$ 0

Multiplying both the sides of equation (i) by b, we get

$abx^2 + 2hbxy + b^2y^2 = 0$

$2hbxy + b^2y^2 = -abx^2$

To make LHS a complete square, we add h^2x^2 on both the sides.

$b^2y^2 + 2hbxy + h^2x^2 = -abx^2 + h^2x^2$

$$(by+hx)^2=\left(h^2-ab\right)x^2$$

$$(by+hx)^2=\left[\left(\sqrt{h^2-ab}\right)x\right]^2$$

$$(by+hx)^2-\left[\left(\sqrt{h^2-ab}\right)x\right]^2=0$$

$$\left[(by+hx)+\left(\sqrt{h^2-ab}\right)x\right]\left[(by+hx)-\left(\sqrt{h^2-ab}\right)x\right]=0$$

It is the joint equation of two lines

$(by+hx)+\left(\sqrt{h^2-ab}\right)x=0$ and $(by+hx)-\left(\sqrt{h^2-ab}\right)x=0$

$\left(h+\sqrt{h^2-ab}\right)x+by=0$ and $\left(h-\sqrt{h^2-ab}\right)x+by=0$

These lines pass through the origin when $h^2-ab>0$.

From the above two cases we conclude that the equation $ax^2+2hxy+by^2=0$ represents a pair of lines passing through the origin.

(B)

(i)

Let M be the foot of the perpendicular drawn from the point A$(1, 2, 1)$ to the line joining P$(1, 4, 6)$ and Q$(5, 4, 4)$.

Equation of a line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Equation of the required line passing through P$(1, 4, 6)$ and Q$(5, 4, 4)$ is

$$\frac{x-1}{4}=\frac{y-4}{0}=\frac{z-6}{-2}$$

Coordinates of any point on the line are given by

$$\frac{x-1}{4}=\frac{y-4}{0}=\frac{z-6}{-2}=\lambda$$

$x=4\lambda+1;\ y=4;\ z=-2\lambda+6$

$\therefore$ Coordinates of M are $(4\lambda+1, 4, -2\lambda+6)$...........$(i)$

The direction ratios of AM are

$4\lambda+1-1,\ 4-2, -2\lambda+6-1$

i.e. $4\lambda, 2, -2\lambda+5$

The direction ratios of given line are 4,0,-2.

Since AM is perpendicular to the given line

$\therefore 4(4\lambda)+0(2)+(-2)(-2\lambda+5)=0$

$\therefore \lambda=\frac{1}{2}$

Putting $\lambda=\frac{1}{2}$ in (i), the coordinates of M are $(3,4,5)$.

$\therefore$ Length of perpendicular from A on the given line

AM = $\sqrt{(3-1)^2+(4-2)^2+(5-1)^2}=\sqrt{24}$ units.

(ii)

Let $\bar{a}=\hat{i}+\hat{j}-2\hat{k},\ \ \bar{b}=\hat{i}+2\hat{j}+\hat{k},\ \ \bar{c}=2\hat{i}-\hat{j}+\hat{k}$

$\overline{AB}=\bar{b}-\bar{a}=\left(\hat{i}+2\hat{j}+\hat{k}\right)-\left(\hat{i}+\hat{j}-2\hat{k}\right)=\hat{j}+3\hat{k}$

$\overline{AC}=\bar{c}-\bar{a}=\left(2\hat{i}-\hat{j}+\hat{k}\right)-\left(\hat{i}+\hat{j}-2\hat{k}\right)=\hat{i}-2\hat{j}+3\hat{k}$

$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 3\\ 1 & -2 & 3\end{vmatrix}$$

$=\hat{i}(3+6)-\hat{j}(0-3)+\hat{k}(0-1)$

$=9\hat{i}+3\hat{j}-\hat{k}$

Let $\bar{n}=\overline{AB}\times\overline{AC}$

Then the equation of required plane is,

$\bar{r}\cdot\bar{n}=\bar{a}\cdot\bar{n}$

$\bar{r}\cdot\left(9\hat{i}+3\hat{j}-\hat{k}\right)=\left(\hat{i}+\hat{j}-2\hat{k}\right)\cdot\left(9\hat{i}+3\hat{j}-\hat{k}\right)$

$\bar{r}\cdot\left(9\hat{i}+3\hat{j}-\hat{k}\right)=9+3+2$

$\bar{r}\cdot\left(9\hat{i}+3\hat{j}-\hat{k}\right)=14$

The cartesian equation of the plane is given by,

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\left(9\hat{i}+3\hat{j}-\hat{k}\right)=14,\quad$ where $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$9x+3y-z=14$

The cartesian equation of the plane is $9x+3y-z=14$.

