Mathematics Class-X

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Foreword

Education is a process of human enlightenment and empowerment. Recognizing the enormous potential of education, all progressive societies have committed themselves to the Universalization of Elementary Education with a strong determination to provide quality education to all. As a part of its continuation, universalization of Secondary Education has gained momentum.

In the secondary stage, the beginning of the transition from functional mathematics studied upto the primary stage to the study of mathematics as a discipline takes place. The logical proofs of propositions, theorems etc. are introduced at this stage. Apart from being a specific subject, it is connected to other subjects involving analysis and through concomitant methods. It is important that children finish the secondary level with the sense of confidence to use mathematics in organising experience and motivation to continue learning in High level and become good citizens of India.

I am confident that the children in our state Andhra Pradesh learn to enjoy mathematics, make mathematics a part of their life experience, pose and solve meaningful problems, understand the basic structure of mathematics by reading this text book.

For teachers, to understand and absorb critical issues on curricular and pedagogic perspectives duly focusing on learning in place of marks, is the need of the hour. Also coping with a mixed class room environment is essentially required for effective transaction of curriculum in teaching learning process. Nurturing class room culture to inculcate positive interest among children with difference in opinions and presumptions of life style, to infuse life in to knowledge is a thrust in the teaching job.

The afore said vision of mathematics teaching presented in Andhra Pradesh State Curriculum Frame work (APSCF -2011) has been elaborated in its mathematics position paper which also clearly lays down the academic standards of mathematics teaching in the state. The text books make an attempt to concretize all the sentiments.

The State Council for Education Research and Training Andhra Pradesh appreciates the hard work of the text book development committee and several teachers from all over the state who have contributed to the development of this text book. I am thankful to the District Educational Officers, Mandal Educational Officers and head teachers for making this possible. I also thank the institutions and organizations which have given their time in the development of this text book. I am grateful to the office of the Commissioner and Director of School Education for extending co-operation in developing this text book. In the endeavor to continuously improve the quality of our work, we welcome your comments and suggestions in this regard.

Place : Hyderabad
Date : 17 October, 2013

Director
SCERT, A.P., Hyderabad

Preface

With this Mathematics book, children would have completed the three years of learning in the elimentary classes and one year of secondary class. We hope that Mathematics learning continues for all children in class X also however, there may be some children from whom this would be the last year of school. It is, therefore, important that children finish the secondary level with a sense of confidence to use Mathematics in organizing experience and motivation to continue learning.

Mathematics is essential for everyone and is a part of the compulsory program for school education till the secondary stage. However, at present, Mathematics learning does not instill a feeling of comfort and confidence in children and adults. It is considered to be extremely difficult and only for a few. The fear of Mathematics pervades not just children and teachers but our entire society. In a context where Mathematics is an increasing part of our lives and is important for furthering our learning, this fear has to be removed. The effort in school should be to empower children and make them feel capable of learning and doing Mathematics. They should not only be comfortable with the Mathematics in the classroom but should be able to use it in the wider world by relating concepts and ideas of Mathematics to formulate their understanding of the world.

One of the challenges that Mathematics teaching faces is in the way it is defined. The visualization of Mathematics remains centered around numbers, complicated calculations, algorithms, definitions and memorization of facts, short-cuts and solutions including proofs. Engaging with exploration and new thoughts is discouraged as the common belief is that there can be only one correct way to solve a problem and that Mathematics does not have possibilities of multiple solutions.

Through this book we want to emphasize the need for multiple ways of attempting problems, understanding that Mathematics is about exploring patterns, discovering relationships and building logic. We would like all teachers to engage students in reading the book and help them in formulating and articulating their understanding of different concepts as well as finding a variety of solutions for each problem. The emphasis in this book is also on allowing children to work with each other in groups and make an attempt to solve problems collectively. We want them to talk to each other about Mathematics and create problems based on the concepts that have learnt. We want everybody to recognize that Mathematics is not only about solving problems set by others or learning proofs and methods that are developed by others, but is about exploration and building new arguments. Doing and learning Mathematics is therefore about each person coming up with her own methods and own rules.

Class X is the final year of secondary level students and their have already dealt about the consolidation of initiations. They have already learnt also to understand that Mathematics consists of ideas that are applied in life situations but do not necessarily arise from life. We would also like children to be exposed to the notion of proof and recognize that presenting examples is not equivalent to proof with modeling aspects.

The purpose of Mathematics as we have tried to indicate in the preface as well as in the book has widened to include exploring mathematization of experiences. This means that students can begin to relate the seemingly abstract ideas they learn in the classrooms to their own experiences and organize their experiences using these ideas. This requires them to have opportunity to reflect and express both their new formulations as well as their hesitant attempt on mathematizing events around them. We have always emphasized the importance of language and Mathematics interplay. While we have tried to indicate at many places the opportunity that has to be provided to children to reflect and use language. We would emphasise the need to make more of this possible in the classrooms. We have also tried to keep the language simple and close to the language that the child normally uses. We hope that teachers and those who formulate assessment tasks would recognize the spirit of the book. The book has been developed with wide consultations and I must thank all those who have contributed to its development. The group of authors drawn from different experiences have worked really hard and together as a team. I salute each of them and look forward to comments and suggestions of those who would be users of this book.

Text Book Development Committee

Mathematics Class-X

Our National Anthem

- Rabindranath Tagore

Jana-gana-mana-adhinayaka, jaya he
Bharata-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga.
Tava shubha name jage,
Tava shubha asisa mage,
Gahe tava jaya gatha,
Jana-gana-mangala-dayaka jaya he
Bharata-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

Pledge

"India is my country. All Indians are my brothers and sisters.
I love my country, and I am proud of its rich and varied heritage.
I shall always strive to be worthy of it.
I shall give my parents, teachers and all elders respect,
and treat everyone with courtesy. I shall be kind to animals
To my country and my people, I pledge my devotion.
In their well-being and prosperity alone lies my happiness."

CHAPTER 1

Real Numbers

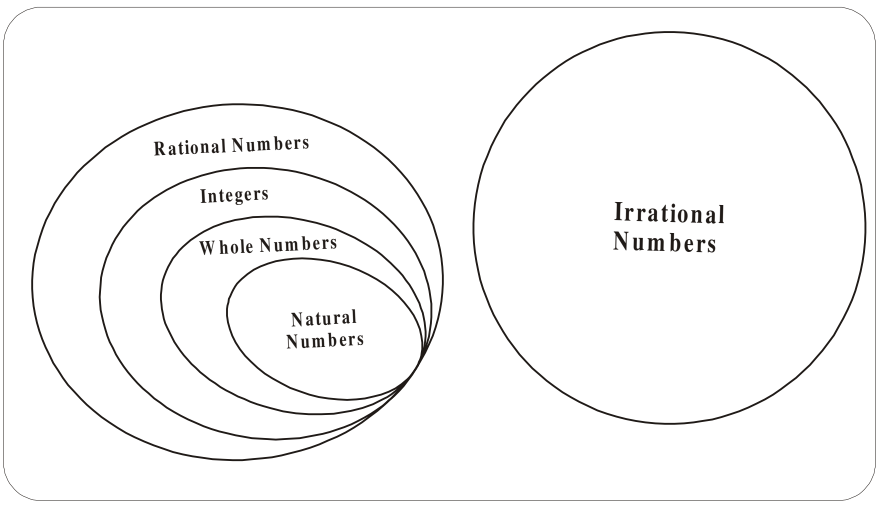
1.1 INTRODUCTION

We have studied different types of numbers in earlier classes. We have learnt about natural numbers, whole numbers, integers, rational numbers and irrational numbers. Let us recall a little bit about rational numbers and irrational numbers.

Rational numbers are numbers which can be written in the form of $\frac{p}{q}$ where both p and q are integers and $q \neq 0$. They are a bigger collection than integers as there can be many rational numbers between two integers. All rational numbers can be written either in the form of terminating decimals or non-terminating repeating decimals.

Numbers which cannot be expressed in the form of $\frac{p}{q}$ are irrational. These include numbers like $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and mathematical quantities like π. When these are written as decimals, they are non-terminaing, non-recurring. For example, $\sqrt{2} = 1.41421356...$ and $\pi = 3.14159...$ These numbers can be located on the number line.

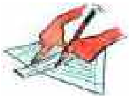
The set of rational and irrational numbers together are called real numbers. We can show them in the form of a diagram:



Real Numbers

In this chapter, we will see some theorems and the different ways in which we can prove them. We will use the theorems to explore properties of rational and irrational numbers. Finally, we will study about a type of function called logarithms (in short logs) and see how they are useful in science and everyday life.

But before exploring real numbers a little more, let us solve some questions.



Exercise - 1.1

1. Which of the following rational numbers are terminating and which are non-terminating, repeating in their deimenal form?

   (i) $\frac{2}{5}$ (ii) $\frac{17}{18}$ (iii) $\frac{15}{16}$ (iv) $\frac{7}{40}$ (v) $\frac{9}{11}$

2. Find any rational number between the pair of numbers given below:

   (i) $\frac{1}{2}$ and $\sqrt{1}$ (ii) $3\frac{1}{3}$ and $3\frac{2}{3}$ (iii) $\sqrt{\frac{4}{9}}$ and $\sqrt{2}$

3. Classify the numbers given below as rational or irrational.

   (i) $2\frac{1}{2}$ (ii) $\sqrt{24}$ (iii) $\sqrt{16}$ (iv) $7.\bar{7}$ (v) $\sqrt{\frac{4}{9}}$ (vi) $-\sqrt{30}$ (vii) $-\sqrt{81}$

4. Represent the following real numbers on the number line. (If necessary make a seperate number line for each number).

   (i) $\frac{3}{4}$ (ii) $\frac{-9}{10}$ (iii) $\frac{27}{3}$ (iv) $\sqrt{5}$ (v) $-\sqrt{16}$

Think - Discuss

Are all integers also in real numbers? Why?

1.2 Exploring Real Numbers

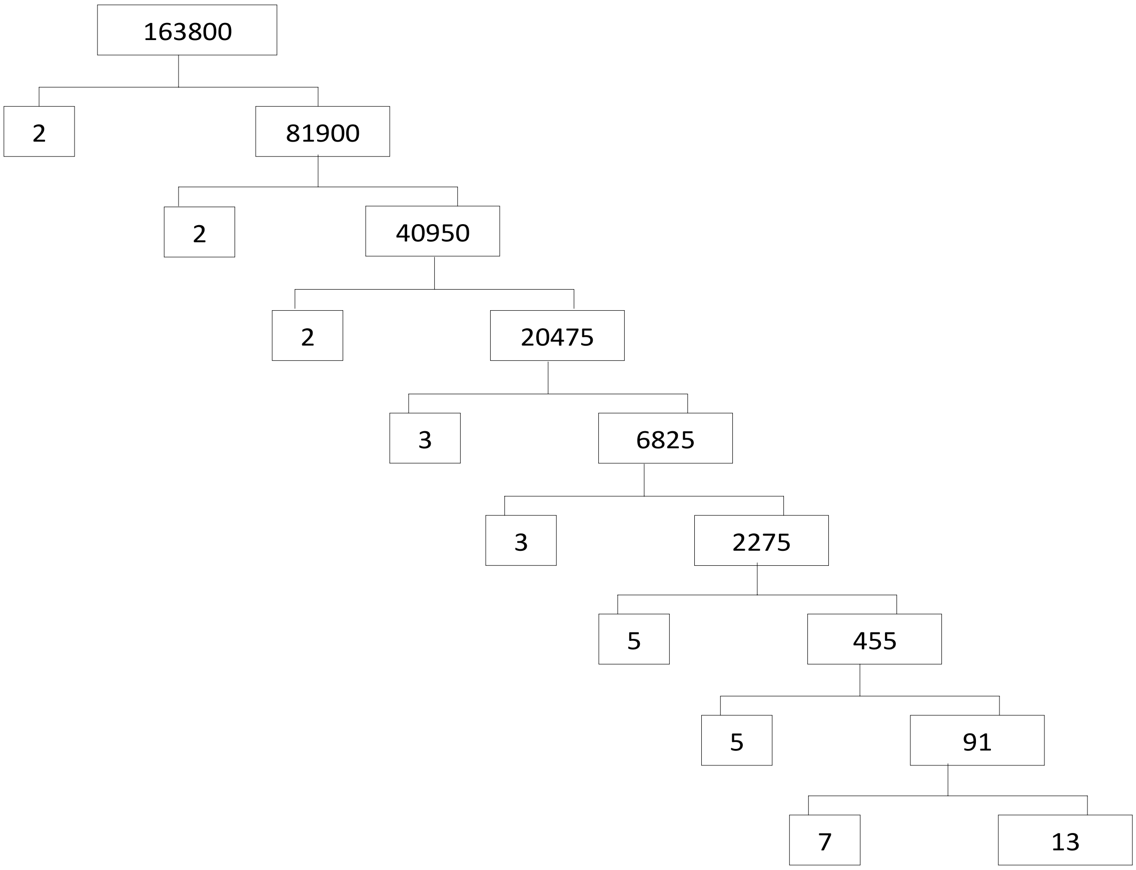
Let us explore real numbers more in this section. We know that natural numbers are also in real numbers. So, we will start with them.

1.2.1 The Fundamental Theorem of Arithmetic

In earlier classes, we have seen that all natural numbers, except 1, can be written as a product of their prime factors. For example, 3 = 3, 6 as 2×3, 253 as 11×23 and so on. (Remember: 1 is neither a composite nor a prime).

Do you think that there may be a composite number which is not the product of the powers of primes? To answer this, let us factorize a natural number as an example.

We are going to use the factor tree which you all are familiar with. Let us take some large number, say 163800, and factorize it as shown :



So we have factorized 163800 as $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 13$. So 163800 $= 2^3 \times 3^2 \times 5^2 \times 7 \times 13$, when we write it as a product of power of primes.

Try another number, 123456789. This can be written as $3^2 \times 3803 \times 3607$. Of course, you have to check that 3803 and 3607 are primes! (Try it out for several other natural numbers yourself.) This leads us to a conjecture that ***every composite number can be written as the product of powers of primes.***

Now, let us try and look at natural numbers from the other direction. Let us take any collection of prime numbers, say 2, 3, 7, 11 and 23. If we multiply some or all of these numbers, allowing them to repeat as many times as we wish, we can produce infinitely many large positive integers. Let us list a few :

$2 \times 3 \times 11 = 66$ $\qquad$ $7 \times 11 = 77$

$7 \times 11 \times 23 = 1771$ $\quad$ $3 \times 7 \times 11 \times 23 = 5313$

$2 \times 3 \times 7 \times 11 \times 23 = 10626$ $\quad$ $2^3 \times 3 \times 7^3 = 8232$

$2^2 \times 3 \times 7 \times 11 \times 23 = 21252$

Now, let us suppose your collection of primes includes all the possible primes. What is your guess about the size of this collection? Does it contain only a finite number of primes or infinitely many? In fact, there are infinitely many primes. So, if we multiply all these primes in all possible ways, we will get an infinite collection of composite numbers.

This gives us the Fundamental Theorem of Arithmetic which says that every composite number can be factorized as a product of primes. Actually, it says more. It says that given any composite number it can be factorized as a product of prime numbers in a **'unique'** way, except for the order in which the primes occur. For example, when we factorize 210, we regard $2 \times 3 \times 5 \times 7$ as same as $3 \times 5 \times 7 \times 2$, or any other possible order in which these primes are written. That is, given any composite number there is one and only one way to write it as a product of primes, as long as we are not particular about the order in which the primes occur. Let us now formally state this theorem.

Theorem-1.1 : (Fundamental Theorem of Arithmetic) : Every composite number can be expressed (factorised) as a product of primes, and this factorization is unique, apart from the order in which the prime factors occur.

In general, given a composite number x, we factorize it as $x = p_1 p_2 ... p_n$, where $p_1, p_2, ..., p_n$ are primes and written in ascending order, i.e., $p_1 \leq p_2 \leq ... \leq p_n$. If we use the same primes, we will get powers of primes. Once we have decided that the order will be ascending, then the way the number is factorised, is unique. For example,

$$163800 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5^2 \times 7 \times 13$$

TRY THIS

Express 2310 as a product of prime factors. Also see how your friends have factorized the number. Have they done it like you? Verify your final product with your friend's result. Try this for 3 or 4 more numbers. What do you conclude?

While this is a result that is easy to state and understand, it has some very deep and significant applications in the field of mathematics. Let us see two examples.

You have already learnt how to find the HCF (Highest Common Factor) and LCM (Lowest Common Multiple) of two positive integers using the Fundamental Theorem of Arithmetic in earlier classes, without realizing it! This method is also called the *prime factorization method*. Let us recall this method through the following example.

Example-1. Find the HCF and LCM of 12 and 18 by the prime factorization method.

Solution : We have $12 = 2 \times 2 \times 3 = 2^2 \times 3^1$

$18 = 2 \times 3 \times 3 = 2^1 \times 3^2$

Note that HCF $(12, 18) = 2^1 \times 3^1 = 6$ = **Product of the smallest power of each common prime factors in the numbers.**

LCM $(12, 18) = 2^2 \times 3^2 = 36$ = **Product of the greatest power of each prime factors, in the numbers.**

From the example above, you might have noticed that HCF $(12, 18) \times$ LCM $(12, 18) = 12 \times 18$. In fact, we can verify that for any two positive integers a and b, HCF $(a,b) \times$ LCM $(a, b) = a \times b$. We can use this result to find the LCM of two positive integers, if we have already found the HCF of the two positive integers.

Example 2. Consider the numbers 4^n, where *n* is a natural number. Check whether there is any value of *n* for which 4^n ends with the digit zero?

Solution : For the number 4^n to end with digit zero for any natural number n, it should be divisible by 5. This means that the prime factorisation of 4^n should contain the prime number 5. But it is not possible because $4^n = (2)^{2n}$ so 2 is the only prime in the factorisation of 4^n. Since 5 is not present in the prime factorization, so there is no natural number n for which 4^n ends with the digit zero.

Try This

Show that 12^n cannot end with the digit 0 or 5 for any natural number '*n*'.

Exercise - 1.2

1. Express each number as a product of its prime factors.

   (i) 140 (ii) 156 (iii) 3825 (iv) 5005 (v) 7429

2. Find the LCM and HCF of the following integers by the prime factorization method.

   (i) 12, 15 and 21 (ii) 17, 23, and 29 (iii) 8, 9 and 25

   (iv) 72 and 108 (v) 306 and 657

3. Check whether 6^n can end with the digit 0 for any natural number n.
4. Explain why $7 \times 11 \times 13 + 13$ and $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 + 5$ are composite numbers.
5. How will you show that $(17 \times 11 \times 2) + (17 \times 11 \times 5)$ is a composite number? Explain.

Now, let us use the Fundamental Theorem of Arithmetic to explore real numbers further. First, we apply this theorem to find out when the decimal expansion of a rational number is terminating and when it is non-terminating, repeating. Second, we use it to prove the irrationality of many numbers such as $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.

1.2.2 Rational numbers and their decimal expansions

In this section, we are going to explore when their decimal expansions of rational numbers are terminating and when they are non-terminating, repeating.

Let us consider the following terminating decimal forms of some rational numbers:

(i) 0.375 (ii) 1.04 (iii) 0.0875 (iv) 12.5 (v) 0.00025

Now let us express them in the form of $\frac{p}{q}$.

(i) $0.375 = \frac{375}{1000} = \frac{375}{10^3}$ (ii) $1.04 = \frac{104}{100} = \frac{104}{10^2}$

(iii) $0.0875 = \frac{875}{10000} = \frac{875}{10^4}$ (iv) $12.5 = \frac{125}{10} = \frac{125}{10^1}$

(v) $0.00025 = \frac{25}{100000} = \frac{25}{10^5}$

We see that all terminating decimals taken by us can be expressed as rational numbers whose denominators are powers of 10. Let us now prime factorize the numerator and denominator and then express in the simplest rational form :

Now (i) $0.375 = \frac{375}{10^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{3}{2^3} = \frac{3}{8}$

(ii) $$1.04 = \frac{104}{10^2} = \frac{2^3 \times 13}{2^2 \times 5^2} = \frac{26}{5^2} = \frac{26}{25}$$

(iii) $$0.0875 = \frac{875}{10^4} = \frac{5^3 \times 7}{2^4 \times 5^4} = \frac{7}{2^4 \times 5}$$

(iv) $$12.5 = \frac{125}{10} = \frac{5^3}{2 \times 5} = \frac{25}{2}$$

(v) $$0.00025 = \frac{25}{10^5} = \frac{5^2}{2^5 \times 5^5} = \frac{1}{2^5 \times 5^3} = \frac{1}{4000}$$

Do you see a pattern in the denominators? It appears that when the decimal expression is expressed in its simplest rational form then p and q are coprime and the denominator (i.e., q) has only powers of 2, or powers of 5, or both. This is because the powers of 10 can only have powers of 2 and 5 as factors.

Do This

Write the following terminating decimals in the form of $\frac{p}{q}$, $q \neq 0$ and p, q are co-primes

(i) 15.265 (ii) 0.1255 (iii) 0.4 (iv) 23.34 (v) 1215.8

What can you conclude about the denominators through this process?

Let us conclude

Even though, we have worked only with a few examples, you can see that any rational number which has a decimal expansion that terminates can be expressed as a rational number whose denominator is a power of 10. The only prime factors of 10 are 2 and 5. So, when we simplyfy the rational number, we find that the number is of the form $\frac{p}{q}$, where the prime factorization of q is of the form 2^n5^m, and n, m are some non-negative integers.

We can write our result formally :

Theorem-1.2 : Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers.

You are probably wondering what happens the other way round. That is, if we have a rational number in the form $\frac{p}{q}$, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers, then does $\frac{p}{q}$ have a terminating decimal expansion?

So, it seems to make sense to convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Let us go back to our examples above and work backwards.

(i) $\frac{25}{2}=\frac{5^3}{2\times5}=\frac{125}{10}=12.5$

(ii) $\frac{26}{25}=\frac{26}{5^2}=\frac{13\times2^3}{2^2\times5^2}=\frac{104}{10^2}=1.04$

(iii) $\frac{3}{8}=\frac{3}{2^3}=\frac{3\times5^3}{2^3\times5^3}=\frac{375}{10^3}=0.375$

(iv) $\frac{7}{80}=\frac{7}{2^4\times5}=\frac{7\times5^3}{2^4\times5^4}=\frac{875}{10^4}=0.0875$

(v) $\frac{1}{4000}=\frac{1}{2^5\times5^3}=\frac{5^2}{2^5\times5^5}=\frac{25}{10^5}=0.00025$

So, these examples show us how we can convert a rational number of the form $\frac{p}{q}$, where q is of the form 2^n5^m, to an equivalent rational number of the form $\frac{a}{b}$, where b is a power of 10. Therefore, the decimal expansion of such a rational number terminates. We find that a rational number of the form $\frac{p}{q}$, where q is a power of 10, will have terminating decimal expansion.

So, we find that the converse of theorem 12 is also true and can be formally stated as :

Theorem 1.3 : Let $x=\frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

Do This

Write the following rational numbers in the form of $\frac{p}{q}$, where q is of the form 2^n5^m where n, m are non-negative integers and then write the numbers in their decimal form

(i) $\frac{3}{4}$ (ii) $\frac{7}{25}$ (iii) $\frac{51}{64}$ (iv) $\frac{14}{23}$ (v) $\frac{80}{81}$

1.2.3 Non-terminating, recurring decimals in rational numbers

Let us now consider rational numbers whose decimal expansions are non-terminating and recurring. Once again, let us look at an example to see what is going on-

Let us look at the decimal conversion of $\frac{1}{7}$.

$\frac{1}{7} = 0.1428571428571$ ..... which is a non-terminating and recurring decimal. Notice, the block of digits '142857' is repeating in the quotient.

Notice that the denominator here, i.e., 7 is not of the form 2^n5^m.

$$
\begin{array}{r}
0.1428571 \\
7\overline{)1.0000000} \\
7 \\
\hline
30 \\
28 \\
\hline
20 \\
14 \\
\hline
60 \\
56 \\
\hline
40 \\
35 \\
\hline
50 \\
49 \\
\hline
10 \\
7 \\
\hline
30
\end{array}
$$

Do This

Write the following rational numbers as decimals and find out the block of digits, repeating in the quotient.

(i) $\frac{1}{3}$ (ii) $\frac{2}{7}$ (iii) $\frac{5}{11}$ (iv) $\frac{10}{13}$

From the 'do this exercise' and from the example taken above, we can formally state:

Theorem-1.4 : Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is not of the form 2^n5^m, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).

From the discussion above, we can conclude that the decimal form of every rational number is either terminating or non-terminating repeating.

Example-3. Using the above theorems, without actual division, state whether the following rational numbers are terminating or non-terminating, repeating decimals.

