3. The larger of two supplementary angles exceeds the smaller by 18°. Find the angles.

4. The taxi charges in Hyderabad are fixed, along with the charge for the distance covered. For a distance of 10 *km*., the charge paid is ₹220. For a journey of 15 *km*. the charge paid is ₹310.

   i. What are the fixed charges and charge per km?

   ii. How much does a person have to pay for travelling a distance of 25 km?

5. A fraction becomes $\frac{4}{5}$ if 1 is added to both numerator and denominator. If, however, 5 is subtracted from both numerator and denominator, the fraction becomes $\frac{1}{2}$. What is the fraction?

6. Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time at different speeds. If the cars travel in the same direction, they meet in 5 hours. If they travel towards each other, they meet in 1 hour. What are the speeds of the two cars?

7. Two angles are complementary. The larger angle is 3° less than twice the measure of the smaller angle. Find the measure of each angle.

8. An algebra textbook has a total of 1382 pages. It is broken up into two parts. The second part of the book has 64 pages more than the first part. How many pages are in each part of the book?

9. A chemist has two solutions of hydrochloric acid in stock. One is 50% solution and the other is 80% solution. How much of each should be used to obtain 100ml of a 68% solution.

10. Suppose you have ₹12000 to invest. You have to invest some amount at 10% and the rest at 15%. How much should be invested at each rate to yield 12% on the total amount invested?

4.4 Equations Reducible to a Pair of Linear Equations in Two Variables

Now we shall discuss the solution of pairs of equations which are not linear but can be reduced to linear form by making suitable substitutions. Let us see an example:

Example-12. Solve the following pair of equations.

$$\frac{2}{x}+\frac{3}{y} = 13$$

$$\frac{5}{x}-\frac{4}{y} = -2$$

Solution : Observe the given pair of equations. They are not linear equations. (Why?)

We have $2\left(\frac{1}{x}\right) + 3\left(\frac{1}{y}\right) = 13$ (1)

$$5\left(\frac{1}{x}\right) - 4\left(\frac{1}{y}\right) = -2 \qquad (2)$$

If we substitute $\frac{1}{x} = p$ and $\frac{1}{y} = q$, we get the following pair of linear equations:

$$2p + 3q = 13 \qquad (3)$$

$$5p - 4q = -2 \qquad (4)$$

Coefficients of q are 3 and 4 and their l.c.m. is 12. Using the elimination method:

Equation (3) $\times$ 4 $\quad 8p + \cancel{12q} = 52$

Equation (4) $\times$ 3 $\quad \underline{15p - \cancel{12q} = -6}$ 'q' terms have opposite sign, so we add the two equations.

$$23p = 46$$

$$p = \frac{46}{23} = 2$$

Substitute the value of p in equation (3)

$$2(2) + 3q = 13$$

$$3q = 13 - 4 = 9$$

$$q = \frac{9}{3} = 3$$

But, $\frac{1}{x} = p = 2$ $\qquad \Rightarrow x = \frac{1}{2}$

$\frac{1}{y} = q = 3$ $\qquad \Rightarrow y = \frac{1}{3}$

Example-13. Kavitha thought of constructing 2 more rooms in her house. She enquired about the labour. She came to know that 6 men and 8 women could finish this work in 14 days. But she wanted the work completed in only 10 days. When she enquired, she was told that 8 men and 12 women could finish the work in 10 days. Find out that how much time would be taken to finish the work if one man or one woman worked alone?

Solution : Let the time taken by one man to finish the work = x days.

Work done by one man in one day $= \frac{1}{x}$

Let the time taken by one woman to finish the work $= y$ days.

Work done by one woman in one day $= \frac{1}{y}$

Now, 8 men and 12 women can finish the work in 10 days.

So work done by 8 men and 12 women in one day $= \frac{1}{10}$ (1)

Also, work done by 8 men in one day is $8 \times \frac{1}{x}$. $= \frac{8}{x}$

Similarly, work done by 12 women in one day is $12 \times \frac{1}{y}$ $= \frac{12}{y}$

Total work done by 8 men and 12 women in one day $= \frac{8}{x} + \frac{12}{y}$ (2)

Equating equations (1) and (2) $\left(\frac{8}{x} + \frac{12}{y}\right) = \frac{1}{10}$

$$10\left(\frac{8}{x} + \frac{12}{y}\right) = 1$$

$$\frac{80}{x} + \frac{120}{y} = 1 \qquad (3)$$

Also, 6 men and 8 women can finish the work in 14 days.

Work done by 6 men and 8 women in one day $= \frac{6}{x} + \frac{8}{y} = \frac{1}{14}$

$$\Rightarrow \qquad 14\left(\frac{6}{x} + \frac{8}{y}\right) = 1$$

$$\left(\frac{84}{x} + \frac{112}{y}\right) = 1 \qquad (4)$$

Observe equations (3) and (4). Are they linear equations? How do we solve them then? We can convert them into linear equations by substituting $\frac{1}{x} = u$ and $\frac{1}{y} = v$.

Equation (3) becomes $\quad 80u + 120v = 1 \quad (5)$

Equation (4) becomes $\quad 84u + 112v = 1 \quad (6)$

L.C.M. of 80 and 84 is 1680. Using the elimination method,

Equation (3) $\times$ 21 $\quad (21 \times 80)u + (21 \times 120)v = 21$

Equation (4) $\times$ 20 $\quad (20 \times 84)u + (20 \times 112)v = 20$

$$\begin{array}{r} \cancel{1680u} + 2520v = 21 \\ \cancel{1680u} + 2240v = 20 \\ \underline{(-) \quad (-) \quad (-)} \\ 280v = 1 \end{array}$$

Same sign for u, so subtract

$$v = \frac{1}{280}$$

Substitute in equation (5) $\quad 80u + 120 \times \frac{1}{280} = 1$

$$80u = 1 - \frac{3}{7} = \frac{7-3}{7} = \frac{4}{7}$$

$$u = \frac{\overset{1}{\cancel{4}}}{7} \times \frac{1}{\underset{20}{\cancel{80}}} = \frac{1}{140}$$

So one man alone can finish the work in 140 days and one woman alone can finish the work in 280 days.

Example-14. A man travels 370 km partly by train and partly by car. If he covers 250 km by train and the rest by car, it takes him 4 hours. But if he travels 130 km by train and the rest by car, it takes 18 minutes more. Find the speed of the train and that of the car.

Solution : Let the speed of the train be x *km*. per hour and that of the car be y *km*. per hour.

Also, we know that time $= \frac{\text{Distance}}{\text{Speed}}$

In situation 1, time spent travelling by train $= \frac{250}{x}$ hrs.

And time spent travelling by car $= \frac{120}{y}$ hrs.

So, total time taken = time spent in train + time spent in car = $\frac{250}{x} + \frac{120}{y}$

But, total time of journey is 4 hours, so

$$\frac{250}{x} + \frac{120}{y} = 4$$

$$\frac{125}{x} + \frac{60}{y} = 2 \qquad \rightarrow (1)$$

Again, when he travels 130 km by train and the rest by car

Time taken by him to travel 130 km by train = $\frac{130}{x}$ hrs.

Time taken by him to travel 240 km (370 - 130) by car = $\frac{240}{y}$ hrs.

Total time taken = $\frac{130}{x} + \frac{240}{y}$

But given, time of journey is 4 hrs 18 min i.e., $4\frac{\cancel{18}^{3}}{\cancel{60}_{10}}$ hrs. = $4\frac{3}{10}$ hrs.

So, $$\frac{130}{x} + \frac{240}{y} = \frac{43}{10} \qquad (2)$$

Substitute $\frac{1}{x} = a$ and $\frac{1}{y} = b$ in equations (1) and (2)

$$125a + 60b = 2 \qquad (3)$$

$$130a + 240b = 43/10 \qquad (4)$$

For 60 and 240, l.c.m. is 240. Using the elimination method,

Equation (3) × 4 $\qquad 500a + \cancel{240b} = 8$

Equation (4) × 1 $\qquad 130a + \cancel{240b} = \frac{43}{10}$ (Same sign, so subtract)

$\qquad\qquad (-) \quad (-) \qquad (-)$

$$370a = 8 - \frac{43}{10} = \frac{80-43}{10} = \frac{37}{10}$$

$$a = \frac{\cancel{37}}{10} \times \frac{1}{\underset{10}{\cancel{370}}} = \frac{1}{100}$$

Substitute $a = \dfrac{1}{100}$ in equation (3)

$$\left(\overset{5}{\cancel{125}} \times \frac{1}{\underset{4}{\cancel{100}}}\right) + 60b = 2$$

$$60b = 2 - \frac{5}{4} = \frac{8-5}{4} = \frac{3}{4}$$

$$b = \frac{\cancel{3}}{4} \times \frac{1}{\underset{20}{\cancel{60}}} = \frac{1}{80}$$

So $a = \dfrac{1}{100}$ and b $= \dfrac{1}{80}$

So $\dfrac{1}{x} = \dfrac{1}{100}$ and $\dfrac{1}{y} = \dfrac{1}{80}$

$x = 100$ km/hr and $y = 80$ km/hr.

So, speed of train was 100 km/hr and speed of car was 80 km/hr.

Exercise - 4.3

Solve each of the following pairs of equations by reducing them to a pair of linear equations.

i) $\dfrac{5}{x-1} + \dfrac{1}{y-2} = 2$

$\dfrac{6}{x-1} - \dfrac{3}{y-2} = 1$

ii) $\dfrac{x+y}{xy} = 2$

$\dfrac{x-y}{xy} = 6$

iii) $\dfrac{2}{\sqrt{x}} + \dfrac{3}{\sqrt{y}} = 2$

$\dfrac{4}{\sqrt{x}} - \dfrac{9}{\sqrt{y}} = -1$

iv) $6x+3y = 6xy$

$2x + 4y = 5xy$

v) $\dfrac{5}{x+y} - \dfrac{2}{x-y} = -1$

vi) $\dfrac{2}{x} + \dfrac{3}{y} = 13$

$\dfrac{15}{x+y} + \dfrac{7}{x-y} = 10$ where $x \neq 0, y \neq 0$

$\dfrac{5}{x} - \dfrac{4}{y} = -2$ where $x \neq 0, y \neq 0$

vii) $\dfrac{10}{x+y} + \dfrac{2}{x-y} = 4$

$\dfrac{15}{x+y} - \dfrac{5}{x-y} = -2$

viii) $\dfrac{1}{3x+y} + \dfrac{1}{3x-y} = \dfrac{3}{4}$

$\dfrac{1}{2(3x+y)} - \dfrac{1}{2(3x-y)} = \dfrac{-1}{8}$

2. Formulate the following problems as a pair of equations and then find their solutions.

   i. A boat goes 30 km upstream and 44 km downstream in 10 hours. In 13 hours it can go 40 km upstream and 55 km downstream. Determine the speed of the stream and that of the boat in still water.

   ii. Rahim travels 600 km to his home partly by train and partly by car. He takes 8 hours if he travels 120 km by train and rest by car. He takes 20 minutes more if he travels 200 km by train and rest by car. Find the speed of the train and the car.

   iii. 2 women and 5 men can together finish an embroidery work in 4 days while 3 women and 6 men can finish it in 3 days. Find the time taken by 1 woman alone and 1 man alone to finish the work.

Optional Exercise

[This exercise is not meant for examination]

1. Solve the following equations:-

(i) $\dfrac{2x}{a} + \dfrac{y}{b} = 2$

$\dfrac{x}{a} - \dfrac{y}{b} = 4$

(ii) $\dfrac{x+1}{2} + \dfrac{y-1}{3} = 8$

$\dfrac{x-1}{3} + \dfrac{y+1}{2} = 9$

(iii) $\dfrac{x}{7} + \dfrac{y}{3} = 5$

$\dfrac{x}{2} - \dfrac{y}{9} = 6$

(iv) $\sqrt{3}x + \sqrt{2}y = \sqrt{3}$

$\sqrt{5}x + \sqrt{3}y = \sqrt{3}$

(v) $\dfrac{ax}{b} - \dfrac{by}{a} = a + b$

$ax - by = 2ab$

(vi) $2^x + 3^y = 17$

$2^{x+2} - 3^{y+1} = 5$

2. Animals in an experiment are to be kept on a strict diet. Each animal is to receive among other things 20g of protein and 6g of fat. The laboratory technicians purchased two food mixes, A and B. Mix A has 10% protein and 6% fat. Mix B has 20% protein and 2% fat. How many grams of each mix should be used?

What We Have Discussed

1. Two linear equations in the same two variables are called a pair of linear equations in two variables.

   $a_1x + b_1y + c_1 = 0 \ (a_1^2 + b_1^2 \neq 0)$

   $a_2x + b_2y + c_2 = 0 \ (a_2^2 + b_2^2 \neq 0)$

   Where $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers.

2. A pair of linear equations in two variables can be solved using various methods.

3. The graph of a pair of linear equations in two variables is represented by two lines.

   i. If the lines intersect at a point then the point gives the unique solution of the two equations. In this case, the pair of equations is consistent.

   ii. If the lines coincide, then there are infinitely many solutions - each point on the line being a solution. In this case, the pair of equations is dependent.

   iii. If the lines are parallel then the pair of equations has no solution. In this case, the pair of equations is inconsistent.

4. We have discussed the following methods for finding the solution(s) of a pair of linear equations.

   i. Model Method.

   ii. Graphical Method

   iii. Algebraic methods - Substitution method and Elimination method.

5. There exists a relation between the coefficients and nature of system of equations.

   i. If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ then the pair of linear equations is consistent.

   ii. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$ then the pair of linear equations is inconsistent.

   iii. If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ then the pair of linear equations is dependent and consistent.

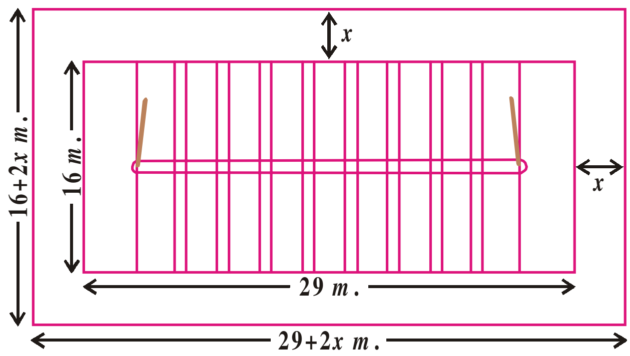
6. There are several situations which can be mathematically represented by two equations that are not linear to start with. But we can alter them so that they will be reduced to a pair of linear equations.

CHAPTER 5

Quadratic Equations

5.1 INTRODUCTION

Sports committee of Kaspa Municipal High School wants to construct a Kho-Kho court of dimension $29\,m. \times 16\,m$. This is to be a rectangular enclosure of area $558\,m^2$. They want to leave space of equal width all around the court for the spectators. What would be the width of the space for spectators? Would it be enough?



Suppose the width of the space be x meter. So from the figure length of the plot would be $(29 + 2x)$ meter.

And, breath of the rectangular plot would be $= (16 + 2x)\ m.$

Therefore, area of the rectangular plot will be $=$ length $\times$ breadth

$$= (29 + 2x)(16 + 2x)$$

Since the area of the plot is $= 558\ m^2$

$$\therefore \quad (29 + 2x)(16 + 2x) = 558$$

$$\therefore \quad 4x^2 + 90x + 464 = 558$$

$$4x^2 + 90x - 94 = 0 \qquad \text{(dividing by 2)}$$

$$2x^2 + 45x - 47 = 0$$

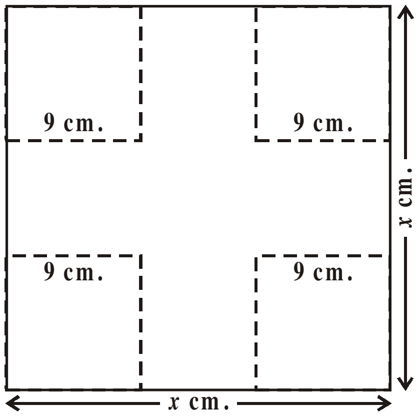
$$2x^2 + 45\,x - 47 = 0 \qquad \text{..... (1)}$$

In previous class we solve the linear equations of the form $ax + b = c$ to find the value of 'x'. Similarly, the value of x from the above equation will give the possible width of the space for spectators.

Can you think of more such examples where we have to find the quantities like in above example and get such equations.

Let us consider another example:

Rani has a square metal sheet. She removed squares of side 9 *cm*. from each corner of this sheet. Of the remaining sheet, she turned up the sides to form an open box as shown. The capacity of the box is 144 cc. Can we find out the dimensions of the metal sheet?



Suppose the side of the square piece of metal sheet be 'x' *cm*.

Then, the dimensions of the box are

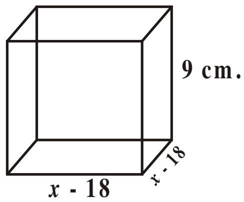
$$9 \; cm. \times (x\text{-}18) \; cm. \times (x\text{-}18) \; cm.$$

Since volume of the box is 144 cc

$$9\,(x-18)\,(x-18) = 144$$
$$(x-18)^2 = 16$$
$$x^2 - 36x + 308 = 0$$

So, the side 'x' of the metal sheet will satisfy the equation.

$$x^2 - 36x + 308 = 0 \qquad ..... (2)$$



Let us observe the L.H.S of equation (1) and (2)

Are they quadratic polynominals?

We studied such quadratic polynomials of the form $ax^2 + bx + c$, $a \neq 0$ in the previous chapter.

Since, the LHS of the above equations are quadratic polynomials they are called quadratic equations.

In this chapter we will study quadratic equations and methods to find their roots.

5.2 Quadratic Equations

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$. For example, $2x^2 + x - 300 = 0$ is quadratic equation, Similarly, $2x^2 - 3x + 1 = 0$, $4x - 3x^2 + 2 = 0$ and $1 - x^2 + 300 = 0$ are also quadratic equations.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is polynomial of degree 2, is a quadratic equation. But when we write the terms of $p(x)$ in descending order of their degrees, then we get the standard form of the equation. That is, $ax^2 + bx + c = 0$, $a \neq 0$ is called the standard form of a quadratic equation and $y = ax^2 + bx + c$ is called a quadratic function.

Try This

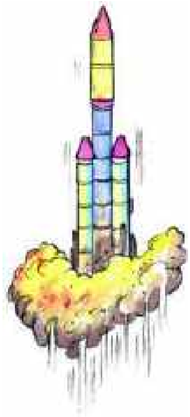
Check whether the following equations are quadratic or not ?

(i) $x^2 - 6x - 4 = 0$ (ii) $x^3 - 6x^2 + 2x - 1 = 0$

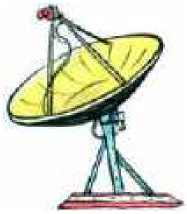
(iii) $7x = 2x^2$ (iv) $x^2 + \dfrac{1}{x^2} = 2$

(v) $(2x + 1)(3x + 1) = b(x - 1)(x - 2)$ (vi) $3y^2 = 192$

There are various uses of Quadratic functions. Some of them are:-



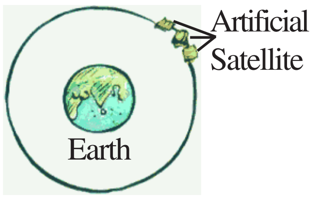
1. When the rocket is fired upward, then the height of the rocket is defined by a 'quadratic function.'
2. Shapes of the satellite dish, reflecting mirror in a telescope, lens of the eye glasses and orbits of the celestial objects are defined by the quadratic equations.

