Chapter 4

DETERMINANTS

4.1 Overview

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number (real or complex) called determinant of the matrix A, written as det A, where a_{ij} is the $(i, j)th$ element of A.

If $A = \begin{matrix} a & b \\ c & d \end{matrix}$, then determinant of A, denoted by |A| (or det A), is given by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Remarks

(i) Only square matrices have determinants.

(ii) For a matrix A, $|A|$ is read as determinant of A and not, as modulus of A.

4.1.1 *Determinant of a matrix of order one*

Let $A = [a]$ be the matrix of order 1, then determinant of A is defined to be equal to a.

4.1.2 *Determinant of a matrix of order two*

Let $A = [a_{ij}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as: det (A) = |A| = $ad - bc$.

4.1.3 *Determinant of a matrix of order three*

The determinant of a matrix of order three can be determined by expressing it in terms of second order determinants which is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows (R_1, R_2 and R_3) and three columns (C_1, C_2 and C_3) and each way gives the same value.

Consider the determinant of a square matrix $A = [a_{ij}]_{3\times3}$, i.e.,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding |A| along C_1, we get

$$|A| = a_{11}\ (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21}\ (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\ (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$= a_{11}(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{21}\, (a_{12}\, a_{33} - a_{13}\, a_{32}) + a_{31}\, (a_{12}\, a_{23} - a_{13}\, a_{22})$$

Remark In general, if $A = kB$, where A and B are square matrices of order n, then $|A| = k^n\ |B|$, $n = 1, 2, 3$.

4.1.4 *Properties of Determinants*

For any square matrix A, |A| satisfies the following properties.

(i) $|A'| = |A|$, where A' = transpose of matrix A.

(ii) If we interchange any two rows (or columns), then sign of the determinant changes.

(iii) If any two rows or any two columns in a determinant are identical (or proportional), then the value of the determinant is zero.

(iv) Multiplying a determinant by k means multiplying the elements of only one row (or one column) by k.

(v) If we multiply each element of a row (or a column) of a determinant by constant k, then value of the determinant is multiplied by k.

(vi) If elements of a row (or a column) in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.

(vii) If to each element of a row (or a column) of a determinant the equimultiples of corresponding elements of other rows (columns) are added, then value of determinant remains same.

Notes:

(i) If all the elements of a row (or column) are zeros, then the value of the determinant is zero.

(ii) If value of determinant 'Δ' becomes zero by substituting $x = \alpha$, then $x - \alpha$ is a factor of 'Δ'.

(iii) If all the elements of a determinant above or below the main diagonal consists of zeros, then the value of the determinant is equal to the product of diagonal elements.

4.1.5 *Area of a triangle*

Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

4.1.6 *Minors and co-factors*

(i) Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting i^{th} row and j^{th} column, and it is denoted by M_{ij}.

(ii) Co-factor of an element a_{ij} is given by $A_{ij} = (-1)^{i+j} M_{ij}$.

(iii) Value of determinant of a matrix A is obtained by the sum of products of elements of a row (or a column) with corresponding co-factors. For example

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

(iv) If elements of a row (or column) are multiplied with co-factors of elements of any other row (or column), then their sum is zero. For example,

$$a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0.$$

4.1.7 *Adjoint and inverse of a matrix*

(i) The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix

$[a_{ij}]_{n\times n}$, where A_{ij} is the co-factor of the element a_{ij}. It is denoted by *adj* A.

If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then $adj\ A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}$, where A_{ij} is co-factor of a_{ij}.

(ii) A (*adj* A) = (*adj* A) A = |A| I, where A is square matrix of order n.

(iii) A square matrix A is said to be singular or non-singular according as |A| = 0 or |A| ≠ 0, respectively.

(iv) If A is a square matrix of order n, then $|adj\ A| = |A|^{n-1}$.

(v) If A and B are non-singular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

(vi) The determinant of the product of matrices is equal to product of their respective determinants, that is, |AB| = |A| |B|.

(vii) If AB = BA = I, where A and B are square matrices, then B is called inverse of A and is written as $B = A^{-1}$. Also $B^{-1} = (A^{-1})^{-1} = A$.

(viii) A square matrix A is invertible if and only if A is non-singular matrix.