(i) $\frac{16}{125}$ (ii) $\frac{25}{32}$ (iii) $\frac{100}{81}$ (iv) $\frac{41}{75}$

Solution : (i) $\frac{16}{125}=\frac{16}{5\times5\times5}=\frac{16}{5^3}$ is terminating decimal.

(ii) $\frac{25}{32}=\frac{25}{2\times2\times2\times2\times2}=\frac{25}{2^5}$ is terminating decimal.

(iii) $\frac{100}{81}=\frac{100}{3\times3\times3\times3}=\frac{10}{3^4}$ is non-terminating, repeating decimal.

(iv) $\frac{41}{75}=\frac{41}{3\times5\times5}=\frac{41}{3\times5^2}$ is non-terminating, repeating decimal.

Example-4. Write the decimal expansion of the following rational numbers without actual division.

(i) $\frac{35}{50}$ (ii) $\frac{21}{25}$ (iii) $\frac{7}{8}$

Solution : (i) $\frac{35}{50}=\frac{7\times5}{2\times5\times5}=\frac{7}{2\times5}=\frac{7}{10^1}=0.7$

(ii) $\frac{21}{25}=\frac{21}{5\times5}=\frac{21\times2^2}{5\times5\times2^2}=\frac{21\times4}{5^2\times2^2}=\frac{84}{10^2}=0.84$

(iii) $\frac{7}{8}=\frac{7}{2\times2\times2}=\frac{7}{2^3}=\frac{7\times5^3}{\left(2^3\times5^3\right)}=\frac{7\times25}{\left(2\times5\right)^3}=\frac{875}{\left(10\right)^3}=0.875$

Exercise - 1.3

1. Write the following rational numbers in their decimal form and also state which are terminating and which have non-terminating, repeating decimal.

   (i) $\frac{3}{8}$ (ii) $\frac{229}{400}$ (iii) $4\frac{1}{5}$ (iv) $\frac{2}{11}$ (v) $\frac{8}{125}$

2. Without actually performing division, state whether the following rational numbers will have a terminating decimal form or a non-terminating, repeating decimal form.

(i) $\frac{13}{3125}$ (ii) $\frac{11}{12}$ (iii) $\frac{64}{455}$ (iv) $\frac{15}{1600}$ (v) $\frac{29}{343}$

(vi) $\frac{23}{2^3 5^2}$ (vii) $\frac{129}{2^2 5^7 7^5}$ (viii) $\frac{9}{15}$ (ix) $\frac{36}{100}$ (x) $\frac{77}{210}$

3. Write the following rationals in decimal form using Theorem 1.1.

(i) $\frac{13}{25}$ (ii) $\frac{15}{16}$ (iii) $\frac{23}{2^3.5^2}$ (iv) $\frac{7218}{3^2.5^2}$ (v) $\frac{143}{110}$

4. The decimal form of some real numbers are given below. In each case, decide whether the number is rational or not. If it is rational, and expressed in form $\frac{p}{q}$, what can you say about the prime factors of q?

(i) 43.123456789 (ii) 0.120120012000120000… (iii) $43.\overline{123456789}$

1.3 More about irrational numbers

Recall, a real number ("Q^1" or "S") is called *irrational* if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Some examples of irrational numbers, with which you are already familiar, are :

$$\sqrt{2},\ \sqrt{3},\ \sqrt{15},\ \pi,\ -\frac{\sqrt{2}}{\sqrt{3}},\ 0.10110111011110\ldots, \text{etc.}$$

In this section, we will prove some real numbers are irrationals with the help of the fundamental theorem of arithmetic. We will prove that $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and in general, $\sqrt{p}$ is irrational, where p is a prime.

Before we prove that $\sqrt{2}$ is irrational, we will look at a statement, the proof of which is based on the Fundamental Theorem of Arithmetic.

Statement-1 : ***Let p be a prime number. If p divides a^2, (where a is a positive integer), then p divides a.***

Proof : Let a be any positive integer. Then the prime factorization of a is as follows :

$a = p_1p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes, not necessarily distinct.

Therefore $a^2 = (p_1p_2 \ldots p_n)(p_1p_2 \ldots p_n) = p^2{}_1p^2{}_2 \ldots p^2{}_n$.

Now, here we have been given that p divides a^2. Therefore, from the Fundamental Theorem of Arithmetic, it follows that p is one of the prime factors of a^2. Also, using the uniqueness part of the Fundamental Theorem of Arithmetic, we realise that the only prime factors of a^2 are $p_1\ p_2 \ldots p_n$. So p is one of $p_1, p_2, \ldots p_n$.

Now, since p is one of $p_1p_2 \ldots p_n$, it divides a.

DO THIS

Verify the statement proved above for p= 2, p = 5 and for a^2 = 1, 4, 9, 25, 36, 49, 64 and 81.

We are now ready to give a proof that $\sqrt{2}$ is irrational. We will use a technique called proof by contradiction.

Example-5. Prove that $\sqrt{2}$ is irrational.

Proof : Since we are using proof by contradiction, let us assume the contrary, i.e., $\sqrt{2}$ is rational.

If it is rational, then there must exist two integers r and s ($s \neq 0$) such that $\sqrt{2} = \frac{r}{s}$.

Suppose r and s have a common factor other than 1. Then, we divide by the common factor to get $\sqrt{2} = \frac{a}{b}$, where a and b are co-prime.

So, $b\sqrt{2} = a$.

On squaring both sides and rearranging, we get $2b^2 = a^2$. Therefore, 2 divides a^2.

Now, by statement 1, it follows that if 2 divides a^2 it also divides a.

So, we can write $a = 2c$ for some integer c.

Substituting for a, we get $2b^2 = 4c^2$, that is, $b^2 = 2c^2$.

This means that 2 divides b^2, and so 2 divides b (again using statement 1 with $p = 2$).

Therefore, both a and b have 2 as a common factor.

But this contradicts the fact that *a* and *b* are co-prime and have no common factors other than 1.

This contradiction has arisen because of our assumption that $\sqrt{2}$ is rational. So, we conclude that $\sqrt{2}$ is irrational.

In general, it can be shown that $\sqrt{d}$ is irrational whenever d is a positive integer which is not the square of an integer. As such, it follows that $\sqrt{6}$, $\sqrt{8}$, $\sqrt{15}$ $\sqrt{24}$ etc. are all irrational numbers.

In earlier classes, we mentioned that :

- the sum or difference of a rational and an irrational number is irrational and
- the product or quotient of a non-zero rational and irrational number is irrational.

We prove some particular cases here.

Example-6. Show that $5 - \sqrt{3}$ is irrational.

Solution : Let us assume, to the contrary, that $5 - \sqrt{3}$ is rational.

That is, we can find coprimes *a* and *b* ($b \neq 0$) such that $5 - \sqrt{3} = \frac{a}{b}$.

Therefore, $5 - \frac{a}{b} = \sqrt{3}$

Rearranging this equation, we get $\sqrt{3} = 5 - \frac{a}{b} = \frac{5b - a}{b}$

Since a and b are integers, we get $5 - \frac{a}{b}$ is rational so $\sqrt{3}$ is rational.

But this contradicts the fact that $\sqrt{3}$ is irrational.

This contradiction has arisen because of our incorrect assumption that $5 - \sqrt{3}$ is rational.

So, we conclude that $5 - \sqrt{3}$ is irrational.

Example-7. Show that $3\sqrt{2}$ is irrational.

Solution : Let us assume, the contrary, that $3\sqrt{2}$ is rational.

i.e., we can find co-primes *a* and *b* ($b \neq 0$) such that $3\sqrt{2} = \frac{a}{b}$.

Rearranging, we get $\sqrt{2} = \frac{a}{3b}$.

Since 3, a and b are integers, $\frac{a}{3b}$ is rational, and so $\sqrt{2}$ is rational.

But this contradicts the fact that $\sqrt{2}$ is irrational.

So, we conclude that 3 $\sqrt{2}$ is irrational.

Example-8. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Solution : Let us suppose that $\sqrt{2} + \sqrt{3}$ is rational.

Let $\sqrt{2} + \sqrt{3} = \frac{a}{b}$, where a, b are integers and $b \neq 0$

Therefore, $\sqrt{2} = \frac{a}{b} - \sqrt{3}$.

Squarring on both sides, we get

$$2 = \frac{a^2}{b^2} + 3 - 2\frac{a}{b}\sqrt{3}$$

Rearranging

$$\frac{2a}{b}\sqrt{3} = \frac{a^2}{b^2} + 3 - 2$$

$$= \frac{a^2}{b^2} + 1$$

$$\sqrt{3} = \frac{a^2 + b^2}{2ab}$$

Since a, b are integers, $\frac{a^2 + b^2}{2ab}$ is rational, and so, $\sqrt{3}$ is rational.

This contradicts the fact that $\sqrt{3}$ is irrational. Hence, $\sqrt{2} + \sqrt{3}$ is irrational.

Note **:**

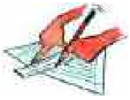
1. The sum of the two irrational numbers need not be irrational.

For example, if a = $\sqrt{2}$ and b = $-\sqrt{2}$, then both a and b are irrational, but a + b = 0 which is rational.

2. The product of two irrational numbers need not be irrational.

For example, $a = \sqrt{2}$ and $b = \sqrt{8}$, then both a and b are irrational, but

$ab = \sqrt{16} = 4$ which is rational.



Exercise - 1.4

1. Prove that the following are irrational.

(i) $\dfrac{1}{\sqrt{2}}$ (ii) $\sqrt{3} + \sqrt{5}$ (iii) $6 + \sqrt{2}$ (iv) $\sqrt{5}$ (v) $3 + 2\sqrt{5}$

2. Prove that $\sqrt{p} + \sqrt{q}$ is irrational, where p, q are primes.

Try This

Properties of real numbers

In this chapter, you have seen many examples to show whether a number is rational or irrational. Now assuming that a, b and c represent real numbers, use your new knowledge to find out whether all the properties listed below hold for real numbers. Do they hold for the operations of subtraction and division? Take as many real numbers you want and investigate.

	Property	Addition	Multiplication
1.	Closure	$a + b = c$	$a \,.\, b = c$
2.	Commutative	$a + b = b + a$	$a \,.\, b = b.a$
3.	Associative	$a + (b + c) = (a + b) + c$	$a(bc) = (ab).c$
4.	Identity	$a + 0 = 0 + a = a$	$a.1 = 1.a = a$
5.	Inverse	$a + (-a) = 0$	$a.\dfrac{1}{a} = 1, (a \neq 0)$
6.	Distributive	$a\,(b + c) = ab + ac$	

1.5 Understanding logarithms

In this section, we are going to learn about logarithms. Logarithms are used for all sorts of calculations in engineering, science, business, economics and include calcuating compound interest, exponential growth and decay, pH value in chemistry, measurement of the magnitude of earthquakes etc.

However, before we can deal with logarithms, we need to revise the laws of exponents as logarithms and laws of exponents are closely related.

1.5.1 Exponents revisted

We know that when 81 is written as 3^4 it is said to be written in its exponential form. That is, in $81 = 3^4$, the number 4 is the exponent or **index** and 3 is the **base**. We say that -

81 is the 4^{th} **power** of the **base** 3 or 81 is the 4^{th} power of 3. Similarly, $27 = 3^3$.

Now, suppose we want to multiply 27 and 81; one way of doing this is by directly multiplying. But multiplication could get long and tedious if the numbers were much larger than 81 and 27. Can we use powers to makes our work easier?

We know that $81 = 3^4$. We also know that $27 = 3^3$.

Using the Law of exponents $a^m \times a^n = a^{m+n}$, we can write

$$27 \times 81 = 3^3 \times 3^4 = 3^7$$

Now, if we had a table containing the values for the powers of 3, it would be straight forward task to find the value of 3^7 and obtain the result of $81 \times 27 = 2187$.

Similaly, if we want to divide 81 by 27 we can use the law of exponents $a^m \div a^n = a^{m-n}$ where $m > n$. Then, $81 \div 27 = 3^4 \div 3^3 = 3^1$ or simply 3

Notice that by using powers, we have changed a multiplication problem into one involving addition and a division problem into one of subtration i.e., the addition of powers, 4 and 3 and the subtraction of the powers 4 and 3.

Do this

Try to write the numbers 10, 100, 1000, 10000 and 100000 in exponential forms. Identify the base and index in each case.

Try this

(i) Find 16 x 64, without actual multiplication, using exponents.

(ii) Find 25 x 125, without actual multiplication, using exponents.

(iii) Express 128 and 32 as powers of 2 and find $128 \div 32$.

1.5.2 Writing exponents as logarithms

We know that $10000 = 10^4$. Here, 10 is the base and 4 is the exponent. Writing a number in the form of a base raised to a power is known as exponentiation. We can also write this in another way called logarithms as

$$\log_{10} 10000 = 4.$$

This is stated as "log of 10000 to the base 10 is equal to 4".

We observe that the base in the original expression becomes the base of the logarithmic form. Thus,

$$10000 = 10^4 \text{ is the same as } \log_{10} 10000 = 4.$$

In general, if $a^n = x$; we write it as $\log_a x = n$ where a and x are positive numbers and $a \neq 1$.

Let us understand this better through examples.

Example-9. Write i) $64 = 8^2$ ii) $64 = 4^3$ in logarithmic form.

Solution : (i) The logarithmic form of $64 = 8^2$ is $\log_8 64 = 2$.

(ii) The logarithmic form of $64 = 4^3$ is $\log_4 64 = 3$.

In this example, we find that log base 8 of 64 is 2 and log base 4 of 64 is 3. So, the logarithms of the same number to different bases are different.

Do this

Write $16 = 2^4$ in logarithmic form. Is it the same as $\log_4 16$?

Example-10. Write the exponential form of the following .

(i) $\log_{10} 100 = 2$ (ii) $\log_5 25 = 2$ (iii) $\log_2 2 = 1$ (iv) $\log_{10} 10 = 1$

Solution : (i) Exponential form of $\log_{10} 100 = 2$ is $10^2 = 100$.

(ii) Exponential form of $\log_5 25 = 2$ is $5^2 = 25$.

(iii) Exponential form of $\log_2 2 = 1$ is $2^1 = 2$.

(iv) Exponential form of $\log_{10} 10 = 1$ is $10^1 = 10$.

In cases (iii) and (iv), we notice that $\log_{10} 10 = 1$ and $\log_2 2 = 1$. In general, for any base a, $a^1 = a$ so $\log_a a = 1$

Try This

Show that $a^0 = 1$ so $\log_a 1 = 0$.

Do This

1. Write the following in logarithmic form.

   (i) $11^2 = 121$ (ii) $(0.1)^2 = 0.01$ (iii) $a^x = b$

2. Write the following in exponential form.

   (i) $\log_5 125 = 3$ (ii) $\log_4 64 = 3$ (iii) $\log_a x = b$ (iv) $\log_2 2 = 1$

Example-11. Determine the value of the following logarithms.

(i) $\log_3 9$ (ii) $\log_8 2$ (iii) $\log_c \sqrt{c}$

Solution : (i) Let $\log_3 9 = x$, then the exponential form is $3^x = 9 \Rightarrow 3^x = 3^2 \Rightarrow x=2$

(ii) Let $\log_8 2=y$, then the exponential form is $8^y=2 \Rightarrow (2^3)^y=2 \Rightarrow 3y=1 \Rightarrow y = \frac{1}{3}$

(iii) Let $\log_c \sqrt{c} =z$, then the exponential form is $c^z = \sqrt{c} \Rightarrow c^z = c^{\frac{1}{2}} \Rightarrow z = \frac{1}{2}$

1.5.3 Laws of Logarithms

Just like we have rules or laws of exponents, we have three laws of logarithms. We will try to prove them in the coming sections

1.5.3a The first law of logarithms

Suppose $x = a^n$ and $y = a^m$ where a>0 and $a \neq 1$. Then we know that we can write:

$\log_a x = n$ and $\log_a y = m$ .... (1)

Using the first law of exponents we know that $a^n \times a^m = a^{n+m}$

So, $xy = a^n \times a^m = a^{n+m}$ i.e. $xy = a^{n+m}$

Writing in the logarithmic form, we get

$\log_a xy = n+m$ .... (2)

But from (1), $n = \log_a x$ and $m=\log_a y$.

So, $\log_a xy = \log_a x + \log_a y$

So, if we want to multiply two numbers and find the logarithm of the product, we can do this by adding the logarithms of the two numbers. This is the first law of logarithms.

$$\mathbf{\log_a xy = \log_a x + \log_a y}$$

1.5.3b The second law of logarithms states $\log_a \frac{x}{y} = \log_a x - \log_a y$

TRY THIS

Prove the second law of logarithms by using the law of exponents $\frac{a^n}{a^m} = a^{n-m}$

1.5.3c The third law of logarithms

Let $x = a^n$ so $\log_a x = n$. Suppose, we raise both sides of $x = a^n$ to the power m, we get-

$$x^m = (a^n)^m$$

Using the laws of exponents-

$$x^m = a^{nm}$$

If we think of x^m as a single quantity, the logarithmic form of it, is

$$\log_a x^m = nm$$

$$\log_a x^m = m \log_a x \qquad (a^n = x \text{ so } \log_a x = n)$$

This is the third law. It states that the logarithm of a power number can be obtained by multiplying the logarithm of the number by that power.

$$\mathbf{\log_a x^m = m \log_a x}$$

Example-12. Expand log15

Solution : As you know, $\log_a xy = \log_a x + \log_a y$.

So, $\log 15 = \log (3 \times 5)$

$= \log 3 + \log 5$

Example-13. Expand $\log \frac{343}{125}$

Solution : As you know, $\log_a \frac{x}{y} = \log_a x - \log_a y$

So, $\log \frac{343}{125} = \log 343 - \log 125$

$= \log 7^3 - \log 5^3$

Since, $\log_a x^m = m \log_a x$

$= 3\log 7 - 3\log 5$

So $\log \frac{343}{125} = 3(\log 7 - \log 5)$.

Example-14. Write $2\log 3 + 3\log 5 - 5\log 2$ as a single logarithm.

Solution : $2\log 3 + 3\log 5 - 5\log 2$

$= \log 3^2 + \log 5^3 - \log 2^5$ (since in $m \log_a x = \log_a x^m$)

$= \log 9 + \log 125 - \log 32$

$= \log (9 \times 125) - \log 32$ (Since $\log_a x + \log_a y = \log_a xy$)

$= \log 1125 - \log 32$

$= \log \frac{1125}{32}$ (Since $\log_a x - \log_a y = \log_a \frac{x}{y}$)

Do This

1. Write the logarithms following in the form $\log_a x + \log_a y$
   - (i) 8×32 (ii) 49×343 (iii) 81×729
2. Write the logarithms following in the form $\log_a x - \log_a y$
   - (i) $8 \div 64$ (ii) $81 \div 27$
3. Write the logarithms following in logarithmic forms
   - (i) $4^3 = (2^2)^3$ (ii) $36^2 = (6^2)^2$

Exercise - 1.5

1. Write the following in logarithmic form.

(i) $3^5 = 243$ (ii) $2^{10} = 1024$ (iii) $10^6 = 1000000$

(iv) $10^{-3} = 0.001$ (v) $3^{-2} = \frac{1}{9}$ (vi) $6^0 = 1$

(vii) $5^{-1} = \frac{1}{5}$ (viii) $\sqrt{49} = 7$ (ix) $27^{\frac{2}{3}} = 9$ (x) $32^{-\frac{2}{5}} = \frac{1}{4}$

2. Write the following in exponemtial form

(i) $\log_{18}324 = 2$ (ii) $\log_{10}10000 = 4$ (iii) $\log_a \sqrt{x} = b$

(iv) $\log_4{}^8 = x$ (v) $\log_3\left(\frac{1}{27}\right) = y$

3. Determine the value of the following.

(i) $\log_{25}5$ (ii) $\log_{81}3$ (iii) $\log_2\left(\frac{1}{16}\right)$

(iv) $\log_7 1$ (v) $\log_x \sqrt{x}$ (vi) $\log_2 512$

(vii) $\log_{10}0.01$ (viii) $\log_{\frac{2}{3}}\left(\frac{8}{27}\right)$

4. Write each of the following expressions as log N. Determine the value of N. (You can assume the base is 10, but the results are identical which ever base is used).

(i) $\log 2 + \log 5$ (ii) $\log 16 - \log 2$ (iii) $3 \log 4$

(iv) $2 \log 3 - 3 \log 2$ (v) $\log 243 + \log 1$ (vi) $\log 10 + 2 \log 3 - \log 2$

5. Expand the following.

(i) $\log 1000$ (ii) $\log\left(\frac{128}{625}\right)$ (iii) $\log x^2y^3z^4$

(iv) $\log \frac{p^2q^3}{r}$ (v) $\log\sqrt{\frac{x^3}{y^2}}$

1.5.4 Standard bases of a logarithm (not meant for examination purpose)

There are two bases which are used more commonly than any others and deserve special mention. They are **base 10** and **base e**

Usually the expression log x implies that the base is 10. In calculators, the button marked log is pre-programmed to evaluate logarithms to base '10'.

For example,

$\log 2 = 0.301029995664\ldots$

$\log 3 = 0.4771212547197\ldots$

Are log 2 and log 3 irrational?

The second common base is 'e'. The symbol 'e' is called the exponential constant. This is an irrational number with an infinite, non-terminating non-recurring decimal expansion. It is usually approximated as 2.718. Base 'e' is used frequently in scientific and mathematical applications. Logarithms to base e or $\log_e$, are often written simply as 'ln'. So, "ln x" implies the base is 'e'. Such logarithms are also called natural logarithms. In calculators, the button marked 'ln' gives natural logs.

For example

$$\ln 2 = 0.6931471805599\ldots$$

$$\ln 3 = 1.0986122886681\ldots$$

Are ln(2) and ln(3) irrational?

1.5.5 Application of logarithms (Not meant for examination purpose)

Let us understand applications of logarithms with some examples.

Example-15. The magnitude of an earthquake was defined in 1935 by Charles Richer with the expression $M=\log \frac{I}{S}$; where I is the intensity of the earthquake tremor and S is the intensity of a "threshold earthquake".

(a) If the intensity of an earthquake is 10 times the intensity of a threshold earthquake, then what is its magnitude?

(b) If the magnitude of an earthquake registers 10 on the Richter scale, how many times is the intensity of this earthquake to that of a threshold earthquake?

Solution :

(a) Let the intensity of the earthquake be I, then we are given

I = 10 S

The magnitude of an earthquake is given by-

$$M = \log \frac{10S}{S}$$

$\therefore$ The magnitude of the Delhi earthquake will be-

$$M = \log \frac{I}{S}$$

$$= \log 10$$

$$= 1$$

(b) Let x be the number of times the intensity of the earthquake to that of a threshold earthquake. So the intensity of earthquake is-

$$I = xS$$

We know that-

$$M = \log \frac{I}{S}$$

So, the magnitude of the earthquake is-

$$M = \log \frac{xs}{s}$$

or $\quad M = \log x$

We know that $\quad M = 10$

So $\log x = 10$ and therefore $x = 10^{10}$

Try This

The formula for calculating pH is pH = $-\log_{10}$ [H+] where pH is the acidity or basicity of the solution and [H^+] is the hydrogen ion concentration.

(i) If Shankar's Grandma's Lux Soap has a hydrogen ion concentration of $9.2 \times 10^{(-12)}$. What is its pH?

(ii) If the pH of a tomato is 4.2, what is its hydrogen ion concentration?

Optional Exercise

[This exercise is not meant for examination]

1. Can the number 6^n, n being a natural number, end with the digit 5? Give reason.
2. Is $7 \times 5 \times 3 \times 2 + 3$ a composite number? Justify your answer.
3. Check whether 12^n can end with the digit 0 for any natural number n?
4. Show that one and only one out of n, $n + 2$ or $n + 4$ is divisible by 3, where n is any positive integer.
5. Prove that $\left(2\sqrt{3}+\sqrt{5}\right)$ is an irrational number. Also check whether $\left(2\sqrt{3}+\sqrt{5}\right)\left(2\sqrt{3}-\sqrt{5}\right)$ is rational or irrational.

6. Without actual division, find after how many places of decimals in the decimal expansion of the following rational numbers terminates. Verify by actual division. What do you infer?

(i) $\frac{5}{16}$ (ii) $\frac{13}{2^2}$ (iii) $\frac{17}{125}$ (iv) $\frac{13}{80}$ (v) $\frac{15}{32}$ (vi) $\frac{33}{2^2 \times 5}$

7. If $x^2 + y^2 = 6xy$, prove that $2 \log (x + y) = \log x + \log y + 3 \log 2$

8. Find the number of digits in 4^{2013}, if $\log_{10} 2 = 0.3010$.

Note : Ask your teacher about integral part and decimal part of a logarithm of number.