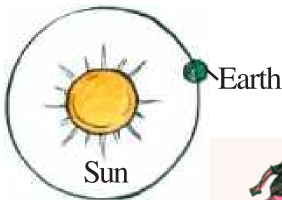


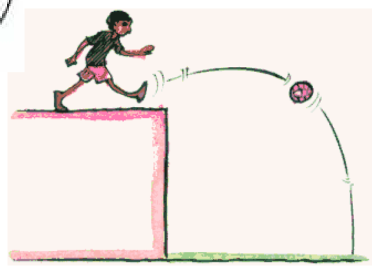
Satellite Dish

Reflecting Mirror

Lens of Eye Glasses







3. The path of a projectile is defined by quadratic function.
4. When the breaks are applied to a vehicle, the stopping distance is calculated by using quadratic equation.

Example-1. Represent the following situations mathematically:

i. Raju and Rajendar together have 45 marbles. Both of them lost 5 marbles each, and the product of the number of marbles now they have is 124. We would like to find out how many marbles they had previously.

ii. The hypotenuse of a right triangle is 25 *cm*. We know that the difference in lengths of the other two sides is 5 *cm*. We would like to find out the length of the two sides?

Solution : i. Let the number of marbles Raju had be x.

Then the number of marbles Rajendar had = $45 - x$ (Why?).

The number of marbles left with Raju, when he lost 5 marbles = $x - 5$

The number of marbles left with Rajendar, when he lost 5 marbles = $(45 - x) - 5$

$= 40 - x$

Therefore, their product = $(x - 5)(40 - x)$

$= 40x - x^2 - 200 + 5x$

$= -x^2 + 45x - 200$

So, $-x^2 + 45x - 200 = 124$ (Given that product = 124)

i.e., $-x^2 + 45x - 324 = 0$

i.e., $x^2 - 45x + 324 = 0$ (Multiply -ve sign)

Therefore, the number of marbles Raju had 'x', satisfies the quadratic equation

$x^2 - 45x + 324 = 0$

which is the required representation of the problem mathematically.

Let the length of smaller side be x *cm*.

Then length of larger side = $(x + 5)$ *cm*.

Given length of hypotenuse = 25 *cm*.

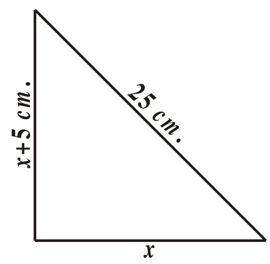
ii. In a right angle triangle we know that $(\text{hypotenuse})^2 = (\text{side})^2 + (\text{side})^2$

So, $x^2 + (x + 5)^2 = (25)^2$

$x^2 + x^2 + 10x + 25 = 625$

$2x^2 + 10x - 600 = 0$

$x^2 + 5x - 300 = 0$



Value of x from the above equation will give the possible value of length of sides of the given right angled triangle.

Example-2. Check whether the following are quadratic equations:

i. $(x - 2)^2 + 1 = 2x - 3$ ii. $x(x + 1) + 8 = (x + 2)(x - 2)$

iii. $x(2x + 3) = x^2 + 1$ iv. $(x + 2)^3 = x^3 - 4$

Solution : i. LHS = $(x - 2)^2 + 1 = x^2 - 4x + 4 + 1 = x^2 - 4x + 5$

Therefore, $(x - 2)^2 + 1 = 2x - 3$ can be written as

$x^2 - 4x + 5 = 2x - 3$

i.e., $x^2 - 6x + 8 = 0$

It is in the form of $ax^2 + bx + c = 0$.

Therefore, the given equation is a quadratic equation.

ii. Here LHS $= x(x + 1) + 8 = x^2 + x + 8$

and RHS $= (x + 2)(x - 2) = x^2 - 4$

Therefore, $x^2 + x + 8 = x^2 - 4$

$x^2 + x + 8 - x^2 + 4 = 0$

i.e., $x + 12 = 0$

It is not in the form of $ax^2 + bx + c = 0$.

Therefore, the given equation is not a quadratic equation.

iii. Here, LHS $= x\,(2x + 3) = 2x^2 + 3x$

So, $x\,(2x + 3) = x^2 + 1$ can be rewritten as

$2x^2 + 3x = x^2 + 1$

Therefore, we get $x^2 + 3x - 1 = 0$

It is in the form of $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

iv. Here, LHS $= (x + 2)^3$ $= (x + 2)^2\,(x + 2)$

$= (x^2 + 4x + 4)\,(x + 2)$

$= x^3 + 2x^2 + 4x^2 + 8x + 4x + 8$

$= x^3 + 6x^2 + 12x + 8$

Therefore, $(x + 2)^3 = x^3 - 4$ can be rewritten as

$x^3 + 6x^2 + 12x + 8 = x^3 - 4$

i.e., $6x^2 + 12x + 12 = 0$ or, $x^2 + 2x + 2 = 0$

It is in the form of $ax^2 + bx + c = 0$.

So, the given equation is a quadratic equation.

Remark : In (ii) above, the given equation appears to be a quadratic equation, but it is not a quadratic equation.

In (iv) above, the given equation appears to be a cubic equation (an equation of degree 3) and not a quadratic equation. But it turns out to be a quadratic equation. As you can see, often we need to simplify the given equation before deciding whether it is quadratic or not.

Exercise - 5.1

1. Check whether the following are quadratic equations :

i. $(x+1)^2 = 2(x-3)$

ii. $x^2 - 2x = (-2)(3-x)$

iii. $(x-2)(x+1) = (x-1)(x+3)$

iv. $(x-3)(2x+1) = x(x+5)$

v. $(2x-1)(x-3) = (x+5)(x-1)$

vi. $x^2 + 3x + 1 = (x-2)^2$

vii. $(x+2)^3 = 2x(x^2-1)$

viii. $x^3 - 4x^2 - x + 1 = (x-2)^3$

2. Represent the following situations in the form of quadratic equations :

i. The area of a rectangular plot is 528 m^2. The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

ii. The product of two consecutive positive integers is 306. We need to find the integers.

iii. Rohan's mother is 26 years older than him. The product of their ages after 3 years will be 360 years. We need to find Rohan's present age.

iv. A train travels a distance of 480 km at a uniform speed. If the speed had been 8 km/h less, then it would have taken 3 hours more to cover the same distance. We need to find the speed of the train.

5.3 Solution of a Quadratic Equation by Factorisation

We have learned to represent some of the daily life situations mathematically in the form of quadratic equation with an unknown variable 'x'.

Now we need to find the value of x.

Consider the quadratic equation $2x^2 - 3x + 1 = 0$. If we replace x by 1. Then, we get $(2 \times 1^2) - (3 \times 1) + 1 = 0 =$ RHS of the equation. Since 1 satisfies the equation , we say that 1 is a root of the quadratic equation $2x^2 - 3x + 1 = 0$.

$\therefore$ $x = 1$ is a solution of the quadratic equation.

This also means that 1 is a zero of the quadratic polynomial $2x^2 - 3x + 1$.

In general, a real number α is called a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + b\alpha + c = 0$. We also say that **$x = \alpha$ is a solution of the quadratic equation**, or α **satisfies the quadratic equation**.

Note that **the zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.**

We have observed, in Chapter 3, that a quadratic polynomial can have at most two zeroes. So, any quadratic equation can have atmost two roots. (Why?)

We have learnt in Class-IX, how to factorise quadratic polynomials by splitting their middle terms. We shall use this knowledge for finding the roots of a quadratic equation. Let us see.

Example-3. Find the roots of the equation $2x^2 - 5x + 3 = 0$, by factorisation.

Solution : Let us first split the middle term. Recall that if $ax^2 + bx + c$ is a quadratic equation polynomial then to split the middle term we have to find two numbers p and q such that $p + q = b$ and $p \times q = a \times c$. So to split the middle term of $2x^2 - 5x + 3$, we have to find two numbers p and q such that $p + q = -5$ and $p \times q = 2 \times 3 = 6$.

For this we have to list out all possible pairs of factors of 6. They are (1, 6), (–1, –6); (2, 3); (–2, –3). From the list it is clear that the pair (–2, –3) will satisfy our condition $p + q = -5$ and $p \times q = 6$.

The middle term '$-5x$' can be written as '$-2x - 3x$'.

So, $2x^2 - 5x + 3 = 2x^2 - 2x - 3x + 3 = 2x(x - 1) - 3(x - 1) = (2x - 3)(x - 1)$

Now, $2x^2 - 5x + 3 = 0$ can be rewritten as $(2x - 3)(x - 1) = 0$.

So, the values of x for which $2x^2 - 5x + 3 = 0$ are the same for which $(2x - 3)(x - 1) = 0$,

i.e., either $2x - 3 = 0$ or $x - 1 = 0$.

Now, $2x - 3 = 0$ gives $x = \frac{3}{2}$ and $x - 1 = 0$ gives $x = 1$.

So, $x = \frac{3}{2}$ and $x = 1$ are the solutions of the equation.

In other words, 1 and $\frac{3}{2}$ are the roots of the equation $2x^2 - 5x + 3 = 0$.

TRY THIS

Verify that 1 and $\frac{3}{2}$ are the roots of the equation $2x^2 - 5x + 3 = 0$.

Note that we have found the roots of $2x^2 - 5x + 3 = 0$ by factorising $2x^2 - 5x + 3$ into two linear factors and equating each factor to zero.

Example 4 : Find the roots of the quadratic equation $x - \frac{1}{3x} = \frac{1}{6}$

Solution : We have $x - \frac{1}{3x} = \frac{1}{6} \Rightarrow 6x^2 - x - 2 = 0$

$$6x^2 - x - 2 = 6x^2 + 3x - 4x - 2$$
$$= 3x\,(2x + 1) - 2\,(2x + 1)$$
$$= (3x - 2)(2x + 1)$$

The roots of $6x^2 - x - 2 = 0$ are the values of x for which $(3x - 2)(2x + 1) = 0$

Therefore, $3x - 2 = 0$ or $2x + 1 = 0$,

i.e., $x = \frac{2}{3}$ or $x = -\frac{1}{2}$

Therefore, the roots of $6x^2 - x - 2 = 0$ are $\frac{2}{3}$ and $-\frac{1}{2}$.

We verify the roots, by checking that $\frac{2}{3}$ and $-\frac{1}{2}$ satisfy $6x^2 - x - 2 = 0$.

Example-5. Find the width of the space for spectators discussed in section 5.1.

Solution : In Section 5.1, we found that if the width of the space for spectators is x *m*., then x satisfies the equation $2x^2 + 45x - 47 = 0$. Applying the factorisation method we write this equation as:-

$$2x^2 - 2x + 47x - 47 = 0$$
$$2x\,(x - 1) + 47\,(x - 1) = 0$$

i.e., $(x - 1)\;(2x + 47) = 0$

So, the roots of the given equation are $x = 1$ or $x = \frac{-47}{2}$. Since 'x' is the width of space of the spectators it cannot be negative.

Thus, the width is 1 *m*.

Exercise - 5.2

1. Find the roots of the following quadratic equations by factorisation:

i. $x^2 - 3x - 10 = 0$ ii. $2x^2 + x - 6 = 0$ iii. $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

iv. $2x^2 - x + \frac{1}{8} = 0$ v. $100x^2 - 20x + 1 = 0$ vi. $x(x + 4) = 12$

vii. $3x^2 - 5x + 2 = 0$ viii. $x - \frac{3}{x} = 2$ ix. $3(x - 4)^2 - 5(x - 4) = 12$

2. Find two numbers whose sum is 27 and product is 182.

3. Find two consecutive positive integers, sum of whose squares is 613.

4. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.

5. A cottage industry produces a certain number of pottery articles in a day. It was observed on a particular day that the cost of production of each article (in rupees) was 3 more than twice the number of articles produced on that day. If the total cost of production on that day was Rs 90, find the number of articles produced and the cost of each article.

6. Find the dimensions of a rectangle whose perimeter is 28 meters and whose area is 40 square meters.

7. The base of a triangle is 4cm longer than its altitude. If the area of the triangle is 48 sq.cm then find its base and altitude.

8. Two trains leave a railway station at the same time. The first train travels towards west and the second train towards north. The first train travels 5 km/hr faster than the second train. If after two hours they are 50 *km.* apart find the average speed of each train.

9. In a class of 60 students, each boy contributed rupees equal to the number of girls and each girl contributed rupees equal to the number of boys. If the total money then collected was ₹1600. How many boys are there in the class?

10. A motor boat heads upstream a distance of 24km on a river whose current is running at 3 km per hour. The trip up and back takes 6 hours. Assuming that the motor boat maintained a constant speed, what was its speed?

5.4 Solution of a Quadratic Equation by Completing the Square

In the previous section, we have learnt method of factorisation for obtaining the roots of a quadratic equation. Is method of factorization applicable to all types of quadratic equation? Let us try to solve $x^2 + 4x - 4 = 0$ by factorisation method

To solve the given equation $x^2 + 4x - 4 = 0$ by factorization method.

We have to find 'p' and 'q' such that $p + q = 4$ and

$$p \times q = -4$$

But it is not possible. So by factorization method we cannot solve the given equation.

Therefore, we shall study another method.

Consider the following situation

The product of Sunita's age (in years) two years ago and her age four years from now is one more than twice her present age. What is her present age?

To answer this, let her present age (in years) be x years. Age before two year = x - 2 & age after four years = $x + 4$ then the product of both the ages is $(x - 2)(x + 4)$.

Therefore, $(x - 2)(x + 4) = 2x + 1$

i.e., $x^2 + 2x - 8 = 2x + 1$

i.e., $x^2 - 9 = 0$

So, Sunita's present age satisfies the quadratic equation $x^2 - 9 = 0$.

We can write this as $x^2 = 9$. Taking square roots, we get $x = 3$ or $x = -3$. Since the age is a positive number, $x = 3$.

So, Sunita's present age is 3 years.

Now consider another quadratic equation $(x + 2)^2 - 9 = 0$. To solve it, we can write it as $(x + 2)^2 = 9$. Taking square roots, we get $x + 2 = 3$ or $x + 2 = -3$.

Therefore, $x = 1$ or $x = -5$

So, the roots of the equation $(x + 2)^2 - 9 = 0$ are 1 and – 5.

In both the examples above, the term containing x is completely a square, and we found the roots easily by taking the square roots. But, what happens if we are asked to solve the equation $x^2 + 4x - 4 = 0$. And it cannot be solved by factorisation also.

So, we now introduce the method of completing the square. The idea behind this method is to adjust the left side of the quadratic equation so that it becomes a perfect square.

The process is as follows:

$$x^2 + 4x - 4 = 0$$

$$\Rightarrow \quad x^2 + 4x = 4$$

$$x^2 + 2.\, x\, .\, 2 = 4$$

Now, the LHS is in the form of $a^2 + 2ab$. If we add b^2 it becomes as $a^2 + 2ab + b^2$ which is perfect square. So, by adding $b^2 = 2^2 = 4$ to both sides we get,

$$x^2 + 2.x.2 + 2^2 = 4 + 4$$

$$\Rightarrow \quad (x + 2)^2 = 8 \Rightarrow x + 2 = \pm\sqrt{8}$$

$$\Rightarrow \quad x = -2 \pm 2\sqrt{2}$$

Now consider the equation $3x^2 - 5x + 2 = 0$. Note that the coefficient of x^2 is not 1. So we divide the entire equation by 3 so that the coefficient of x^2 is 1

$$\therefore \quad x^2 - \frac{5}{3}x + \frac{2}{3} = 0$$

$$\Rightarrow \quad x^2 - \frac{5}{3}x = \frac{-2}{3}$$

$$\Rightarrow \quad x^2 - 2.x.\frac{5}{6} = \frac{-2}{3}$$

$$\Rightarrow \quad x^2 - 2.x.\frac{5}{6} + \left(\frac{5}{6}\right)^2 = \frac{-2}{3} + \left(\frac{5}{6}\right)^2 \qquad \left(\text{add } \left(\frac{5}{6}\right)^2 \text{ both side}\right)$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{-2}{3} + \frac{25}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{(12 \times -2) + (25 \times 1)}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{-24 + 25}{36}$$

$$\left(x - \frac{5}{6}\right)^2 = \frac{1}{36} \qquad \text{(take both side square root)}$$

$$x - \frac{5}{6} = \pm\frac{1}{6}$$

So, $x = \frac{5}{6} + \frac{1}{6}$ or $x = \frac{5}{6} - \frac{1}{6}$

Therefore, $x = 1$ or $x = \frac{4}{6}$

i.e., $x = 1$ or $x = \frac{2}{3}$

Therefore, the roots of the given equation are 1 and $\frac{2}{3}$.

From the above examples we can deduce the following algorithm for completing the square.

Algorithm : Let the quadratic equation by $ax^2 + bx + c = 0$

Step-1 : Divide each side by 'a'

Step-2 : Rearrange the equation so that constant term c/a is on the right side. (RHS)

Step-3 : Add $\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2$ to both sides to make LHS, a perfect square.

Step-4 : Write the LHS as a square and simplify the RHS.

Step-5 : Solve it.

Example-6. Find the roots of the equation $5x^2 - 6x - 2 = 0$ by the method of completing the square.

Solution : Given : $5x^2 - 6x - 2 = 0$

Now we follow the Algoritm

Step-1 : $x^2 - \frac{6}{5}x - \frac{2}{5} = 0$ (Dividing both sides by 5)

Step-2 : $x^2 - \frac{6}{5}x = \frac{2}{5}$

Step-3 : $x^2 - \frac{6}{5}x + \left(\frac{3}{5}\right)^2 = \frac{2}{5} + \left(\frac{3}{5}\right)^2$ $\left(\text{Adding } \left(\frac{3}{5}\right)^2 \text{ to both sides}\right)$

Step-4 : $\left(x - \frac{3}{5}\right)^2 = \frac{2}{5} + \frac{9}{25}$

Step-5 : $\left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$

$$x - \frac{3}{5} = \pm\sqrt{\frac{19}{25}}$$

$$x = \frac{3}{5} + \frac{\sqrt{19}}{5} \quad \text{or} \quad x = \frac{3}{5} - \frac{\sqrt{19}}{5}$$

$$\therefore \quad x = \frac{3+\sqrt{19}}{5} \quad \text{or} \quad x = \frac{3-\sqrt{19}}{5}$$

Example-7. Find the roots of $4x^2 + 3x + 5 = 0$ by the method of completing the square.

Solution : Given $4x^2 + 3x + 5 = 0$

$$x^2 + \frac{3}{4}x + \frac{5}{4} = 0$$

$$x^2 + \frac{3}{4}x = \frac{-5}{4}$$

$$x^2 + \frac{3}{4}x + \left(\frac{3}{8}\right)^2 = \frac{-5}{4} + \left(\frac{3}{8}\right)^2$$

$$\left(x + \frac{3}{8}\right)^2 = \frac{-5}{4} + \frac{9}{64}$$

$$\left(x + \frac{3}{8}\right)^2 = \frac{-71}{64} < 0$$

But $\left(x + \frac{3}{8}\right)^2$ cannot be negative for any real value of x (Why?). So, there is no real value of x satisfying the given equation. Therefore, the given equation has no real roots.

Do This

Solve the equations by completing the square

(i) $x^2 - 10x + 9 = 0$ (ii) $x^2 - 5x + 5 = 0$

(iii) $x^2 + 7x - 6 = 0$

We have solved several examples with the use of the method of 'completing the square.' Now, let us apply this method in standard form of quadratic equation $ax^2+bx+c=0$ ($a \neq 0$).