(iii)

$$\sin x + \sin 3x + \sin 5x = 0$$
$$\therefore (\sin x + \sin 5x) + \sin 3x = 0$$
$$\therefore 2\sin\left(\frac{x+5x}{2}\right)\cos\left(\frac{5x-x}{2}\right) + \sin 3x = 0$$
$$\therefore 2\sin 3x \cos 2x + \sin 3x = 0$$
$$\therefore (2\cos 2x + 1)\sin 3x = 0$$
$$\therefore (2\cos 2x + 1) = 0 \quad \text{or} \quad \sin 3x = 0$$
$$\therefore \cos 2x = -\frac{1}{2} \quad \text{or} \quad \sin 3x = 0$$
$$\therefore \cos 2x = -\cos\frac{\pi}{3} \quad \text{or} \quad \sin 3x = 0$$
$$\therefore \cos 2x = \cos\left(\pi - \frac{\pi}{3}\right) \quad \text{or} \quad \sin 3x = 0$$
$$\therefore 2x = 2n\pi \pm \frac{2\pi}{3} \quad \text{or} \quad 3x = m\pi$$

where n,m $\in$ Z

$$\therefore x = n\pi \pm \frac{\pi}{3} \quad \text{or} \quad x = \frac{m\pi}{3}$$

The required solution is $x = n\pi \pm \frac{\pi}{3}$ or $x = \frac{m\pi}{3}$, where n,m $\in$ Z.

SECTION – II

4. (A)

(i) (c)

$f(1) = 4(1) + 3 = 7$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{h \to 0} f(1-h) = \lim\limits_{h \to 0} k + 1 - h = k + 1$

For the function to be continuous at $x = 1$,

$f(1) = \lim\limits_{x \to 1^-} f(x)$

$\Rightarrow 7 = k + 1$

$\Rightarrow k = 6$

(ii) (a)

$y = x^2 + 4x + 1$

Differentiating w.r.t 'x', we get

$$\frac{dy}{dx} = 2x + 4$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=-1} = 2(-1) + 4 = 2$$

Hence, slope of tangent at $(-1, \ -2)$ is 2.

So equation of tangent line is

$y - (-2) = 2(x - (-1))$

$\Rightarrow 2x - y = 0$

(iii) (c)

Since $X \sim B(n = 10, p)$,

$E(x) = np$

$\Rightarrow 10p = 8$

$\Rightarrow p = 0.8$

(B)

(i)

$y = x^x$

$\Rightarrow \ln y = x \ln x$

Differentiating both sides with respect to 'x', we get

$$\left(\frac{1}{y}\right)\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x(1)$$

$$\Rightarrow \frac{dy}{dx} = y(1 + \ln x)$$

$$\Rightarrow \frac{dy}{dx} = x^x(1 + \ln x)$$

(ii)

$s = 5 + 20t - 2t^2$

$v = \frac{ds}{dt} = 20 - 4t$

$v = 0 \Rightarrow 20 - 4t = 0 \Rightarrow t = 5$

$a = \frac{dv}{dt} = -4,$ which is a constant.

Hence, acceleration is -4 when velocity is zero.

(iii)

Area bounded $= \int_0^a y dx$

$= \int_0^a \sqrt{4ax}\, dx$

$= 2\sqrt{a}\int_0^a x^{\frac{1}{2}} dx$

$= 2\sqrt{a}\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^a$

$= 2\sqrt{a} \times \frac{2}{3} \times a^{\frac{3}{2}}$

$= \frac{4}{3} a^{\frac{1}{2}+\frac{3}{2}}$

$= \frac{4}{3} a^2$

(iv)

$P(X \le 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= k + 2k + 3k + 4k$

$= 10k$

(v)

$\int \frac{\sin x}{\sqrt{36 - \cos^2 x}} dx$

Substitute $\cos x = t$

$\Rightarrow -\sin x dx = dt$

The integral becomes

$\int \frac{-dt}{\sqrt{36 - t^2}} = -\int \frac{dt}{\sqrt{6^2 - t^2}}$

$= -\sin^{-1}\frac{t}{6} + C = -\sin^{-1}\frac{\cos x}{6} + C$

5. (A)

(i)

Let δx be a small increment in x.

Let δy and δu be the corresponding increments in y and u respectively.

As $\delta x \to 0, \delta y \to 0,\ \delta u \to 0$.

As u is differentiable function, it is continuous.