What we have discussed

1. The Fundamental Theorem of Arithmetic states that every composite number can be expressed (factorized) as a product of its primes, and this factorization is unique, apart from the order in which the prime factors occur.
2. If p is a prime and p divides a^2, where a is a positive integer, then p divides a.
3. Let x be a rational number whose decimal expansion terminates. Then we can express x in the form $\frac{p}{q}$, where p and q are coprime, and the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers.
4. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which terminates.
5. Let $x = \frac{p}{q}$ be a rational number, such that the prime factorization of q is of the form 2^n5^m, where n, m are non-negative integers. Then x has a decimal expansion which is non-terminating, repeating (recurring).
6. We define $\log_a x = n$, if $a^n = x$, where a and x are positive numbers and $a \neq 1$.
7. Laws of logarithms :

   (i) $\log_a xy = \log_a x + \log_a y$ (ii) $\log_a \frac{x}{y} = \log_a x - \log_a y$

   (iii) $\log_a x^m = m \log_a x$
8. Logarithms are used for all sorts of calculations in engineering, science, business and economics.

CHAPTER 2

Sets

2.1 INTRODUCTION

Observe the examples given below:

1. Euclid, Pythagoras, Gauss, Leibnitz, Aryabhata, Bhaskar.
2. a,e,i,o,u
3. Happy, sad, angry, anxious, joyful, confused.
4. Cricket, football, kho-kho, kabaddi, basketball.
5. 1, 3, 5, 7, 9.....

What do you observe? Example 1 is a collection of names of some mathematicians, For example 2 is the collection of vowel letters in the English alphabet and example 3 is a collection of feelings. We see that the names/ items/ objects in each example have something in common., i.e. they form a collection. Can you tell what are the collections in examples 4 and 5?

We come across collections in mathematics too. For example, natural numbers, prime numbers, quadrilaterals in a plane etc. All examples seen so far are well defined collection of objects or ideas. A well defined collection of objects or ideas is known as a **set.** Set theory is a comparitively new concept in mathematics. It was developed by Georg Cantor (1845-1918). In this chapter, we will learn about sets and their properties, and what we mean when we say well-defined, elements of a set, types of sets etc.

2.2 WELL DEFINED SETS

What do we mean when we say that a set is a well defined collection of objects. Well defined means that:

1. All the objects in the set should have a common feature or property; and
2. It should be possible to decide whether any given object belongs to the set or not.

Let us understand 'well defined' through some examples. Consider the statement : The collection of all tall students in your class.

What difficulty is caused by this statement? Here, who is tall is not clear. Richa decides that all students taller than her are tall. Her set has five students. Yashodhara also decides that tall means all students taller than her. Her set has ten students. Ganapati decides that tall means every student whose height is more than 5 feet. His set has 3 students. We find that different people get different collections. So, this collection is not well defined.

Now consider the following statement : The collection of all students in your class who are taller than 5 feet 6 inches.

In this case, Richa, Yashodhara and Ganapati, all will get the same collection. So, this collection forms a well defined set.

Do This

1. Write 3 examples of 'sets' from your daily life.
2. Some collections are given below. Tick the ones that form well defined sets.
   (i) Collection of all good students in your class.
   (ii) Red, blue, green, yellow, block.
   (iii) 1,2,3,4,5,6,7,....
   (iv) 1, 8, 27, 64, 125, ...

Try This

State which of the following collections are sets.

(i) All even numbers
(ii) Stars in the sky
(iii) The collection of odd positive integers. 1, 3, 5, …..

2.3 Naming of Sets and Elements of a Set

We usually denote a set by upper case letters, A, B, C, X, Y, Z etc. A few examples of sets in mathematics are given below.

- The set of all Natural numbers is denoted by N.
- The set of all Integers is denoted by Z.
- The set of all Rational numbers is denoted by Q.
- The set of all Real numbers is denoted by R.

Notice that all the sets given above are well defined collections because given a number we can decide whether it belongs to the set or not. Let us see some more examples of elements.

Suppose we define a set as all days in a week, whose name begins with T. Then we know that Tuesday and Thursday are part of the set but Monday is not. We say that Tuesday and Thursday are **elements** of the set of all days in a week starting with T.

Consider some more examples:

(i) We know that N usually stands for the set of all natural numbers. Then 1,2, 3... are elements of the set. But 0 is not an element of N.

(ii) Let us consider the set B, of quadrilaterals

B = {square, rectangle, rhombus, parallelogram}

Can we put triangle, trapezium or cone in the above set, B? No, a triangle and cone are can not be members of B. But a trapezium can be a member of the set B.

So, we can say that an object belonging to a set is known as a member/ element of the set. We use the symbol $\in$ to denote 'belongs to'. So $1 \in N$ means that 1 belongs to N. Similarly $0 \notin N$ means that 0 does not belong to N.

There are various ways in which we can write sets. For example, we have the set of all vowel letters in the English alphabet. Then, we can write:

(i) V = {a, e, i, o, u}. Here, we list down all the elements of the set between chain/ curly brackets. This is called the **roster** form of writing sets. In roster form, all elements of the set are written, separated by commas, within curly brackets.

(ii) V = {x : x is a vowel letter in the English alphabet}

or V = {x / x is a vowel letter in the English alphabet}

This way of writing a set is known as the set builder form. Here , we use a symbol x (or any other symbol y, z etc.,) for the element of the set. This is is followed by a colon (or a vertical line), after which we write the characteristic property possessed by the elements of the set. The whole is enclosed within curly brackets.

Let C = {2, 3, 5, 7, 11}, a set of prime numbers less than 13. This set can be denoted as:

C = {x / x is a prime number less than 13}or

C = {x: x is a prime number less than 13}.

Example-1. Write the following in roster and set builder forms.

(i) The set of all natural numbers which divide 42.

(ii) The set of natural numbers which are less than 10.

Solution :

(i) Let B be the set of all natural numbers which divide 42. Then, we can write:

B = {1, 2, 3, 6, 7, 14, 21, 42} Roster form

B = {x : x is a natural number which divides 42} Set builder form

(ii) Let A be the set of all natural numbers which are less than 10. Then, we can write:

A = {1, 2, 3, 4, 5, 6, 7, 8, 9} (Roster form)

B = {x : x is a natural number which is less than 10} (Set builder form)

Note : (i) In roster form, the order in which the elements are listed is **immaterial**. Thus, in example 1, we can also write {1, 3, 7, 21, 2, 6, 4, 42}.

(ii) While writing the elements of a set in roster form, an element is not repeated. For example, the set of letters forming the word "SCHOOL" is {S, C, H, O, L} and not {S, C, H, O, O, L}

Example-2. Write the set B = {x : x is a natural number and $x^2 < 40$} in the roster form.

Solution : We look at natural numbers and their squares starting from 1. When we reach 7, the sqaure is 49 which is greater than 40. The required numbers are 1, 2, 3, 4, 5, 6.

So, the given set in the roster form is B = {1, 2, 3, 4, 5, 6}.

DO THIS

1. List the elements of the following sets.
   (i) G = all the factors of 20
   (ii) F = the multiples of 4 between 17 and 61 which are divisible by 7
   (iii) S = {x : x is a letter in the word 'MADAM'}
   (iv) P = {x : x is a whole number between 3.5 and 6.7}

2. Write the following sets in the roster form.
   (i) B is the set of all months in a year having 30 days.
   (ii) P is the set of all prime numbers less than 10.
   (iii) X is the set of the colours of the rainbow

3. A is the set of factors of 12. Which one of the following is not a member of A.
   (A) 1 (B) 4 (C) 5 (D) 12

Try This

1. Formulate sets of your choice, involving algebraic and geometrical ideas.
2. Match roster forms with the set builder form.

(i)	{P, R, I, N, C, A, L}	(a)	$\{x : x$ is a positive integer and is a divisor of 18$\}$
(ii)	{0}	(b)	$\{x : x$ is an integer and $x^2 - 9 = 0\}$
(iii)	{1, 2, 3, 6, 9, 18}	(c)	$\{x : x$ is an integer and $x + 1 = 1\}$
(iv)	{3, -3}	(d)	$\{x : x$ is a letter of the word PRINCIPAL$\}$

Exercise - 2.1

1. Which of the following are sets? Justify your answer?
   (i) The collection of all the months of a year begining with the letter "J".
   (ii) The collection of ten most talented writers of India.
   (iii) A team of eleven best cricket batsmen of the world.
   (iv) The collection of all boys in your class.
   (v) The collection of all even integers.

2. If A={0, 2, 4, 6}, B = {3, 5, 7} and C = {p, q, r} then fill the appropriate symbol, $\in$ or $\notin$ in the blanks.
   (i) 0 ….. A (ii) 3 ….. C (iii) 4 ….. B
   (iv) 8 ….. A (v) p ….. C (vi) 7 ….. B

3. Express the following statements using symbols.
   (i) The elements 'x' does not belong to 'A'.
   (ii) 'd' is an element of the set 'B'.
   (iii) '1' belongs to the set of Natural numbers N.
   (iv) '8' does not belong to the set of prime numbers P.

4. State whether the following statements are true or false.
   (i) $5 \notin$ {Prime numbers}
   (ii) S = {5, 6, 7} implies $8 \in$ S.
   (iii) $-5 \notin$ W where 'W' is the set of whole numbers
   (iv) $\frac{8}{11} \in$ Z where 'Z' is the set of integers.

5. Write the following sets in roster form.

   (i) B = {x : x is a natural number less than 6}

   (ii) C = {x : x is a two-digit natural number such that the sum of its digits is 8}.

   (iii) D = {x : x is a prime number which is a divisor of 60}.

   (iv) E = {the set of all letters in the word BETTER}.

6. Write the following sets in the set-builder form.

   (i) {3, 6, 9, 12} (ii) {2, 4, 8, 16, 32}

   (iii) {5, 25, 125, 625} (iv) {1, 4, 9, 25, ….. 100}

7. List all the elements of the following sets in roster form.

   (i) A = {x : x is a natural number greater than 50 but less than 100}

   (ii) B = {x : x is an integer, $x^2 = 4$}

   (iii) D = {x : x is a letter in the word "LOYAL"}

8. Match the roster form with set-builder form.

(i)	{1, 2, 3, 6}	(a)	{x : x is prime number and a divisor of 6}
(ii)	{2, 3}	(b)	{x : x is an odd natural number less than 10}
(iii)	{M, A, T, H, E, I, C, S}	(c)	{x : x is a natural number and divisor of 6}
(iv)	{1, 3, 5, 7, 9}	(d)	{x : x is a letter of the word MATHEMATICS}

2.4 Types of Set

Let us consider the following examples of sets:

(i) A = {x : x is natural number smaller than 1}

(ii) D = {x : x is a odd prime number divisible by 2}

How many elements are there in A and D? We find that there is no natural number which is smaller than 1. So set A contains no elements or we say that A is an empty set.

Similarly, there are no prime numbers that are divisible by 2. So, D is also an empty set.

A set which does not contain any element is called an **empty** set, or a **Null** set, or a **void** set. Empty set is denoted by the symbol ϕ or { }.

Here are some more examples of empty sets.

(i) A = $\{x : 1 < x < 2, x$ is a natural number$\}$

(ii) B = $\{x : x^2 - 2 = 0$ and x is a rational number$\}$

(iii) D = $\{x : x^2 = 4, x$ is odd$\}$

Note : ϕ and $\{0\}$ are two different sets. $\{0\}$ is a set containing the single element 0 while { } is null set.

Finite & Infinite sets

Now consider the following sets:

(i) A = {the students of your school}

(ii) L = {p,q,r,s}

(iii) B = $\{x : x$ is an even number$\}$

(iv) J = $\{x : x$ is a multiple of 7$\}$

Can you list the number of elements in each of the sets given above? In (i), the number of elements will be the number of students in your school. In (ii), the number of elements in set L is 4. We find that it is possible to count the number of elements of sets A and L or that they contain a finite number of elements. Such sets are called **finite sets**.

Now, consider the set B of all even numbers. We cannot count all of them i.e., we see that the number of elements of this set is not finite. Similarly, all the elements of J cannot be listed. We find that the number of elements in B and J is infinite. Such sets are called **infinite sets**.

We can draw many number of straight lines passing though a given point. So this set is infinite. Similarly, it is not possible to find out the last even number or odd number among the collection of all integers. Thus, we can say a set is infinite if it is not finite.

Consider some more examples :

(i) Let 'W' be the set of the days of the week. Then W is finite.

(ii) Let 'S' be the set of solutions of the equation $x^2 - 16 = 0$. Then S is finite.

(iii) Let 'G' be the set of points on a line. Then G is infinite.

Example-3. State which of the following sets are finite or infinite.

(i) $\{x : x \in$ N and $(x - 1)(x - 2) = 0\}$

(ii) $\{x : x \in$ N and $x^2 = 4\}$

(iii) $\{x : x \in$ N and $2x - 2 = 0\}$

(iv) $\{x : x \in$ N and x is prime$\}$

(v) $\{x : x \in$ N and x is odd$\}$

Solution :

(i) x can take the values 1 or 2 in the given case. The set is $\{1,2\}$. Hence, it is finite.

(ii) $x^2 = 4$, implies that $x = +2$ or -2. But $x \in$ N or x is a natural number so the set is$\{2\}$. Hence, it is finite.

(iii) In a given set $x = 1$ and $1 \in$ N. Hence, it is finite.

(iv) The given set is the set of all prime numbers. There are infinitely many prime numbers. Hence, set is infinite.

(v) Since there are infinite number of odd numbers, hence the set is infinite.

Now, consider the following finite sets :

A = {1, 2, 4}; B = {6, 7, 8, 9, 10}; C = {x : x is a alphabet in the word "INDIA"}

Here,

Number of elements in set A = 3.

Number of elements in set B = 5.

Number of elements in set C = 4 (In the set C, the element 'I' repeats twice. We know that the elements of a given set should be distinct. So, the number of distinct elements in set C is 4).

The number of elements in a set is called the cardinal number of the set. The cardinal number of the set A is denoted as n(A) = 3.

Similarly, n(B) = 5 and n(C) = 4.

Note : There are no elements in a null set. The cardinal number of that set is 0. $\therefore$ n(ϕ) = 0

Example-4. If A = {1, 2, 3}; B = {a, b, c} then find n(A) and n(B).

Solution : The set A contains three distinct elements $\therefore$ n(A) = 3

and the set B contains three distinct elements $\therefore$ n(B) = 3

Do These

1. Which of the following are empty sets? Justify your answer.

(i) Set of integers which lie between 2 and 3.

(ii) Set of natural numbers that are less than 1.

(iii) Set of odd numbers that have remainder zero, when divided by 2.

2. State which of the following sets are finite and which are infinite. Give reasons for your answer.

(i) A = $\{x : x \in \text{N and } x < 100\}$ (ii) B = $\{x : x \in \text{N and } x \leq 5\}$

(iii) C = $\{1^2, 2^2, 3^2, \ldots..\}$ (iv) D = $\{1, 2, 3, 4\}$

(v) $\{x : x$ is a day of the week$\}$.

3. Tick the set which is infinite

(A) The set of whole numbers < 10 (B) The set of prime number < 10

(C) The set of integers < 10 (D) The set of factors of 10

Try This

1. Which of the following sets are empty sets? Justify your answer.

(i) A = $\{x : x^2 = 4 \text{ and } 3x = 9\}$.

(ii) The set of all triangles in a plane having the sum of their three angles less than 180.

2. B = $\{x : x + 5 = 5\}$ is not an empty set. Why?

Think - Discuss

An empty set is a finite set. Is this statement true or false? Why?

Exercise - 2.2

1. State which of the following sets are empty and which are not?

(i) The set of straight lines passing through a point.

(ii) Set of odd natural numbers divisible by 2.

(iii) $\{x : x$ is a natural number, $x < 5$ and $x > 7\}$

(iv) $\{x : x$ is a common point to any two parallel lines$\}$

(v) Set of even prime numbers.

2. Which of the following sets are finite or infinite.

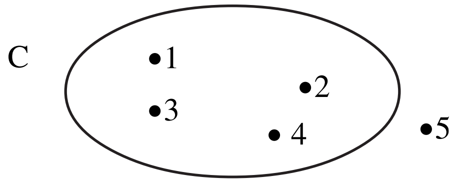
(i) The set of months in a year. (ii) $\{1, 2, 3, \ldots, 99, 100\}$

(iii) The set of prime numbers less than 99.

3. State whether each of the following set is finite or infinite.
   (i) The set of letters in the English alphabet.
   (ii) The set of lines which are parallel to the X-Axis.
   (iii) The set of numbers which are multiples of 5.
   (iv) The set of circles passing through the origin (0, 0).

2.5 Using Diagrams to Represent Sets

If S is a **set and x is an object then either** $x \in$ S or $x \notin$ S. Every set can be represented by a drawing a closed curve C where elements of C are represented by points within C and elements not in the set by points outside C. For example, the set C = {1, 2, 3, 4} can be represented as shown below:



2.6 Universal Set and Subsets

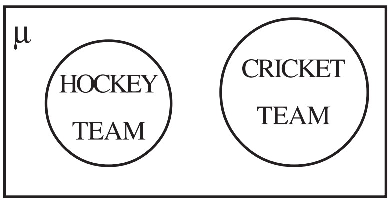
Let us consider that a cricket team is to be selected from your school. What is the set from which the team can be selected? It is the set of all students in your school. Now, we want to select the hockey team. Again, the set from which the team will be selected is the set of all students in your school. So, for selection of any school team, the students of your school are considered as the universal set.

Let us see some more examples of universal sets:

(i) If we want to study the various groups of people of our state, universal set is the set of all people in Andhra Pradesh.

(ii) If we want to study the various groups of people in our country, universal set is the set of all people in India.

The universal set is denoted by 'μ'. The Universal set is usually represented by rectangles.



If the set of real numbers R; is the universal set then what about rational and irrational numbers?

Let us consider the set of rational numbers,

$$Q = \{x : x = \frac{p}{q}, \text{p, q} \in \text{z and q} \neq 0\}$$

which is read as 'Q' is the set of all numbers such that x equals $\frac{p}{q}$, where p and q are integers and q is not zero. Then we know that every element of Q is also an element of R. So, we can say that Q is a subset of R. If Q is a subset of R, then we write is as $Q \subset R$.

Note **:** It is often convenient to use the symbol '$\Rightarrow$' which means implies.

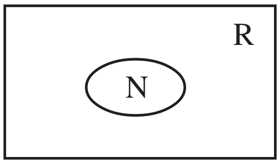
Using this symbol, we can write the definition of subset as follows:

$A \subset B$ if $a \in A \Rightarrow a \in B$, where A, B are two sets.

We read the above statement as "A is a subset of B if 'a' is an element of A implies that 'a' is also an element of B".

Real numbers R; has many subsets. For example,

The set of natural numbers $N = \{1, 2, 3, 4, 5, \ldots\ldots\}$



The set of whole number $W = \{0, 1, 2, 3, \ldots\ldots\}$

The set of integers $Z = \{\ldots., -3, -2, -1, 0, 1, 2, 3, \ldots..\}$

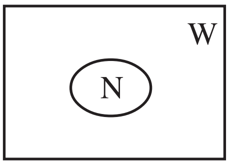
The set of irrational numbers Q', is composed of all real numbers that are not rational.

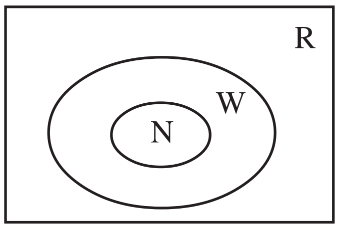
Thus, $Q' = \{x : x \in R \text{ and } x \notin Q\}$ i.e., all real numbers that are not rational. e.g. $\sqrt{2}$, $\sqrt{5}$ and π.

Similarly, the set of natural numbers, N is a subset of the set of whole numbers W and we can write $N \subset W$. Also W is a subet of R.

That is $N \subset W$ and $W \subset R$

$\Rightarrow N \subset W \subset R$





Some of the obvious relations among these subsets are $N \subset Z \subset Q$, $Q \subset R$, $Q' \subset R$, and $N \not\subset Q'$.

Example-5. Consider a set of vowels letters , $V = \{a, e, i, o, u\}$. Also consider the set A, of all letters in the English alphabet. $A = \{a, b, c, d, \ldots.., z\}$. Identify the universal set and the subset in the given example.

Solution : We can see that every element of set V is also an element A. But every element of A is not a part of V. In this case, V is the subset of A.

In other words $V \subset A$ since whenever $a \in V$, then $a \in A$.

Note **:** Since the empty set ϕ has no elements, we consider that ϕ is a subset of every set.

If A is not a subset of B ($A \not\subset B$), that means there is at least one element in A that is not a member of B.

Let us consider some more examples of subsets.

- The set C = {1, 3, 5} is a subset of D = {5, 4, 3, 2, 1}, since each number 1, 3, and 5 belonging to C also belongs to D.
- Let A = {a, e, i, o, u} and B = {a, b, c, d} then A is not a subset of B. Also B is not a subset of A.

2.6.1 Equal Sets

Consider the following sets.

A = {Sachin, Dravid, Kohli}

B = {Dravid, Sachin, Dhoni}

C = {Kohli, Dravid, Sachin}

What do you observe in the above three sets A, B and C? All the players that are in A are in C but not in B. Thus, A and C have same elements but some elements of A and B are different. So, the sets A and C are equal sets but sets A and B are not equal.

Two sets A and C are said to be **equal** if every element in A belongs to C and every element in C belongs to A. If A and C are equal sets, than we write A = C.

Example-6. Consider the following sets:

$$A = \{p, q, r\} \qquad B = \{q, p, r\}$$

In the above sets, every element of A is also an element of B. $\therefore A \subset B$.

Similarly every element of B is also in A. $\therefore B \subset A$.

Thus, we can also write that if $B \subset A$ and $A \subset B \Leftrightarrow A = B$. Here $\Leftrightarrow$ is the symbol for two way implication and is usually read as, **if and only if** (briefly written as "iff").

Examples-7. If A = {1, 2, 3, ….} and N is a set of natural numbers, check whether A and N are equal?

Solution : The elements are same in both the sets. Therefore, both A and N are the set of Natural numbers. Therefore the sets A and N are equal sets or A = N.

Example-8. Consider the sets $A = \{p, q, r, s\}$ and $B = \{1, 2, 3, 4\}$. Are they equal?

Solution : A and B do not contain the same elements. So, $A \neq B$.

Example-9. Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Check if A and P are equal.

Solution : The set of prime numbers less than 6, $A = \{2,3,5\}$

The prime factors of 30 are 2, 3 and 5. So, $P = \{2,3,5\}$

Since the elements of A are the same as the elements of P, therefore, A and P are equal.

Example-10. Show that the sets A and B are equal, where

$A = \{x : x$ is a letter in the word 'ASSASSINATION'$\}$

$B = \{x : x$ is a letter in the word STATION$\}$

Solution : Given, $A = \{x : x$ is a letter in the word 'ASSASSINATION'$\}$

This set A can also be written as $A = \{A,S,I,N,T,O\}$ since generally elements in a set are not repeated.

Also given that $B = \{x : x$ is a letter in the word STATION$\}$

'B' can also be written as $B = \{A,S,I,N,T,O\}$

So, the elements of A nd B are same and $A = B$

Exercise - 2.3

1. Which of the following sets are equal?

   (i) $A = \{x : x$ is a letter in the word FOLLOW$\}$

   (ii) $B = \{x : x$ is a letter in the word FLOW$\}$

   (iii) $C = \{x : x$ is a letter in the word WOLF$\}$

2. Consider the following sets and fill up the blank in the statement given below with = or $\neq$ so as to make the statement true.

$A = \{1, 2, 3\}$;	$B = \{$The first three natural numbers$\}$
$C = \{a, b, c, d\}$;	$D = \{d, c, a, b\}$
$E = \{a, e, i, o, u\}$;	$F = \{$set of vowels in English Alphabet$\}$

(i) A ···· B (ii) A ···· E (iii) C ···· D

(iv) D ···· F (v) F ···· A (vi) D ···· E

(vii) F ···· B

3. In each of the following, state whether A = B or not.

(i) A = {a, b, c, d} B = {d, c, a, b}

(ii) A = {4, 8, 12, 16} B = {8, 4, 16, 18}

(iii) A = {2, 4, 6, 8, 10} B = $\{x : x$ is a positive even integer and $x < 10\}$

(iv) A = $\{x : x$ is a multiple of $10\}$ B = {10, 15, 20, 25, 30, …}

Consider the set E = {2, 4, 6} and F = {6, 2, 4}. Note that E = F. Now, since each element of E also belongs to F, therefore E is a subset of F. But each element of F is also an element of E. So F is a subset of E. In this manner it can be shown that every set is a subset of itself.

If A and B contain the same elements, they are equal i.e. A = B. By this observation we can say that "Every set is subset of itself".