(ix) If A is an invertible matrix, then $A^{-1} = \frac{1}{|A|}(adj\ A)$

4.1.8 *System of linear equations*

(i) Consider the equations:

$$a_1x + b_1\,y + c_1\,z = d_1$$
$$a_2x + b_2\,y + c_2\,z = d_2$$
$$a_3x + b_3\,y + c_3\,z = d_3,$$

In matrix form, these equations can be written as A X = B, where

$$A = \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}, X = \begin{matrix} x \\ y \\ z \end{matrix} \text{ and } B = \begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix}$$

(ii) Unique solution of equation AX = B is given by $X = A^{-1}B$, where |A| ≠ 0.

(iii) A system of equations is consistent or inconsistent according as its solution exists or not.

(iv) For a square matrix A in matrix equation AX = B

(a) If $|A| \neq 0$, then there exists unique solution.

(b) If $|A| = 0$ and (*adj* A) B $\neq 0$, then there exists no solution.

(c) If $|A| = 0$ and (*adj* A) B = 0, then system may or may not be consistent.

4.2 Solved Examples

Short Answer (S.A.)

Example 1 If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$, then find x.

Solution We have $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$. This gives

$$2x^2 - 40 = 18 - 40 \quad \Rightarrow x^2 = 9 \Rightarrow \quad x = \pm 3.$$

Example 2 If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then prove that $\Delta + \Delta_1 = 0$.

Solution We have $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

Interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix} = \frac{1}{xyz} \begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$$

$$= \frac{xyz}{xyz}\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} \quad \text{Interchanging } C_1 \text{ and } C_2$$

$$= (-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = -\Delta$$

$$\Rightarrow \quad \Delta_1 + \Delta = 0$$

Example 3 Without expanding, show that

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \operatorname{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0.$$

Solution Applying $C_1 \to C_1 - C_2 - C_3$, we have

$$\Delta = \begin{vmatrix} \operatorname{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \operatorname{cosec}^2\theta + 1 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0$$

Example 4 Show that $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x - p)\,(x^2 + px - 2q^2)$

Solution Applying $C_1 \to C_1 - C_2$, we have

$$\Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} = (x-p)\begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix}$$

$$= (x-p)\begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix} \quad \text{Applying } R_1 \to R_1 + R_2$$

Expanding along C_1, we have

$$\Delta = (x-p)(px + x^2 - 2q^2) = (x-p)(x^2 + px - 2q^2)$$

Example 5 If $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$, then show that Δ is equal to zero.

Solution Interchanging rows and columns, we get $\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$

Taking '–1' common from R_1, R_2 and R_3, we get

$$\Delta = (-1)^3 \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = -\Delta$$

$$\Rightarrow \quad 2\Delta = 0 \quad \text{or} \quad \Delta = 0$$

Example 6 Prove that $(A^{-1})' = (A')^{-1}$, where A is an invertible matrix.

Solution Since A is an invertible matrix, so it is non-singular.

We know that $|A| = |A'|$. But $|A| \neq 0$. So $|A'| \neq 0$ i.e. A' is invertible matrix.

Now we know that $AA^{-1} = A^{-1} A = I$.

Taking transpose on both sides, we get $(A^{-1})' A' = A' (A^{-1})' = (I)' = I$

Hence $(A^{-1})'$ is inverse of A', i.e., $(A')^{-1} = (A^{-1})'$

Long Answer (L.A.)

Example 7 If $x = -4$ is a root of $\Delta = \begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0$, then find the other two roots.

Solution Applying $R_1 \rightarrow (R_1 + R_2 + R_3)$, we get

$$\begin{vmatrix} x+4 & x+4 & x+4 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}.$$

Taking $(x + 4)$ common from R_1, we get

$$\Delta = (x+4)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = (x+4)\begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & 0 \\ 3 & -1 & x-3 \end{vmatrix}.$$

Expanding along R_1,

$\Delta = (x + 4)\ [(x - 1)\ (x - 3) - 0]$. Thus, $\Delta = 0$ implies

$$x = -4, 1, 3$$

Example 8 In a triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0,$$

then prove that ΔABC is an isoceles triangle.