Consider the incrementary ratio $\frac{\delta y}{\delta x}$.

We have, $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

Taking limit as $\delta x \to 0$, on both sides,

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}\right)$$

$$\Rightarrow \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \quad ...(1)$$

Since y is a differentiable function of u, $\lim_{\delta u \to 0} \frac{\delta y}{\delta u}$ exists

and $\lim_{\delta x \to 0} \frac{\delta u}{\delta x}$ exists as u is a differentiable function of x.

Hence, R.H.S. of (1) exists.

Now, $\lim_{\delta u \to 0} \frac{\delta y}{\delta u} = \frac{dy}{du}$ and $\lim_{\delta x \to 0} \frac{\delta u}{\delta x} = \frac{du}{dx}$

$$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{du} \times \frac{du}{dx}$$

Since R.H.S. exists, L.H.S. of (1) also exists and

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

(ii)

Probability of recovery=$P(R) = 0.5$

Probability of non-recovery $= P(\bar{R}) = 1 - 0.5 = 0.5$

(a) If there are six patients, the probability that none recovers

$$= {}^6C_0 \times [P(R)]^0 \times [P(\bar{R})]^6 = (0.5)^6 = \frac{1}{64}$$

(b) Of the six patients, the probability that half will recover

$$= {}^6C_3 \times [P(R)]^3 \times [P(\bar{R})]^3 = \frac{6!}{3!3!} \times 0.5^3 \times 0.5^3 = 20 \times \frac{1}{64} = \frac{5}{16}$$

(iii)

$$I=\int_0^{\pi}\frac{x}{a^2\cos^2x+b^2\sin^2x}dx\text{...(i)}$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi-x}{a^2\cos^2(\pi-x)+b^2\sin^2(\pi-x)}dx$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi-x}{a^2\cos^2x+b^2\sin^2x}dx\text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi}\frac{x}{a^2\cos^2x+b^2\sin^2x}dx+\int_0^{\pi}\frac{\pi-x}{a^2\cos^2x+b^2\sin^2x}dx$$

$$\Rightarrow 2I=\int_0^{\pi}\frac{\pi}{a^2\cos^2x+b^2\sin^2x}dx$$

$$\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{1}{a^2\cos^2x+b^2\sin^2x}dx$$

$$\Rightarrow I=\frac{\pi}{2}\int_0^{\pi}\frac{\sec^2x}{a^2+b^2\tan^2x}dx \text{ (dividing numerator and denominator by } \cos^2x)$$

Substitute $\tan x=t\Rightarrow \sec^2x\,dx=dt$

$t=\tan x=0$ at $x=0$, $t=\tan x=0$ at $x=\pi$

$$\Rightarrow I=\frac{\pi}{2}\int_0^{0}\frac{dt}{a^2+b^2t^2}=0$$

(B)

(i)

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\frac{4^x-e^x}{6^x-1}$$

$$=\lim_{h\to0}\frac{\frac{4^{0-h}-1}{0-h}-\frac{e^{0-h}-1}{0-h}}{\frac{6^{0-h}-1}{0-h}}$$

$$=\frac{\log4-\log e}{\log6}$$

$$=\frac{\log\left(\frac{4}{e}\right)}{\log6}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{4^x - e^x}{6^x - 1}$$

$$= \lim_{h\to 0} \frac{\frac{4^{0+h}-1}{0+h} - \frac{e^{0+h}-1}{0+h}}{\frac{6^{0+h}-1}{0+h}}$$

$$= \frac{\log 4 - \log e}{\log 6}$$

$$= \frac{\log\left(\frac{4}{e}\right)}{\log 6}$$

$\text{LHL} = \text{RHL}$ at $x = 0$.

But $f(0) \neq \lim_{x\to 0} f(x)$.

Hence, the given function has removable discontinuity at $x = 0$.

To remove the discontinuity, we define $f(0) = \dfrac{\log\left(\frac{4}{e}\right)}{\log 6}$

So the revised function becomes

$$f(x) = \begin{cases} \dfrac{4^x - e^x}{6^x - 1}, & x \neq 0 \\ \dfrac{\log\left(\frac{4}{e}\right)}{\log 6}, & x = 0 \end{cases}$$

(ii)

$$\int \sqrt{a^2 - x^2}\,dx$$

Substitute $x = a\sin\theta$...(i)