Example-11. Consider the sets ϕ, A = {1, 3}, B = {1, 5, 9}, C = {1, 3, 5, 7, 9}. Insert the symbol $\subset$ or $\not\subset$ between each of the following pair of sets.

(i) ϕ ….. B (ii) A ….. B (iii) A ….. C (iv) B ….. C

Solution : (i) $\phi \subset B$, as ϕ is a subset of every set.

(ii) $A \not\subset B$, for $3 \in A$ but $3 \notin B$.

(iii) $A \subset C$ as $1, 3 \in A$ also belong to C.

(iv) $B \subset C$ as each element of B is also an element of C.

Do This

1. A = {1, 2, 3, 4}, B = {2, 4}, C = {1, 2, 3, 4, 7}, F = { }.

Fill in the blanks with $\subset$ or $\not\subset$.

(i) A ….. B (ii) C ….. A (iii) B ….. A

(iv) A ….. C (v) B ….. C (vi) ϕ ….. B

2. State which of the following statement are true.

(i) { } = ϕ (ii) $\phi = 0$ (iii) 0 = { 0 }

Try This

1. $A = \{\text{quadrilaterals}\}$, $B = \{\text{square, rectangle, trapezium, rhombus}\}$. State whether $A \subset B$ or $B \subset A$. Justify your answer.
2. If $A = \{a, b, c, d\}$. How many subsets does the set A have?
   (A) 5 (B) 6 (C) 16 (D) 65
3. P is the set of factors of 5, Q is the set of factors of 25 and R is the set of factors of 125. Which one of the following is false?
   (A) $P \subset Q$ (B) $Q \subset R$ (C) $R \subset P$ (D) $P \subset R$
4. A is the set of prime numbers less than 10, B is the set of odd numbers < 10 and C is the set of even numbers < 10. How many of the following statements are true?
   (i) $A \subset B$ (ii) $B \subset A$ (iii) $A \subset C$
   (iv) $C \subset A$ (v) $B \subset C$ (vi) $X \subset A$

Consider the following sets:

$A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 2, 3, 4, 5\}$

All the elements of A are in B $\therefore A \subset B$.

All the elements of B are in C $\therefore B \subset C$.

All the elements of A are in C $\therefore A \subset C$.

That is, $A \subset B, B \subset C \Rightarrow A \subset C$.

Exercise - 2.4

1. State which of the following statements are true given that. $A = \{1, 2, 3, 4\}$
   (i) $2 \in A$ (ii) $2 \in \{1, 2, 3, 4\}$
   (iii) $A \subset \{1, 2, 3, 4\}$ (iv) $\{2, 3, 4\} \subset \{1, 2, 3, 4\}$
2. State the reasons for the following :
   (i) $\{1, 2, 3, \ldots, 10\} \neq \{x : x \in N \text{ and } 1 < x < 10\}$
   (ii) $\{2, 4, 6, 8, 10\} \neq \{x : x = 2n+1 \text{ and } x \in N\}$

(iii) {5, 15, 30, 45} $\neq$ $\{x : x$ is a multiple of 15$\}$

(iv) {2, 3, 5, 7, 9} $\neq$ $\{x : x$ is a prime number$\}$

3. List all the subsets of the following sets.

(i) B = {p, q} (ii) C = {x, y, z} (iii) D = {a, b, c, d}

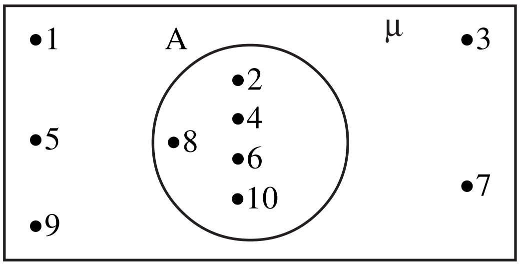
(iv) E = {1, 4, 9, 16} (v) F = {10, 100, 1000}

2.7 Venn Diagrams

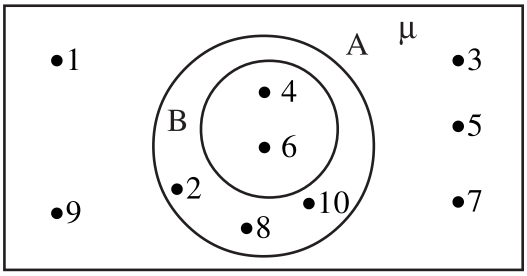
We have already seen some ways of representing sets using diagrams. Let us study it in more detail now. Venn-Euler diagram or simply Venn-diagram is a way of representing the relationships between sets. These diagrams consist of rectangles and closed curves usually circles.

As mentioned earlier in the chapter, the universal set is usually represented by a rectangle.

(i) Consider that $\mu = \{1, 2, 3, \ldots, 10\}$ is the universal set of which, $A = \{2, 4, 6, 8, 10\}$ is a subset. Then the venn-diagrams is as:

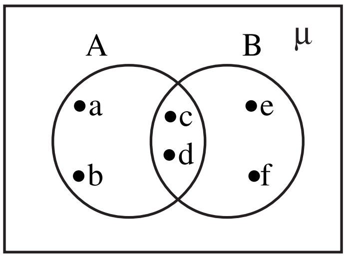


(ii) $\mu = \{1, 2, 3, \ldots, 10\}$ is the universal set of which, $A = \{2, 4, 6, 8, 10\}$ and $B = \{4, 6\}$ are subsets and also $B \subset A$. Then we have the following figure:



(iii) Let A = {a, b, c, d} and B = {c, d, e, f}.

Then we illustrate these sets with a Venn diagram as



2.8 Basic Operations on Sets

We know that arithmetic has operations of additions, subtraction and multiplication of numbers. Similarly in sets, we define the operation of union, intersection and difference of sets.

2.8.1 Union of Sets

Example-12. Suppose A is the set of students in your class who were absent on Tuesday and B the set of students who were absent on Wednesday. Then,

A = {Roja, Ramu, Ravi} and

B = {Ramu, Preethi, Haneef}

Now, we want to find K, the set of students who were absent on either Tuesday or Wednesday. Then, does Roja $\in$ K? Ramu $\in$ K? Ravi $\in$ K? Haneef $\in$ K? Preeti $\in$ K? Akhila $\in$ K?

Roja, Ramu, Ravi, Haneef and Preeti all belong to K but Ganpati does not.

Hence, K = {Roja, Ramu, Raheem, Prudhvi, Preethi}

Here K is the called the union of sets A and B. The union of A and B is the set which consists of all the elements of A and B and the common elements being taken only once. The symbol 'µ' is used to denote the union. Symbolically, we write $A \cup B$ and usually read as 'A union B'.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Example-13. Let A = {2, 5, 6, 8} and B = {5, 7, 9, 1}. Find $A \cup B$.

Solution : We have $A \cup B = \{1, 2, 5, 6, 7, 8, 9\}$.

Note that the common element 5 was taken only once while writing $A \cup B$.

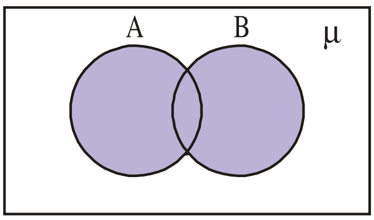
Example-14. Let A = {a, e, i, o, u} and B = {a, i, u}. Show that $A \cup B = A$.

Solution : We have $A \cup B = \{a, e, i, o, u\} = A$.

This example illustrates that union of sets A and its subset B is the Set A itself.

i.e, if $B \subset A$, then $A \cup B = A$.

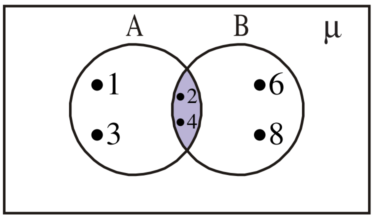
The union of the sets can be represented by a Venn-diagram as shown (shaded portion)



Example-15. Illustrate $A \cup B$ in Venn-diagrams where.

A = {1, 2, 3, 4} and B = {2, 4, 6, 8}

Solution :



$A \cup B = \{1, 2, 3, 4, 6, 8\}$

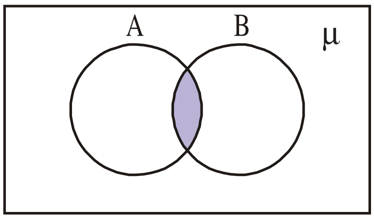
2.8.2 Intersection of Sets

Let us again consider the example of absent students. This time we want to find the set L of students who were absent on both Tuesday and Wednesday. We find that L = {Ramu}. Here, L is called the intersection of sets A and B.

In general, the intersection of sets A and B is the set of all elements which are common to A and B. i.e., those elements which belong to A and also belong to B. We denote intersection by $A \cap B$. (read as "A intersection B"). Symbolically, we write

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

The intersection of A and B can be illustrated in the Venn-diagram as shown in the shaded portion in the adjacent figure.



$A \cap B$

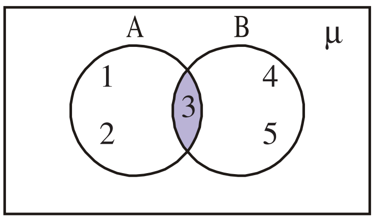
Example-16. Find $A \cap B$ when A = {5, 6, 7, 8} and B = {7, 8, 9, 10}.

Solution : The common elements in both A and B are 7, 8.

$\therefore A \cap B = \{7, 8\}$.

Example-17. Illustrate $A \cap B$ in Venn-diagrams where A = {1, 2, 3} and B = {3, 4, 5}

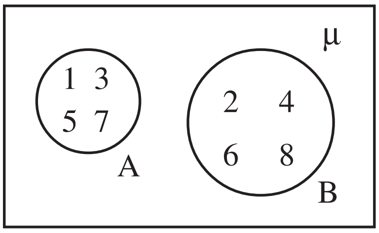
Solution : The intersection of A and B can be illustrated in the Venn-diagram as follows:



$A \cap B = \{3\}$

2.8.3 Disjoint Set

Suppose A {1, 3, 5, 7} and B = {2, 4, 6, 8}. We see that there are no common elements in A and B. Such sets are known as disjoint sets. The disjoint sets can be represented by means of the Venn-diagram as follows:



$A \cap B = \phi$

Do This

1. Let A = {1, 3, 7, 8} and B = {2, 4, 7, 9}. Find $A \cap B$.
2. If A = {6, 9, 11}; B = { }, find $A \cup \phi$.
3. A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}; B = {2, 3, 5, 7}. Find $A \cap B$ and show that $A \cap B = B$.
4. If A = {4, 5, 6}; B = {7, 8} then show that $A \cup B = B \cup A$.

Try This

1. List out some sets A and B and choose their elements such that A and B are disjoint
2. If A = {2, 3, 5}, find $A \cup \phi$ and $\phi \cup A$ and compare.
3. If A = {1, 2, 3, 4}; B = {1, 2, 3, 4, 5, 6, 7, 8} then find $A \cup B$, $A \cap B$. What do you notice about the result?
4. A = {1, 2, 3, 4, 5, 6}; B = {2, 4, 6, 8, 10}. Find the intersection of A and B.

Think - Discuss

The intersection of any two disjoint sets is a null set. Justify your answer.

2.8.4 Difference of Sets

The difference of sets A and B is the set of elements which belong to A but do not belong to B. We denote the difference of A and B by A – B or simply "A Minus B".

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

Example-18. Let A = {1, 2, 3, 4, 5}; B = {4, 5, 6, 7}. Find A – B.

Solution : Given A = {1, 2, 3, 4, 5} and B = {4, 5, 6, 7}. Only the elements which are in A but not in B should be taken.

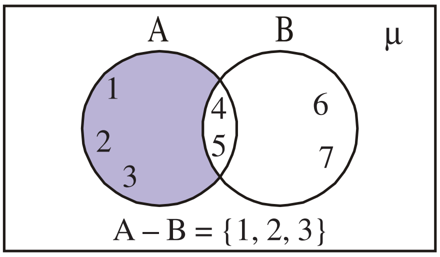
$\therefore$ A – B = {1, 2, 3}. Since 4, 5 are the elements in B they are not taken.

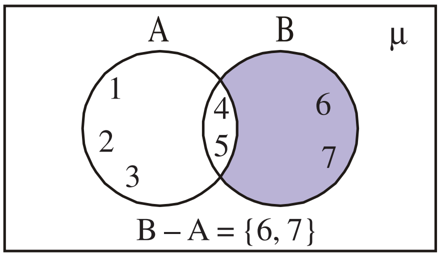
Similarly for B – A, the elements which are only in B are taken.

$\therefore$ B – A = {6, 7} (4, 5 are the elements in A).

Note that A – B $\neq$ B – A

The Venn diagram of A-B is as shown.





Example-19. Observe the following

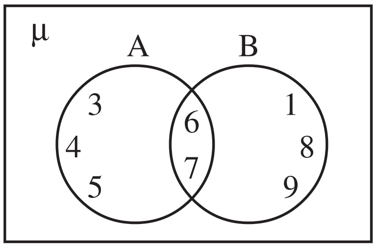
A = {3, 4, 5, 6, 7} $\therefore$ n(A) = 5

B = {1, 6, 7, 8, 9} $\therefore$ n(B) = 5

A $\cup$ B = {1, 3, 4, 5, 6, 7, 8, 9} $\therefore$ n(A $\cup$ B) = 8

A $\cap$ B = {6, 7} $\therefore$ n(A $\cap$ B) = 2

$\therefore$ n(A $\cup$ B) = 5 + 5 – 2 = 8



We observe that **n(A $\cup$ B) = n(A) + n(B) – n(A $\cap$ B)**

Do This

1. If A = {1, 2, 3, 4 ,5}; B = {4, 5, 6, 7}then find A – B and B – A. Are they equal?
2. If V = {a, e, i, o, u} and B = {a, i,k, u}, find V – B and B – V.

Think - Discuss

The sets A – B, B – A and A $\cap$ B are mutually disjoint sets. Use examples to observe if this is true.

Exercise - 2.5

1. If $A = \{1, 2, 3, 4\}$; $B = \{1, 2, 3, 5, 6\}$ then find $A \cap B$ and $B \cap A$. Are they equal?

2. $A = \{0, 2, 4\}$, find $A \cap \phi$ and $A \cap A$. Comment.

3. If $A = \{2, 4, 6, 8, 10\}$ and $B = \{3, 6, 9, 12, 15\}$, find $A - B$ and $B - A$.

4. If A and B are two sets such that $A \subset B$ then what is $A \cup B$?

5. If $A = \{x : x \text{ is a natural number}\}$

   $B = \{x : x \text{ is an even natural number}\}$

   $C = \{x : x \text{ is an odd natural number}\}$

   $D = \{x : x \text{ is a prime number}\}$

   Find $A \cap B$, $A \cap C$, $A \cap D$, $B \cap C$, $B \cap D$, $C \cap D$.

6. If $A = \{3, 6, 9, 12, 15, 18, 21\}$; $B = \{4, 8, 12, 16, 20\}$

   $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$; $D = \{5, 10, 15, 20\}$ find

   (i) $A - B$ (ii) $A - C$ (iii) $A - D$ (iv) $B - A$ (v) $C - A$

   (vi) $D - A$ (vii) $B - C$ (viii) $B - D$ (ix) $C - B$ (x) $D - B$

7. State whether each of the following statement is true or false. Justify you answers.

   (i) $\{2, 3, 4, 5\}$ and $\{3, 6\}$ are disjoint sets.

   (ii) $\{a, e, i, o, u\}$ and $\{a, b, c, d\}$ are disjoint sets.

   (iii) $\{2, 6, 10, 14\}$ and $\{3, 7, 11, 15\}$ are disjoint sets.

   (iv) $\{2, 6, 10\}$ and $\{3, 7, 11\}$ are disjoint sets.

What We Have Discussed

1. A set is a well defined collection of objects where well defined means that:

   (i) All the objects in the set have a common feature or property; and

   (ii) It is possible to decide whether any given object belongs to the set or not.

2. An object belonging to a set is known as an element of the set. We use the symbol '$\in$' to denote 'belongs to'.

3. Sets can be written in the roster form where all elements of the set are written, separated by commas, within { } curly brackets.
4. Sets can also be written in the set-builder form.
5. A set which does not contain any element is called an **empty** set, or a **Null** set, or a **void** set.
6. A set is called a finite set if it is possible to count the number of elements of that set.
7. We can say that a set is infinite if it is not finite.
8. The number of elements in a set is called the cardinal number of the set.
9. The universal set is denoted by 'μ'. The Universal set is usually represented by rectangles.
10. A is a subset of B if 'a' is an element of A implies that 'a' is also an element of B. This is written as $A \subset B$ if $a \in A \Rightarrow a \in B$, where A, B are two sets.
11. Two sets, A and B are said to be **equal** if every element in A belongs to B and every element in B belongs to A.
12. A union B is written as $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
13. A intersection B is written as $A \cap B = \{x : x \in A \text{ and } x \in B\}$
14. The difference of two sets A, B is denoted as $A - B$ or $B - A$
15. Venn diagrams are a convenient way of showing operations between sets.

CHAPTER 3 Polynomials

3.1 INTRODUCTION

Let us observe two situations

1. A flower bed in a garden is in the shape of a triangle. The longest side is 3 times the middle side and smallest side is 2 units shorter than the middle side. Let **P** represent the length of the middle side, then what's the perimeter in terms of **P?**

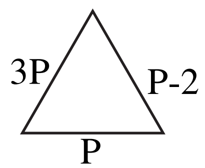
2. The length of a rectangular dining hall is twice its breadth. Let x represent the breadth of the hall. What is the area of the floor of the hall in terms of x?

In the above situations, there is an unknown in each statement. In the first situation, middle side is given as '**P**' units.

Since, Perimeter of triangle = sum of all sides

Perimeter $= P + 3P + P - 2$

$= 5P - 2$



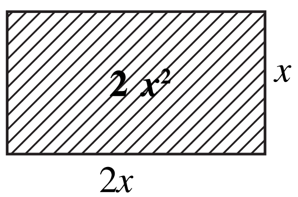
Similarly in the second situation, length is given as twice the breadth.

So, if breadth = x, length = $2x$

Since area of rectangle = lb

Area = $(2x)\ (x)$

$= 2\,x^2$



As you know, the perimeter, $5P - 2$ of the triangle and area $2x^2$ of the rectangle are in the form of polynomials of different degrees.

3.2 What are Polynomials?

Polynomials are algebraic expressions constructed using constants and variables. Coefficients operate on variables, which can be raised to various powers of non-negative integer exponents. For example, $2x + 5$, $3x^2 + 5x + 6$, $- 5y$, x^3 are some polynomials.

$\frac{1}{x^2}, \frac{1}{\sqrt{2x}}, \frac{1}{y-1}, \sqrt{3x^3}$ etc. are not polynomials.

Why is $\frac{1}{y-1}$ not a polynomial? Discuss with your friends and teacher.

Do This

State which of the following are polynomials and which are not? Give reasons.

(i) $2x^3$ (ii) $\frac{1}{x-1}$ (iii) $4z^2 + \frac{1}{7}$ (iv) $m^2 - \sqrt{2}m + 2$ (v) $P^{-2} + 1$

3.2.1 Degree of a Polynomial

Recall that if p(x) is a polynomial in x, the highest power of x in p(x) is called the degree of the polynomial p(x). For example, $3x + 5$ is a polynomial in the variable x. It is of degree 1 and is called a linear polynomial. $5x$, $\sqrt{2}y+5$, $\frac{1}{3}$P, $m + 1$ etc. are some more linear polynomials.

A polynomial of degree 2 is called a quadratic polynomial. For example, $x^2 + 5x + 4$ is a quadratic polynomial in the variable x. $2x^2 + 3x - \frac{1}{2}$, $p^2 - 1$, $3 - z - z^2$, $y^2 - \frac{y}{3} + \sqrt{2}$ are some examples of quadratic polynomials.

The expression $5x^3 - 4x^2 + x - 1$ is a polynomial in the variable x of degree 3, and is called a cubic polynomial. Some more examples of cubic polynomials are $2 - x^3$, p^3, $\ell^3 - \ell^2 - \ell + 5$.

Try This

Write 3 different quadratic, cubic and 2 linear polynomials with different number of terms.

We can write polynomials of any degree. $7u^6 - \frac{3}{2}u^4 + 4u^2 - 8$ is polynomial of degree 6 and $x^{10} - 3x^8 + 4x^5 + 2x^2 - 1$ is a polynomial of degree 10.

We can write a polynomial in a variable x of a degree n where n is any natural number.

Generally, we say

$p(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} x + a_n$ is a polynomial of n^{th} degree, where $a_0, a_1, a_2 \ldots\ldots a_{n-1}, a_n$ are real coefficients and $a_0 \neq 0$

For example, the general form of a first degree polynomial in one variable x is $ax+b$, where a and b are real numbers and $a \neq 0$.

TRY THIS

1. Write a quadratic polynomial and a cubic polynomial in variable x in the general form.
2. Write a general polynomial $q(z)$ of degree n with coefficients that are $b_0 \ldots b_n$. What are the conditions on $b_0 \ldots b_n$?.

3.2.2 VALUE OF A POLYNOMIAL

Now consider the polynomial $p(x) = x^2 - 2x - 3$. What is the value of the polynomial at any point? For example, what is the value at $x = 1$? Putting $x = 1$, in the polynomial, we get $p(1) = (1)^2 - 2(1) - 3 = -4$. The value -4, is obtained by replacing x by 1 in the given polynomial $p(x)$. This is the value of $x^2 - 2x - 3$ at $x = 1$.

Similarly, $p(0) = -3$ is the value of $p(x)$ at $x = 0$.

Thus, if $p(x)$ is a polynomial in x, and if k is a real number, then the value obtained by replacing x by k in $p(x)$, is called the value of $p(x)$ at $x = k$, and is denoted by $p(k)$.

DO THIS

(i) $p(x) = x^2 - 5x - 6$, find the values of $p(1), p(2), p(3), p(0), p(-1), p(-2), p(-3)$.

(ii) $p(m) = m^2 - 3m + 1$, find the value of $p(1)$ and $p(-1)$.

3.2.3 ZEROES OF A POLYNOMIAL

What are values of $p(x) = x^2 - 2x - 3$ at $x = 3, -1$ and 2?

We have, $p(3) = (3)^2 - 2(3) - 3 = 9 - 6 - 3 = 0$

Also $p(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$

and $p(2) = (2)^2 - 2(2) - 3 = 4 - 4 - 3 = -3$

We see that $p(3) = 0$ and $p(-1) = 0$. These points, $x = 3$ and $x = -1$, are called **Zeroes** of the polynomial $p(x) = x^2 - 2x - 3$.

As $p(2) \neq 0$, 2 is not the zero of $p(x)$.

More generally, a real number k is said to be a zero of a polynomial $p(x)$, if $p(k) = 0$.

Do this

(i) Let $p(x) = x^2 - 4x + 3$. Find the value of $p(0)$, $p(1)$, $p(2)$, $p(3)$ and obtain zeroes of the polynomial $p(x)$.

(ii) Check whether -3 and 3 are the zeroes of the polynomial $x^2 - 9$.

Exercise - 3.1

1. (a) If $p(x) = 5x^7 - 6x^5 + 7x - 6$, find

   (i) coefficient of x^5 (ii) degree of $p(x)$ (iii) constant term.

   (b) Write three more polynomials and create three questions for each of them.

2. State which of the following statements are true and which are false? Give reasons for your choice.

   (i) The degree of the polynomial $\sqrt{2}\,x^2 - 3x + 1$ is $\sqrt{2}$.

   (ii) The coefficient of x^2 in the polynomial $p(x) = 3x^3 - 4x^2 + 5x + 7$ is 2.

   (iii) The degree of a constant term is zero.

   (iv) $\dfrac{1}{x^2 - 5x + 6}$ is a quadratic polynomial.

   (v) The degree of a polynomial is one more than the number of terms in it.

3. If $p(t) = t^3 - 1$, find the values of $p(1)$, $p(-1)$, $p(0)$, $p(2)$, $p(-2)$.

4. Check whether -2 and 2 are the zeroes of the polynomial $x^4 - 16$

5. Check whether 3 and -2 are the zeroes of the polynomial $p(x)$ when $p(x) = x^2 - x - 6$.

3.3 Working with Polynomials

You have already studied how to find the zeroes of a linear polynomial.

For example, if k is a zero of $p(x) = 2x + 5$, then $p(k) = 0$ gives $2k + 5 = 0$ i.e., $k = \frac{-5}{2}$.

In general, if k is a zero of $p(x) = ax+b$, $a \neq 0$.

then $p(k) = ak + b = 0$,

i.e., $k = \frac{-b}{a}$, or the zero of the linear polynomial $ax + b$ is $\frac{-b}{a}$.

Thus, the zero of a linear polynomial is related to its coefficients, including the constant term.

Are the zeroes of higher degree polynomials also related to their coefficients? Think about this and discuss with friends. We will come to this later.