Step 1 : Dividing the equation through out by 'a' we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Step 2 : $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Step 3 : $x^2+\frac{b}{a}x+\left[\frac{1}{2}\frac{b}{a}\right]^2=-\frac{c}{a}+\left[\frac{1}{2}\frac{b}{a}\right]^2$ $\left[\text{adding } \left[\frac{1}{2}\frac{b}{a}\right]^2 \text{ both sides}\right]$

$$\Rightarrow x^2+2\cdot x\frac{b}{2a}+\left[\frac{b}{2a}\right]^2=-\frac{c}{a}+\left[\frac{b}{2a}\right]^2$$

Step 4 : $\left[x+\frac{b}{2a}\right]^2=\frac{b^2-4ac}{4a^2}$

Step 5 : If $b^2-4ac\geq 0$, then by taking the square roots, we get

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

Therefore, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

So, the roots of $ax^2 + bx + c = 0$ are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, if $b^2-4ac\geq 0$.

If $b^2-4ac<0$, the equation will have no real roots. (Why?)

Thus, if $b^2-4ac\geq 0$, then the roots of the quadratic equation $ax^2+bx+c=0$ are given by $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

This formula for finding the roots of a quadratic equation is known as the **quadratic formula**.

Let us consider some examples by using quadratic formula.

Example-8. Solve Q. 2(i) of Exercise 5.1 by using the quadratic formula.

Solution : Let the breadth of the plot be x metres.

Then the length is $(2x+1)$ metres.

Since area of rectangular plot is 528 m^2

We can write $x(2x + 1) = 528$, i.e., $2x^2 + x - 528 = 0$.

This is in the form of $ax^2 + bx + c = 0$, where $a = 2$, $b = 1$, $c = -528$.

So, the quadratic formula gives us the solution as

$$x = \frac{-1 \pm \sqrt{1+4(2)(528)}}{4} = \frac{-1 \pm \sqrt{4225}}{4} = \frac{-1 \pm 65}{4}$$

i.e., $x = \frac{64}{4}$ or $x = \frac{-66}{4}$

i.e., $x = 16$ or $x = -\frac{33}{2}$

Since x cannot be negative. So, the breadth of the plot is 16 metres and hence, the length of the plot is $(2x + 1) = 33$m.

You should verify that these values satisfy the conditions of the problem.

THINK - DISCUSS

We have three methods to solve a quadratic equation. Among these three, which method would you like to use? Why?

Example-9. Find two consecutive odd positive integers, sum of whose squares is 290.

Solution : Let first odd positive integers be x. Then, the second integer will be $x + 2$. According to the question,

$$x^2 + (x + 2)^2 = 290$$

i.e., $x^2 + x^2 + 4x + 4 = 290$

i.e., $2x^2 + 4x - 286 = 0$

i.e., $x^2 + 2x - 143 = 0$

which is a quadratic equation in x.

Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2}$

we get, $x = \frac{-2 \pm \sqrt{4+572}}{2} = \frac{-2 \pm \sqrt{576}}{2} = \frac{-2 \pm 24}{2}$

i.e., $x = 11$ or $x = -13$

But x is given to be an odd positive integer. Therefore, $x \neq -13$, $x = 11$.

Thus, the two consecutive odd integers are 11 and $(x + 2) = 11 + 2 = 13$.

Check : $11^2 + 13^2 = 121 + 169 = 290$.

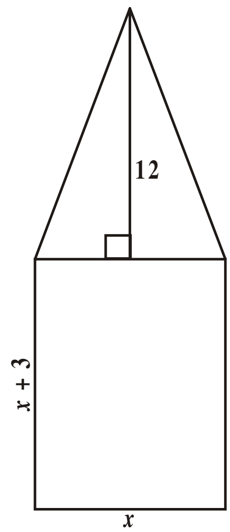
Example-10. A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m (see Fig. 5.3). Find its length and breadth.

Solution : Let the breadth of the rectangular park be x m.

So, its length = $(x + 3)$ m.

Therefore, the area of the rectangular park = $x(x + 3)\ \text{m}^2 = (x^2 + 3x)\ \text{m}^2$.

Now, base of the isosceles triangle = x m.



Therefore, its area = $\frac{1}{2} \times x \times 12 = 6\,x\ \text{m}^2$.

According to our requirements,

$$x^2 + 3x = 6x + 4$$

i.e., $$x^2 - 3x - 4 = 0$$

Using the quadratic formula, we get

$$x = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2} = \ = 4 \text{ or } -1$$

But $x \neq -1$ (Why?). Therefore, $x = 4$.

So, the breadth of the park = 4m and its length will be $x + 3 = 4 + 3 = 7$m.

Verification : Area of rectangular park = 28 m^2,

area of triangular park = 24 m^2 = $(28 - 4)\ \text{m}^2$

Example-11. Find the roots of the following quadratic equations, if they exist, using the quadratic formula:

(i) $x^2 + 4x + 5 = 0$ (ii) $2x^2 - 2\sqrt{2x} + 1 = 0$

Solution :

(i) $x^2 + 4x + 5 = 0$. Here, $a = 1$, $b = 4$, $c = 5$. So, $b^2 - 4ac = 16 - 20 = -4 < 0$.

Since the square of a real number cannot be negative, therefore $\sqrt{b^2 - 4ac}$ will not have any real value.

So, there are no real roots for the given equation.

(ii) $2x^2 - 2\sqrt{2}\,x + 1 = 0$. Here, $a = 2$, $b = -2\sqrt{2}$, $c = 1$.

So, $b^2 - 4ac = 8 - 8 = 0$

Therefore, $x = \dfrac{2\sqrt{2} \pm \sqrt{0}}{4} = \dfrac{\sqrt{2}}{2} \pm 0$ *i.e.*, $x = \dfrac{1}{\sqrt{2}}$.

So, the roots are $\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}$.

Example-12. Find the roots of the following equations:

(i) $x + \dfrac{1}{x} = 3,\ x \neq 0$ (ii) $\dfrac{1}{x} - \dfrac{1}{x-2} = 3,\ x \neq 0, 2$

Solution :

(i) $x + \dfrac{1}{x} = 3$. Multiplying whole by x, we get

$x^2 + 1 = 3x$

i.e., $x^2 - 3x + 1 = 0$, which is a quadratic equation.

Here, $a = 1, b = -3, c = 1$

So, $b^2 - 4ac = 9 - 4 = 5 > 0$

Therefore, $x = \dfrac{3 \pm \sqrt{5}}{2}$ (why ?)

So, the roots are $\dfrac{3+\sqrt{5}}{2}$ and $\dfrac{3-\sqrt{5}}{2}$.

(ii) $\dfrac{1}{x} - \dfrac{1}{x-2} = 3,\ x \neq 0, 2.$

As $x \neq 0, 2$, multiplying the equation by $x(x-2)$, we get

$(x - 2) - x = 3x(x - 2)$

$= 3x^2 - 6x$

So, the given equation reduces to $3x^2 - 6x + 2 = 0$, which is a quadratic equation.

Here, $a = 3, b = -6, c = 2$. So, $b^2 - 4ac = 36 - 24 = 12 > 0$

Therefore, $x = \dfrac{6 \pm \sqrt{12}}{6} = \dfrac{6 \pm 2\sqrt{3}}{6} = \dfrac{3 \pm \sqrt{3}}{3}$.

So, the roots are $\dfrac{3+\sqrt{3}}{3}$ and $\dfrac{3-\sqrt{3}}{3}$.

Example-13. A motor boat whose speed is 18 km/h in still water. It takes 1 hour more to go 24 km upstream than to return downstream to the same spot. Find the speed of the stream.

Solution : Let the speed of the stream be x km/h.

Therefore, the speed of the boat upstream = $(18 - x)$ km/h and the speed of the boat downstream = $(18 + x)$ km/h.

The time taken to go upstream = $\dfrac{\text{distance}}{\text{speed}} = \dfrac{24}{18-x}$ hours.

Similarly, the time taken to go downstream = $\dfrac{24}{18+x}$ hours.

According to the question,

$$\frac{24}{18-x} - \frac{24}{18+x} = 1$$

i.e., $24(18 + x) - 24(18 - x) = (18 - x)(18 + x)$

i.e., $x^2 + 48x - 324 = 0$

Using the quadratic formula, we get

$$x = \frac{-48 \pm \sqrt{48^2 + 1296}}{2} = \frac{-48 \pm \sqrt{3600}}{2}$$

$$= \frac{-48 \pm 60}{2} = 6 \text{ or } -54$$

Since x is the speed of the stream, it cannot be negative. So, we ignore the root $x = -54$. Therefore, $x = 6$ gives the speed of the stream as 6 km/h.

Exercise - 5.3

1. Find the roots of the following quadratic equations, if they exist, by the method of completing the square:

i. $2x^2 + x - 4 = 0$ ii. $4x^2 + 4\sqrt{3}x + 3 = 0$

iii. $5x^2 - 7x - 6 = 0$ iv. $x^2 + 5 = -6x$

2. Find the roots of the quadratic equations given in Q.1 above by applying the quadratic formula.

3. Find the roots of the following equations:

(i) $x - \frac{1}{x} = 3,\ x \neq 0$ (ii) $\frac{1}{x+4} - \frac{1}{x-7} = \frac{11}{30},\ x \neq -4, 7$

4. The sum of the reciprocals of Rehman's ages, (in years) 3 years ago and 5 years from now is $\frac{1}{3}$. Find his present age.

5. In a class test, the sum of Moulika's marks in Mathematics and English is 30. If she got 2 marks more in Mathematics and 3 marks less in English, the product of her marks would have been 210. Find her marks in the two subjects.

6. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

7. The difference of squares of two numbers is 180. The square of the smaller number is 8 times the larger number. Find the two numbers.

8. A train travels 360 km at a uniform speed. If the speed had been 5 km/h more, it would have taken 1 hour less for the same journey. Find the speed of the train.

9. Two water taps together can fill a tank in $9\frac{3}{8}$ hours. The tap of larger diameter takes 10 hours less than the smaller one to fill the tank separately. Find the time in which each tap can separately fill the tank.

10. An express train takes 1 hour less than a passenger train to travel 132 km between Mysore and Bangalore (without taking into consideration the time they stop at intermediate stations). If the average speed of the express train is 11km/h more than that of the passenger train, find the average speed of the two trains.

11. Sum of the areas of two squares is 468 m^2. If the difference of their perimeters is 24 m, find the sides of the two squares.

12. A ball is thrown vertically upward from the top of a building 96 feet tall with an initial velocity 80 m/second. The distance 's' of the ball from the ground after t seconds is S = 96 + 80t – 16t^2. After how may seconds does the ball strike the ground.

13. If a polygon of 'n' sides has $\frac{1}{2}n(n-3)$ diagonals. How many sides will a polygon having 65 diagonals? Is there a polygon with 50 diagonals?

5.5 Nature of Roots

In the previous section, we have seen that the roots of the equation $ax^2+bx+c=0$ are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Now let us try to understand the nature of roots.

Remember that zeros are those points where value of polynomial becomes zero or we can say that the curve of quadratic polynomial cuts the X-axis.

Similarly, roots of a quadratic equation are those points where the curve cuts the X-axis.

Case-1 : If $b^2 - 4ac > 0$;

We get two distinct real roots $\frac{-b+\sqrt{b^2-4ac}}{2a}$, $\frac{-b-\sqrt{b^2-4ac}}{2a}$

In such case if we draw graph for the given quadratic equation we get the following figures.

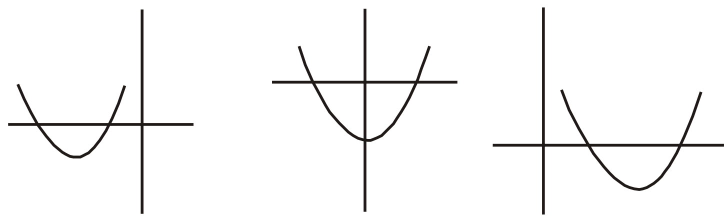


Figure shows that the curve of the quadratic equation cuts the x-axis at two distinct points

Case-2 : If $b^2 - 4ac = 0$

$$x = \frac{-b+0}{2a}$$

So, $x = \frac{-b}{2a}, \frac{-b}{2a}$

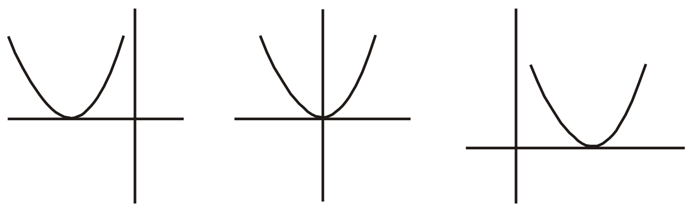
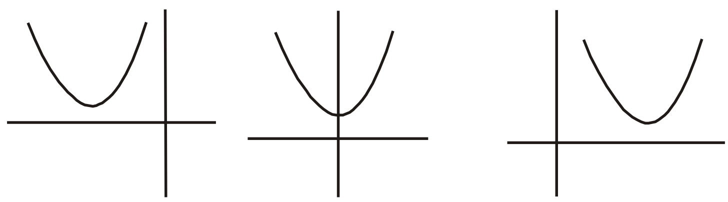


Figure shows that the curve of the quadratic equation touching X-axis at one point.

Case-3 : $b^2 - 4ac < 0$

There are no real roots. Roots are imaginary.



In this case graph neither intersects nor touches the X-axis at all. So, there are no real roots.

Since $b^2 - 4ac$ determines whether the quadratic equation $ax^2 + bx + c = 0$ has real roots or not, $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

So, a quadratic equation $ax^2 + bx + c = 0$ has

i. two distinct real roots, if $b^2 - 4ac > 0$,

ii. two equal real roots, if $b^2 - 4ac = 0$,

iii. no real roots, if $b^2 - 4ac < 0$.

Let us consider some examples.

Example-14. Find the discriminant of the quadratic equation $2x^2 - 4x + 3 = 0$, and hence find the nature of its roots.

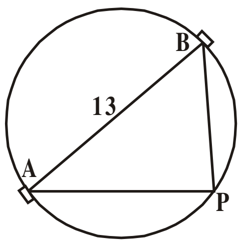
Solution : The given equation is in the form of $ax^2 + bx + c = 0$, where $a = 2$, $b = -4$ and $c = 3$. Therefore, the discriminant

$$b^2 - 4ac = (-4)^2 - (4 \times 2 \times 3) = 16 - 24 = -8 < 0$$

So, the given equation has no real roots.

Example-15. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the differences of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

Solution : Let us first draw the diagram.



Let P be the required location of the pole. Let the distance of the pole from the gate B be x m, i.e., BP = x m. Now the difference of the distances of the pole from the two gates = AP – BP (or, BP – AP)= 7 m. Therefore, AP = $(x + 7)$ m.

Now, AB = 13m, and since AB is a diameter,

$\angle APB = 90^0$ (Why?)

Therefore, $AP^2 + PB^2 = AB^2$ (By Pythagoras theorem)

i.e., $(x + 7)^2 + x^2 = 13^2$

i.e., $x^2 + 14x + 49 + x^2 = 169$

i.e., $2x^2 + 14x - 120 = 0$

So, the distance 'x' of the pole from gate B satisfies the equation

$$x^2 + 7x - 60 = 0$$

So, it would be possible to place the pole if this equation has real roots. To see if this is so or not, let us consider its discriminant. The discriminant is

$$b^2 - 4ac = 7^2 - 4 \times 1 \times (-60) = 289 > 0.$$

So, the given quadratic equation has two real roots, and it is possible to erect the pole on the boundary of the park.

Solving the quadratic equation $x^2 + 7x - 60 = 0$, by the quadratic formula, we get

$$x = \frac{-7 \pm \sqrt{289}}{2} = \frac{-7 \pm 17}{2}$$

Therefore, $x = 5$ or -12.

Since x is the distance between the pole and the gate B, it must be positive.

Therefore, $x = -12$ will have to be ignored. So, $x = 5$.

Thus, the pole has to be erected on the boundary of the park at a distance of 5m from the gate B and 12m from the gate A.

TRY THIS

1. Explain the benefits of evaluating the discriminant of a quadritic equation before attempting to solve it. What does its value signifies?
2. Write three quadratic equations one having two distinct real solutions, one having no real solution and one having exactly one real solution.

Example-16. Find the discriminant of the equation $3x^2 - 2x + \frac{1}{3} = 0$ and hence find the nature of its roots. Find them, if they are real.

Solution : Here $a = 3$, $b = -2$ and $c = \frac{1}{3}$

Therefore, discriminant $b^2 - 4ac = (-2)^2 - 4 \times 3 \times \frac{1}{3} = 4 - 4 = 0.$

Hence, the given quadratic equation has two equal real roots.

The roots are $\frac{-b}{2a}, \frac{-b}{2a}$, i.e., $\frac{2}{6}, \frac{2}{6}$, i.e., $\frac{1}{3}, \frac{1}{3}$.

Exercise - 5.4

1. Find the nature of the roots of the following quadratic equations. If real roots exist, find them:

   (i) $2x^2 - 3x + 5 = 0$ (ii) $3x^2 - 4\sqrt{3}x + 4 = 0$

   (iii) $2x^2 - 6x + 3 = 0$

2. Find the values of k for each of the following quadratic equations, so that they have two equal roots.

   (i) $2x^2 + kx + 3 = 0$ (ii) $kx\,(x - 2) + 6 = 0$

3. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2? If so, find its length and breadth.

4. The sum of the ages of two friends is 20 years. Four years ago, the product of their ages in years was 48. Is the situation possible? If so, determine their present ages.

5. Is it possible to design a rectangular park of perimeter 80 m. and area 400 m^2? If so, find its length and breadth.

Optional Exercise

[This exercise is not meant for examination]

1. Some points are plotted on a plane. Each point joined with all remaining points by line segments. Find the number of points if the number of line segments are 10.

2. A two digit number is such that the product of the digits is 8. When 18 is added to the number they interchange their places. Determine the number.

3. A piece of wire 8 m. in length, cut into two pieces, and each piece is bent into a square. Where should the cut in the wire be made if the sum of the areas of these squares is to be 2 m^2?

$$\left[\text{Hint} : x + y = 8, \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 = 2 \Rightarrow \left(\frac{x}{4}\right)^2 + \left(\frac{8-x}{4}\right)^2 = 2\right].$$

4. Vinay and Praveen working together can paint the exterior of a house in 6 days. Vinay by himself can complete the job in 5 days less than Praveen. How long will it take vinay to complete the job by himself.

5. Show that the sum of roots of a quadratic equation is '$\frac{-b}{a}$'.

6. Show that the product of the roots of a quadratic equation is '$\frac{c}{a}$'.

7. The denominator of a fraction is one more than twice the numerator. If the sum of the fraction and its reciprocal is $2\frac{16}{21}$, find the fraction.

What We Have Discussed

In this chapter, we have studied the following points:

1. Standard form of quadratic equation in variable x is $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

2. A real number α is said to be a root of the quadratic equation $ax^2 + bx + c = 0$, if $a\alpha^2 + + b\alpha + c = 0$. The zeroes of the quadratic polynomial $ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$ are the same.

3. If we can factorise $ax^2 + bx + c, a \neq 0,$ into a product of two linear factors, then the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by equating each factor to zero.

4. A quadratic equation can also be solved by the method of completing the square.

5. Quadratic formula: The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ provided } b^2 - 4ac \geq 0.$$

6. A quadratic equation $ax^2 + bx + c = 0$ has

   (i) two distinct real roots, if $b^2 - 4ac > 0$,

   (ii) two equal roots (i.e., coincident roots), if $b^2 - 4ac = 0$, and

   (iii) no real roots, if $b^2 - 4ac < 0$.

CHAPTER 6

Progressions

6.1 INTRODUCTION

You must have observed that in nature, many things follow a certain pattern such as the petals of a sunflower, the holes of a honeycomb, the grains on a maize cob, the spirals on a pineapple and on a pine cone etc.