$\Rightarrow dx = a\cos\theta d\theta$

The integral becomes

$$\int \sqrt{a^2 - a^2\sin^2\theta}\, a\cos\theta d\theta$$

$$= \int a\sqrt{1-\sin^2\theta}\, a\cos\theta d\theta$$

$$= a^2\int \cos^2\theta d\theta$$

$$= a^2\int \frac{1+\cos 2\theta}{2} d\theta$$

$$= a^2\left[\int \frac{1}{2} d\theta + \int \frac{\cos 2\theta}{2} d\theta\right]$$

$$= \frac{a^2\theta}{2} + \frac{a^2}{4}\sin 2\theta + C$$

From (i), $\theta = \sin^{-1}\left(\frac{x}{a}\right)$, $\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{x}{a}\right)\sqrt{1-\frac{x^2}{a^2}} = \frac{2x}{a^2}\sqrt{a^2-x^2}$

Substituting these values, we get

$$\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{4}\times\frac{2x}{a^2}\sqrt{a^2-x^2}+C$$

$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C$ (Proved)

(iii)

Let θ be the temperature of the body at any time t.

Temperature of air is given to be $10^\circ C$.

According to Newton's law of cooling, we have

$$\frac{d\theta}{dt} \propto \theta - 10^\circ$$

$$\Rightarrow \frac{d\theta}{dt} = -k\left(\theta - 10^\circ\right),\ k>0$$

$$\Rightarrow \frac{d\theta}{\theta - 10^\circ} = -k\,dt$$

Integrating both sides, we get

$$\int\frac{d\theta}{\theta-10^\circ} = -k\int dt$$

$$\Rightarrow \ln\left(\theta-10^\circ\right) = -kt + C$$

$$\Rightarrow \theta = 10^\circ + e^{-kt+C}\ldots(1)$$

When $t=0$, $\theta = 110^\circ$.

Substituting in the equation, we get

$$110^\circ = 10^\circ + e^{-k(0)+C}$$

$$\Rightarrow e^{C} = 100^\circ$$

Substituting the above in (1), we get

$$\theta = 10^\circ + 100^\circ e^{-kt}\ldots(2)$$

As per the data in the question,

$$60^\circ = 10^\circ + 100^\circ e^{-k(1)}$$

$$\Rightarrow 50^\circ = 100^\circ e^{-k(1)}\ldots(3)$$

$$\Rightarrow e^{-k} = \frac{1}{2}$$

$$\Rightarrow k = \ln 2$$

$$35^\circ = 10^\circ + 100^\circ e^{-kt}$$

$$\Rightarrow 25^\circ = 100^\circ e^{-kt}\ldots(4)$$

Dividing (4) by (3), we get

$2 = e^{-k(1-t)}$

$\Rightarrow \ln 2 = -k(1-t)$

$\Rightarrow t - 1 = \dfrac{\ln 2}{k} = \dfrac{\ln 2}{\ln 2} = 1$

Hence, additional time required for cooling from 60° to 35° is 1 hour.

6. (A)

(i)

$$\text{LHS} = \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx \ldots(1)$$

Substitute $x = a + t$ in the second integral.

$\Rightarrow dx = dt$

When x = a, t = 0.

When x = 2a, t = a.

$$\therefore \int_a^{2a} f(x)dx = \int_0^a f(a+t)dt$$

$$= \int_0^a f(a+(a-t))dt \quad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$= \int_0^a f(2a-t)dt$$

$$\Rightarrow \int_a^{2a} f(x)dx = \int_0^a f(2a-x)dx \quad \left(\because \int_0^a f(t)\,dt = \int_0^a f(x)\,dx\right)$$

Using the above in (1), we get

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

$$= \int_0^a f(x)dx + \int_0^a f(2a-x)dx = \text{RHS (Proved)}$$

(ii)

$$\int \frac{1+\log x}{x(2+\log x)(3+\log x)}dx$$

Substitute $\log x = t \ldots(1)$

$\Rightarrow \dfrac{1}{x}dx = dt$

Hence, the integral becomes

$$\int \frac{1+t}{(2+t)(3+t)}dt$$

$$= \int \frac{2+t-1}{(2+t)(3+t)} dt$$

$$= \int \frac{2+t}{(2+t)(3+t)} dt - \int \frac{1}{(2+t)(3+t)} dt$$

$$= \int \frac{1}{3+t} dt - \int \frac{(t+3)-(t+2)}{(2+t)(3+t)} dt$$

$$= \int \frac{1}{3+t} dt - \left[\int \frac{(t+3)}{(2+t)(3+t)} dt - \int \frac{(t+2)}{(2+t)(3+t)} dt \right]$$