Solution Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ -\cos^2 A & -\cos^2 B & -\cos^2 C \end{vmatrix} \quad R_3 \to R_3 - R_2$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin B \\ -\cos^2 A & \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C \end{vmatrix}. \ (C_3 \to C_3 - C_2 \text{ and } C_2 \to C_2 - C_1)$$

Expanding along R_1, we get

Δ = (sinB – sinA) (sin²C – sin²B) – (sinC – sin B) (sin²B – sin²A)

= (sinB – sinA) (sinC – sinB) (sinC – sin A) = 0

$\Rightarrow$ either sinB – sinA = 0 or sinC – sinB or sinC – sinA = 0

$\Rightarrow$ A = B or B = C or C = A

i.e. triangle ABC is isoceles.

Example 9 Show that if the determinant $\Delta = \begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0$, then $\sin\theta = 0$ or $\frac{1}{2}$.

Solution Applying $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 + 7R_1$, we get

$$\begin{vmatrix} 3 & -2 & \sin 3\theta \\ 5 & 0 & \cos 2\theta + 4\sin 3\theta \\ 10 & 0 & 2 + 7\sin 3\theta \end{vmatrix} = 0$$

or $2\,[5\,(2 + 7\sin 3\theta) - 10\,(\cos 2\theta + 4\sin 3\theta)] = 0$

or $2 + 7\sin 3\theta - 2\cos 2\theta - 8\sin 3\theta = 0$

or $2 - 2\cos 2\theta - \sin 3\theta = 0$

$\sin\theta\ (4\sin^2\theta + 4\sin\theta - 3) = 0$

or $\sin\theta = 0$ or $(2\sin\theta - 1) = 0$ or $(2\sin\theta + 3) = 0$

or $\sin\theta = 0$ or $\sin\theta = \dfrac{1}{2}$ (Why ?).

Objective Type Questions

Choose the correct answer from the given four options in each of the Example 10 and 11.

Example 10 Let $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$, then

(A) $\Delta_1 = -\Delta$ (B) $\Delta \neq \Delta_1$

(C) $\Delta - \Delta_1 = 0$ (D) None of these

Solution (C) is the correct answer since $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \begin{vmatrix} A & x & yz \\ B & y & zx \\ C & z & xy \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} Ax & x^2 & xyz \\ By & y^2 & xyz \\ Cz & z^2 & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta$$

Example 11 If $x, y \in \mathbf{R}$, then the determinant $\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x+y) & -\sin(x+y) & 0 \end{vmatrix}$ lies in the interval

(A) $-\sqrt{2}, \sqrt{2}$ (B) $[-1, 1]$

(C) $-\sqrt{2}, 1$ (D) $-1, -\sqrt{2},$

Solution The correct choice is A. Indeed applying $R_3 \to R_3 - \cos y R_1 + \sin y R_2$, we get

$$\Delta = \begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ 0 & 0 & \sin y - \cos y \end{vmatrix}.$$

Expanding along R_3, we have

$$\Delta = (\sin y - \cos y)\ (\cos^2 x + \sin^2 x)$$

$$= (\sin y - \cos y) = \sqrt{2}\ \frac{1}{\sqrt{2}}\sin y - \frac{1}{\sqrt{2}}\cos y$$

$$= \sqrt{2}\ \cos\frac{\pi}{4}\sin y - \sin\frac{\pi}{4}\cos y \qquad = \sqrt{2}\ \sin\left(y - \frac{\pi}{4}\right)$$

Hence $-\sqrt{2} \le \Delta \le \sqrt{2}$.

Fill in the blanks in each of the Examples 12 to 14.

Example 12 If A, B, C are the angles of a triangle, then

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} = ...$$

Solution Answer is 0. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$.

Example 13 The determinant $\Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$ is equal to ....

Solution Answer is 0.Taking $\sqrt{5}$ common from C_2 and C_3 and applying $C_1 \to C_3 - \sqrt{3}\ C_2$, we get the desired result.

Example 14 The value of the determinant

$$\Delta = \begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix} = \text{..........}$$

Solution $\Delta = 0$. Apply $C_1 \to C_1 + C_2 + C_3$.

State whether the statements in the Examples 15 to 18 is **True** or **False**.