Can you see a pattern in each of the given example? We can see the natural patterns have a repetition which is not progressive. The identical petals of the sunflower are equidistantly grown. In a honeycomb identical hexagonal shaped holes are arranged symmetrically around each hexagon. Similarly, you can find out other natural patterns in spirals of pineapple....

You can look for some other patterns which occur in our day-to-day life. Some examples are:

(i) List of the last digits (digits in unit place) taken from the values of $4, 4^2, 4^3, 4^4, 4^5, 4^6$ ..... is

4, 6, 4, 6, 4, 6, ...

(ii) Mary is doing problems on patterns as part of preparing for a bank exam. One of them is "find the next two terms in the following pattern".

1, 2, 4, 8, 10, 20, 22 ....

(iii) Usha applied for a job and got selected. She has been offered a job with a starting monthly salary of ₹ 8000, with an annual increment of ₹500. Her salary (in rupees) for to 1st, 2nd, 3rd ... years will be 8000, 8500, 9000 ..... respectively.

(iv) The lengths of the rungs of a ladder decrease uniformly by 2 cm from bottom to top. The bottom rung is 45 cm in length. The lengths (in cm) of the 1st, 2nd, 3rd, ... 8th rung from the bottom to the top are 45, 43, 41, 39, 37, 35, 33, 31 respectively.

Can you see any relationship between the terms in the pattern of numbers written above?

Pattern given in example (i) has a relation of two numbers one after the other i.e. 4 and 6 are repeating alternatively.

Now try to find out pattern in exampled (ii). In examples (iii) and (iv), the relationship between the numbers in each list is constantly progressive. In the given list 8000, 8500, 9000, ... each succeeding term is obtained by adding 500 to the preceding term.

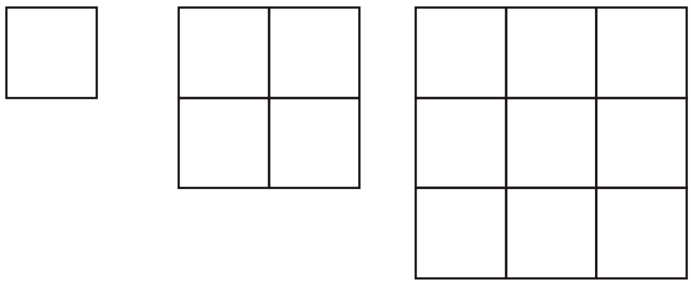
Where as in 45, 43, 41, ... each succeeding term is obtained by adding '-2' to each preceding term. Now we can see some more examples of progressive patterns.

(a) In a savings seheme, the amount becomes $\frac{5}{4}$ times of itself after 3 years.

The maturity amount (in Rupees) of an investment of ₹8000 after 3, 6, 9 and 12 years will be respectively. 10000, 12500, 15625, 19531.25.

(b) The number of unit squares in squares with sides 1, 2, 3, ... units are respectively.

$1^2, 2^2, 3^2, \ldots$

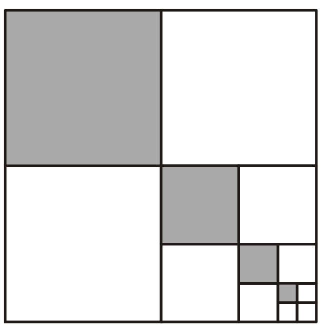


(c) Hema put Rs. 1000 into her daughter's money box when she was one year old and increased the amount by Rs. 500 every year. The amount of money (in Rs.) in the box on her 1st, 2nd, 3rd, 4th .... birthday would be.

1000, 1500, 2000, 2500, ... respectively.

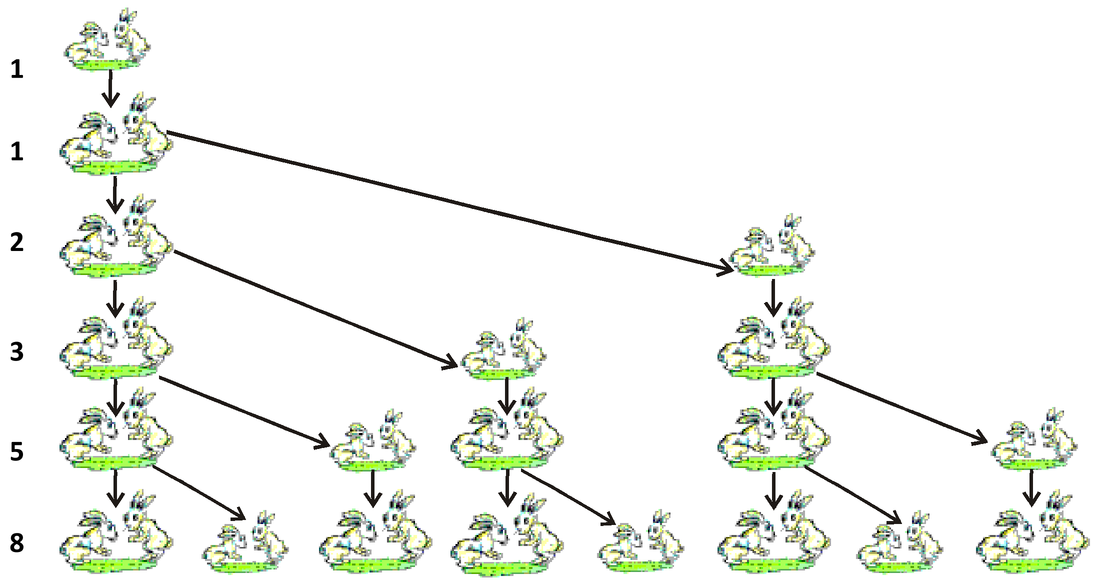
(d) The fraction of first, second, third ..... shaded regions of the squares in the following figure will be respectively.

$\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \ldots$



(e) A pair of rabbits are too young to produce in their first month. In the second, and every subsequent month, they produce a new pair. Each new pair of rabbits produce a new pair in their second month and in every subsequent month (see the figure below). Assuming no rabbit dies, the number of pairs of rabbits at the start of the 1^{st}, 2^{nd}, 3^{rd}, ..., 6^{th} month, respectively are :

1, 1, 2, 3, 5, 8



In the examples above, we observe some patterns. In some of them, we find that the succeeding terms are obtained by adding a fixed number or in other by multiplying with a fixed number or in another, we find that they are squares of consecutive numbers and so on.

In this chapter, we shall discuss some of these patterns in which succeeding terms are obtained by adding a fixed number to the preceding terms or multiplying preceding terms by a fixed number. We shall also see how to find their n^{th} term and the sum of n consecutive terms, and use this knowledge in solving some daily life problems.

History : Evidence is found that Babylonians some 400 years ago, knew of Arithmetic and geometric progressions. According to Boethins (570 AD), these progressions were known to early Greek writers. Among the Indian mathematicians, Aryabhata (470 AD) was the first to give formula for the sum of squares and cubes of natural number in his famous work Aryabhatiyam written around 499 A.D. He also gave the formula for finding the sum of n terms of an Arithmetic Progression starting with p^{th} term. Indian mathematician Brahmagupta (598 AD), Mahavira (850 AD) and Bhaskara (1114-1185 AD) also considered the sums of squares and cubes.

6.2 Arithmetic Progressions

Consider the following lists of numbers :

(i) 1, 2, 3, 4, . . .

(ii) 100, 70, 40, 10, . . .

(iii) – 3, –2, –1, 0, . . .

(iv) 3, 3, 3, 3, . . .

(v) –1.0, –1.5, –2.0, –2.5, . . .

Each of the numbers in the list is called a **term**.

Given a term, can you write the next term in each of the lists above? If so, how will you write it? Perhaps by following a pattern or rule, let us observe and write the rule.

In (i), each term is 1 more than the term preceding it.

In (ii), each term is 30 less than the term preceding it.

In (iii), each term is obtained by adding 1 to the term preceding it.

In (iv), all the terms in the list are 3 , i.e., each term is obtained by adding (or subtracting) 0 to the term preceding it.

In (v), each term is obtained by adding – 0.5 to (i.e., subtracting 0.5 from) the term preceding it.

In all the lists above, we can observe that successive terms are obtained by adding or substracting a fixed number to the preceding terms. Such list of numbers is said to form an **Arithmetic Progression (AP)**.

Try This

(i) Which of these are Arithmetic Progressions and why?

(a) 2, 3, 5, 7, 8, 10, 15, ...

(b) 2, 5, 7, 10, 12, 15, ...

(c) -1, -3, -5, -7, ...

(ii) Write 3 more Arithmetic Progressions.

6.2.1 What is an Arithmetic Progression?

We observe that an arithmetic progression is a list of numbers in which each term is obtained by adding a fixed number to the preceding term except the first term.

This fixed number is called the **common difference** of the AP. Remember that it **can be positive, negative or zero**.

Let us denote the first term of an AP by a_1, second term by a_2, . . ., nth term by a_n and the common difference by d. Then the AP becomes $a_1, a_2, a_3, \ldots, a_n$.

So, $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$.

Let us see some more examples of AP :

(a) Heights (in cm) of some students of a school standing in a queue in the morning assembly are 147 , 148, 149, . . ., 157.

(b) Minimum temperatures (in degree celsius) recorded for a week, in the month of January in a city, arranged in ascending order are

$$-3.1, -3.0, -2.9, -2.8, -2.7, -2.6, -2.5$$

(c) The balance money (in ₹) after paying 5% of the total loan of ₹1000 every month is 950, 900, 850, 800, . . ., 50.

(d) Cash prizes (in ₹) given by a school to the toppers of Classes I to XII are 200, 250, 300, 350, . . ., 750 respectively.

(e) Total savings (in ₹) after every month for 10 months when Rs 50 are saved each month are 50, 100, 150, 200, 250, 300, 350, 400, 450, 500.

Think - Discuss

1. Think how each of the list given above form an AP Discuss with your friends.
2. Find the common difference of each of the above lists? Think when is it positive?
3. Make a positive Arithmetic progression in which the common difference is a small positive quantity.
4. Make an AP in which the common difference is big(large) positive quantity.
5. Make an AP in which the common difference is negative.

General form of AP : Can you see that all AP's can be written as.

$$a, a + d, a + 2d, a + 3d, . . .$$

This is called general form of an A.P where 'a' is the first term and 'd' is the common difference

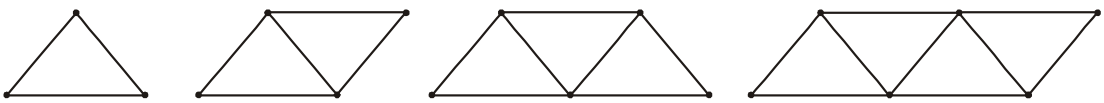
For example in 1, 2, 3, 4, 5, ...

The first terms is 1 and the common difference is also 1.

In 2, 4, 6, 8, 10 ..... What is the first term and common difference?

Activity

(i) Make the following figures with match sticks



(iii) Write down the number of match sticks required for each figure.

(iv) Can you find a common difference in members of the list?

(v) Does the list of these numbers form an AP?

6.2.2 Parameters of a Arithmetic Progressions

Note that in examples (a) to (e) above, in section 6.2.1 there are only a finite number of terms. Such an AP is called a **finite AP.** Also note that each of these Arithmetic Progressions (APs) has a last term. The APs in examples (i) to (v) in the section 6.2, are not finite APs and so they are called **infinite Arithmetic Progressions**. Such APs are never ending and do not have a last term.

Do this

Write three examples for finite AP and three for infinite AP.

Now, to know about an AP, what is the minimum information that you need? Is it enough to know the first term? Or, is it enough to know only the common difference?

We can see that we will need to know both – the first term a and the common difference d. These two parameters are sufficient for us to complete the Arithmetic Progression.

For instance, if the first term a is 6 and the common difference d is 3, then the AP is

$$6, 9, 12, 15, \ldots$$

and if a is 6 and d is – 3, then the AP is

$$6, 3, 0, -3, \ldots$$

Similarly, when

$a = -7, \quad d = -2, \qquad$ the AP is $\quad -7, -9, -11, -13, \ldots$

$a = 1.0, \quad d = 0.1, \qquad$ the AP is $\quad 1.0, 1.1, 1.2, 1.3, \ldots$

$a = 0, \quad d = 1\frac{1}{2}, \qquad$ the AP is $0, 1\frac{1}{2}, 3, 4\frac{1}{2}, 6, \ldots$

$a = 2, \quad d = 0, \qquad$ the AP is $\quad 2, 2, 2, 2, \ldots$

So, if you know what a and d are, you can list the AP.

Let us try other way. If you are given a list of numbers, how can you say whether it is an A.P. or not?

For example, for any list of numbers :

$$6, 9, 12, 15, \ldots,$$

We check the difference of the succeeding terms. In the given list we have $a_2 - a_1 = 9 - 6 = 3$,

$$a_3 - a_2 = 12 - 9 = 3,$$
$$a_4 - a_3 = 15 - 12 = 3$$

We see that $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 \ldots = 3$

Here the difference of any two consecutive terms in each case is 3. So, the given list is an AP whose first term *a* is 6 and common difference *d* is 3.

For the list of numbers : 6, 3, 0, – 3, . . .,

$$a_2 - a_1 = 3 - 6 = -3,$$
$$a_3 - a_2 = 0 - 3 = -3$$
$$a4 - a_3 = -3 - 0 = -3$$

$$a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = -3$$

Similarly, this is also an AP whose first term is 6 and the common difference is –3.

So, we see that if the difference between succeeding terms is constant then it is an Arithmetic Progression.

In general, for an AP $a_1, a_2, \ldots, a_n$, we can say

$$d = a_{k+1} - a_k \quad \text{where } k \in \mathrm{N};\ k \geq 1$$

where a_{k+1} and a_k are the $(k + 1)$th and the kth terms respectively.

Consider the list of numbers 1, 1, 2, 3, 5, ... By looking at it, you can tell that the difference between any two consecutive terms is not the same. So, this is not an AP.

Note : To find *d* in the AP : 6, 3, 0, – 3, . . ., we have subtracted 6 from 3 and not 3 from 6. We have to subtract the k^{th} term from the $(k+1)$ th term even if the $(k+1)^{th}$ term is smaller and to find 'd' in a given A.P. we need not find all of $a_2 - a_1, a_1 - a_2 \ldots.$ It is enough to find only one of them

Do This

1. Take any Arithmetic Progression.
2. Add a fixed number to each and every term of AP. Write the resulting numbers as a list.
3. Similarly subtract a fixed number from each and every term of AP. Write the resulting numbers as a list.
4. Multiply and divide each term of AP by a fixed number and write the resulting numbers as a list.
5. Check whether the resulting lists are AP in each case.

6. What is your conclusion?

Let us consider some examples

Example-1. For the AP : $\frac{1}{4}, \frac{-1}{4}, \frac{-3}{4}, \frac{-5}{4}$ ...., write the first term a and the common difference d. And find the 7[th] term

Solution : Here, $a = \frac{1}{4}$; $d = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2}$

Remember that we can find d using any two consecutive terms, once we know that the numbers are in AP.

The seventh term would be $\frac{-5}{4} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = \frac{-11}{4}$

Example-2. Which of the following forms an AP? If they form AP then write next two terms?

(i) 4, 10, 16, 22, . . . (ii) 1, – 1, – 3, – 5, . . . (iii) – 2, 2, – 2, 2, – 2, . . .

(iv) 1, 1, 1, 2, 2, 2, 3, 3, 3, . . . (v) $x, 2x, 3x, 4x$ ......

Solution : (i) We have $a_2 - a_1 = 10 - 4 = 6$

$$a_3 - a_2 = 16 - 10 = 6$$

$$a_4 - a_3 = 22 - 16 = 6$$

i.e., $a_{k+1} - a_k$ is same every time.

So, the given list of numbers forms an AP with the common difference $d = 6$.

The next two terms are: 22 + 6 = 28 and 28 + 6 = 34.

(ii) $a_2 - a_1 = -1 - 1 = -2$

$a_3 - a_2 = -3 - (-1) = -3 + 1 = -2$

$a_4 - a_3 = -5 - (-3) = -5 + 3 = -2$

i.e., $a_{k+1} - a_k$ is same every time.

So, the given list of numbers forms an AP with the common difference $d = -2$.

The next two terms are:

$$-5 + (-2) = -7 \text{ and } -7 + (-2) = -9$$

(iii) $a_2 - a_1 = 2 - (-2) = 2 + 2 = 4$

$a_3 - a_2 = -2 - 2 = -4$

As $a_2 - a_1 \neq a_3 - a_2$, the given list of numbers do not form an AP.

(iv) $a_2 - a_1 = 1 - 1 = 0$

$a_3 - a_2 = 1 - 1 = 0$

$a_4 - a_3 = 2 - 1 = 1$

Here, $a_2 - a_1 = a_3 - a_2 \neq a_4 - a_3$.

So, the given list of numbers do not form an AP.

(v) We have $a_2 - a_1 = 2x - x = x$

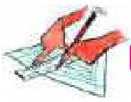
$a_3 - a_2 = 3x - 2x = x$

$a_4 - a_3 = 4x - 3x = x$

i.e., $a_{k+1} - a_k$ is same every time.

$\therefore$ So, the given list form an AP.

The next two terms are $4x + x = 5x$ and $5x + x = 6x$.



Exercise - 6.1

1. In which of the following situations, does the list of numbers involved make an arithmetic progression, and why?

   (i) The taxi fare after each km when the fare is ₹ 20 for the first km and rises by ₹ 8 for each additional km.

   (ii) The amount of air present in a cylinder when a vacuum pump removes $\frac{1}{4}$ of the air remaining in the cylinder at a time.

   (iii) The cost of digging a well, after every metre of digging, when it costs ₹ 150 for the first metre and rises by ₹ 50 for each subsequent metre.

   (iv) The amount of money in the account every year, when ₹10000 is deposited at compound interest at 8 % per annum.

2. Write first four terms of the AP, when the first term a and the common difference d are given as follows:

   (i) $a = 10, d = 10$ (ii) $a = -2, d = 0$

   (iii) $a = 4, d = -3$ (iv) $a = -1, d = \frac{1}{2}$

   (v) $a = -1.25, d = -0.25$

3. For the following APs, write the first term and the common difference:

(i) 3, 1, – 1, – 3, . . .

(ii) – 5, – 1, 3, 7, . . .

(iii) $\frac{1}{3}, \frac{5}{3}, \frac{9}{3}, \frac{13}{3}, ...$

(iv) 0.6, 1.7, 2.8, 3.9, . . .

4. Which of the following are APs ? If they form an AP, find the common difference *d* and write three more terms.

(i) 2, 4, 8, 16, . . .

(ii) $2, \frac{5}{2}, 3, \frac{7}{2}, ...$

(iii) – 1.2, – 3.2, – 5.2, – 7.2, . . .

(iv) – 10, – 6, – 2, 2, . . .

(v) $3, 3+\sqrt{2}, 3+2\sqrt{2}, 3+3\sqrt{2}, ...$

(vi) 0.2, 0.22, 0.222, 0.2222, . . .

(vii) 0, – 4, – 8, –12, . . .

(viii) $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, ...$

(ix) 1, 3, 9, 27, . . .

(x) a, $2a$, $3a$, $4a$, . . .

(xi) $a, a^2, a^3, a^4, \ldots$

(xii) $\sqrt{2}, \sqrt{8}, \sqrt{18}, \sqrt{32}, ...$

(xiii) $\sqrt{3}, \sqrt{6}, \sqrt{9}, \sqrt{12}, ...$

6.3 n^{th} Term of an Arithmetic Progression

Let us consider the offer to Usha who applied for a job and got selected. She has been offered a starting monthly salary of ₹ 8000, with an annual increment of ₹ 500. What would be her monthly salary of the fifth year?