3.4 Geometrical Meaning of the Zeroes of a Polynomial

You know that a real number k is a zero of the polynomial $p(x)$ if $p(k) = 0$. Let us see the **graphical** representations of linear and quadratic polynomials and the geometrical meaning of their zeroes.

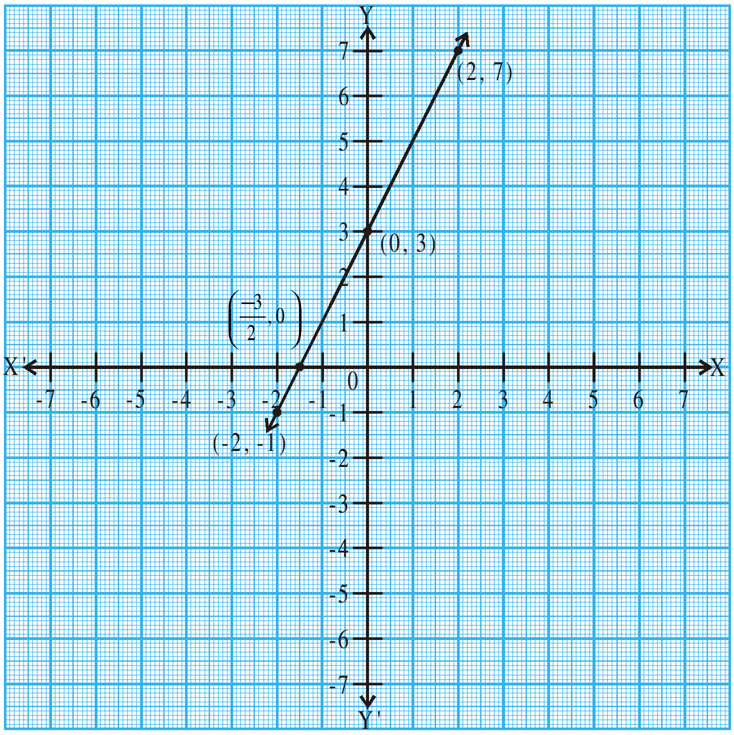
3.4.1. Graphical representation of a linear polynomial

Consider first a linear polynomial $ax + b$, $a \neq 0$. You have studied in Class-IX that the graph of $y = ax + b$ is a straight line. For example, the graph of $y = 2x + 3$ is a straight line intersecting the y-axis at $(0, 3)$ and passing through the points $(-2, -1)$ and $(2, 7)$.

Table 3.1

x	-2	0	2
$y = 2x + 3$	-1	3	7
(x, y)	$(-2, -1)$	$(0, 3)$	$(2, 7)$

From the graph, you can see that the graph of $y = 2x+3$ intersects the x-axis between $x = -1$ and $x = -2$, that is, at the point $\left(\frac{-3}{2}, 0\right)$. But $x = \frac{-3}{2}$ is also the zero of the polynomial $2x + 3$. Thus, the zero of the polynomial $2x + 3$ is the x-coordinate of the point where the graph of $y = 2x + 3$ intersects the x-axis.



Do This

Draw the graph of (i) $y = 2x + 5$, (ii) $y = 2x - 5$, (iii) $y = 2x$ and find the point of intersection on x-axis. Is the x-coordinates of these points also the zero of the polynomial?

In general, for a linear polynomial $ax + b$, $a \neq 0$, the graph of $y = ax + b$ is a straight line which intersects the x-axis at exactly one point, namely, $\left(\frac{-b}{a}, 0\right)$.

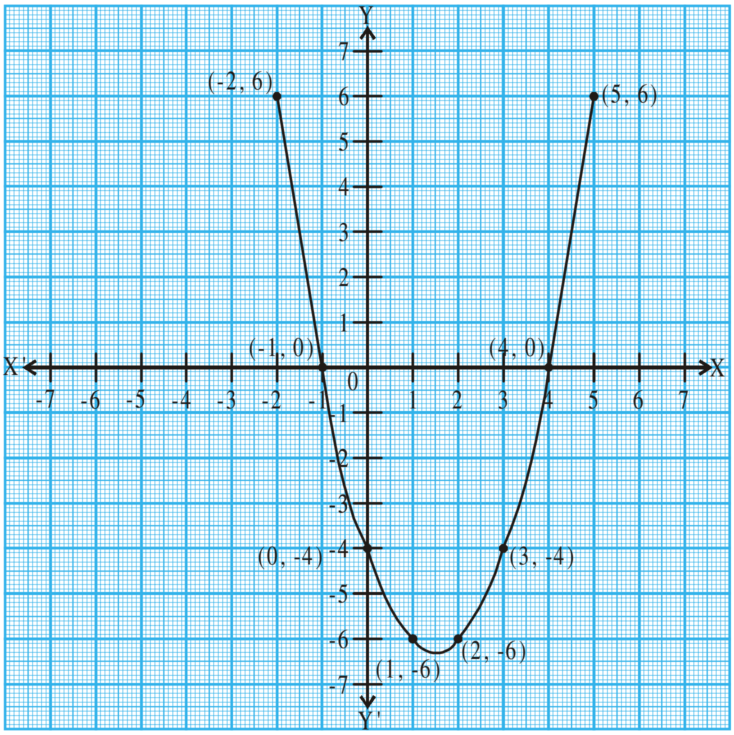
Therefore, the linear polynomial $ax + b$, $a \neq 0$, has exactly one zero, namely, the x-coordinate of the point where the graph of $y = ax + b$ intersects the x-axis.

3.4.2. Graphical Representation of a Quadratic Polynomial

Now, let us look for the geometrical meaning of a zero of a quadratic polynomial. Consider the quadratic polynomial $x^2 - 3x - 4$. Let us see how the graph of $y = x^2 - 3x - 4$ looks like. Let us list a few values of $y = x^2 - 3x - 4$ corresponding to a few values for x as given in Table 3.2.

Table 3.2

x	– 2	– 1	0	1	2	3	4	5
$y = x^2 - 3x - 4$	6	0	– 4	– 6	– 6	– 4	0	6
(x, y)	(– 2, 6)	(– 1, 0)	(0, 4)	(1, – 6)	(2, – 6)	(3, – 4)	(4, 0)	(5, 6)



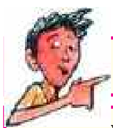
We locate the points listed above on a graph paper and draw the graph.

Is the graph of this quadratic polynomial a straight line? It is like a $\bigcup$ shaped curve. It intersects the x-axis at two points.

In fact, for any quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, the graph of the corresponding equation $y = ax^2 + bx + c$ either opens upwards like $\bigcup$ or opens downwards like $\bigcap$. This depends on whether $a > 0$ or $a < 0$. (The shape of these curves are called **parabolas.**)

We observe that -1 and 4 are zeroes of the quadratic polynomial and -1 and 4 are intersection points of x-axis. Zeroes of the quadratic polynomial $x^2 - 3x - 4$ are the x-coordinates of the points where the graph of $y = x^2 - 3x - 4$ intersects the x-axis.

This is true for any quadratic polynomial, i.e., the zeroes of a quadratic polynomial $ax^2 + bx + c$, are precisely the x-coordinates of the points where the parabola representing $y = ax^2 + bx + c$ intersects the x-axis.

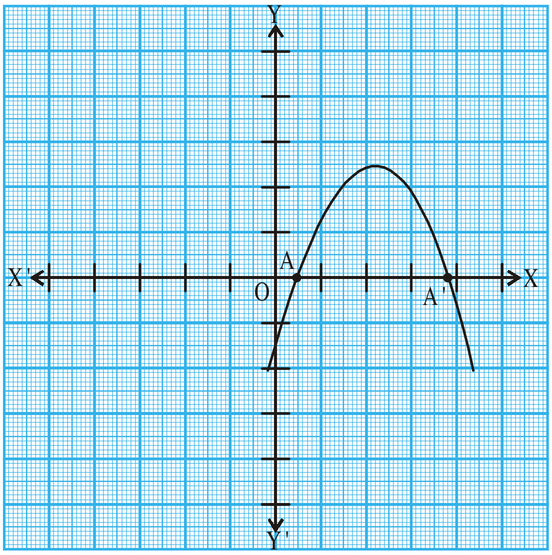


TRY THIS

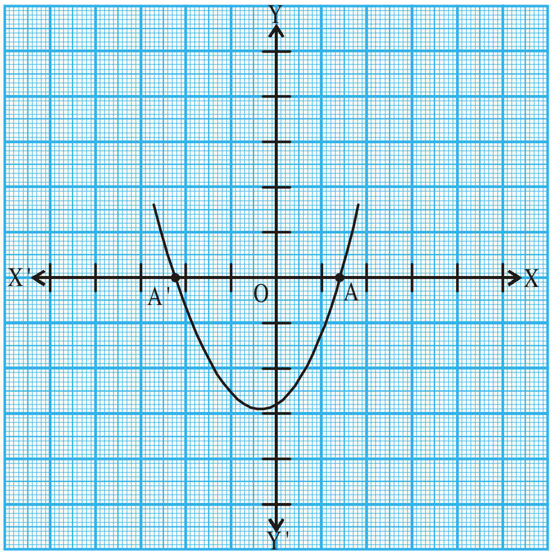
Draw the graphs of (i) $y = x^2 - x - 6$ (ii) $y = 6 - x - x^2$ and find zeroes in each case. What do you notice?

From our observation earlier about the shape of the graph of $y = ax^2 + bx + c$, the following three cases can happen:

Case (i) : Here, the graph cuts x-axis at two distinct points A and A'. In this case, the x-coordinates of A and A' are the **two zeroes** of the quadratic polynomial ax^2+bx+c . The parabola can open either upward or downward.

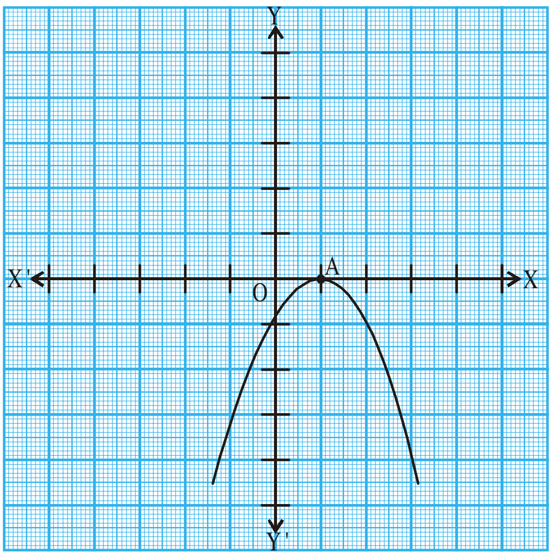


(i)

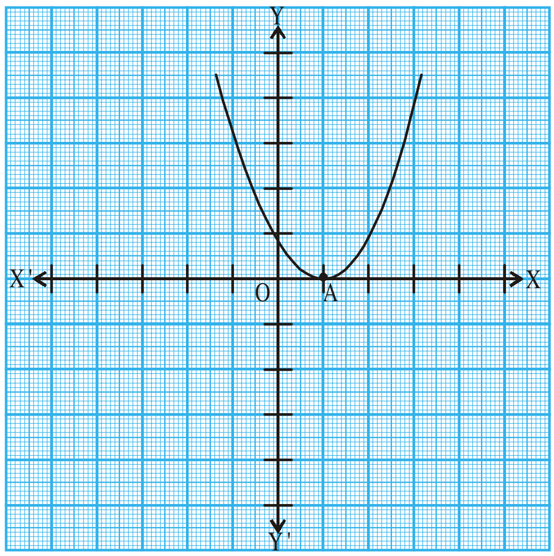


(ii)

Case (ii) : Here, the graph touches x-axis at exactly one point, i.e., at two coincident points. So, the two points A and A' of Case (i) coincide here to become one point A.



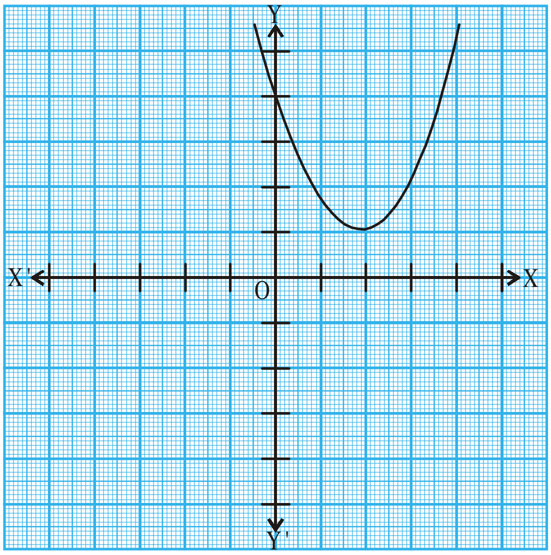
(i)



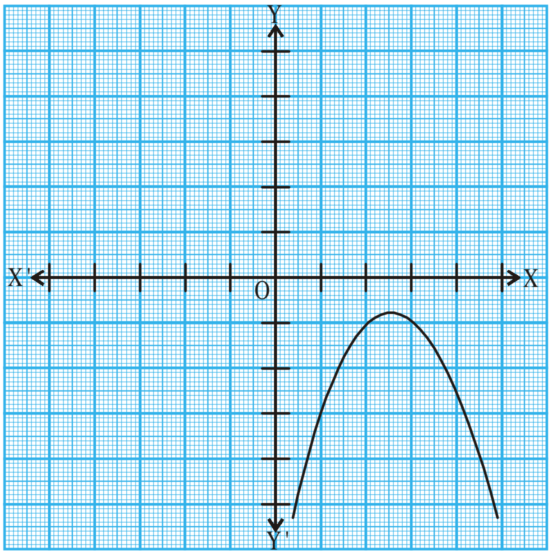
(ii)

The x-coordinate of A is the **only zero** for the quadratic polynomial $ax^2 + bx + c$ in this case.

Case (iii) : Here, the graph is either completely above the x-axis or completely below the x-axis. So, it does not cut the x-axis at any point.



(i)



(ii)

So, the quadratic polynomial $ax^2 + bx + c$ has **no zero** in this case.

So, you can see geometrically that a quadratic polynomial can have either two distinct zeroes or two equal zeroes (i.e., one zero), or no zero. This also means that a polynomial of degree 2 has atmost two zeroes.

TRY THIS

1. Write three polynomials that have 2 zeros each.
2. Write one polynomial that has one zero.
3. How will you verify if it has only one zero.
4. Write three polynomials that have no zeroes for x that are real numbers.

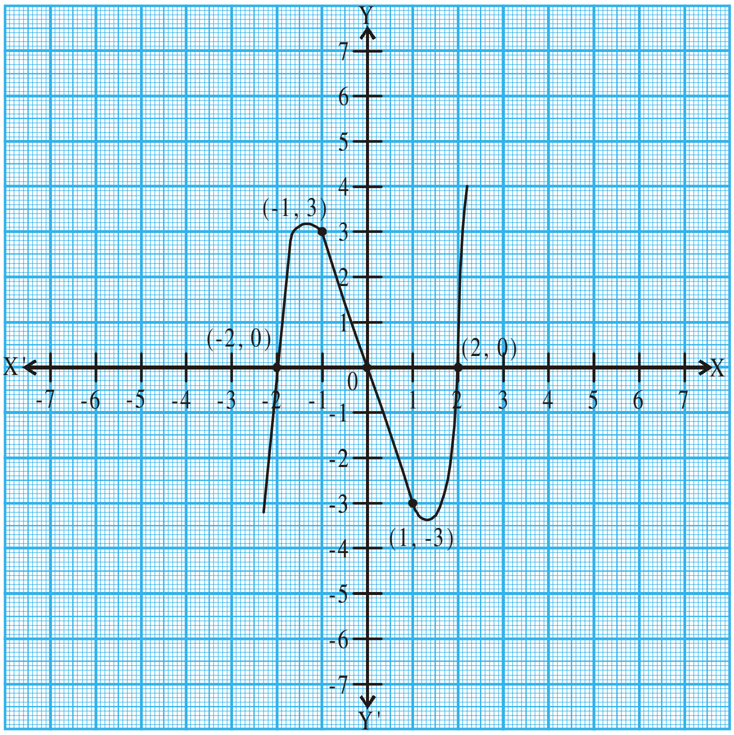
3.4.3 Geometrical Meaning of Zeroes of a Cubic Polynomial

What could you expect the geometrical meaning of the zeroes of a cubic polynomial to be? Let us find out. Consider the cubic polynomial $x^3 - 4x$. To see how the graph of $y = x^3 - 4x$ looks like, let us list a few values of y corresponding to a few values for x as shown in Table 3.3.

Table 3.3

x	-2	-1	0	1	2
$y = x^3 - 4x$	0	3	0	-3	0
(x, y)	$(-2, 0)$	$(-1, 3)$	$(0, 0)$	$(1, -3)$	$(2, 0)$

On drawing the graph, we see that the graph of $y = x^3 - 4x$ looks like the one given in the figure.



We see from the table above that -2, 0 and 2 are zeroes of the cubic polynomial $x^3 - 4x$. -2, 0 and 2 are the x-coordinates of the points where the graph of $y = x^3 - 4x$ intersects the x-axis. So this polynomial has three zeros.

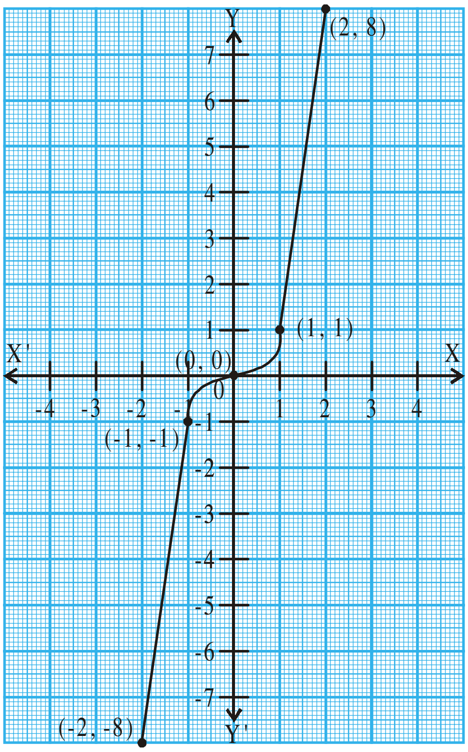
Let us take a few more examples. Consider the cubic polynomials x^3 and $x^3 - x^2$ respectively. See Table 3.4 and 3.5.

Table 3.4

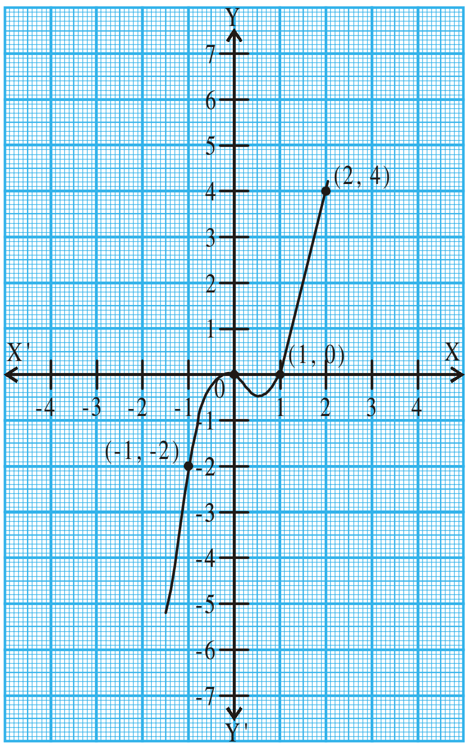
x	-2	-1	0	1	2
$y = x^3$	-8	-1	0	1	8
(x, y)	$(-2, -8)$	$(-1, -1)$	$(0, 0)$	$(1, 1)$	$(2, 8)$

Table 3.5

x	-2	-1	0	1	2
$y = x^3 - x^2$	-12	-2	0	0	4
(x, y)	$(-2, -12)$	$(-1, -2)$	$(0, 0)$	$(1, 0)$	$(2, 4)$



$y = x^3$



$y = x^3 - x^2$

In $y = x^3$, you can see that 0 is the x-coordinate of the only point where the graph of $y = x^3$ intersects the x-axis. So, the polynomial has only one distinct zero. Similarly, 0 and 1 are the x-coordinates of the only points where the graph of $y = x^3 - x^2$ intersects the x-axis. So, the cubic polynomial has two distinct zeros.

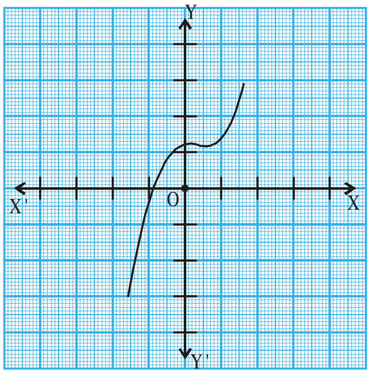
From the examples above, we see that there are at most 3 zeroes for any cubic polynomial. In other words, any polynomial of degree 3 can have at most three zeroes.

Try This

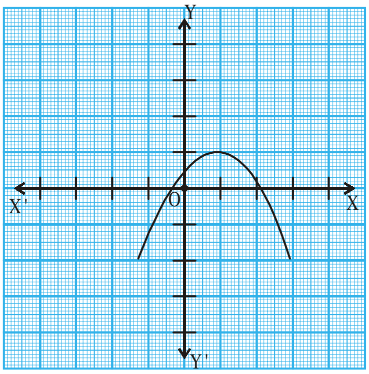
Find the zeroes of cubic polynomials (i) $-x^3$ (ii) $x^2 - x^3$ (iii) $x^3 - 5x^2 + 6x$ without drawing the graph of the polynomial.

Remark : In general, given a polynomial $p(x)$ of degree n, the graph of $y = p(x)$ intersects the x-axis at at most n points. Therefore, a polynomial $p(x)$ of degree n has at most n zeroes.

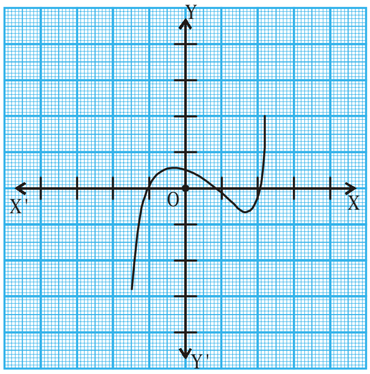
Example-1. Look at the graphs in the figures given below. Each is the graph of $y = p(x)$, where $p(x)$ is a polynomial. In each of the graphs, find the number of zeroes of $p(x)$ in the given range of x.



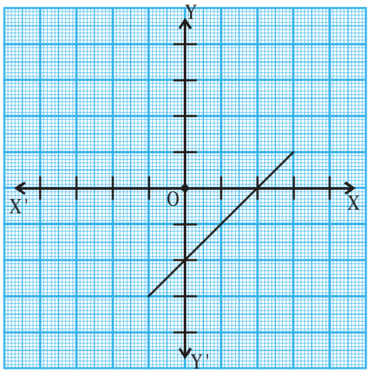
(i)



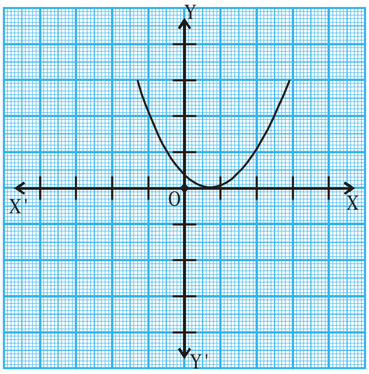
(ii)



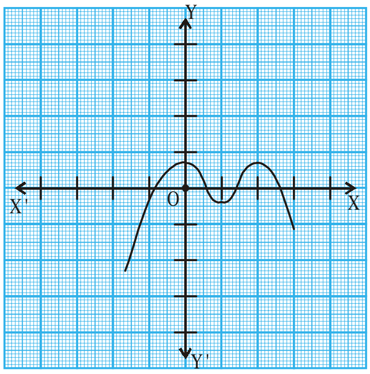
(iii)



(iv)



(v)



(vi)

Solution : In the given range of x in respective graphs :

(i) The number of zeroes is 1 as the graph intersects the x-axis at one point only.

(ii) The number of zeroes is 2 as the graph intersects the x-axis at two points.

(iii) The number of zeroes is 3. (Why?)

(iv) The number of zeroes is 1. (Why?)

(v) The number of zeroes is 1. (Why?)

(vi) The number of zeroes is 4. (Why?)

Example-2. Find the number of zeroes of the given polynomials. And also find their values.

(i) $p(x) = 2x + 1$ (ii) $q(y) = y^2 - 1$ (iii) $r(z) = z^3$

Solution : We will do this without plotting the graph.

(i) $p(x) = 2x + 1$ is a linear polynomial. It has only one zero.