Example 15 The determinant

$$\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & \cos y \end{vmatrix}$$

is independent of x only.

Solution True. Apply $R_1 \to R_1 + \sin y R_2 + \cos y\, R_3$, and expand

Example 16 The value of

$$\begin{vmatrix} 1 & 1 & 1 \\ {}^{n}C_1 & {}^{n+2}C_1 & {}^{n+4}C_1 \\ {}^{n}C_2 & {}^{n+2}C_2 & {}^{n+4}C_2 \end{vmatrix} \text{ is 8.}$$

Solution True

Example 17 If $A = \begin{matrix} x & 5 & 2 \\ 2 & y & 3 \\ 1 & 1 & z \end{matrix}$, $xyz = 80$, $3x + 2y + 10z = 20$, then

$$A\ adj.\ A = \begin{matrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{matrix}.$$

Solution : False.

Example 18 If $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{1}{2} & -4 & \frac{5}{2} \\ -\frac{1}{2} & 3 & -\frac{3}{2} \\ \frac{1}{2} & y & \frac{1}{2} \end{bmatrix}$

then $x = 1$, $y = -1$.

Solution True

4.3 EXERCISE

Short Answer (S.A.)

Using the properties of determinants in Exercises 1 to 6, evaluate:

1. $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

2. $\begin{vmatrix} a + x & y & z \\ x & a + y & z \\ x & y & a + z \end{vmatrix}$

3. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2 y & 0 & yz^2 \\ x^2 z & zy^2 & 0 \end{vmatrix}$

4. $\begin{vmatrix} 3x & -x + y & -x + z \\ x - y & 3y & z - y \\ x - z & y - z & 3z \end{vmatrix}$

5. $\begin{vmatrix} x + 4 & x & x \\ x & x + 4 & x \\ x & x & x + 4 \end{vmatrix}$

6. $\begin{vmatrix} a - b - c & 2a & 2a \\ 2b & b - c - a & 2b \\ 2c & 2c & c - a - b \end{vmatrix}$

Using the proprties of determinants in Exercises 7 to 9, prove that:

7. $\begin{vmatrix} y^2 z^2 & yz & y + z \\ z^2 x^2 & zx & z + x \\ x^2 y^2 & xy & x + y \end{vmatrix} = 0$

8. $\begin{vmatrix} y + z & z & y \\ z & z + x & x \\ y & x & x + y \end{vmatrix} = 4xyz$

9. $$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

10. If A + B + C = 0, then prove that $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$

11. If the co-ordinates of the vertices of an equilateral triangle with sides of length 'a' are (x_1, y_1), (x_2, y_2), (x_3, y_3), then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \dfrac{3a^4}{4}$.

12. Find the value of θ satisfying $\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0$.

13. If $\begin{matrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{matrix} = 0$, then find values of x.

14. If $a_1, a_2, a_3, \ldots, a_r$ are in G.P., then prove that the determinant

$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ is independent of r.

15. Show that the points $(a + 5, a - 4)$, $(a - 2, a + 3)$ and (a, a) do not lie on a straight line for any value of a.

16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta=\begin{bmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{bmatrix}=0 .$$

17. Find A^{-1} if $A=\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$ and show that $A^{-1}=\dfrac{A^2-3I}{2}$.

Long Answer (L.A.)

18. If $A=\begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}.

Using A^{-1}, solve the system of linear equations $x-2y=10$, $2x-y-z=8$, $-2y+z=7$.

19. Using matrix method, solve the system of equations $3x+2y-2z=3$, $x+2y+3z=6$, $2x-y+z=2$.

20. Given $A=\begin{matrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{matrix}$, $B=\begin{matrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{matrix}$, find BA and use this to solve the system of equations $y+2z=7$, $x-y=3$, $2x+3y+4z=17$.

21. If $a+b+c\neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$, then prove that $a=b=c$.

22. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ is divisible by $a+b+c$ and find the quotient.