To answer this, let us first see what her monthly salary for the second year would be.

It would be ₹ (8000 + 500) = ₹ 8500.

In the same way, we can find the monthly salary for the 3^{rd}, 4^{th} and 5^{th} year by adding ₹ 500 to the salary of the previous year.

So, the salary for the 3^{rd} year = ₹(8500 + 500)

= ₹ (8000 + 500 + 500)

= ₹ (8000 + 2 × 500)

= ₹ [8000 + **(3 – 1)** × 500] (for the **3^{rd} year**)

= ₹ 9000

Salary for the 4^{th} year = ₹ (9000 + 500)

= ₹ (8000 + 500 + 500 + 500)

$= ₹\ (8000 + 3 \times 500)$

$= ₹\ [8000 + \mathbf{(4 - 1)} \times 500]$ (for the $\mathbf{4^{th}}$ **year**)

$= ₹\ 9500$

Salary for the 5^{th} year $= ₹\ (9500 + 500)$

$= ₹\ (8000+500+500+500 + 500)$

$= ₹\ (8000 + 4 \times 500)$

$= ₹\ [8000 + \mathbf{(5 - 1)} \times 500]$ (for the $\mathbf{5^{th}}$ **year**)

$= ₹\ 10000$

Observe that we are getting a list of numbers

8000, 8500, 9000, 9500, 10000, . . .

These numbers are in Arithmetic Progression.

Looking at the pattern above, can we find her monthly salary in the 6^{th} year? The 15^{th} year? And, assuming that she is still working in the same job, what would be her monthly salary in the 25^{th} year? Here we can calculate the salary of the present year by adding ₹ 500 to the salary of previous year. Can we make this process shorter? Let us see. You may have already got some idea from the way we have obtained the salaries above.

Salary for the 15^{th} year = Salary for the 14^{th} year + ₹ 500

$$= ₹\left[8000 + \underbrace{500 + 500 + 500 + \ldots + 500}_{13 \text{ times}}\right] + ₹500$$

$= ₹\ [8000 + 14 \times 500]$

$= ₹\ [8000 + \mathbf{(15 - 1)} \times 500] = ₹\ 15000$

i.e., First salary + (15 – 1) × Annual increment.

In the same way, her monthly salary for the 25^{th} year would be

$₹\ [8000 + \mathbf{(25 - 1)} \times 500] = ₹\ 20000$

= First salary + **(25 – 1)** × Annual increment

This example has given us an idea about how to write the 15^{th} term, or the 25^{th} term. By using the same idea, now let us find the n^{th} term of an AP.

Let $a_1, a_2, a_3, \ldots$ be an AP whose first term a_1 is a and the common difference is d.

Then,

the **second term** $a_2 = a + d = a + \mathbf{(2 - 1)}\ d$

the **third** term $a_3 = a_2 + d = (a + d) + d = a + 2d = a + \mathbf{(3 - 1)}\, d$
the **fourth** term $a_4 = a3 + d = (a + 2d) + d = a + 3d = a + \mathbf{(4 - 1)}\, d$

. . . .

. . . .

Looking at the pattern, we can say that the ***n*th** term $a_n = a + (n - 1)\, d$.

So, **the n^{th} term *an* of the AP with first term *a* and common difference *d* is** given by $a_n = a + (n - 1)\, d$.

$\boldsymbol{a_n}$ is also called the **general term of the AP**.

If there are *m* terms in the AP, then $\boldsymbol{a_m}$ represents the **last term which is sometimes also denoted by *l***.

Finding terms of an AP : Using the above formula we can find different terms of an arithemetic progression.

Let us consider some examples.

Example-3. Find the 10th term of the AP : 5, 1, –3, –7 . . .

Solution : Here, $a = 5$, $d = 1 - 5 = -4$ and $n = 10$.

We have $a_n = a + (n - 1)\, d$

So, $a_{10} = 5 + (10 - 1)\,(-4) = 5 - 36 = -31$

Therefore, the 10th term of the given AP is – 31.

Example-4. Which term of the AP : 21, 18, 15, . . . is – 81?

Is there any term 0? Give reason for your answer.

Solution : Here, $a = 21$, $d = 18 - 21 = -3$ and if $a_n = -81$, we have to find n.

As $\quad a_n = a + (n - 1)\, d,$

we have $\quad -81 = 21 + (n - 1)(-3)$

$$-81 = 24 - 3n$$

$$-105 = -3n$$

So, $\quad n = 35$

Therefore, the 35th term of the given AP is – 81.

Next, we want to know if there is any n for which $a_n = 0$. If such an n is there, then

$$21 + (n - 1)\,(-3) = 0,$$

i.e., $\quad 3(n - 1) = 21$

i.e., $\quad n = 8$

So, the eighth term is 0.

Example-5. Determine the AP whose 3rd term is 5 and the 7th term is 9.

Solution : We have

$$a_3 = a + (3 - 1)\, d = a + 2d = 5 \qquad (1)$$

and $$a_7 = a + (7 - 1)\, d = a + 6d = 9 \qquad (2)$$

Solving the pair of linear equations (1) and (2), we get

$$a = 3,\ d = 1$$

Hence, the required AP is 3, 4, 5, 6, 7, . . .

Example-6. Check whether 301 is a term of the list of numbers 5, 11, 17, 23, . . .

Solution : We have :

$a_2 - a_1 = 11 - 5 = 6,\ a_3 - a_2 = 17 - 11 = 6,\ a_4 - a_3 = 23 - 17 = 6$

As $(a_{k+1} - a_k)$ is the same for $k = 1, 2, 3$, etc., the given list of numbers is an AP.

Now, for this AP we have $a = 5$ and $d = 6$.

We choose to begin with the assumption that 301 is a term, say, the nth term of the this AP. We will see if an 'n' exists for which $a_n = 301$.

We know

$$a_n = a + (n - 1)\, d$$

So, for 301 to be a term we must have

$$301 = 5 + (n - 1) \times 6$$

or $$301 = 6n - 1$$

So, $$n = \frac{302}{6} = \frac{151}{3}$$

But n should be a positive integer (Why?).

So, 301 is not a term of the given list of numbers.

Example-7. How many two-digit numbers are divisible by 3?

Solution : The list of two-digit numbers divisible by 3 is :

$$12,\ 15,\ 18,\ . . . ,\ 99$$

Is this an AP? Yes it is. Here, $a = 12$, $d = 3$, $a_n = 99$.

As $$a_n = a + (n - 1)\, d,$$

we have $99 = 12 + (n-1) \times 3$

i.e., $87 = (n-1) \times 3$

i.e., $n - 1 = \frac{87}{3} = 29$

i.e., $n = 29 + 1 = 30$

So, there are 30 two-digit numbers divisible by 3.

Example-8. Find the 11^{th} term from the last of the the AP series given below :

AP : 10, 7, 4, . . ., – 62.

Solution : Here, $a = 10$, $d = 7 - 10 = -3$, $l = -62$,

where $l = a + (n-1)d$

To find the 11th term from the last term, we will find the total number of terms in the AP.

So, $-62 = 10 + (n-1)(-3)$

i.e., $-72 = (n-1)(-3)$

i.e., $n - 1 = 24$

or $n = 25$

So, there are 25 terms in the given AP.

The 11^{th} term from the last will be the 15^{th} term of the series. (Note that it will not be the 14^{th} term. Why?)

So, $a_{15} = 10 + (15-1)(-3) = 10 - 42 = -32$

i.e., the 11^{th} term from the end is – 32.

Note : The 11^{th} term from the last is also equal to 11^{th} term of the AP with first term – 62 and the common difference 3.

Example-9. A sum of ₹ 1000 is invested at 8% simple interest per year. Calculate the interest at the end of each year. Do these interests form an AP? If so, find the interest at the end of 30 years.

Solution : We know that the formula to calculate simple interest is given by

$$\text{Simple Interest} = \frac{P \times R \times T}{100}$$

So, the interest at the end of the 1^{st} year = ₹ $\frac{1000 \times 8 \times 1}{100}$ = ₹ 80

The interest at the end of the 2^{nd} year = ₹ $\frac{1000 \times 8 \times 2}{100}$ = ₹ 160

The interest at the end of the 3rd year = $\dfrac{1000 \times 8 \times 3}{100}$ = ₹ 240

Similarly, we can obtain the interest at the end of the 4th year, 5th year, and so on. So, the interest (in Rs) at the end of the 1st, 2nd, 3rd, . . . years, respectively are

80, 160, 240, . . .

It is an AP as the difference between the consecutive terms in the list is 80,

i.e., $d = 80$. Also, $a = 80$.

So, to find the interest at the end of 30 years, we shall find a_{30}.

Now, $a_{30} = a + (30 - 1)\,d = 80 + 29 \times 80 = 2400$

So, the interest at the end of 30 years will be ₹ 2400.

Example-10. In a flower bed, there are 23 rose plants in the first row, 21 in the second, 19 in the third, and so on. There are 5 rose plants in the last row. How many rows are there in the flower bed?

Solution : The number of rose plants in the 1st, 2nd, 3rd, . . ., rows are :

23, 21, 19, . . ., 5

It forms an AP (Why?).

Let the number of rows in the flower bed be n.

Then $a = 23$, $d = 21 - 23 = -2$, $a_n = 5$

As, $a_n = a + (n - 1)\,d$

We have, $5 = 23 + (n - 1)(-2)$

i.e., $-18 = (n - 1)(-2)$

i.e., $n = 10$

So, there are 10 rows in the flower bed.

Exercise - 6.2

1. Fill in the blanks in the following table, given that a is the first term, d the common difference and a_n the nth term of the AP:

S. No.	a	d	n	a_n
(i)	7	3	8	. . .
(ii)	– 18	. . .	10	0

(iii)	. . .	– 3	18	– 5
(iv)	– 18.9	2.5	. . .	3.6
(v)	3.5	0	105	. . .

2. Find the
   (i) 30^{th} term of the A.P. 10, 7, 4 ......
   (ii) 11^{th} term of the A.P. : $-3, \frac{-1}{2}, 2, ...$
3. Find the respective terms for the following APs.
   (i) $a_1 = 2;\ a_3 = 26$ find a_2 (ii) $a_2 = 13;\ a_4 = 3$ find a_1, a_3
   (iii) $a_1 = 5;\ a_4 = 9\frac{1}{2}$ find a_2, a_3 (iv) $a_1 = -4;\ a_6 = 6$ find a_2, a_3, a_4, a_5
   (v) $a_2 = 38;\ a_6 = -22$ find a_1, a_3, a_4, a_5
4. Which term of the AP : 3, 8, 13, 18, . . . ,is 78?
5. Find the number of terms in each of the following APs :
   (i) 7, 13, 19, . . . , 205 (ii) $18, 15\frac{1}{2}, 13, ..., -47$
6. Check whether, –150 is a term of the AP : 11, 8, 5, 2 . . .
7. Find the 31^{st} term of an AP whose 11^{th} term is 38 and the 16^{th} term is 73.
8. If the 3^{rd} and the 9^{th} terms of an AP are 4 and – 8 respectively, which term of this AP is zero?
9. The 17^{th} term of an AP exceeds its 10^{th} term by 7. Find the common difference.
10. Two APs have the same common difference. The difference between their 100th terms is 100, what is the difference between their 1000th terms?
11. How many three-digit numbers are divisible by 7?
12. How many multiples of 4 lie between 10 and 250?
13. For what value of n, are the nth terms of two APs: 63, 65, 67, ... and 3, 10, 17, . . . equal?
14. Determine the AP whose third term is 16 and the 7^{th} term exceeds the 5^{th} term by 12.
15. Find the 20^{th} term from the end of the AP : 3, 8, 13, . . ., 253.

16. The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

17. Subba Rao started work in 1995 at an annual salary of ₹ 5000 and received an increment of ₹ 200 each year. In which year did his income reach ₹ 7000?

6.4 Sum of First *n* Terms in Arithmetic Progression

Let us consider the situation again given in Section 6.1 in which Hema put ₹ 1000 money box when her daughter was one year old, ₹ 1500 on her second birthday, ₹ 2000 on her third birthday and will continue in the same way. How much money will be collected in the money box by the time her daughter is 21 years old?

Here, the amount of money (in Rupees) put in the money box on her first, second, third, fourth . . . birthday were respectively 1000, 1500, 2000, 2500, . . . till her 21st birthday. To find the total amount in the money box on her 21st birthday, we will have to write each of the 21 numbers in the list above and then add them up. Don't you think it would be a tedious and time consuming process? Can we make the process shorter?

This would be possible if we can find a method for getting this sum. Let us see.

6.4.1 How 'Gauss' find the sum of terms

Carl Fredrich Gauss (1777-1855) is a great German Mathematician

We consider the problem given to Gauss, to solve when he was just 10 years old. He was asked to find the sum of the positive integers from 1 to 100. He immediately replied that the sum is 5050. Can you guess how can he do? He wrote :

$$S = 1 + 2 + 3 + . . . + 99 + 100$$

And then, reversed the numbers to write

$$S = 100 + 99 + . . . + 3 + 2 + 1$$

When he added these two he got 25 as both the sums have to be equal. So he work,

$$2S = (100 + 1) + (99 + 2) + . . . + (3 + 98) + (2 + 99) + (1 + 100)$$

$$= 101 + 101 + . . . + 101 + 101 \text{ (100 times) (check this out and discuss)}$$

So, $S = \dfrac{100 \times 101}{2} = 5050$, i.e., the sum = 5050.

6.4.2 Sum of *n* terms of an AP.

We will now use the same technique that was used by transs to find the sum of the first n terms of an AP :

$$a, a + d, a + 2d, \ldots$$

The nth term of this AP is $a + (n - 1)\,d$.

Let S_n denote the sum of the first n terms of the A.P. Whose n^{th} term is

$$a_n = a + (n - 1)\,d$$

$$\therefore \quad S_n = a + (a + a) + (a + 2d) + \ldots + a + (n - 1)d$$

$$S_n = (a + (n - 1)d) + (a + (n - 2)d + \ldots + a$$

Adding $2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + \ldots. + (2a + (n - 1)d)$ (n times)

$$= n(2a + (n - 1)d)$$

$$\therefore S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}\ [\text{first term} + \text{nth term}] = \frac{n}{2}(a + a_n)$$

If the first and last terms of an A.P. are given and the common difference is not given then

$S_n = \frac{n}{2}(a + a_n)$ is very useful to find S_n.

Money for Hema's daughter

Now we return to the question that was posed to us in the beginning. The amount of money (in Rs) in the money box of Hema's daughter on 1^{st}, 2^{nd}, 3^{rd}, 4th birthday, . . ., were 1000, 1500, 2000, 2500, . . ., respectively.

This is an AP. We have to find the total money collected on her 21st birthday, i.e., the sum of the first 21 terms of this AP.

Here, $a = 1000$, $d = 500$ and $n = 21$. Using the formula :

$$S_n = \frac{n}{2}[2a + (n - 1)d],$$

we have

$$S = \frac{21}{2}[2 \times 1000 + (21 - 1) \times 500]$$

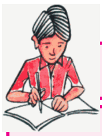
$$= \frac{21}{2}[2000 + 10000]$$

$$= \frac{21}{2}[12000] = 126000$$

So, the amount of money collected on her 21st birthday is ₹ 12600.

We use S_n in place of S to denote the sum of first n terms of the AP so that we know how many terms we have added. We write S_{20} to denote the sum of the first 20 terms of an AP. The formula for the sum of the first n terms involves four quantities S_n, a, d and n. If we know any three of them, we can find the fourth.

Remark : The n^{th} term of an AP is the difference of the sum to first n terms and the sum to first $(n - 1)$ terms of it, i.e., $a_n = S_n - S_{n-1}$.



Do This

Find the sum of indicated number of terms in each of the following A.P.s

(i) 16, 11, 6 .....; 23 terms (ii) –0.5, –1.0, –1.5, ...; 10 terms

(iii) $-1, \frac{1}{4}, \frac{3}{2}$....., 10 terms

Let us consider some examples.

Example-11. If the sum of the first 14 terms of an AP is 1050 and its first term is 10, find the 20th term.

Solution : Here, $S_n = 1050$; $n = 14$, $a = 10$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$1050 = \frac{14}{2}[2a + 13d] = 140 + 91d$$

$$910 = 91d$$

$$\therefore \quad d = 10$$

$$\therefore \quad a_{20} = 10 + (20 - 1)\,10 = 200$$

Example-12. How many terms of the AP : 24, 21, 18, . . . must be taken so that their sum is 78?

Solution : Here, $a = 24$, $d = 21 - 24 = -3$, $Sn = 78$. We need to find n.

We know that $S_n = \frac{n}{2}[2a + (n-1)d]$

So, $78 = \frac{n}{2}[48 + (n-1(-3)] = \frac{n}{2}[51 - 3n]$

or $3n^2 - 51n + 156 = 0$

or $\quad n^2 - 17n + 52 = 0$

or $\quad (n-4)(n-13) = 0$

or $\quad n = 4$ or 13

Both values of n are admissible. So, the number of terms is either 4 or 13.

Remarks :

1. In this case, the sum of the first 4 terms = the sum of the first 13 terms = 78.
2. Two answers are possible because the sum of the terms from 5^{th} to 13^{th} will be zero. This is because a is positive and d is negative, so that some terms are positive and some are negative, and will cancel out each other.

Example-13. Find the sum of :

(i) the first 1000 positive integers (ii) the first n positive integers

Solution :

(i) Let S = 1 + 2 + 3 + . . . + 1000

Using the formula $S_n = \frac{n}{2}(a+l)$ for the sum of the first n terms of an AP, we have

$$S_{1000} = \frac{1000}{2}(1+1000) = 500 \times 1001 = 500500$$

So, the sum of the first 1000 positive integers is 500500.

(ii) Let $S_n = 1 + 2 + 3 + . . . + n$

Here $a = 1$ and the last term l is n.

Therefore, $S_n = \frac{n(1+n)}{2}$ (or) $S_n = \frac{n(n+1)}{2}$

So, **the sum of first *n* positive integers is given by**

$$\mathbf{S_n = \frac{n(n+1)}{2}}$$

Example-14. Find the sum of first 24 terms of the list of numbers whose nth term is given by

$$a_n = 3 + 2n$$

Solution : As $\quad a_n = 3 + 2n,$

so, $\quad a_1 = 3 + 2 = 5$

$a_2 = 3 + 2 \times 2 = 7$

$a_3 = 3 + 2 \times 3 = 9$

$\vdots$

List of numbers becomes 5, 7, 9, 11, . . .

Here, $7 - 5 = 9 - 7 = 11 - 9 = 2$ and so on.

So, it forms an AP with common difference $d = 2$.

To find S_{24}, we have $n = 24$, $a = 5$, $d = 2$.

Therefore, $$S_{24} = \frac{24}{2}[2\times5+(24-1)\times2] = 12(10+46) = 672$$

So, sum of first 24 terms of the list of numbers is 672.

Example-15. A manufacturer of TV sets produced 600 sets in the third year and 700 sets in the seventh year. Assuming that the production increases uniformly by a fixed number every year, find :

(i) the production in the 1^{st} year (ii) the production in the 10^{th} year

(iii) the total production in first 7 years

Solution : (i) Since the production increases uniformly by a fixed number every year, the number of TV sets manufactured in 1^{st}, 2^{nd}, 3^{rd}, . . ., years will form an AP.

Let us denote the number of TV sets manufactured in the nth year by a_n.

Then, $a_3 = 600$ and $a_7 = 700$

or, $a + 2d = 600$

and $a + 6d = 700$

Solving these equations, we get $d = 25$ and $a = 550$.