To find zeroes,

Let $p(x) = 0$

So, $2x+1=0$

Therefore $x = \frac{-1}{2}$

The zero of the given polynomial is $\frac{-1}{2}$.

(ii) $q(y) = y^2 - 1$ is a quadratic polynomial.

It has atmost two zeroes.

To find zeroes, let $q(y) = 0$

$\Rightarrow y^2 - 1 = 0$

$\Rightarrow (y + 1)(y - 1) = 0$

$\Rightarrow y = -1$ or $y = 1$

Therefore the zeroes of the polynomial are -1 and 1.

(iii) $r(z) = z^3$ is a cubic polynomial. It has at most three zeroes.

Let $r(z) = 0$

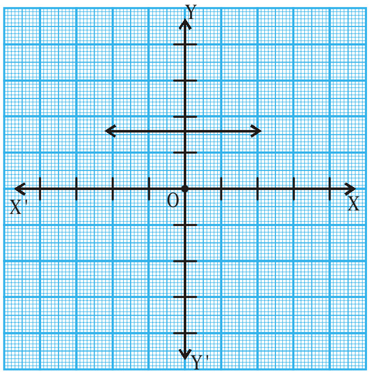
$\Rightarrow z^3 = 0$

$\Rightarrow z = 0$

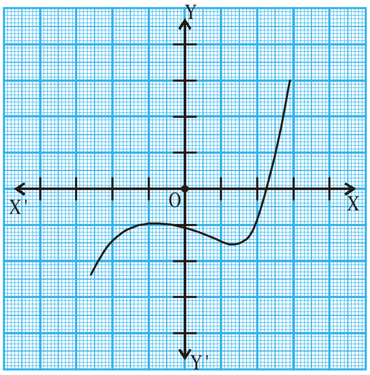
So, the zero of the polynomial is 0.

Exercise – 3.2

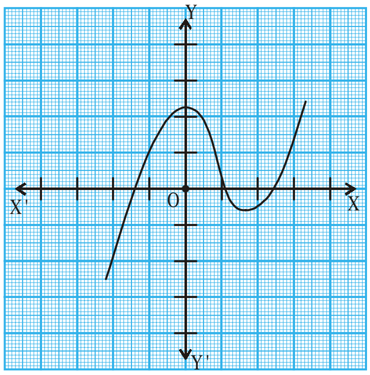
1. The graphs of $y = p(x)$ are given in the figure below, for some polynomials $p(x)$. In each case, find the number of zeroes of $p(x)$.



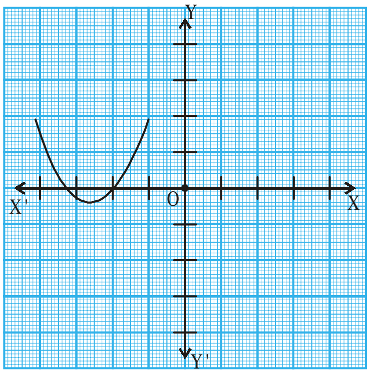
(i)



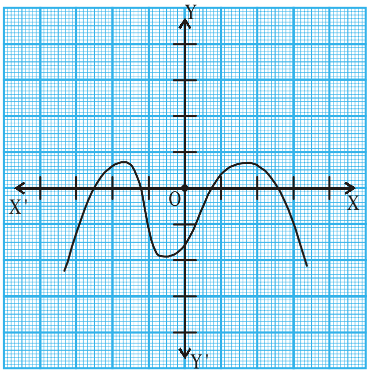
(ii)



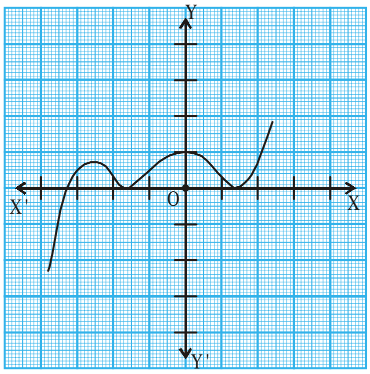
(iii)



(iv)



(v)



(vi)

2. Find the zeroes of the given polynomials.

   (i) $p(x) = 3x$ (ii) $p(x) = x^2 + 5x + 6$

   (iii) $p(x) = (x+2)(x+3)$ (iv) $p(x) = x^4 - 16$

3. Draw the graphs of the given polynomial and find the zeroes. Justify the answers.

   (i) $p(x) = x^2 - x - 12$ (ii) $p(x) = x^2 - 6x + 9$

   (iii) $p(x) = x^2 - 4x + 5$ (iv) $p(x) = x^2 + 3x - 4$

   (v) $p(x) = x^2 - 1$

4. Why are $\frac{1}{4}$ and -1 zeroes of the polynomials $p(x) = 4x^2 + 3x - 1$?

3.5 Relationship between Zeroes and Coefficients of a Polynomial

You have already seen that zero of a linear polynomial $ax + b$ is $-\frac{b}{a}$. We will now try to explore the relationship between zeroes and coefficients of a quadratic polynomial. For this, let us take the quadratic polynomial $p(x) = 2x^2 - 8x + 6$.

In Class-IX, we have learnt how to factorise quadratic polynomials by splitting the middle term. So, here we split the middle term '$-8x$' as a sum of two terms, whose product is $6 \times 2x^2 = 12\,x^2$. So, we write

$$\begin{aligned}2x^2 - 8x + 6 &= 2x^2 - 6x - 2x + 6\\ &= 2x(x-3) - 2(x-3)\\ &= (2x-2)\,(x-3) = 2(x-1)\,(x-3)\end{aligned}$$

$p(x) = 2x^2 - 8x + 6$ is zero when $x - 1 = 0$ or $x - 3 = 0$, i.e., when $x = 1$ or $x = 3$. So, the zeroes of $2x^2 - 8x + 6$ are 1 and 3. We now try and see if these zeroes have some relationship to the coefficients of terms in the polynomial. The coefficient of x^2 is 2; of x is -8 and the constant is 6, which is the coefficient of x^0. (i.e. $6x^0 = 6$)

We see that the sum of the zeroes $= 1 + 3 = 4 = \frac{-(-8)}{2} = \frac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of the zeroes $= 1 \times 3 = 3 = \frac{6}{2} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Let us take one more quadratic polynomial:

$$p(x) = 3x^2 + 5x - 2.$$

By splitting the middle term we see,

$$\begin{aligned}3x^2 + 5x - 2 &= 3x^2 + 6x - x - 2 = 3x(x+2) - 1(x+2)\\ &= (3x-1)\,(x+2)\end{aligned}$$

$3x^2 + 5x - 2$ is zero when either $3x - 1 = 0$ or $x + 2 = 0$

i.e., when $x = \frac{1}{3}$ or $x = -2$.

The zeroes of $3x^2 + 5x - 2$ are $\frac{1}{3}$ and -2. We can see that the :

Sum of its zeroes $= \frac{1}{3} + (-2) = \frac{-5}{3} = \frac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of its zeroes $= \frac{1}{3} \times (-2) = \frac{-2}{3} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

Do This

Find the zeroes of the quadratic polynomials given below. Find the sum and product of the zeroes and verify relationship to the coefficients of terms in the polynomial.

(i) $p(x) = x^2 - x - 6$ (ii) $p(x) = x^2 - 4x + 3$

(iii) $p(x) = x^2 - 4$ (iv) $p(x) = x^2 + 2x + 1$

In general, if α and β are the zeroes of the quadratic polynomial $p(x) = ax^2 + bx + c$, where $a \neq 0$, then $(x - \alpha)$ and $(x - \beta)$ are the factors of $p(x)$. Therefore,

$$ax^2 + bx + c = k(x - \alpha)(x - \beta), \text{ where } k \text{ is a constant}$$

$$= k[x^2 - (\alpha + \beta)x + \alpha\beta]$$

$$= kx^2 - k(\alpha + \beta)x + k\alpha\beta$$

Comparing the coefficients of x^2, x and constant terms on both the sides, we get

$$a = k,\ b = -k(\alpha+\beta) \text{ and } c = k\alpha\beta.$$

This gives $\alpha + \beta = \dfrac{-b}{a}$,

$$\alpha\beta = \frac{c}{a}$$

Note : α and β are Greek letters pronounced as 'alpha' and 'beta' respectively. We will use later one more letter 'γ' pronounced as 'gamma'.

So, sum of zeroes of a quadratic polynomial $= \alpha + \beta = \dfrac{-b}{a} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of zeroes of a quadratic polynomial $= \alpha\beta = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Let us consider some examples.

Example-3. Find the zeroes of the quadratic polynomial $x^2 + 7x + 10$, and verify the relationship between the zeroes and the coefficients.

Solution : We have

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

So, the value of $x^2 + 7x + 10$ is zero when $x + 2 = 0$ or $x + 5 = 0$,

i.e., when $x = -2$ or $x = -5$.

Therefore, the zeroes of $x^2 + 7x + 10$ are -2 and -5.

Now, sum of the zeroes $= -2 + (-5) = -(7) = \dfrac{-(7)}{1} = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of the zeroes $= -2 \times (-5) = 10 = \dfrac{10}{1} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Example-4. Find the zeroes of the polynomial $x^2 - 3$ and verify the relationship between the zeroes and the coefficients.

Solution : Recall the identity $a^2 - b^2 = (a - b)(a + b)$.

Using it, we can write:

$$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3})$$

So, the value of $x^2 - 3$ is zero when $x = \sqrt{3}$ or $x = -\sqrt{3}$.

Therefore, the zeroes of $x^2 - 3$ are $\sqrt{3}$ and $-\sqrt{3}$.

Sum of the zeroes $= \sqrt{3} + (-\sqrt{3}) = 0 = \dfrac{-(\text{coefficient of } x)}{\text{coefficient of } x^2}$

Product of zeroes $= (\sqrt{3}) \times (-\sqrt{3}) = -3 = \dfrac{-3}{1} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$

Example-5. Find a quadratic polynomial, the sum and product of whose zeroes are -3 and 2, respectively.

Solution : Let the quadratic polynomial be $ax^2 + bx + c$, and its zeroes be α and β. We have

$$\alpha + \beta = -3 = \frac{-b}{a},$$

and $$\alpha\beta = 2 = \frac{c}{a}.$$

If we take $a = 1$, then $b = 3$ and $c = 2$

So, one quadratic polynomial which fits the given conditions is $x^2 + 3x + 2$.

Similarly, we can take 'a' to be any real number. Let us say it is k. This gives $\frac{-b}{k} = -3$ or $b = 3k$ and $\frac{c}{k} = 2$ or $c = 2k$. Putting the values of a, b and c, we get the polynomial is $kx^2 + 3kx + 2k$.

Example-6. Find a quadratic polynomial if the zeroes of it are 2 and $\frac{-1}{3}$ respectively.

Solution : Let the quadratic polynomial be

$ax^2 + bx + c$, $a \neq 0$ and its zeroes be α and β.

Here $\alpha = 2$, $\beta = \frac{-1}{3}$

Sum of the zeroes $= (\alpha + \beta) = 2 + \left(\frac{-1}{3}\right) = \frac{5}{3}$

Product of the zeroes $= (\alpha\beta) = 2\left(\frac{-1}{3}\right) = \frac{-2}{3}$

Therefore the quadratic polynomial $ax^2 + bx + c$ is

$k[x^2 - (\alpha+\beta)x + \alpha\,\beta]$, where k is a constant

$$= k[x^2 - \frac{5}{3}x - \frac{2}{3}]$$

We can put different values of k.

When $k = 3$, the quadratic polynomial will be $3x^2 - 5x - 2$.

TRY THIS

(i) Find a quadratic polynomial with zeroes -2 and $\frac{1}{3}$.

(ii) What is the quadratic polynomial whose sum of zeroes is $\frac{-3}{2}$ and the product of zeroes is -1.

3.6 Cubic Polynomials

Let us now look at cubic polynomials. Do you think some relation holds between the zeroes of a cubic polynomial and its coefficients as well?

Let us consider $p(x) = 2x^3 - 5x^2 - 14x + 8$.

We see that $p(x) = 0$ for $x = 4, -2, \frac{1}{2}$.

Since $p(x)$ can have at most three zeroes, these are the zeroes of $2x^3 - 5x^2 - 14x + 8$.

Sum of its zeroes $= 4 + (-2) + \frac{1}{2} = \frac{5}{2} = \frac{-(-5)}{2} = \frac{-(\text{coefficient of } x^2)}{\text{coefficient of } x^3}$

Product of its zeroes $= 4 \times (-2) \times \frac{1}{2} = -4 = \frac{-8}{2} = \frac{-(\text{constant term})}{\text{coefficient of } x^3}$

However, there is one more relationship here. Consider the sum of the products of the zeroes taken two at a time. We have:

$$= \{4 \times (-2)\} + \left\{(-2) \times \frac{1}{2}\right\} + \left\{\frac{1}{2} \times 4\right\}$$

$$= -8 - 1 + 2 = -7 = \frac{-14}{2} = \frac{\text{constant of } x}{\text{coefficient of } x^3}$$

In general, it can be proved that if α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, then

$\alpha + \beta + \gamma = \frac{-b}{a}$,

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$,

$\alpha\beta\gamma = \frac{-d}{a}$.

$ax^3 + bx^2 + cx + d$ is a polynomial with zeroes α, β, γ. Let us see how α, β, γ relate to a, b, c, d.

Since α, β, γ are the zeroes, the polynomial can be written as $(x-\alpha)(x-\beta)(x-\gamma)$

$$= x^3 - x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\beta\gamma+\alpha\gamma) - \alpha\beta\gamma$$

To compare with the polynomial, we multiply by 'a' and get

$ax^3 - x^2a(\alpha+\beta+\gamma) + xa(\alpha\beta+\beta\gamma+\alpha\gamma) - a\alpha\beta\gamma$.

$\therefore\ b = -a(\alpha+\beta+\gamma), c = a(\alpha\beta+\beta\gamma+\alpha\gamma), d = -a\alpha\beta\gamma$

Do This

If α, β, γ are the zeroes of the given cubic polynomials, find the values as given in the table.

S.No.	Cubic Polynomial	$\alpha+\beta+\gamma$	$\alpha\beta+\beta\alpha+\gamma\alpha$	$\alpha\beta\gamma$
1	x^3+3x^2-x-2			
2	$4x^3+8x^2-6x-2$			
3	x^3+4x^2-5x-2			
4	x^3+5x^2+4			

Let us consider an example.

Example-7. Verify that $3, -1, -\frac{1}{3}$ are the zeroes of the cubic polynomial $p(x) = 3x^3 - 5x^2 - 11x - 3$, and then verify the relationship between the zeroes and the coefficients.

Solution : Comparing the given polynomial with $ax^3 + bx^2 + cx + d$, we get

$a = 3,\ b = -5,\ c = -11,\ d = -3$. Further

$$p(3) = 3 \times 3^3 - (5 \times 3^2) - (11 \times 3) - 3 = 81 - 45 - 33 - 3 = 0,$$

$$p(-1) = 3 \times (-1)^3 - 5 \times (-1)^2 - 11 \times (-1) - 3 = -3 - 5 + 11 - 3 = 0,$$

$$p\left(-\frac{1}{3}\right) = 3 \times \left(-\frac{1}{3}\right)^3 - 5 \times \left(-\frac{1}{3}\right)^2 - 11 \times \left(-\frac{1}{3}\right) - 3,$$

$$= -\frac{1}{9} - \frac{5}{9} + \frac{11}{3} - 3 = -\frac{2}{3} + \frac{2}{3} = 0$$

Therefore, 3, –1, and $-\frac{1}{3}$ are the zeroes of $3x^3 - 5x^2 - 11x - 3$.

So, we take $\alpha = 3$, $\beta = -1$ and $\gamma = -\frac{1}{3}$.

Now,

$$\alpha + \beta + \gamma = 3 + (-1) + \left(-\frac{1}{3}\right) = 2 - \frac{1}{3} = \frac{5}{3} = \frac{-(-5)}{3} = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times (-1) + (-1) \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 3 = -3 + \frac{1}{3} - 1 = \frac{-11}{3} = \frac{c}{a},$$

$$\alpha\beta\gamma = 3 \times (-1) \times \left(-\frac{1}{3}\right) = 1 = \frac{-(-3)}{3} = \frac{-d}{a}.$$

Exercise – 3.3

1. Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.

   (i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$ (iii) $6x^2 - 3 - 7x$

   (iv) $4u^2 + 8u$ (v) $t^2 - 15$ (vi) $3x^2 - x - 4$

2. Find the quadratic polynomial in each case, with the given numbers as the sum and product of its zeroes respectively.

   (i) $\frac{1}{4}, -1$ (ii) $\sqrt{2}, \frac{1}{3}$ (iii) $0, \sqrt{5}$

   (iv) $1, 1$ (v) $-\frac{1}{4}, \frac{1}{4}$ (vi) $4, 1$

3. Find the quadratic polynomial, for the zeroes α, β given in each case.

   (i) $2, -1$ (ii) $\sqrt{3}, -\sqrt{3}$ (iii) $\frac{1}{4}, -1$ (iv) $\frac{1}{2}, \frac{3}{2}$

4. Verify that $1, -1$ and -3 are the zeroes of the cubic polynomial $x^3 + 3x^2 - x - 3$ and check the relationship between zeroes and the coefficients.

3.7 Division Algorithm for Polynomials

You know that a cubic polynomial has at most three zeroes. However, if you are given only one zero, can you find the other two? For this, let us consider the cubic polynomial $x^3 + 3x^2 - x - 3$. If we tell you that one of its zeroes is 1, then you know that this polynomial is divisible by $x - 1$. Dividing by $x - 1$ we would get the quotient $x^2 - 2x - 3$.

We get the factors of $x^2 - 2x - 3$ by splitting the middle term. The factors are $(x + 1)$ and $(x - 3)$. This gives us

$$x^3 + 3x^2 - x - 3 = (x - 1)(x^2 - 2x - 3)$$

$$= (x - 1)(x + 1)(x - 3)$$

So, the three zeroes of the cubic polynomial are $1, -1, 3$.

Let us discuss the method of dividing one polynomial by another in some detail. Before doing the steps formally, consider a particular example.

Example-8. Divide $2x^2 + 3x + 1$ by $x + 2$.

Solution : Note that we stop the division process when either the remainder is zero or its degree is less than the degree of the divisor. So, here the quotient is $2x - 1$ and the remainder is 3. Also,

$$\begin{array}{r|l} & 2x - 1 \\ \hline x+2 & 2x^2 + 3x + 1 \\ & 2x^2 + 4x \\ & -\quad\ - \\ \hline & -x + 1 \\ & -x - 2 \\ & +\quad + \\ \hline & 3 \end{array}$$

$$(2x - 1)(x + 2) + 3 = 2x^2 + 3x - 2 + 3 = 2x^2 + 3x + 1$$

i.e., $2x^2 + 3x + 1 = (x + 2)(2x - 1) + 3$

Therefore, Dividend = Divisor × Quotient + Remainder

Let us now extend this process to divide a polynomial by a quadratic polynomial.

Example-9. Divide $3x^3 + x^2 + 2x + 5$ by $1 + 2x + x^2$.

Solution : We first arrange the terms of the dividend and the divisor in the decreasing order of their degrees. Arranging the terms in this order is termed as writing the polynomials in its standard form. In this example, the dividend is already in its standard form, and the divisor, also in standard form, is $x^2 + 2x + 1$.

$$\begin{array}{r|l} & 3x - 5 \\ \hline x^2+2x+1 & 3x^3 + x^2 + 2x + 5 \\ & 3x^3 + 6x^2 + 3x \\ & -\quad\ -\quad\ - \\ \hline & -5x^2 - x + 5 \\ & -5x^2 - 10x - 5 \\ & +\quad\ +\quad\ + \\ \hline & 9x + 10 \end{array}$$

Step 1 : To obtain the first term of the quotient, divide the highest degree term of the dividend (i.e., $3x^3$) by the highest degree term of the divisor (i.e., x^2). This is $3x$. Then carry out the division process. What remains is $-5x^2 - x + 5$.

Step 2 : Now, to obtain the second term of the quotient, divide the highest degree term of the new dividend (i.e., $-5x^2$) by the highest degree term of the divisor (i.e., x^2). This gives -5. Again carry out the division process with $-5x^2 - x + 5$.

Step 3 : What remains is $9x + 10$. Now, the degree of $9x + 10$ is less than the degree of the divisor $x^2 + 2x + 1$. So, we cannot continue the division any further.

So, the quotient is $3x - 5$ and the remainder is $9x + 10$. Also,

$$\begin{aligned}(x^2 + 2x + 1) \times (3x - 5) + (9x + 10) &= (3x^3 + 6x^2 + 3x - 5x^2 - 10x - 5 + 9x + 10) \\ &= 3x^3 + x^2 + 2x + 5\end{aligned}$$

Here again, we see that

Dividend = Divisor × Quotient + Remainder

We are applying here an algorithm called Euclid's division algorithm.

This says that

If $p(x)$ and $g(x)$ are any two polynomials with $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x) \times q(x) + r(x),$$

where either $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$ if $r(x) \neq 0$

This result is known as the **Division Algorithm** for polynomials.

Now, we have the following results from the above discussions

(i) If $q(x)$ is linear polynomial then $r(x) = r$ is a constant.

(ii) If degree of $q(x) = 1$, then degree of $p(x) = 1 +$ degree of $g(x)$.

(iii) If $p(x)$ is divided by $(x - a)$, then the remainder is $p(a)$.

(iv) If $r = 0$, we say $q(x)$ divides $p(x)$ exactly or $q(x)$ is a factor of $p(x)$.

Let us now take some examples to illustrate its use.

Example-10. Divide $3x^2 - x^3 - 3x + 5$ by $x - 1 - x^2$, and verify the division algorithm.

Solution : Note that the given polynomials are not in standard form. To carry out division, we first write both the dividend and divisor in decreasing orders of their degrees.

So, dividend $= -x^3 + 3x^2 - 3x + 5$ and

divisor $= -x^2 + x - 1$.

Division process is shown on the right side.

$$\begin{array}{r|l} & \quad x-2 \\ \hline -x^2+x-1 & -x^3+3x^2-3x+5 \\ & -x^3+x^2-x \\ & +\quad -\quad + \\ \hline & \quad 2x^2-2x+5 \\ & \quad 2x^2-2x+2 \\ & \quad -\quad +\quad - \\ \hline & \quad\quad 3 \end{array}$$

We stop here since degree of the remainder is less than the degree of $(-x^2 + x - 1)$ the divisor.

So, quotient $= x - 2$, remainder $= 3$.

Now,

Dividend = Divisor × Quotient + Remainder

$= (-x^2 + x - 1)(x - 2) + 3$

$= -x^3 + x^2 - x + 2x^2 - 2x + 2 + 3$

$= -x^3 + 3x^2 - 3x + 5$

In this way, the division algorithm is verified.

Example-11. Find all the zeroes of $2x^4 - 3x^3 - 3x^2 + 6x - 2$, if you know that two of its zeroes are $\sqrt{2}$ and $-\sqrt{2}$.

Solution : Since two of the zeroes are $\sqrt{2}$ and $-\sqrt{2}$, therefore we can divide by $(x - \sqrt{2})$ $(x + \sqrt{2}) = x^2 - 2$.

$$
\begin{array}{r|l}
 & 2x^2 - 3x + 1 \\
\hline
x^2 - 2 & 2x^4 - 3x^3 - 3x^2 + 6x - 2 \\
 & 2x^4 \qquad\quad - 4x^2 \\
 & - \qquad\qquad + \\
\hline
 & \quad -3x^3 + x^2 + 6x - 2 \\
 & \quad -3x^3 \qquad + 6x \\
 & \quad + \qquad\quad - \\
\hline
 & \qquad\qquad x^2 - 2 \\
 & \qquad\qquad x^2 - 2 \\
 & \qquad\qquad - \quad + \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

First term of quotient is $\dfrac{2x^4}{x^2} = 2x^2$

Second term of quotient is $\dfrac{-3x^3}{x^2} = -3x$

Third term of quotient is $\dfrac{x^2}{x^2} = 1$

So, $2x^4 - 3x^3 - 3x^2 + 6x - 2 = (x^2 - 2)(2x^2 - 3x + 1)$.

Now, by splitting $-3x$, we factorize $2x^2 - 3x + 1$ as $(2x - 1)(x - 1)$. So, its zeroes are given by $x = \dfrac{1}{2}$ and $x = 1$. Therefore, the zeroes of the given polynomial are $\sqrt{2}, -\sqrt{2}$, 1 and $\dfrac{1}{2}$.