$$= \int \frac{1}{3+t} dt - \int \frac{1}{2+t} dt + \int \frac{1}{3+t} dt$$

$$= 2\int \frac{1}{3+t} dt - \int \frac{1}{2+t} dt$$

$$= 2\ln(3+t) - \ln(2+t) + C$$

Substituting the value of 't' from (1), we get

$$\int \frac{1+\log x}{x(2+\log x)(3+\log x)} dx$$

$$= 2\ln(3+\log x) - \ln(2+\log x) + C$$

(iii)

$$y = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$$

Substitute $x = \sin\theta$

$$y = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$= \cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$= \cos^{-1}\left(2\sin\theta\cos\theta\right)$$

$$= \cos^{-1}\left(\sin 2\theta\right)$$

$$= \cos^{-1}\left(\cos\left(\frac{\pi}{2} - 2\theta\right)\right)$$

$$\Rightarrow y = \frac{\pi}{2} - 2\theta = \frac{\pi}{2} - 2\sin^{-1} x$$

Differentiating with respect to 'x', we get

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-x^2}}$$

(B)

(i)

$$\frac{dy}{dx}=\frac{1}{\cos(x+y)}$$

Substitute $x+y=v$...(1)

Differentiating w.r.t. 'x', we get

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$$

The original differential equation becomes

$$\frac{dv}{dx}-1=\frac{1}{\cos v}$$

$$\Rightarrow \frac{dv}{dx}=\frac{1+\cos v}{\cos v}$$

$$\Rightarrow \int\frac{\cos v dv}{1+\cos v}=\int dx$$

$$\Rightarrow \int\frac{(\cos v+1-1)dv}{1+\cos v}=x+C$$

$$\Rightarrow \int\frac{1+\cos v}{1+\cos v}dv-\int\frac{1}{1+\cos v}dv=x+C$$

$$\Rightarrow \int dv-\int\frac{1}{2\cos^2\frac{v}{2}}dv=x+C$$

$$\Rightarrow v-\frac{1}{2}\int\sec^2\frac{v}{2}dv=x+C$$

$$\Rightarrow v-\frac{1}{2}\tan\frac{v}{2}=x+C$$

Resubstituting the value of 'v' from (1), we get

$$x+y-\frac{1}{2}\tan\frac{x+y}{2}=x+C$$

$$\Rightarrow y=\frac{1}{2}\tan\frac{x+y}{2}+C$$

For $x=0$ and $y=0$, we get C=0

Hence, particular solution is $y=\frac{1}{2}\tan\frac{x+y}{2}$

(ii)

Length of the wire is 'l'.

Let the part bent to make circle is of length 'x',

and the part bent to make square is of length '$l-x$'.

Circumference of the circle $=2\pi r=x$

$$\Rightarrow r=\frac{x}{2\pi}$$

Area of the circle $= \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$

Perimeter of the square $= 4a = l - x \Rightarrow a = \frac{l-x}{4}$

Area of the square $= \left(\frac{l-x}{4}\right)^2 = \frac{(l-x)^2}{16}$

Sum of the areas $A(x) = \frac{x^2}{4\pi} + \frac{(l-x)^2}{16}$

For extrema, $\frac{dA(x)}{dx} = 0$

$\Rightarrow \frac{2x}{4\pi} + \frac{2(l-x)(-1)}{16} = 0$

$\Rightarrow \frac{4(2x) + 2\pi(x-l)}{16\pi} = 0$

$\Rightarrow 4x + \pi x - \pi l = 0$

$\Rightarrow x = \frac{\pi l}{4+\pi}$

Since there is one point of extremum, it has to be the minimum in this case.

$r = \frac{x}{2\pi} = \frac{l}{2(4+\pi)}$...(1)

Side of the square $a = \frac{l-x}{4} = \frac{l - \frac{\pi l}{4+\pi}}{4} = \frac{l}{4+\pi}$...(2)

From (1) and (2), we get that the radius of the circle is half the side of the square, for least sum of areas. (Proved)

(iii)

(a) c.d.f. of a continuous random variable X is given by

$F(x) = \int_{-\infty}^{x} f(y)dy$

In the given density function f(x), range of X starts at '0'.

$\therefore F(x) = \int_0^x f(y)dy = \int_0^x \frac{y}{32}dy = \left[\frac{y^2}{64}\right]_0^x = \frac{x^2}{64}$

Thus, $F(x) = \frac{x^2}{64},\ \forall x \in R$

(b) $F(0.5) = \frac{0.5^2}{64} = \frac{1}{256}$

For any value of $x \geq 8,\ F(x)=1$

$\therefore F(9)=1$