Therefore, production of TV sets in the first year is 550.

(ii) Now $a_{10} = a + 9d = 550 + 9 \times 25 = 775$

So, production of TV sets in the 10^{th} year is 775.

(iii) Also, $$S_7 = \frac{7}{2}[2\times550+(7-1)\times25]$$

$$= \frac{7}{2}[1100+150] = 4375$$

Thus, the total production of TV sets in first 7 years is 4375.

Exercise - 6.3

1. Find the sum of the following APs:

   (i) 2, 7, 12, . . ., to 10 terms.

   (ii) –37, –33, –29, . . ., to 12 terms.

   (iii) 0.6, 1.7, 2.8, . . ., to 100 terms.

   (iv) $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$, ...,to 11terms.

2. Find the sums given below :

   (i) $7 + 10\frac{1}{2} + 14 + ... + 84$

   (ii) 34 + 32 + 30 + . . . + 10

   (iii) –5 + (–8) + (–11) + . . . + (–230)

3. In an AP:

   (i) given $a = 5$, $d = 3$, $a_n = 50$, find n and S_n.

   (ii) given $a = 7$, $a_{13} = 35$, find d and S_{13}.

   (iii) given $a_{12} = 37$, $d = 3$, find a and S_{12}.

   (iv) given $a_3 = 15$, $S_{10} = 125$, find d and a_{10}.

   (v) given $a = 2$, $d = 8$, $S_n = 90$, find n and a_n.

   (vi) given $a_n = 4$, $d = 2$, $S_n = -14$, find n and a.

   (vii) given $l = 28$, $S = 144$, and there are total 9 terms. Find a.

4. The first and the last terms of an AP are 17 and 350 respectively. If the common difference is 9, how many terms are there and what is their sum?

5. Find the sum of first 51 terms of an AP whose second and third terms are 14 and 18 respectively.

6. If the sum of first 7 terms of an AP is 49 and that of 17 terms is 289, find the sum of first n terms.

7. Show that $a_1, a_2, \ldots, a_n, \ldots$ form an AP where a_n is defined as below :

   (i) $a_n = 3 + 4n$

   (ii) $a_n = 9 - 5n$

   Also find the sum of the first 15 terms in each case.

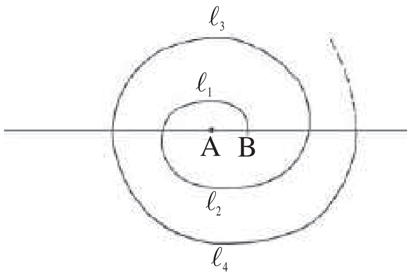
8. If the sum of the first n terms of an AP is $4n - n^2$, what is the first term (remember the first term is S_1)? What is the sum of first two terms? What is the second term? Similarly, find the 3rd, the 10th and the nth terms.

9. Find the sum of the first 40 positive integers divisible by 6.

10. A sum of ₹ 700 is to be used to give seven cash prizes to students of a school for their overall academic performance. If each prize is ₹ 20 less than its preceding prize, find the value of each of the prizes.

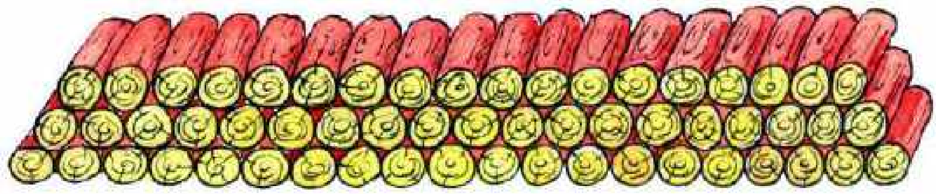
11. In a school, students thought of planting trees in and around the school to reduce air pollution. It was decided that the number of trees, that each section of each class will plant, will be the same as the class, in which they are studying, e.g., a section of Class I will plant 1 tree, a section of Class II will plant 2 trees and so on till Class XII. There are three sections of each class. How many trees will be planted by the students?

12. A spiral is made up of successive semicircles, with centres alternately at A and B, starting with centre at A, of radii 0.5 cm, 1.0 cm, 1.5 cm, 2.0 cm, . . . as shown in Fig. 5.4. What is the total length of such a spiral made up of thirteen consecutive semicircles? (Take $\pi = \frac{22}{7}$)



[**Hint :** Length of successive semicircles is $l_1, l_2, l_3, l_4, \ldots$ with centres at A, B, A, B, . . ., respectively.]

13. 200 logs are stacked in the following manner: 20 logs in the bottom row, 19 in the next row, 18 in the row next to it and so on. In how may rows are the 200 logs placed and how many logs are in the top row?



14. In a bucket and ball race, a bucket is placed at the starting point, which is 5 m from the first ball, and the other balls are placed 3 m apart in a straight line. There are ten balls in the line.

A competitor starts from the bucket, picks up the nearest ball, runs back with it, drops it in the bucket, runs back to pick up the next ball, runs to the bucket to drop it in, and she continues in the same way until all the balls are in the bucket. What is the total distance the competitor has to run?

[**Hint :** To pick up the first ball and the second ball, the total distance (in metres) run by a competitor is $2 \times 5 + 2 \times (5 + 3)$]

6.5 Geometric Progressions

Consider the lists

(i) 30, 90, 270, 810 ..... (ii) $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}$.....

(iii) 30, 24, 19.2, 15.36, 12.288

Given a term, can we write the next term in each of the lists above ?

in (i) each term is obtained by multiplying the preceeding term by 3.

in (ii) each term is obtained by multiplying the preceeding term by $\frac{1}{4}$.

in (iii) each term is obtained by multiplying the preceeding term by 0.8.

In all the lists above, we see that successive terms are obtained by multiplying the preceeding term by a fixed number. Such a list of numbers is said to form **Geometric Progression (GP).** This fixed number is called the common ration 'r' of GP. So in the above example (i), (ii), (iii) the common ratios are 3, $\frac{1}{4}$, 0.8 respectively.

Let us denote the first term of a GP by a and common ratio r. To get the second term according to the rule of Geometric Progression, we have to multiply the first term by the common ratio r.

$\therefore$ The second term $= ar$

Third term $= ar \cdot r = ar^2$

$\therefore$ a, ar, ar^2 ..... is called the general form of a GP.

in the above GP the ratio between any term (except first term) and its preceding term is 'r'

i.e., $\frac{ar}{a} = \frac{ar^2}{ar} = \text{.........} = r$

If we denote the first term of GP by a_1, second term by a_2 ..... nth term by an

then $\frac{a_2}{a_1} = \frac{a_3}{a_2} = \text{......} = \frac{a_n}{a_{n-1}} = r$

$\therefore$ A list of numbers $a_1, a_2, a_3 \text{ .... } a_n$ ... is called a geometric progression (GP), if each term is non zero and

$$\frac{a_n}{a_{n-1}} = r$$

Where n is a natural number and $n \geq 2$.

Do This

Find which of the following are not G.P.

1. 6, 12, 24, 48, ...
2. 1, 4, 9, 16, ...
3. 1, –1, 1, –1, ...
4. –4, –20, –100, –500, ...

Some more example of GP are :

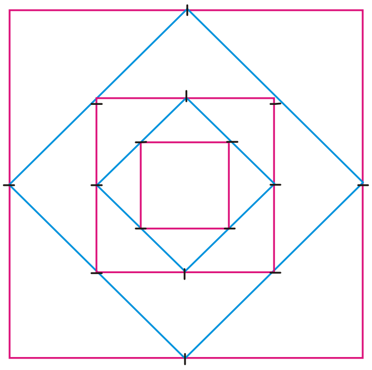
(i) A person writes a letter to four of his friends. He asks each one of them to copy the letter and give it to four different persons with same instructions so that they can move the chain ahead similarly. Assuming that the chain is not broken the number of letters at first, second, third ... stages are

1, 4, 16, 256 .... respectively.

(ii) The total amount at the end of first , second, third .... year if ₹ 500/- is deposited in the bank with annual rate 10% interest compounded annually is

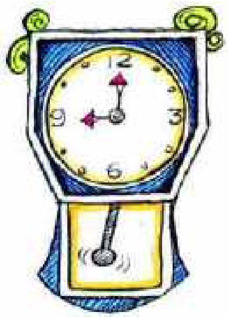
550, 605, 665.5 ......

(iii) A square is drawn by joining the mid points of the sides of a given square. A third square is drawn inside the second square in the same way and this process continues indefinitely. If a side of the first square is 16cm then the area of first, second, third ..... square will be respectively.



256, 128, 64, 32, ...

(iv) Initially a pendulum swings through an arc of 18 cms. On each sucessive swing the length of the arc is 0.9 of the previous length. So the length of the arc at first, second, third....... swing will be resepectively.



18, 16.2, 14.58, 13.122......

Think - Discuss

1. Explain why each of the lists above is a G.P.
2. To know about a G.P. what is minimum information that we need ?

Now let us learn how to construct a GP. when the first term 'a' and common ratio 'r' are given. And also learn how to check whether the given list of numbers is a G.P.

Example-16. Write the GP. if the first term $a = 3$, and the common ratio $r = 2$.

Solution : Since 'a' is the first term it can easily be written

We know that in GP. every succeeding term is obtained by multiplying the preceding term with common ratio 'r'. So to get the second term we have to multiply the first term $a = 3$ by the common ratio $r = 2$.

$\therefore$ Second term $= ar = 3 \times 2 = 6$

Similarly the third term = second term $\times$ common ratio

$= 6 \times 2 = 12$

If we proceed in this way we get the following G.P.

3, 6, 12, 24,..... .

Example-17. Write GP. if $a = 256$, $r = \frac{-1}{2}$

Solution : General form of GP $= a, ar, ar^2, ar^3, \ldots\ldots$

$$= 256, 256\left(\frac{-1}{2}\right), 256\left(\frac{-1}{2}\right)^2, 256\left(\frac{-1}{2}\right)^3$$

$$= 256, -128, 64, -32 \ldots\ldots$$

Example-18. Find the common ratio of the GP $25, -5, 1, \frac{-1}{5}$.

Solution : We know that if the first, second, third .... terms of a GP are a_1, a_2, a_3 .... respectively the common ratio $r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots\ldots$

Here $a_1 = 25$, $a_2 = -5$, $a_3 = 1$.

So common ratio $r = \frac{-5}{25} = \frac{1}{-5} = \frac{-1}{5}$.

Example-19. Which of the following list of numbers form GP. ?

(i) 3, 6, 12, ...

(ii) 64, –32, 16,

(iii) $\frac{1}{64}, \frac{1}{32}, \frac{1}{8}, \ldots\ldots$

Solution : (i)We know that a list of numbers $a_1, a_2, a_3, ...a_n .....$ is called a GP if each term is non zero and $\frac{a_2}{a_1} = \frac{a_3}{a_2} = ... \frac{a_n}{a_{n-1}} = r$

Here all the terms are non zero. Further

$$\frac{a_2}{a_1} = \frac{6}{3} = 2 \quad \text{and}$$

$$\frac{a_3}{a_2} = \frac{12}{6} = 2$$

i.e., $$\frac{a_2}{a_1} = \frac{a_3}{a_2} = 2$$

So, the given list of number form a G.P. which contain ratio 2.

(ii) All the terms are non zero.

$$\frac{a_2}{a_1} = \frac{-32}{64} = \frac{-1}{2}$$

and $$\frac{a_3}{a_1} = \frac{16}{-32} = \frac{-1}{2}$$

$$\therefore \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{-1}{2}$$

So, the given list of numbers form a GP with common ratio $\frac{-1}{2}$.

(iii) All the terms are non zero.

$$\frac{a_2}{a_1} = \frac{\frac{1}{32}}{\frac{1}{64}} = 2$$

$$\frac{a_3}{a_2} = \frac{\frac{1}{8}}{\frac{1}{32}} = 4$$

Here $\frac{a_2}{a_1} \neq \frac{a_3}{a_2}$

So, the given list of numbers does not form GP.

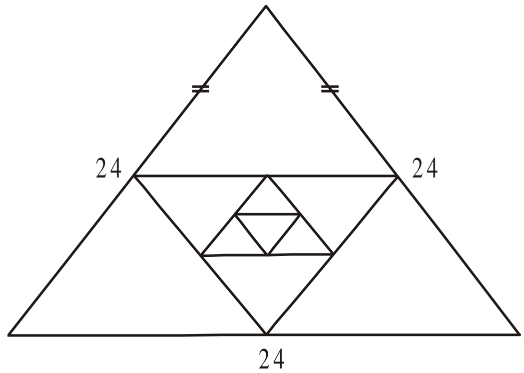
Exercise - 6.4

1. In which of the following situations, does the list of numbers involved in form a GP.?

   (i) Salary of Sharmila, when her salary is ₹ 5,00,000 for the first year and expected to receive yearly increase of 10% .

   (ii) Number of bricks needed to make each step, if the stair case has total 30 steps. Bottom step needs 100 bricks and each successive step needs 2 brick less than the previous step.

   (iii) Perimeter of the each triangle, when the mid points of sides of an equilateral triangle whose side is 24 cm are joined to form another triangle, whose mid points in turn are joined to form still another triangle and the process continues indefinitely.



2. Write three terms of the GP when the first term 'a' and the common ratio 'r' are given?

   (i) $a = 4;\quad r = 3$ (ii) $a = \sqrt{5}\,;\ r = \dfrac{1}{5}$

   (iii) $a = 81;\ r = \dfrac{-1}{3}$ (iv) $a = \dfrac{1}{64}\,;\ r = 2$

3. Which of the following are GP? If they are GP. Write three more terms?

   (i) 4, 8, 16 ..... (ii) $\dfrac{1}{3}, \dfrac{-1}{6}, \dfrac{1}{12}$.....

   (iii) 5, 55, 555, ... (iv) $-2, -6, -18$ .....

   (v) $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{6}$..... (vi) $3, -3^2, 3^3$, ...

   (vii) $x, 1, \dfrac{1}{x}$,..... (viii) $\dfrac{1}{\sqrt{2}}, -2, \dfrac{8}{\sqrt{2}}$.....

   (ix) 0.4, 0.04, 0.004, ...

4. Find x so that $x, x+2, x+b$ are conseecitive terms of a geometric progression.

6.6 n^{th} TERM OF A GP

Let us examine a problem the number of bacteria in a certain culture triples every hour. If there were 30 bacteria present in the culture originally. Then, what would be number of bacteria in fourth hour?

To answer this let us first see what the number of bacteria in second hour would be.

Since for every hour it triples

No.of bacteria in Second hour $= 3 \times$ no.of bacteria in first hour

$= 3 \times 30 = 30 \times 3^1$

$= 30 \times 3^{(2-1)}$

$= 90$

No.of bacteria in third hour $= 3 \times$ no.of bacteria in second hour

$= 3 \times 90 \qquad = 30 \times (3\times3)$

$= 30 \times 3^2 \qquad = 30 \times 3^{(3-1)}$

$= 270$

No.of bacteria in fourth hour $= 3 \times$ no.of bacteria in third hour

$= 3 \times 270 \qquad = 30 \times (3\times3\times3)$

$= 30 \times 3^3 \qquad = 30 \times 3^{(4-1)}$

$= 810$

Observe that we are getting a list of numbers

30, 90, 270, 810, ...

These numbers are in GP (why ?)

Now looking at the pattern formed above, can you find number of bacteria in 20th hour?

You may have already got some idea from the way we have obtained the number of bacteria as above. By using the same pattern, we can compute that Number of bacteria in 20^{th} hour.

$$= 30\times\underbrace{(3\times3\times\ldots\times3)}_{19\text{ terms}}$$

$$= 30 \times 3^{19} \qquad = 30 \times 3^{(20-1)}$$

This example would have given you some idea about how to write the 25th term. 35^{th} term and more generally the nth term of the GP.

Let a_1, a_2, a_3 ..... be in GP whose 'first term' a_1 is a and the common ratio is $\boldsymbol{r}$

then the second term $a_2 = ar = ar^{(2-1)}$

the third term $a_3 = a_2 \times r = (ar) \times r = ar^2 = ar^{(3-1)}$

the fourth term $a_4 = a_3 \times r = ar^2 \times r = ar^3 = ar^{(4-1)}$

....

....

Looking at the pattern we can say that nth term $a_n = ar^{n-1}$

So nth term an of a GP with first term 'a' and common ratio 'r' is given by $a_n = ar^{n-1}$.

Let us consider some examples

Example-20. Find the 20th and n^{th} term of the GP.

$$\frac{5}{2}, \frac{5}{4}, \frac{5}{8} ....$$

Solution : Here $a = \frac{5}{2}$ and $r = \dfrac{\frac{5}{4}}{\frac{5}{2}} = \frac{1}{2}$

Then $$a_{20} = ar^{20-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{19} = \frac{5}{2^{10}}$$

and $$a_n = ar^{n-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{5}{2^n}$$

Example-21. Which term of the GP : 2, $2\sqrt{2}$, 4 ..... is 128 ?

Solution : Here $a = 2$ $r = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Let 128 be the n^{th} term of the GP.

Then $a_n = ar^{n-1} = 128$

$$2.(\sqrt{2})^{n-1} = 128$$

$$(\sqrt{2})^{n-1} = 64$$

$$(2)^{\frac{n-1}{2}} = 2^6$$

$$\Rightarrow \qquad \frac{n-1}{2} = 6$$

$$\therefore \qquad n = 13.$$

Hence 128 is the 13^{th} term of the GP.

Example-22. In a GP the 3^{rd} term is 24 and 65^{th} term is 192. Find the 10^{th} term.

Solution : Here $\quad a_3 = ar^2 = 24 \qquad ...(1)$

$\quad a_6 = ar^5 = 195 \qquad ...(2)$

Dividing (2) by (1) we get $\dfrac{ar^5}{ar^2} = \dfrac{195}{24}$

$$\Rightarrow \qquad r^3 = 8 = 2^3$$

$$\Rightarrow \qquad r = 2$$

Substituting $r = 2$ in (1) we get $a = 6$.

$\therefore \; a_{10} = ar^9 = 6(2)^9 = 3072.$

EXERCISE-6.5

1. For each geometric progression find the common ratio 'r', and then find a_n

   (i) $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}$......... (ii) $2, -6, 18, -54$

   (iii) $-1, -3, -9, -18$ .... (iv) $5, 2, \frac{4}{5}, \frac{8}{25}$.........

2. Find the 10^{th} and n^{th} term of GP. : 5, 25, 125, ...

3. Find the indicated term of each geometric Progression

   (i) $a_1 = 9;\; r = \frac{1}{3};$ find a_7 (ii) $a_1 = -12;\; r = \frac{1}{3};$ find a_6

4. Which term of the GP.

   (i) 2, 8, 32, ... is 512 ? (ii) $\sqrt{3}, 3, 3\sqrt{3}$ ...... is 729 ?

   (iii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}$..... is $\frac{1}{2187}$?

5. Find the 12^{th} term of a GP. whose 8^{th} term is 192 and the common ratio is 2.

6. The 4th term of a geometric progression is $\frac{2}{3}$ and the seventh term is $\frac{16}{81}$. Find the geometric series.

7. If the geometric progressions 162, 54, 18 ..... and $\frac{2}{81}, \frac{2}{27}, \frac{2}{9}$.... have their n^{th} term equal. Find the value of n.