Exercise – 3.4

1. Divide the polynomial $p(x)$ by the polynomial $g(x)$ and find the quotient and remainder in each of the following :

   (i) $p(x) = x^3 - 3x^2 + 5x - 3,\ g(x) = x^2 - 2$

   (ii) $p(x) = x^4 - 3x^2 + 4x + 5,\ g(x) = x^2 + 1 - x$

   (iii) $p(x) = x^4 - 5x + 6,\ g(x) = 2 - x^2$

2. Check in which case the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial :

   (i) $t^2 - 3,\ 2t^4 + 3t^3 - 2t^2 - 9t - 12$

   (ii) $x^2 + 3x + 1,\ 3x^4 + 5x^3 - 7x^2 + 2x + 2$

   (iii) $x^3 - 3x + 1,\ x^5 - 4x^3 + x^2 + 3x + 1$

3. Obtain all other zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.

4. On dividing $x^3 - 3x^2 + x + 2$ by a polynomial $g(x)$, the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find $g(x)$.

5. Give examples of polynomials $p(x), g(x), q(x)$ and $r(x)$, which satisfy the division algorithm and

   (i) deg $p(x)$ = deg $q(x)$ (ii) deg $q(x)$ = deg $r(x)$ (iii) deg $r(x) = 0$

Optional Exercise

[This exercise is not meant for examination]

1. Verify that the numbers given alongside the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:

   (i) $2x^3 + x^2 - 5x + 2$; $(\frac{1}{2}, 1, -2)$ (ii) $x^3 + 4x^2 + 5x - 2$; (1, 1, 1)

2. Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, –7, –14 respectively.

3. If the zeroes of the polynomial $x^3 - 3x^2 + x + 1$ are $a - b, a, a + b$ find a and b.

4. If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.

5. If the polynomial $x^4 - 6x^3 - 16x^2 + 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.

What We Have Discussed

In this chapter, you have studied the following points:

1. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.

2. A quadratic polynomial in x with real coefficients is of the form $ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$.

3. The zeroes of a polynomial $p(x)$ are the x-coordinates of the points where the graph of $y = p(x)$ intersects the x-axis.

4. A quadratic polynomial can have at most 2 zeroes and a cubic polynomial can have at the most 3 zeroes.

5. If α and β are the zeroes of the quadratic polynomial $ax^2 + bx + c$, $a \neq 0$, then

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

6. If α, β, γ are the zeroes of the cubic polynomial $ax^3 + bx^2 + cx + d$, $a \neq 0$, then

$$\alpha + \beta + \gamma = \frac{-b}{a},$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a},$$

and $$\alpha\beta\gamma = \frac{-d}{a}.$$

7. The division algorithm states that given any polynomial $p(x)$ and any non-zero polynomial $g(x)$, there are polynomials $q(x)$ and $r(x)$ such that

$$p(x) = g(x)\, q(x) + r(x),$$

CHAPTER

4

Pair of Linear Equations in Two Variables

4.1 INTRODUCTION

One day Siri went to a book shop with her father and bought 3 notebooks and 2 pens. Her father paid ₹80 for them. Her friend Laxmi liked the notebooks and pens so she bought 4 notebooks and 3 pens of the same kind for ₹110 and again her classmates Rubina liked the pens and Joseph liked the notebooks. They asked Siri the cost of one pen and one notebook. But, Siri did not know the cost of one notebook and one pen separately. How can they find the costs of these items?

In this example, the cost of a notebook and a pen are not known. These are unknown quantities. We come across many such situations in our day-to-day life.

THINK - DISCUSS

Two situations are given below:

(i) The cost of 1kg potatoes and 2kg tomatoes was ₹30 on a certain day. After two days, the cost of 2kg potatoes and 4kg tomatoes was found to be ₹66.

(ii) The coach of a cricket team of M.K.Nagar High School buys 3 bats and 6 balls for ₹3900. Later he buys one more bat and 2 balls for ₹1300.

Identify the unknowns in each situation. We observe that there are two unknowns in each case.

4.1.1 HOW DO WE FIND UNKNOWN QUANTITIES?

In the introduction, Siri bought 3 notebooks and 2 pens for ₹80. How can we find the cost of a notebook or the cost of a pen?

Rubina and Joseph tried to guess. Rubina said that price of each notebook could be ₹25. Then three notebooks would cost ₹75, the two pens would cost ₹5 and each pen could be for ₹2.50.

Joseph felt that ₹2.50 for one pen was too little. It should be at least ₹16. Then the price of each notebook would also be ₹16.

We can see that there can be many possible values for the price of a notebook and of a pen so that the total cost is ₹80. So, how do we find cost price at which Siri and Laxmi bought them? By only using Siri's situation, we cannot find the costs. We have to use Laxmi's situation also.

4.1.2 Using Both Equations Together

Laxmi also bought the same types of notebooks and pens as Siri. She paid ₹110 for 4 notebooks and 3 pens.

So, we have two situations which can be represented as follows:

(i) Cost of 3 notebooks + 2 pens = ₹80.

(ii) Cost of 4 notebooks + 3 pens = ₹110.

Does this help us find the cost of a pen and a notebook?

Consider the prices mentioned by Rubina. If the price of one notebook is ₹25 and the price of one pen is ₹2.50 then,

The cost of 4 notebooks would be : $4 \times 25 =$ ₹100

And the cost for 3 pens would be : $3 \times 2.50 =$ ₹7.50

If Rubina is right then Laxmi should have paid ₹ 100 +₹ 7.50 = ₹ 107.50 but she paid ₹110.

Now, consider the prices mentioned by Joseph. Then,

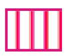
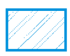
The cost of 4 notebooks, if one is for ₹16, would be : $4 \times 16 =$ ₹64

And the cost for 3 pens, if one is for ₹16, would be : $3 \times 16 =$ ₹ 48

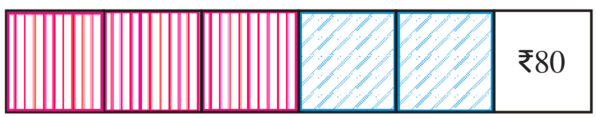
If Joseph is right then Laxmi should have paid ₹64 + ₹48 = ₹112 but this is more than the price she paid.

So what do we do? How to find the exact cost of the notebook and the pen?

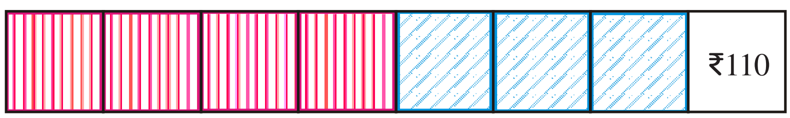
If we have only one equation but two unknowns (variables), we can find many solutions. So, when we have two variables, we need at least two independent equations to get a unique solution. One way to find the values of unknown quantities is by using the Model method. In this method, rectangles or portions of rectangles are often used to represent the unknowns. Let us look at the first situation using the model method:

Step-1 : Represent notebooks by  and pens by .

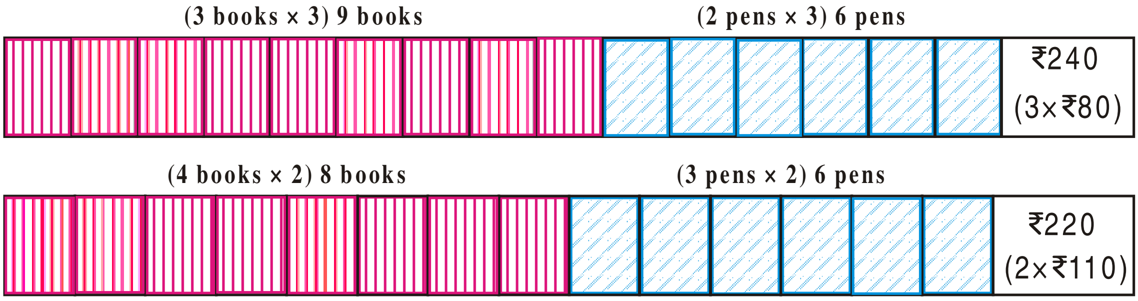
Siri bought 3 books and 2 pens for ₹80.



Laxmi bought 4 books and 3 pens for ₹110.



Step-2 : Increase (or decrease) the quantities in proportion to make one of the quantities equal in both situations. Here, we make the number of pens equal.



In Step 2, we observe a simple proportional reasoning.

Since Siri bought 3 books and 2 pens for ₹80, so for 9 books and 6 pens:

$3 \times 3 = 9$ books and $3 \times 2 = 6$ pens, the cost will be $3 \times 80 =$ ₹240 (1)

Similarly, Laxmi bought 4 books and 3 pens for ₹110, so:

$2 \times 4 = 8$ books and $2 \times 3 = 6$ pens will cost $2 \times 110 =$ ₹220 (2)

After comparing (1) and (2), we can easily observe that 1 extra book costs

₹240 - ₹ 220 = ₹20. So one book is of ₹20.

Siri bought 3 books and 2 pens for ₹80. Since each book costs ₹20, 3 books cost ₹ 60. So the cost of 2 pens become ₹ 80 - ₹ 60 = ₹ 20.

So, cost of each pen is ₹20 ÷ 2 = ₹10.

Let us try these costs in Laxmi's situation. 4 books will cost ₹ 80 and three pens will cost ₹ 30 for a total of ₹ 110, which is true.

From the above discussion and calculation, it is clear that to get exactly one solution (unique solution) we need at least two independent linear equations in the same two variables.

In general, an equation of the form $ax + by + c = 0$ where a, b, c are real numbers and where at least one of a or b is not zero, is called a linear equation in two variables x and y. [We often write this condition as $a^2 + b^2 \neq 0$].

Try This

Mark the correct option in the following questions:

1. Which of the following equations is not a linear equation?

a) $5 + 4x = y + 3$ b) $x+2y = y - x$

c) $3 - x = y^2 + 4$ d) $x + y = 0$

2. Which of the following is a linear equation in one variable?

a) $2x + 1 = y - 3$ b) $2t - 1 = 2t + 5$

c) $2x - 1 = x^2$ d) $x^2 - x + 1 = 0$

3. Which of the following numbers is a solution for the equation $2(x + 3) = 18$?

a) 5 b) 6 c) 13 d) 21

4. The value of x which satisfies the equation $2x - (4 - x) = 5 - x$ is

a) 4.5 b) 3 c) 2.25 d) 0.5

5. The equation $x - 4y = 5$ has

a) no solution b) unique solution

c) two solutions d) infinitely many solutions

4.2 Solutions of Pairs of Linear Equations in Two Variables

In the introductory example of notebooks and pens, how many equations did we have? We had two equations or a pair of linear equations in two variables. What do we mean by the solution for a pair of linear equations?

The pair of values of the variables x and y which together satisfy each one of the equations is called the solution for a pair of linear equations.

4.2.1 Graphical Method of Finding Solution of a Pair of Linear Equations

What will be the number of solutions for a pair of linear equations in two variables? Is the number of solutions infinite or unique or none?

In an earlier section, we used the model method for solving the pair of linear equations. Now we will use graphs to solve the equations.

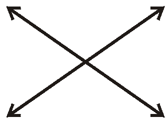
Let: $a_1x + b_1y + c_1 = 0$, $(a_1^2 + b_1^2 \neq 0)$ and $a_2x + b_2y + c_2 = 0$; $(a_2^2 + b_2^2 \neq 0)$ form a pair of linear equation in two variables.

The graph of a linear equation in two variables is a straight line. Ordered pairs of real numbers (x, y) representing points on the line are solutions of the equation and ordered pairs of real numbers (x, y) that do not represent points on the line are not solutions.

When we have a pair of equations, they represent lines in the same plane. So, if we have two lines in the same plane, what can be the possible relations between them? What is the significance of this relation?

When two lines are drawn in the same plane, only one of the following three situations is possible:

i) The two lines may intersect at one point.



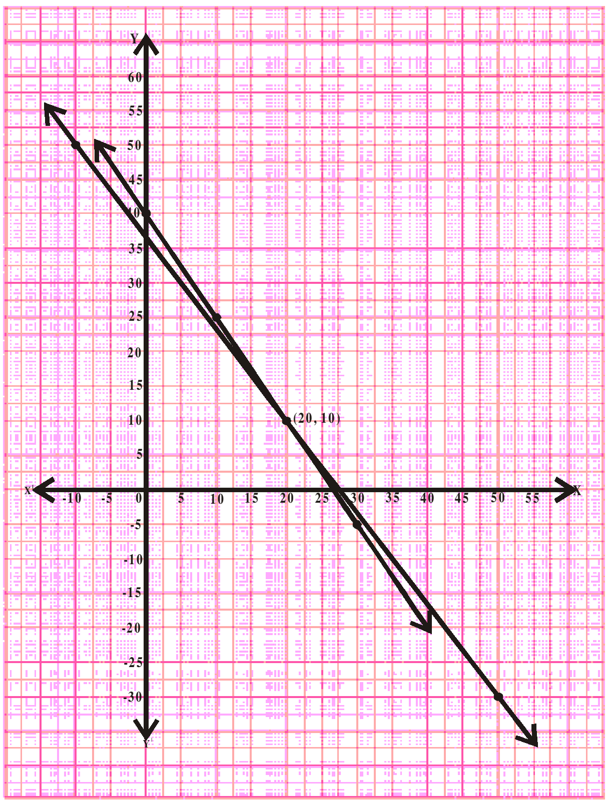
ii) The two lines may not intersect i.e., they are parallel.

iii) The two lines may be coincident.
(actually both are same)

Let us write the equations in the first example in terms of x and y where x is the cost of a notebook and y is the cost of a pen. Then, the equations are $3x + 2y = 80$ and $4x + 3y = 110$.

For the equation $3x + 2y = 80$		
x	$y = \frac{80-3x}{2}$	(x, y)
0	$y = \frac{80-3(0)}{2} = 40$	(0, 40)
10	$y = \frac{80-3(10)}{2} = 25$	(10, 25)
20	$y = \frac{80-3(20)}{2} = 10$	(20, 10)
30	$y = \frac{80-3(30)}{2} = -5$	(30, -5)

For the equation $4x + 3y = 110$		
x	$y = \frac{110-4x}{3}$	(x, y)
-10	$y = \frac{110-4(-10)}{3} = 50$	(-10, 50)
20	$y = \frac{110-4(20)}{3} = 10$	(20, 10)
50	$y = \frac{110-4(50)}{3} = -30$	(50, -30)



After plotting the above points in the Cartesian plane, we observe that the two straight lines are intersecting at the point (20, 10).

Substituting the values of x and y in equation we get $3(20) + 2(10) = 80$ and $4(20) + 3(10) = 110$.

Thus, as determined by the graphical method, the cost of each book is ₹20 and of each pen is ₹10. Recall that we got the same solution using the model method.

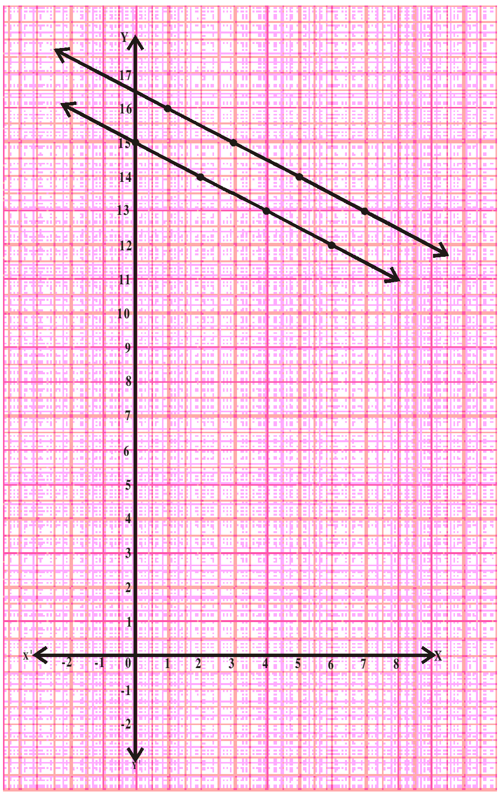
Since (20, 10) is the only common point, there is only one solution for this pair of linear equations in two variables. Such equations are known as consistent pairs of linear equations. They will always have only a unique solution.

Now, let us look at the first example from the think and discuss section. We want to find the cost of 1kg of potatoes and the cost of 1 kg of tomatoes each. Let the cost of 1kg potatoes be ₹x and cost of 1kg of tomato be ₹y. Then, the equations will become $1x+2y=30$ and $2x+4y=66$.

For the equation $x + 2y = 30$		
x	$y = \frac{30-x}{2}$	(x, y)
0	$y = \frac{30-0}{2} = 15$	(0, 15)
2	$y = \frac{30-2}{2} = 14$	(2, 14)
4	$y = \frac{30-4}{2} = 13$	(4, 13)
6	$y = \frac{30-6}{2} = 12$	(6, 12)

For the equation $2x + 4y = 66$		
x	$y = \frac{66-2x}{4}$	(x, y)
1	$y = \frac{66-2(1)}{4} = 16$	(1, 16)
3	$y = \frac{66-2(3)}{4} = 15$	(3, 15)
5	$y = \frac{66-2(5)}{4} = 14$	(5, 14)
7	$y = \frac{66-2(7)}{4} = 13$	(7, 13)

Here, we observe that the situation is represented graphically by two parallel lines. Since the lines do not intersect, the equations have no common solution. This means that the cost of the potato and tomato was different on different days. We see this in real life also. We cannot expect the same cost price of vegetables every day; it keeps changing. Also, the change is independent.



Such pairs of linear equations which have no solution are known as inconsistent pairs of linear equations.

In the second example from the think and discuss section, let the cost of each bat be ₹ x and each ball be ₹ y. Then we can write the equations as $3x + 6y = 3900$ and $x + 2y = 1300$.

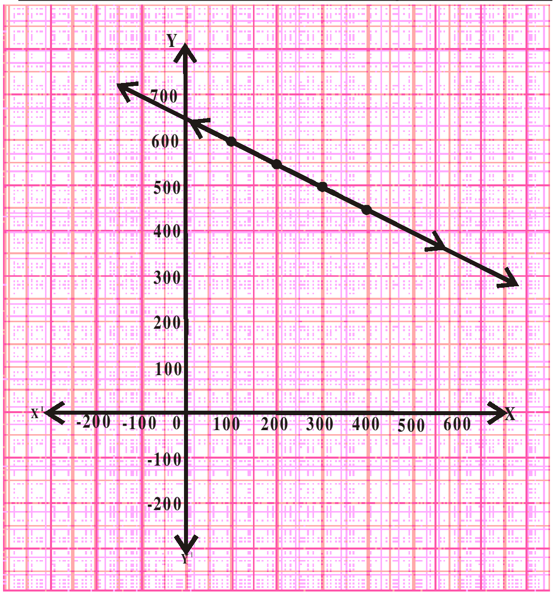
For the equation $3x + 6y = 3900$		
x	$y = \frac{3900-3x}{6}$	(x, y)
100	$y = \frac{3900-3(100)}{6} = 600$	(100, 600)

For the equation $x + 2y = 1300$		
x	$y = \frac{1300-x}{2}$	(x, y)
100	$y = \frac{1300-100}{2} = 600$	(100, 600)

200	$y = \dfrac{3900-3(200)}{6} = 550$	(200, 550)
300	$y = \dfrac{3900-3(300)}{6} = 500$	(300, 500)
400	$y = \dfrac{3900-3(400)}{6} = 450$	(400, 450)

200	$y = \dfrac{1300-200}{2} = 550$	(200, 550)
300	$y = \dfrac{1300-300}{2} = 500$	(300, 500)
400	$y = \dfrac{1300-400}{2} = 450$	(400, 450)

We see that the equations are geometrically shown by a pair of coincident lines. If the solutions of the equations are given by the common points, then what are the common points in this case?



From the graph, we observe that every point on the line is a common solution to both the equations. So, they have infinitely many solutions as both the equations are equivalent. Such pairs of equations are called **dependent** pair of linear equations in two variables.

Try this

In the example given above, can you find the cost of each bat and ball?

Think - discuss

Is a dependent pair of linear equations always consistent. Why or why not?

Do this

1. Solve the following systems of equations :

   i) $x - 2y = 0$, $3x + 4y = 20$

   ii) $x + y = 2$, $2x + 2y = 4$

   iii) $2x - y = 4$, $4x - 2y = 6$

2. Two rails on a railway track are represented by the equations. $x + 2y - 4 = 0$ and $2x + 4y - 12 = 0$. Represent this situation graphically.

4.2.3 Relation Between Coefficients and Nature of System of Equations

Let a_1, b_1, c_1 and a_2, b_2, c_2 denote the coefficients of a given pair of linear equations in two variables. Then, let us write and compare the values of $\dfrac{a_1}{a_2}$, $\dfrac{b_1}{b_2}$ and $\dfrac{c_1}{c_2}$ in the above examples.

Pair of lines	$\frac{a_1}{a_2}$	$\frac{b_1}{b_2}$	$\frac{c_1}{c_2}$	Comparison of ratios	Graphical representation	Algebraic interpretation
1. $3x+2y-80=0$ $4x+3y-110=0$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{-80}{-110}$	$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$	Intersecting lines	Unique solution
2. $1x+2y-30=0$ $2x+4y-66=0$	$\frac{1}{2}$	$\frac{2}{4}$	$\frac{-30}{-66}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$	Parallel lines	No solution
3. $3x+6y=3900$ $x+2y=1300$	$\frac{3}{1}$	$\frac{6}{2}$	$\frac{3900}{1300}$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$	Coincident lines (Dependent lines)	Infinite number of solutions

Let us look few examples.

Example-1. Check whether the given pair of equations repressent intersecting, parallel or coincident lines. Find the solution if the equations are consistent.

$$2x + y - 5 = 0$$

$$3x - 2y - 4 = 0$$

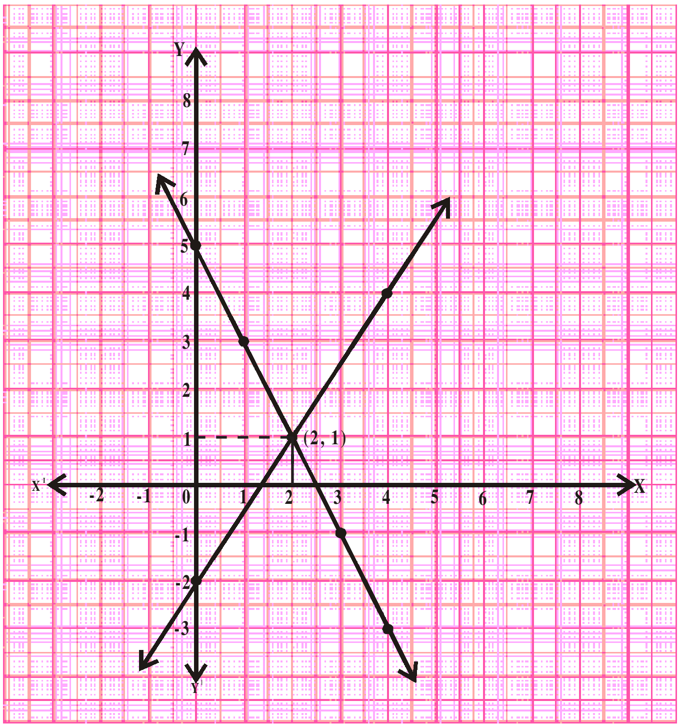
Solution : $\frac{a_1}{a_2} = \frac{2}{3}$ $\frac{b_1}{b_2} = \frac{1}{-2}$ $\frac{c_1}{c_2} = \frac{-5}{-4}$

Since $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, therefore they are intersecting lines and hence, consistent pair of linear equation.

For the equation $2x + y = 5$		
x	$y = 5 - 2x$	(x, y)
0	$y = 5 - 2(0) = 5$	(0, 5)
1	$y = 5 - 2(1) = 3$	(1, 3)
2	$y = 5 - 2(2) = 1$	(2, 1)
3	$y = 5 - 2(3) = -1$	(3, -1)
4	$y = 5 - 2(4) = -3$	(4, -3)

For the equation $3x - 2y = 4$		
x	$y = \frac{4-3x}{-2}$	(x, y)
0	$y = \frac{4-3(0)}{-2} = -2$	(0, -2)
2	$y = \frac{4-3(2)}{-2} = 1$	(2, 1)
4	$y = \frac{4-3(4)}{-2} = 4$	(4, 4)

The unique solution of this pair of equations is (2,1).



Example-2. Check whether the following pair of equations is consistent.

$3x + 4y = 2$ and $6x + 8y = 4$. Verify by a graphical representation.