23. If $x + y + z = 0$, prove that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Objective Type Questions (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

24. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then value of x is

(A) 3 (B) ± 3

(C) ± 6 (D) 6

25. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$

(A) $a^3 + b^3 + c^3$ (B) $3\ bc$

(C) $a^3 + b^3 + c^3 - 3abc$ (D) none of these

26. The area of a triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is 9 sq. units. The value of k will be

(A) 9 (B) 3

(C) -9 (D) 6

27. The determinant $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ equals

(A) $abc\ (b-c)\ (c-a)\ (a-b)$ (B) $(b-c)\ (c-a)\ (a-b)$

(C) $(a+b+c)\ (b-c)\ (c-a)\ (a-b)$ (D) None of these

28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(A) 0 (B) 2

(C) 1 (D) 3

29. If A, B and C are angles of a triangle, then the determinant

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is equal to}$$

(A) 0 (B) – 1

(C) 1 (D) None of these

30. Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then $\lim_{t \to 0} \frac{f(t)}{t^2}$ is equal to

(A) 0 (B) – 1

(C) 2 (D) 3

31. The maximum value of $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$ is (θ is real number)

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$

(C) $\sqrt{2}$ (D) $\frac{2\sqrt{3}}{4}$

32. If $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

(A) $f(a) = 0$ (B) $f(b) = 0$

(C) $f(0) = 0$ (D) $f(1) = 0$

33. If A = $\begin{matrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{matrix}$, then A^{-1} exists if

(A) $\lambda = 2$ (B) $\lambda \neq 2$

(C) $\lambda \neq -2$ (D) None of these

34. If A and B are invertible matrices, then which of the following is not correct?

(A) $adj\ A = |A|.\ A^{-1}$ (B) $\det(A)^{-1} = [\det(A)]^{-1}$

(C) $(AB)^{-1} = B^{-1}A^{-1}$ (D) $(A + B)^{-1} = B^{-1} + A^{-1}$

35. If x, y, z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then value of $x^{-1} + y^{-1} + z^{-1}$ is

(A) $x\,y\,z$ (B) $x^{-1}\ y^{-1}\ z^{-1}$

(C) $-x\ -y\ -z$ (D) -1

36. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(A) $9x^2(x + y)$ (B) $9y^2(x + y)$

(C) $3y^2(x + y)$ (D) $7x^2(x + y)$

37. There are two values of a which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then sum of these number is

(A) 4 (B) 5

(C) – 4 (D) 9

Fill in the blanks

38. If A is a matrix of order 3 × 3, then $|3A|$ = _______ .

39. If A is invertible matrix of order 3 × 3, then $|A^{-1}|$ _______ .

40. If $x, y, z \in$ R, then the value of determinant $\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$ is equal to _______.

41. If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$ = _________.

42. If A is a matrix of order 3 × 3, then $(A^2)^{-1}$ = ________.

43. If A is a matrix of order 3 × 3, then number of minors in determinant of A are ________.

44. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to _________.

45. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are __________.

46. $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$ = __________.

47. If $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = \text{A} + \text{B}x + \text{C}x^2 + \ldots$, then

A = ________.

State True or False for the statements of the following Exercises:

48. $\left(\text{A}^3\right)^{-1} = \left(\text{A}^{-1}\right)^3$, where A is a square matrix and $|\text{A}| \neq 0$.

49. $(a\text{A})^{-1} = \dfrac{1}{a}\text{A}^{-1}$, where a is any real number and A is a square matrix.

50. $|\text{A}^{-1}| \neq |\text{A}|^{-1}$, where A is non-singular matrix.

51. If A and B are matrices of order 3 and $|\text{A}| = 5$, $|\text{B}| = 3$, then $|3\text{AB}| = 27 \times 5 \times 3 = 405$.

52. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.

53. $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, where a, b, c are in A.P.

54. $|adj.\ \text{A}| = |\text{A}|^2$, where A is a square matrix of order two.

55. The determinant $\begin{vmatrix} \sin \text{A} & \cos \text{A} & \sin \text{A} + \cos \text{B} \\ \sin \text{B} & \cos \text{A} & \sin \text{B} + \cos \text{B} \\ \sin \text{C} & \cos \text{A} & \sin \text{C} + \cos \text{B} \end{vmatrix}$ is equal to zero.

56. If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly K determinants of order 3, each element of which contains only one term, then the value of K is 8.

57. Let $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$, then $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$.

58. The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is $\frac{1}{2}$.