Optional Exercise

[This exercise is not meant for examination]

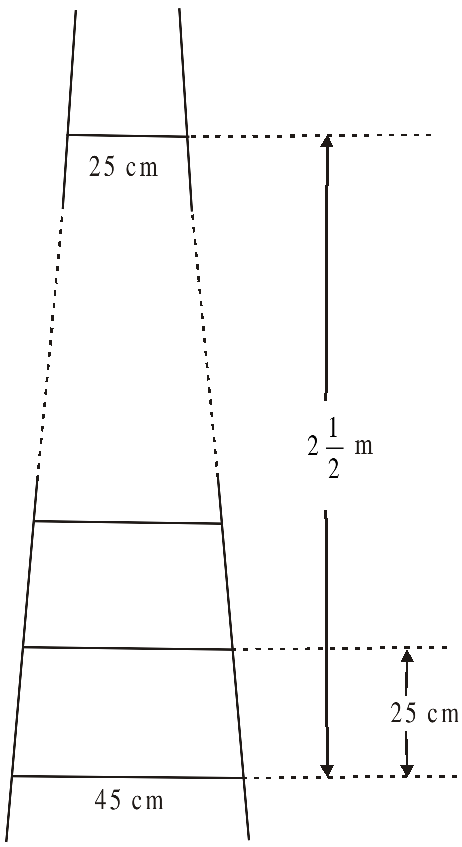
1. Which term of the AP : 121, 117, 113, . . ., is the first negative term?

   [**Hint :** Find n for $a_n < 0$]

2. The sum of the third and the seventh terms of an AP is 6 and their product is 8. Find the sum of first sixteen terms of the AP.

3. A ladder has rungs 25 cm apart. The rungs decrease uniformly in length from 45 cm at the bottom to 25 cm at the top. If the top and the bottom rungs are $2\frac{1}{2}$ m apart, what is the length of the wood required for the rungs?

   [**Hint :** Number of rungs = $\frac{250}{25} + 1$]



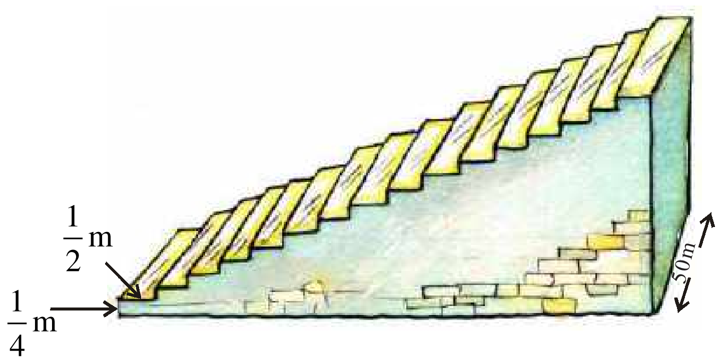
4. The houses of a row are numbered consecutively from 1 to 49. Show that there is a value of x such that the sum of the numbers of the houses preceding the house numbered x is equal to the sum of the numbers of the houses following it. And find this value of x.

   [**Hint :** $S_{x-1} = S_{49} - S_x$]

5. A small terrace at a football ground comprises of 15 steps each of which is 50 m long and built of solid concrete.

Each step has a rise of $\frac{1}{4}$ m and a tread of $\frac{1}{2}$ m. (see Fig. 5.8). Calculate the total volume of concrete required to build the terrace.

[**Hint :** Volume of concrete required to build the first step $= \frac{1}{4}\times\frac{1}{2}\times 50\,\text{m}^3$]



6. 150 workers ware engaged to finish a piece of work in a certain number of days. Four workers dropped from the work in the second day. Four workers dropped in third day and so on. It took 8 more days to finish the work. Find the number of days in which the was and completed.

[let the no.of days to finish the work is 'x' then

$$150x = \frac{x+8}{2}[2\times 150 + (x+8-1)(-4)]$$

[Ans. $x = 17 \Rightarrow x + 8 = 17 + 8 = 25$]

7. A machine costs ₹ 5,00,000. If the value depreciates 15% in the first year, $13\frac{1}{2}$ % in the second year, 12% in the third year and so on. What will be its value at the end of 10 years, when all the percentages will be applied to the original cost?

[Total depreciation = $15 + 13\frac{1}{2} + 12 + \ldots .10$ terms.

$$S_n = \frac{10}{2}[30-13.5] = 82.5\%$$

$\therefore$ after 10 year original cost $= 100 - 82.5 = 17.5$ i.e., 17.5% of 5,00,000

What We Have Discussed

In this chapter, you have studied the following points :

1. An **arithmetic progression** (AP) is a list of numbers in which each term is obtained by adding a fixed number d to the preceding term, except the first term. The fixed number d is called the **common difference**.

   The terms of AP are $a, a + d, a + 2d, a + 3d, \ldots$

2. A given list of numbers $a_1, a_2, a_3, \ldots$ is an AP, if the differences $a_2 - a_1, a_3 - a_2, a_4 - a_3, \ldots$, give the same value, i.e., if $a_{k+1} - a_k$ is the same for different values of k.

3. In an AP with first term a and common difference d, the nth term (or the general term) is given by $a_n = a + (n - 1) d$.

4. The sum of the first n terms of an AP is given by :

   $$S = \frac{n}{2}[2a + (n-1)d]$$

5. If l is the last term of the finite AP, say the nth term, then the sum of all terms of the AP is given by :

   $$S = \frac{n}{2}(a + l).$$

6. A Geometric Progression (GP) is a list of numbers in which each term is obtained by multiplying preceeding term with a fixed number 'r' except first term. This fixed number is called common ratio 'r'.

   The general form of GP is $a, ar, ar^2, ar^3 \ldots.$

7. If the first term and common ratio of a GP are a, r respectively then nth term $a_n = ar^{n-1}$.

CHAPTER

7

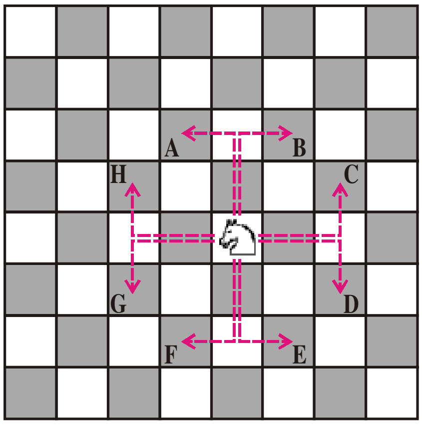
Coordinate Geometry

7.1 Introduction

You know that in chess, the Knight moves in 'L' shape or two and a half steps (see figure). It can jump over other pieces too. A Bishop moves diagonally, as many steps as are free in front of it.

Find out how other pieces move. Also locate Knight, Bishop and other pieces on the board and see how they move.

Consider that the Knight is at the origin (0, 0). It can move in 4 directions as shown by dotted lines in the figure. Find the coordinates of its position after the various moves shown in the figure.



Do This

i. From the figure write coordinates of the points A, B, C, D, E, F, G, H.

ii. Find the distance covered by the Knight in each of its 8 moves i.e. find the distance of A, B, C, D, E, F, G and H from the origin.

iii. What is the distance between two points H and C? and also find the distance between two points A and B

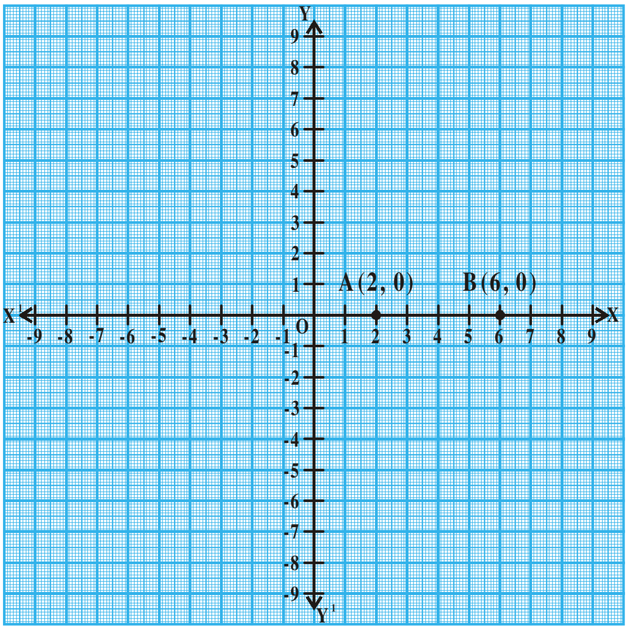
7.2 Distance Between Two Points

The two points (2, 0) and (6, 0) lie on the X-axis as shown in figure.

It is easy to see that the distance between two points A and B as 4 units.

We can say the distance between points lying on X-aixs is the difference between the x-coordinates.

What is the distance between (–2, 0) and (–6, 0)?



The difference in the value of x-coordinates is

$(-6) - (-2) = -4$ (Negative)

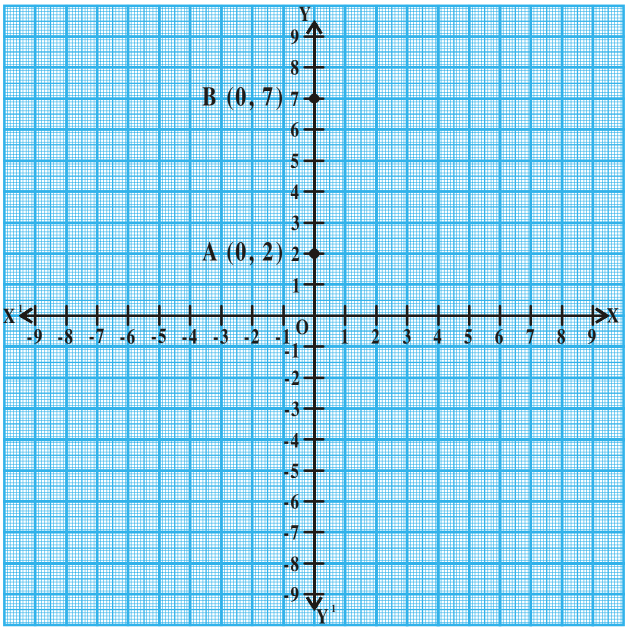
We never say the distance in negative values.

So, we calculate that absolute value of the distance.

Therefore, the distance

$= |(-6) - (-2)| = |-4| = 4$ units.

So, in general for the points $A(x_1, 0)$, $B(x_2, 0)$ on the X-axis, the distance between A and B is $|x_2 - x_1|$



Similarly, if two points lie on Y-axis, then the distance between the points A and B would be the difference between their y coordinates of the points.

The distance between two points $(0, y_1)$ $(0, y_2)$ would be $|y_2 - y_1|$.

For example, Let the points be A(0, 2) and B(0, 7)

Then, the distance between A and B is $|7 - 2| = 5$ units.

Do This

1. Where do these following points lie (–4, 0), (2, 0), (6, 0), (–8, 0).
2. What is the distance between points (–4, 0) and (6, 0)?

Try this

1. Where do these following points lie (0, –3), (0, –8), (0, 6), (0, 4)
2. What is the distance between (0, –3), (0, –8) and justify that the distance between two points on Y-axis is $|y_2 - y_1|$.

Think - Discuss

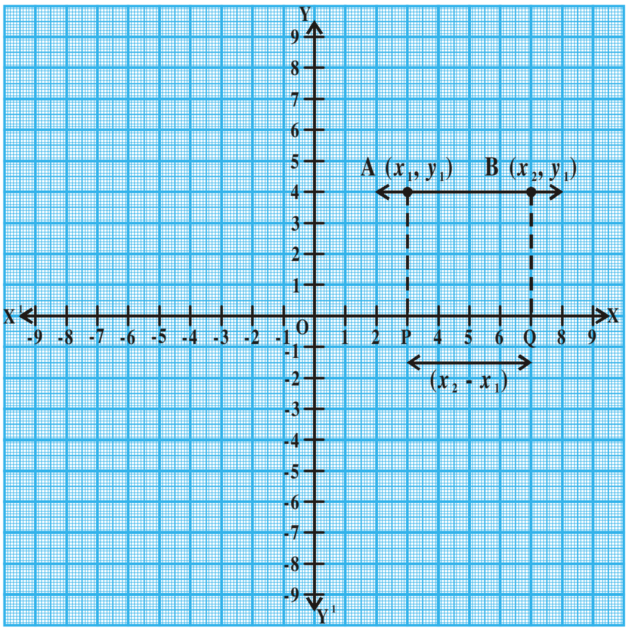
How will you find the distance between two points in which x or y coordinates are same but not zero?

7.3 Distance Between Two Points on a Line Parallel to the Coordinate Axes.

Consider the points A(x_1, y_1) and B(x_2, y_1). Since the y-coordinates are equal, points lie on a line, parallel to X-axis.

AP and BQ are drawn perpendicular to X-axis.

Observe the figure. The distance between two points A and B is equal to the distance between P and Q.



Therefore,

Distance AB = Distance

PQ = $|x_2 - x_1|$ (i.e., The difference between x coordinates)

Similarly, line joining two points A(x_1, y_1) and B(x_1, y_2) parallel to Y-axis, then the distance between these two points is $|y_2 - y_1|$ (i.e. the difference between y coordinates)

Example-1. What is the distance between A (4,0) and B (8, 0).

Solution : The difference in the x coordinates is $|x_2 - x_1| = |8 - 4| = 4$ units.

Example-2. A and B are two points given by (8, 3), (–4, 3). Find the distance between A and B.

Solution : Here x_1 and x_2 are lying in two different quadrants and y-coordinate are equal.

Distance AB $= |x_2 - x_1| = |-4 - 8| = |-12| = 12$ units

i. (3, 8), (6, 8) ii. (–4,–3), (–8,–3)

iii. (3, 4), (3, 8) (iv) (–5, –8), (–5, –12)

Let A and B denote the points (4, 0) and (0, 3) and 'O' be the origin.

The ΔAOB is a right angle triangle.

From the figure

OA = 4 units (x-coordinate)

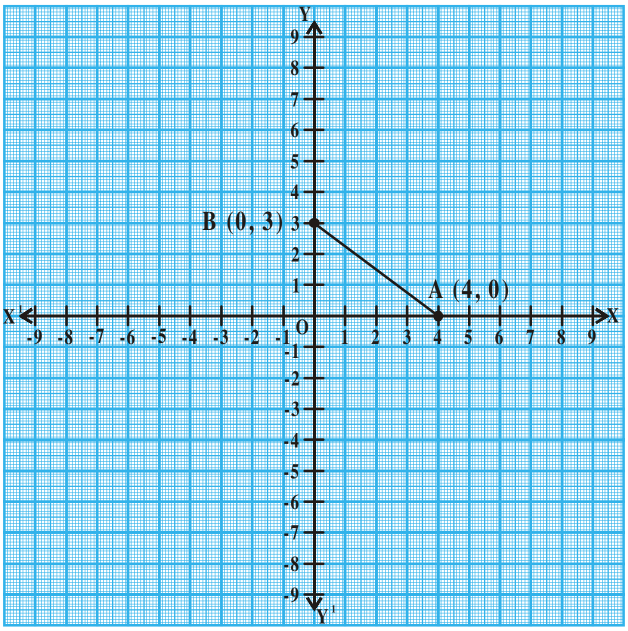
OB = 3 units (y-coordinate)

Then distance AB = ?

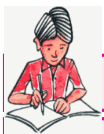
Hence, by using pythagoran theorem

$AB^2 = AO^2 + OB^2$

$AB^2 = 4^2 + 3^2$



$AB = \sqrt{16+9} = \sqrt{25} = 5$ units $\Rightarrow$ is the distance between A and B.



Do This

Find the distance between the following points (i) A = (2, 0) and B(0, 4) (ii) P(0, 5) and Q(12, 0)

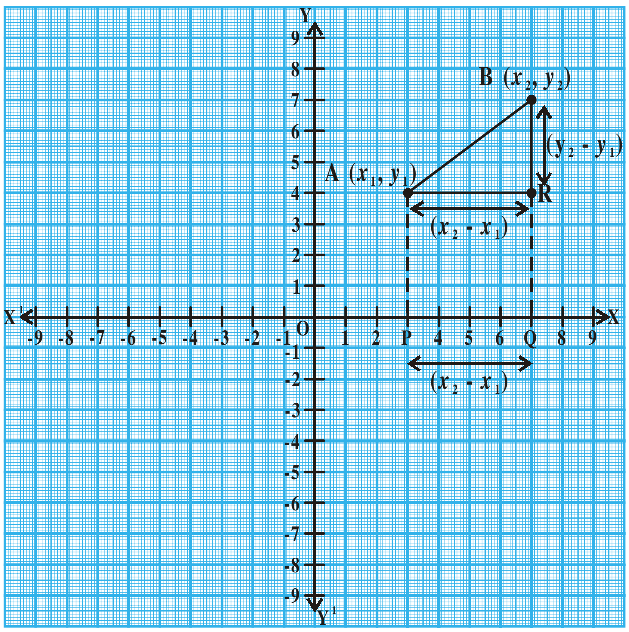
Try This

Find the distance between points 'O' (origin) and 'A' (7, 4).

Think - Discuss

1. Ramu says the distance of a point $P(x_1, y_1)$ from the origin O(0, 0) is $\sqrt{x^2 + y^2}$. Do you agree with Ramu or not? Why?
2. Ramu also writes the distance formulas as $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ (why?)

7.4 Distance Between Any Two Points on a line in the X-Y plane



Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points (on a line) in a plane as shown in figure.

Draw AP and BQ perpendiculars to X-axis

Draw a perpendicular ARfrom the point A on BQ to meet at the point R.

Then $OP = x_1$, $OQ = x_2$

So $PQ = OQ - OP = x_2 - x_1$

Observe the shape of APQR. It is a rectangle.

So $PQ = AR = x_2 - x_1$.

Also $QB = y_2$, $QR = y_1$,

So $BR = QB - QR = y_2 - y_1$

from ΔARB (right angle triangle)

$$AB^2 = AR^2 + RB^2 \qquad \text{(By Pythagoras theorem)}$$

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

i.e., $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Hence, the distance between the points A and B is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

this is called the distance formula.

Example-3. Let's find the distance between two points A(4, 2) and B(8, 6)

Solution : Compare these points with $(x_1, y_1), (x_2, y_2)$

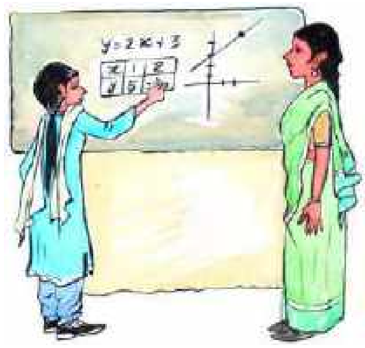
$x_1 = 4,\ x_2 = 8,\ y_1 = 3,\ y_2 = 6$

Using distance formula

$$\text{distance AB} = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(8-4)^2 + (6-3)^2} = \sqrt{4^2 + 3^2}$$

$$= \sqrt{16+9} = \sqrt{25} = 5 \text{ units.}$$



Do This

Find the distance between the following pairs of points

(i) (7, 8) and (–2, 3) (ii) (–8, 6) and (2, 0)

Try This

Find the distance between A(1, –3) and B(–4, 4) and rounded to are decimal

Think - Discuss

Sridhar calculated the distance between T(5, 2) and R(–4, –1) to the nearest tenth is 9.5 units.

Now you find the distance between P (4, 1) and Q (-5, -2). Do you get the same answer that sridhar got? Why?

Let us see some examples

Example-4. Show that the points A (4, 2), B (7, 5) and C (9, 7) are three points lie on a same line.

Solution : Now, we find the distances AB, BC, AC

By using distance formula $= d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

So, $d = AB = \sqrt{(7-4)^2 + (5-2)^2} = \sqrt{3^2 + 3^2} = \sqrt{9+9} = \sqrt{18}$

$= \sqrt{9 \times 2} = 3\sqrt{2}$ units.