Solution : $3x + 4y - 2 = 0$

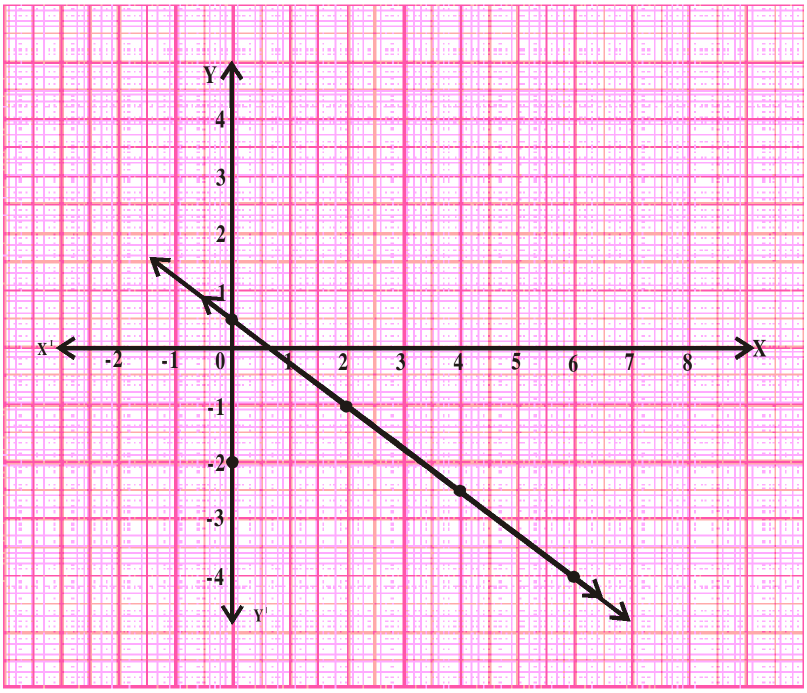
$6x+8y - 4 = 0$

$\frac{a_1}{a_2} = \frac{3}{6} = \frac{1}{2}$ $\qquad$ $\frac{b_1}{b_2} = \frac{4}{8} = \frac{1}{2}$ $\qquad$ $\frac{c_1}{c_2} = \frac{-2}{-4} = \frac{1}{2}$

Since $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, therefore, they are coincident lines. So, the pair of linear equations is dependent and have infinitely many solutions.

For the equation $3x + 4y = 2$		
x	$y = \frac{2-3x}{4}$	(x, y)
0	$y = \frac{2-3(0)}{4} = \frac{1}{2}$	$(0, \frac{1}{2})$
2	$y = \frac{2-3(2)}{4}$ = -1	(2, -1)
4	$y = \frac{2-3(4)}{4}$ = -2.5	(4, -2.5)
6	$y = \frac{2-3(6)}{4}$ = -4	(6, -4)

For the equation $6x + 8y = 4$		
x	$y = \frac{4-6x}{8}$	(x, y)
0	$y = \frac{4-6(0)}{8} = \frac{1}{2}$	$(0, \frac{1}{2})$
2	$y = \frac{4-6(2)}{8}$ = -1	(2, -1)
4	$y = \frac{4-6(4)}{8}$ = -2.5	(4, -2.5)
6	$y = \frac{4-6(6)}{8}$ = -4	(6, -4)



Example-3. Check whether the equations $2x-3y=5$ and $4x-6y=15$ are consistent. Also verify by graphical representation.

Solution : $4x-6y-15=0$

$2x-3y-5=0$

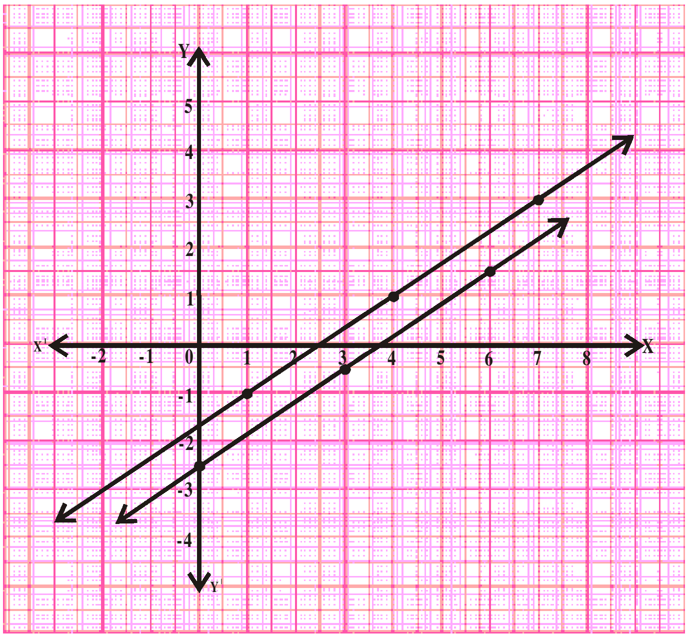
$\frac{a_1}{a_2}=\frac{4}{2}=\frac{2}{1}$ $\qquad$ $\frac{b_1}{b_2}=\frac{-6}{-3}=\frac{2}{1}$

$\frac{c_1}{c_2}=\frac{-15}{-5}=\frac{3}{1}$ $\qquad$ $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$

So the equations are inconsistent. They have no solutions and its graph is of parallel lines.

For the equation $4x - 6y = 9$		
x	$y=\frac{15-4x}{-6}$	(x, y)
0	$y=\frac{15-0}{-6}=\frac{-5}{2}$	(0, -2.5)
3	$y=\frac{15-4(3)}{-6}=\frac{-1}{2}$	(3, -0.5)
6	$y=\frac{15-4(6)}{-6}=\frac{3}{2}$	(6, 1.5)

For the equation $2x - 3y = 5$		
x	$y=\frac{5-2x}{-3}$	(x, y)
1	$y=\frac{5-2(1)}{-3}=-1$	(1, -1)
3	$y=\frac{5-2(4)}{-3}=1$	(4, 1)
6	$y=\frac{5-2(7)}{-3}=3$	(7, 3)



Do This

Check each of the given systems of equations to see if it has a unique solution, infinitely many solutions or no solution. Solve them graphically.

(i) $2x+3y = 1$, $3x-y = 7$ (ii) $x + 2y = 6$, $2x + 4y = 12$ (iii) $3x + 2y = 6$, $6x + 4y = 18$

Try This

1. For what value of 'p' the following pair of equations has a unique solution.
   $2x + py = -5$ and $3x + 3y = -6$
2. Find the value of 'k' for which the pair of equations $2x - ky + 3 = 0$, $4x + 6y - 5 = 0$ represent parallel lines.
3. For what value of 'k', the pair of equation $3x + 4y + 2 = 0$ and $9x + 12y + k = 0$ represent coincident lines.
4. For what positive values of 'p' the following pair of liner equations have infinitely many solutions?
   $px + 3y - (p - 3) = 0$
   $12x + py - p = 0$

Let us look at some more examples.

Example-4. In a garden there are some bees and flowers. If one bee sits on each flower then one bee will be left. If two bees sit on each flower, one flower will be left. Find the number of bees and number of flowers.

Solution : Let the number of bees = x and
the number of flowers = y

If one bee sits on each flower then one bee will be left. So, $x = y + 1$

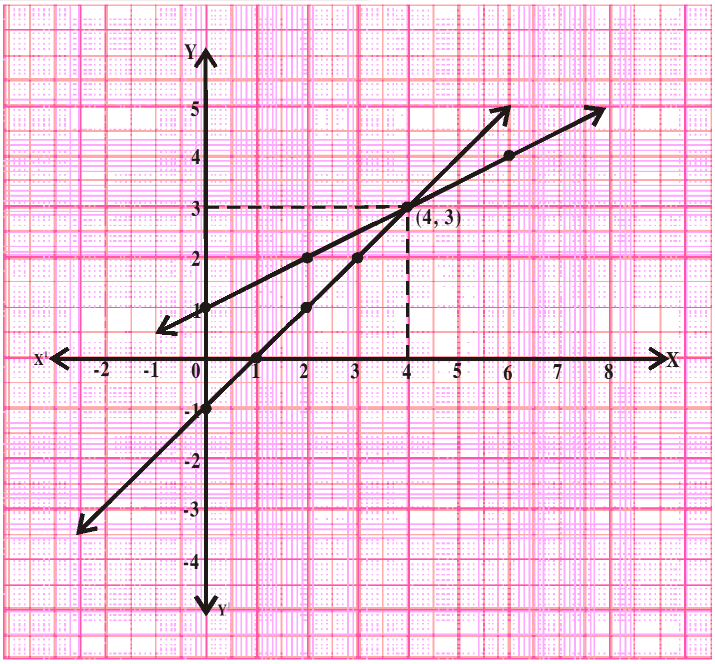
or $\qquad x - y - 1 = 0 \qquad \ldots (1)$

If two bees sit on each flower, one flower will be left. So, $x = 2(y - 1)$

or $\qquad x - 2y + 2 = 0 \qquad \ldots (2)$

For the equation $x - y - 1 = 0$		
x	$y = x - 1$	(x, y)
0	$y = 0 - 1 = -1$	$(0, -1)$
1	$y = 1 - 1 = 0$	$(1, 0)$
2	$y = 2 - 1 = 1$	$(2, 1)$
3	$y = 3 - 1 = 2$	$(3, 2)$
4	$y = 4 - 1 = 3$	$(4, 3)$

For the equation $x - 2y + 2 = 0$		
x	$y = \frac{x+2}{2}$	(x, y)
0	$y = \frac{0+2}{2} = 1$	$(0, 1)$
2	$y = \frac{2+2}{2} = 2$	$(2, 2)$
4	$y = \frac{4+2}{2} = 3$	$(4, 3)$
6	$y = \frac{6+2}{2} = 4$	$(6, 4)$



Therefore, there are 4 bees and 3 flowers.

Example-5. The perimeter of a rectangular plot is 32m. If the length is increased by 2m and the breadth is decreased by 1m, the area of the plot remains the same. Find the length and breadth of the plot.

Solution : Let length and breadth of the rectangular land be l and b respectively. Then,

area = lb $\qquad$ and

Perimeter = $2(l + b) = 32\ m$.

$l + b = 16$ $\qquad$ or $\qquad l + b - 16 = 0 \qquad \ldots (1)$

When length is increased by 2 *m*., then new length is $l+2$. Also breadth is decreased by 1m so new breadth is $b - 1$.

Then, area= $(l + 2)(b - 1)$

Since there is no change in the area,

$(l + 2)(b - 1) = lb$

$lb - l + 2b - 2 = lb$ or $lb - lb = l - 2b + 2$

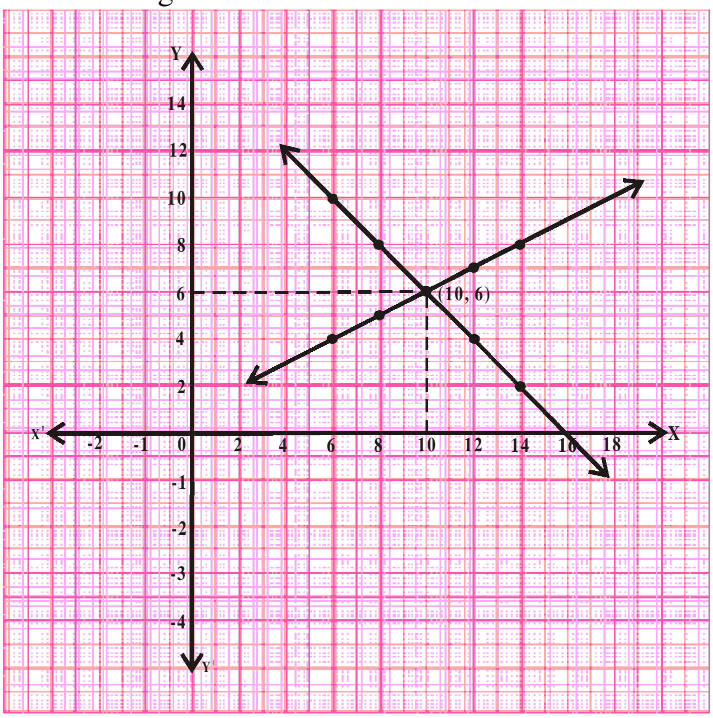
$l - 2b + 2 = 0$ ... (2)

For the equation $l + b - 16 = 0$		
l	$b = 16 - l$	(l, b)
6	$b = 16 - 6 = 10$	(6, 10)
8	$b = 16 - 8 = 8$	(8, 8)
10	$b = 16 - 10 = 6$	(10, 6)
12	$b = 16 - 12 = 4$	(12, 4)
14	$b = 16 - 14 = 2$	(14, 2)

For the equation $l - 2b + 2 = 0$		
l	$b = \frac{l+2}{2}$	(l, b)
6	$b = \frac{6+2}{2} = 4$	(6, 4)
8	$b = \frac{8+2}{2} = 5$	(8, 5)
10	$b = \frac{10+2}{2} = 6$	(10, 6)
12	$b = \frac{12+2}{2} = 7$	(12, 7)
14	$b = \frac{14+2}{2} = 8$	(14, 8)

So, original length of the plot is 10m and its breadth is 6m.

Taking measures of length on X-axis and measure of breadth on Y-axis, we get the graph



Exercise - 4.1

1. By comparing the ratios $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, $\frac{c_1}{c_2}$, find out whether the lines represented by the following pairs of linear equations intersect at a point, are parallel or are coincident.

   a) $5x - 4y + 8 = 0$ $7x + 6y - 9 = 0$

   b) $9x + 3y + 12 = 0$ $18x + 6y + 24 = 0$

   c) $6x - 3y + 10 = 0$ $2x - y + 9 = 0$

2. Check whether the following equations are consistent or inconsistent. Solve them graphically.

   a) $3x + 2y = 5$ $2x - 3y = 7$

   b) $2x - 3y = 8$ $4x - 6y = 9$

   c) $\frac{3}{2}x + \frac{5}{3}y = 7$ $9x - 10y = 14$

   d) $5x - 3y = 11$ $-10x + 6y = -22$

   e) $\frac{4}{3}x + 2y = 8$ $2x + 3y = 12$

   f) $x + y = 5$ $2x + 2y = 10$

   g) $x - y = 8$ $3x - 3y = 16$

   h) $2x + y - 6 = 0$ $4x - 2y - 4 = 0$

   i) $2x - 2y - 2 = 0$ $4x - 4y - 5 = 0$

3. Neha went to a 'sale' to purchase some pants and skirts. When her friend asked her how many of each she had bought, she answered "The number of skirts are two less than twice the number of pants purchased. Also the number of skirts is four less than four times the number of pants purchased."

   Help her friend to find how many pants and skirts Neha bought.

4. 10 students of Class-X took part in a mathematics quiz. If the number of girls is 4 more than the number of boys then, find the number of boys and the number of girls who took part in the quiz.

5. 5 pencils and 7 pens together cost ₹50 whereas 7 pencils and 5 pens together cost ₹46. Find the cost of one pencil and that of one pen.

6. Half the perimeter of a rectangular garden, whose length is 4m more than its width, is 36m. Find the dimensions of the garden.

7. We have a linear equation $2x + 3y - 8 = 0$. Write another linear equation in two variables such that the geometrical representation of the pair so formed is intersecting lines.

   Now, write two more linear equations so that one forms a pair of parallel lines and the second forms coincident line with the given equation.

8. The area of a rectangle gets reduced by 80 sq units if its length is reduced by 5 units and breadth is increased by 2 units. If we increase the length by 10 units and decrease the

breadth by 5 units, the area will increase by 50 sq units. Find the length and breadth of the rectangle.

9. In X class, if three students sit on each bench, one student will be left. If four students sit on each bench, one bench will be left. Find the number of students and the number of benches in that class.

4.3 Algebraic Methods of Finding the Solutions for a Pair of Linear Equations

We have learnt how to solve a pair of linear equations graphically. But, the graphical method is not convenient in cases where the point representing the solution has no integral co-ordinates.

For example, when the solution is of the form $(\sqrt{3}, 2\sqrt{7})$, $(-1.75, 3.3)$, $(\frac{4}{13}, \frac{1}{19})$ etc. There is every possibility of making mistakes while reading such co-ordinates. Is there any alternative method of finding the solution? There are several algebraic methods, which we shall discuss now.

4.3.1 Substitution Method

This method is useful for solving a pair of linear equations in two variables where one variable can easily be written in terms of the other variable. To understand this method, let us consider it step-wise

Step-1 : In one of the equations, express one variable in terms of the other variable. Say y in terms of x.

Step-2 : Substitute the value of y obtained in step 1 in the second equation.

Step-3 : Simplify the equation obtained in step 2 and find the value of x.

Step-4 : Substitute the value of x obtained in step 3 in either of the equations and solve it for y.

Step-5 : Check the obtained solution by substituting the values of x and y in both the original equations.

Example-6. Solve the given pair of equations using substitution method.

$2x - y = 5$

$3x + 2y = 11$

Solution : $2x - y = 5$ (1)

$3x + 2y = 11$ (2)

Equation (1) can be written as (Step 1)

$y = 2x - 5$

Substituting in equation (2) we get (Step 2)

$3x + 2(2x - 5) = 11$

$3x + 4x - 10 = 11$

$7x = 11 + 10 = 21$

$x = 21/7 = 3.$ (Step 3)

Substitute $x = 3$ in equation (1)

$2(3) - y = 5$ (Step 4)

$y = 6 - 5 = 1$

Substitute the values of x and y in equation (2), we get $3(3) + 2(1) = 9 + 6 = 11$

Both the equations are satisfied by $x = 3$ and $y = 1$. (Step 5)

Therefore, required solution is $x = 3$ and $y = 1$.

Do This

Solve each pair of equation by using the substitution method.

1) $3x - 5y = -1$
$x - y = -1$

2) $x + 2y = -1$
$2x - 3y = 12$

3) $2x + 3y = 9$
$3x + 4y = 5$

4) $x + \frac{6}{y} = 6$
$3x - \frac{8}{y} = 5$

5) $0.2x + 0.3y = 13$
$0.4x + 0.5y = 2.3$

6) $\sqrt{2}x + \sqrt{3}y = 0$
$\sqrt{3}x - \sqrt{8}y = 0$

4.3.2 Elimination Method

In this method, first we eliminate (remove) one of the two variables by equating its coefficients. This gives a single equation which can be solved to get the value of the other variable. To understand this method, let us consider it stepwise.

Step-1 : Write both the equations in the form of $ax + by = c$.

Step-2 : Make the coefficients of one of the variables, say 'x', numerically equal by multiplying each equation by suitable real numbers.

Step-3 : If the variable to be eliminated has the same sign in both equations, subtract the two equations to get an equation in one variable. If they have opposite signs then add.

Step-4 : Solve the equation for the remaining variable.

Step-5 : Substitute the value of this variable in any one of the original equations and find the value of the eliminated variable.

Example-7. Solve the following pair of linear equations using elimination method.

$3x + 2y = 11$
$2x + 3y = 4$

Solution : $3x + 2y = 11$ (1)

$2x + 3y = 4$ (2) (Step 1)

Let us eliminate 'y' from the given equations. The coefficients of 'y' in the given equations are 2 and 3. L.C.M.of 2 and 3 is 6. So, multiply equation (1) by 3 and equation (2) by 2.

Equation (1) × 3 $\quad 9x + \cancel{6y} = 33$ (Step 2)

Equation (2) × 2 $\quad 4x + \cancel{6y} = 8$

$\underline{(-)\quad(-)\quad(-)}$ (Step 3)

$5x = 25$

$x = \dfrac{25}{5} = 5$ (Step 4)

Substitute $x = 5$, in equation (1)

$3(5) + 2y = 11$

$2y = 11 - 15 = -4 \Rightarrow y = \dfrac{-4}{2} = -2$ (Step 5)

Therefore, the required solution is $x = 5$, $y = -2$.

Do this

Solve each of the following pairs of equations by the elimination method.

1. $8x + 5y = 9$, $3x + 2y = 4$
2. $2x + 3y = 8$, $4x + 6y = 7$
3. $3x + 4y = 25$, $5x - 6y = -9$

Try This

Solve the given pair of linear equations

$(a - b)x + (a + b)y = a^2 - 2ab - b^2$

$(a + b)(x + y) = a^2 + b^2$

Let us see some more examples:

Example-8. Tabita went to a bank to withdraw ₹2000. She asked the cashier to give the cash in ₹50 and ₹100 notes only. Snigdha got 25 notes in all. Can you tell how many notes each of ₹50 and ₹100 she received?

Solution : Let the number of ₹50 notes be x;

Let the number of ₹100 notes be y;

then, $x + y = 25$ (1)

and $50x + 100y = 2000$ (2)

Kavitha used the substitution method.

From equation (1)) $x = 25 - y$

Substituting in equation (2) $50(25 - y) + 100y = 2000$

$$1250 - 50y + 100y = 2000$$

$$50y = 2000 - 1250 = 750$$

$$y = \frac{750}{50} = 15$$

$$x = 25 - 15 = 10$$

Hence, Tabita received ten ₹50 notes and fifteen ₹100 notes.

Prathyusha used the elimination method to get the solution.

In the equations, coefficients of x are 1 and 50 respectively. So,

Equation (1) × 50 $\cancel{50x} + 50y = 1250$

Equation (2) × 1 $\cancel{50x} + 100y = 2000$ same sign, so subtract

$$\underline{(-) \quad (-) \quad (-)}$$

$$-50y = -750$$

or $y = \dfrac{-750}{-50} = 15$

Substitute y in equation (1) $x + 15 = 25$

$$x = 25 - 15 = 10$$

Hence Snigdha received ten ₹50 notes and fifteen ₹100 rupee notes.

Example-9. In a competitive exam, 3 marks are to be awarded for every correct answer and for every wrong answer, 1 mark will be deducted. Madhu scored 40 marks in this exam. Had 4 marks been awarded for each correct answer and 2 marks deducted for each incorrect answer, Madhu would have scored 50 marks. How many questions were there in the test? (Madhu attempted all the questions)

Solution : Let the number of correct answers be x;

and the number of wrong answers be y.

When 3 marks are given for each correct answer and 1 mark deducted for each wrong answer, his score is 40 marks.

$$3x - y = 40 \qquad (1)$$

His score would have been 50 marks if 4 marks were given for each correct answer and 2 marks deducted for each wrong answer.

$$4x - 2y = 50 \qquad (2)$$

Substitution method

From equation (1), $y = 3x - 40$

Substitute in equation (2) $4x - 2(3x - 40) = 50$

$4x - 6x + 80 = 50$

$-2x = 50 - 80 = -30$

$$x = \frac{-30}{-2} = 15$$

Substitute the value of x in equation (1)

$3(15) - y = 40$

$45 - y = 40$

$y = 45 - 40 = 5$

$\therefore$ Total number of questions = 15 + 5 = 20

Do this

Now use the elimination method to solve the above example-9.

Example-10. Mary told her daughter, "Seven years ago, I was seven times as old as you were then. Also, three years from now, I shall be three times as old as you will be." Find the present age of Mary and her daughter.

Solution : Let Mary's present age be x years and her daughter's age be y years.

Then, seven years ago Mary's age was $x - 7$ and daughter's age was $y - 7$.

$x - 7 = 7(y - 7)$

$x - 7 = 7y - 49$

$x - 7y + 42 = 0$ (1)

Three years hence, Mary's age will be $x + 3$ and daughter's age will be $y + 3$.

$x + 3 = 3(y + 3)$

$x + 3 = 3y + 9$

$x - 3y - 6 = 0$ (2)

Elimination method

Equation 1 $\cancel{x} - 7y = -42$

Equation 2 $\cancel{x} - 3y = 6$

$\underline{(-) \quad (+) \quad (-)}$ same sign for x, so subtract.

$-4y = -48$

$$y = \frac{-48}{-4} = 12$$

Substitute the value of y in equation (2)

$$x - 3\,(12) - 6 = 0$$

$$x = 36 + 6 = 42$$

Therefore, Mary's present age is 42 years and her daughter's age is 12 years.`

Do This

Solve example-10 by the substitution method.

Example-11. A publisher is planning to produce a new textbook. The fixed costs (reviewing, editing, typesetting and so on) are ₹ 31.25 per book. Besides that, he also spends another ₹ 320000 in producing the book. The wholesale price (the amount received by the publisher) is ₹ 43.75 per book. How many books must the publisher sell to break even, i.e., so that the costs will equal revenues?

> The point which corresponds to how much money you have to earn through sales in order to equal the money you spent in production is **break even point**.

Solution : The publisher breaks even when costs equal revenues. If x represents the number of books printed and sold and y be the breakeven point, then the cost and revenue equations for the publisher are

Cost equation is given by $\quad y = 320000 + 31.25x \quad$ (1)

Revenue equation is given by $\quad y = 43.75x \quad$ (2)

Using the second equation to substitute for y in the first equation, we have

$$43.75x = 3{,}20{,}000 + 31.25x$$

$$12.5x = 3{,}20{,}000$$

$$x = \frac{3{,}20{,}000}{12.5} = 25{,}600$$

Thus, the publisher will break even when 25,600 books are printed and sold.

Exercise - 4.2

Form a pair of linear equations for each of the following problems and find their solution.

1. The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹2000 per month, find their monthly income.

2. The sum of a two digit number and the number obtained by reversing the digits is 66. If the digits of the number differ by 2, find the number. How many such numbers are there?