$BC = \sqrt{(9-7)^2 + (7-5)^2} = \sqrt{2^2 + 2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ units

$AC = \sqrt{(9-4)^2 + (7-2)^2} = \sqrt{5^2 + 5^2} = \sqrt{25+25} = \sqrt{50}$

$= \sqrt{25 \times 2} = 5\sqrt{2}$ units.

Now AB + BC = $3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2} = AC$. Therefore, that the three points (4, 2), (7, 5) and (9, 7) lie on a straight line. (Points that lie on the same line are called collinear points).

Example-5. Are the points (3, 2), (–2, –3) and (2, 3) form a triangle?

Solution : Let us apply the distance formula to find the distances PQ, QR and PR, where P(3, 2), Q(–2, –3) and R(2, 3) are the given points. We have

$PQ = \sqrt{(-2-3)^2 + (-3-2)^2} = \sqrt{(-5)^2 + (-5)^2} = \sqrt{25+25} = \sqrt{50} = 7.07$ units (approx)

$QR = \sqrt{(2-(-2))^2 + (3-(-3))^2} = \sqrt{(4)^2 + (6)^2} = \sqrt{52} = 7.21$ units (approx)

$PR = \sqrt{(2-3)^2 + (3-2)^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2} = 1.41$ units (approx)

Since the sum of any two of these distances is greater than the third distance, therefore, the points P, Q and R form a triangle and all the sides of triangle is unequal.

Example-6. Show that the points (1, 7), (4, 2), (–1, –1) and (–4, 4) are the vertices of a square.

Solution : Let A(1, 7), B(4, 2), C(–1, –1)and D(–4, 4) be the given points.

One way of showing that ABCD is a square is to use the property that all its sides should be equal and both its digonals should also be equal. Now

So sides are $AB = d = \sqrt{(1-4)^2 + (7-2)^2} = \sqrt{9+25} = \sqrt{34}$ units

$BC = \sqrt{(4+1)^2 + (2+1)^2} = \sqrt{25+9} = \sqrt{34}$ units

$CD = \sqrt{(-1+4)^2 + (-1-4)^2} = \sqrt{9+25} = \sqrt{34}$ units

$$DA = \sqrt{(-4-1)^2 + (4-7)^2} = \sqrt{25+9} = \sqrt{34} \text{ units}$$

and digonal are $AC = \sqrt{(1+1)^2 + (7+1)^2} = \sqrt{4+64} = \sqrt{68}$ units

$$BD = \sqrt{(4+4)^2 + (2-4)^2} = \sqrt{64+4} = \sqrt{68} \text{ units}$$

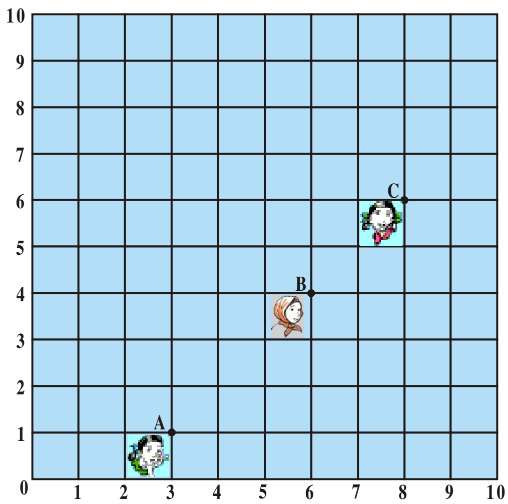
Since $AB = BC = CD = DA$ and $AC = BD$. So all the four sides of the quadrilateral ABCD are equal and its diagonals AC and BD are also equal. Therefore, ABCD is square.

Example-7. Figure shows the arrangement of desks in a class room.

Madhuri, Meena, Pallavi are seated at A(3, 1), B(6, 4) and C(8, 6) respectively.

Do you think they are seated in a line ?

Give reasons for your answer.



Solution : Using the distance formula, we have

$$AB = \sqrt{(6-3)^2 + (4-1)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

units

$$BC = \sqrt{(18-6)^2 + (6-4)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

units

$$AC = \sqrt{(8-3)^2 + (6-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2} \text{ units}$$

Since, $AB + BC = 3\sqrt{2} + 2\sqrt{2} + 5\sqrt{2} = AC$, we can say that the points A, B and C are collinear. Therefore, they are seated in a line.

Example-8. Find a relation between x and y such that the point (x, y) is equidistant from the points (7, 1) and (3, 5).

Solution : Let P(x, y) be equidistant from the points A(7, 1) and B(3, 5).

Given that AP = BP. So, $AP^2 = BP^2$

i.e., $(x-7)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$

i.e., $(x^2 - 14x + 49) + (y^2 - 2y + 1) = (x^2 - 6x + 9) + (y^2 - 10y + 25)$

$(x_2 + y^2 - 14x - 2y + 50) - (x^2 + y^2 - 6x - 10y + 34) = 0$

$-8x + 8y = -16$

i.e., $x - y = 2$ which is the required relation.

Example-9. Find a point on the y-axis which is equidistant from both the points A(6, 5) and B(– 4, 3).

Solution : We know that a point on the Y-axis is of the form (0, y). So, let the point P(0, y) be equidistant from A and B. Then

$$PA = \sqrt{(6-0)^2 + (5-y)^2}$$

$$PB = \sqrt{(-4-0)^2 + (3-4)^2}$$

$$PA^2 = PB^2$$

So, $(6 - 0)^2 + (5 - y)^2 = (-4 - 0)^2 + (3 - y)^2$

i.e., $36 + 25 + y^2 - 10y = 16 + 9 + y^2 - 6y$

i.e., $4y = 36$

i.e., $y = 9$

So, the required point is (0, 9).

Let us check our solution : $AP = \sqrt{(6-0)^2 + (5-9)^2} = \sqrt{36+16} = \sqrt{52}$

$$BP = \sqrt{(-4-0)^2 + (3-9)^2} = \sqrt{16+36} = \sqrt{52}$$

So (0, 9) is equidistant from (6, 5) and (4, 3).

Exercise 7.1

1. Find the distance between the following pairs of points

   (i) (2, 3) and (4, 1)   (ii) (–5, 7) and (–1, 3)

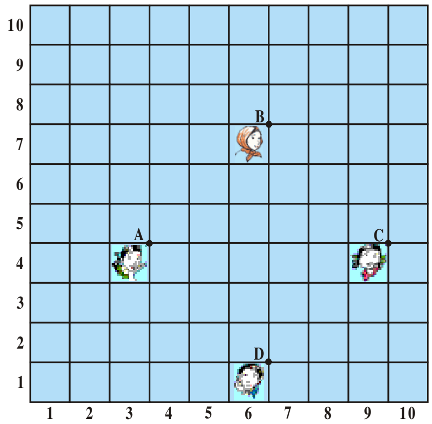
   (iii) (–2, –3) and (3, 2)   (iv) (a, b) and ($-a$, $-b$)

2. Find the distance between the points (0, 0) and (36, 15).

3. Verify that the points (1, 5), (2, 3) and (–2, –1) are collinear or not.

4. Check whether (5, –2), (6, 4) and (7, –2) are the vertices of an isosceles triangle.

5. In a class room, 4 friends are seated at the points A, B, C and D as shown in Figure. Jarina and Phani walk into the class and after observing for a few minutes Jarina asks phani "Don't you think ABCD is a square?" Phani disagrees.
Using distance formula, find which of them is correct. Why ?



6. Show that the following points form a equilateral triangle A(a, 0), B($-a$, 0), C(0, $a\sqrt{3}$)

7. Prove that the points (–7, –3), (5, 10), (15, 8) and (3, –5) taken in order are the corners of a parallelogram. And find its area.

(Hint : Area of rhombus = $\frac{1}{2}\times$ product of its diagonals)

8. Show that the points (–4, –7), (–1, 2), (8, 5) and (5, –4) taken in order are the vertices of a rhombus.

9. Name the type of quadrilateral formed, if any, by the following points, and give reasons for your answer.

(i) (–1, –2), (1, 0), (–1, 2), (–3, 0) (ii) (–3, 5), (3, 1), (0, 3), (–1, –4)

(iii) (4, 5), (7, 6), (4, 3), (1, 2)

10. Find the point on the x-axis which is equidistant from (2, –5) and (–2, 9).

11. If the distance between two points (x, 7) and (1, 15) is 10, find the value of x.

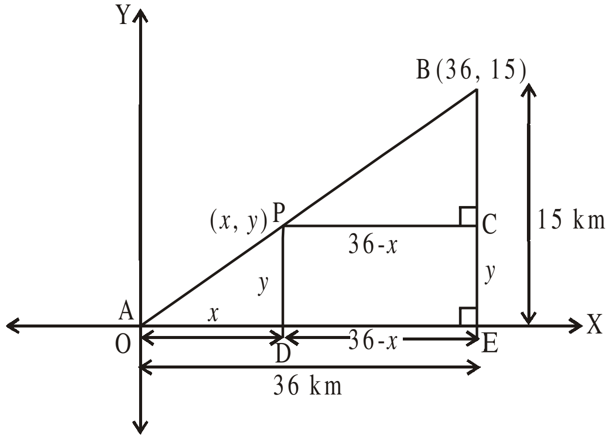
12. Find the values of y for which the distance between the points P(2, –3) and Q(10, y) is 10 units.

13. Find the radius of the circle whose centre is (3, 2) and passes through (–5, 6).

14. Can you draw a triangle with vertices (1, 5), (5, 8) and (13, 14) ? Give reason.

15. Find a relation between x and y such that the point (x, y) is equidistant from the points (–2, 8) and (–3, –5)

7.5 Section Formula



Suppose a telephone company wants to position a relay tower at P between A and B in such a way that the distance of the tower from B is twice its distance from A. If P lies on AB, it will divide AB in the ratio 1 : 2 (See figure). If we take A as the origin O, and 1 km as one unit on both the axis, the coordinates of B will be (36, 15). In order to know the position of the tower, we must know the coordinates of P. How do we find these coordinates?

Let the coordinates of P be (x, y). Draw perpendiculars from P and B to the x-axis, meeting it in D and E, respectively. Draw PC perpendicular to BE. Then, by the AA similarity criterion, studied earlier, ΔPOD and ΔBPC are similar.

Therefore, $\dfrac{OD}{PC} = \dfrac{OP}{PB} = \dfrac{1}{2}$ and $\dfrac{PD}{BC} = \dfrac{OP}{PB} = \dfrac{1}{2}$

So, $\dfrac{x}{36-x} = \dfrac{1}{2}$ $\qquad$ $\dfrac{y}{15-y} = \dfrac{1}{2}$.

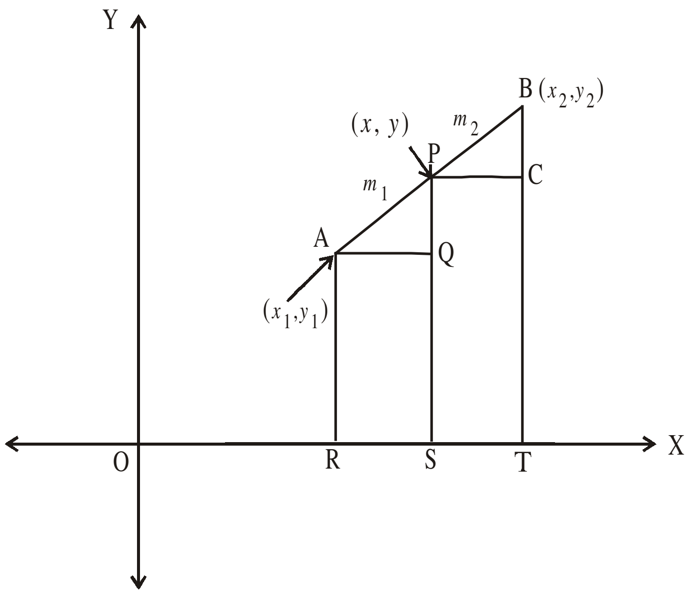
$2x = (36 - x)$ $\qquad$ $2y = 15 - y$

$3x = 36$ $\qquad$ $3y = 15$

$x = 12$ $\qquad$ $y = 5$

These equations give $x = 12$ and $y = 5$.

You can check that P(12, 5) meets the condition that OP : PB = 1 : 2.



Consider any two points A(x_1, y_1) and B(x_2, y_2) and assume that P (x, y) divides AB internally in the ratio $m_1 : m_2$,

i.e., $\dfrac{AP}{PB} = \dfrac{m_1}{m_2}$ ..... (1)

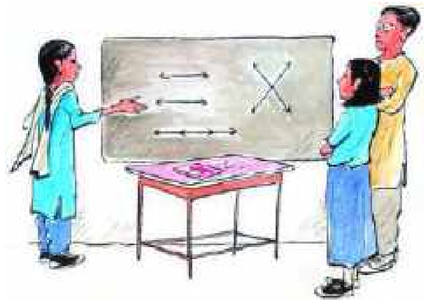
(See figure).

Draw AR, PS and BT perpendicular to the x-axis. Draw

AQ and PC parallel to the X-axis. Then, by the AA similarity criterion,

$$\Delta PAQ \sim \Delta BPC$$

Therefore, $\dfrac{AP}{PB} = \dfrac{AQ}{PC} = \dfrac{PQ}{BC}$ .....(2)



Now, $AQ = RS = OS - OR = x - x_1$

$PC = ST = OT - OS = x_2 - x$

$PQ = PS - QS = PS - AR = y - y_1$

$BC = BT - CT = BT - PS = y_2 - y$

Substituting these values in (1), we get

$$\frac{m_1}{m_2} = \frac{x - x_1}{x_2 - x} = \frac{y - y_1}{y_2 - y} \quad \left[\because \frac{AP}{PB} = \frac{m_1}{m_2} \text{ from}(1)\right]$$

Taking $\dfrac{m_1}{m_2} = \dfrac{x - x_1}{x_2 - x}$, we get $x = \dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}$

Similarly, taking $\dfrac{m_1}{m_2} = \dfrac{y - y_1}{y_2 - y}$, we get $y = \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}$

So, the coordinates of the point P(x, y) which divides the line segment joining the points A(x_1, y_1) and B(x_2, y_2), **internally** in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right) \quad .....(3)$$

This is known as the **section formula**.

This can also be derived by drawing perpendiculars from A, P and B on the Y-axis and proceeding as above.

If the ratio in which P divides AB is $k : 1$, then the coordinates of the point P are

$$\left(\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1}\right).$$

Special Case : The mid-point of a line segment divides the line segment in the ratio 1 : 1. Therefore, the coordinates of the mid-point P of the join of the points A(x_1, y_1) and B(x_2, y_2) are

$$\left(\frac{1.x_1+1.x_2}{1+1}, \frac{1.y_1+1.y_2}{1+1}\right) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right).$$

Let us solve few examples based on the section formula.

Example-10. Find the coordinates of the point which divides the line segement joining the points (4, –3) and (8, 5) in the ratio 3 : 1 internally.

Solution : Let P(x, y) be the required point. Using the section formuls

$$P(x, y) = \left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}\right), \text{ we get}$$

$$x = \frac{3(8)+1(4)}{3+1} = \frac{24+4}{4} = \frac{28}{4} = 7,$$

$$y = \frac{3(5)+1(-3)}{3+1} = \frac{15-3}{4} = \frac{12}{4} = 3$$

P(x, y) = (7, 3) is the required point.

Example-11. Find the mid point of the line segment joining the points (3, 0) and (-1, 4)

Solution : The mid point M(x, y) of the line segment joining the points (x_1, y_1) and (x_2, y_2).

$$M(x, y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$\therefore$ The mid point of the line segment joining the points (3, 0) and (–1, 4) is

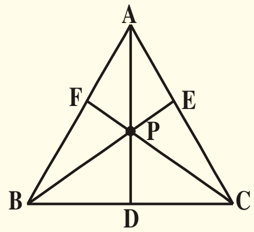
$$M(x, y) = \left(\frac{3+(-1)}{2}, \frac{0+4}{2}\right) = \left(\frac{2}{2}, \frac{4}{2}\right) = (1, 2).$$

Do This

1 Find the point which divides the line segment joining the points (3, 5) and (8, 10) internally in the ratio 2 : 3

2. Find the midpoint of the line segement joining the points (2, 7) and (12, –7).

Try This

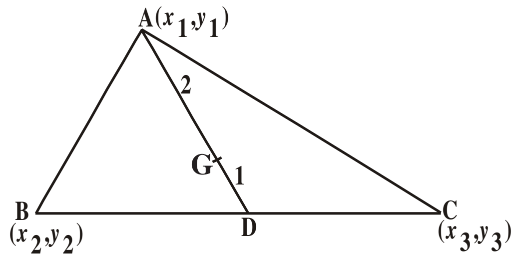


Let A(4, 2), B(6, 5) and C(1, 4) be the vertices of ΔABC

1. The median from A meets BC at D. Find the coordinates of the point D.
2. Find the coordinates of the point P on AD such that AP : PD = 2 : 1.
3. Find the coordinates of points Q and R on medians BE and CF.
4. Find the points which divide the line segment BE in the ratio 2 : 1 and also that divide the line segment CF in the ratio 2 : 1.
5. What do you observe ?

Justify the point that divides each median in the ratio 2 : 1 is the centriod of a triangle.

7.6 Centroid of a Triangle



The centroid of a triangle is the point of intersection of its medians.

Let A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) be the vertices of the triangle ABC.

Let AD be the median bisecting its base. Then,

$$D = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

Now the point G on AD which divides it internally in the ratio 2 : 1, is the centroid. If (x, y) are the coordinates of G, then

$$G(x, y) = \left[\frac{2\left(\frac{x_2 + x_3}{2}\right) + 1(x_1)}{2+1}, \frac{2\left(\frac{y_2 + y_3}{2}\right) + 1(y_1)}{2+1}\right]$$

$$= \left[\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right]$$

Hence, the coordinates of the centroid are given by

$$\left[\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right].$$

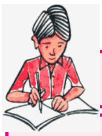
Example-12. Find the centroid of the triangle whose vertices are (3, –5), (–7, 4), (10, –2) respectively.

Solution : The coordinates of the centroid are

$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

$$\left(\frac{3+(-7)+10}{3}, \frac{(-5)+4+(-2)}{3}\right) = (2, -1)$$

$\therefore$ the centroid is (2, –1).



Do This

Find the centroid of the triangle whose vertices are (–4, 6), (2, –2) and (2, 5) respectively.

Try This

The points (2, 3), (x, y), (3, –2) are vertices of a triangle. If the centroid of this triangle is again find (x, y).

Example-13. In what ratio does the point (– 4, 6) divide the line segment joining the points A(– 6, 10) and B(3, – 8)?

Solution : Let (– 4, 6) divide AB internally in the ratio $m_1 : m_2$. Using the section formula, we get

$$(-4, 6) = \left(\frac{3m_1 - 6m_2}{m_1 + m_2}, \frac{-8m_1 + 10m_2}{m_1 + m_2}\right) \qquad .....(1)$$

We know that if $(x, y) = (a, b)$ then $x = a$ and $y = b$.

So, $\quad -4 = \dfrac{3m_1 - 6m_2}{m_1 + m_2}$ and $6 = \dfrac{-8m_1 + 10m_2}{m_1 + m_2}$

Now, $-4 = \frac{3m_1 - 6m_2}{m_1 + m_2}$ gives us

$$-4m_1 - 4m_2 = 3m_1 - 6m_2$$

i.e., $7m_1 = 2m_2$

$$\frac{m_1}{m_2} = \frac{2}{7}$$

i.e., $m_1 : m_2 = 2 : 7$

We should verify that the ratio satisfies the y-coordinate also.

Now, $\frac{-8m_1 + 10m_2}{m_1 + m_2} = \frac{-8\frac{m_1}{m_2} + 10}{\frac{m_1}{m_2} + 1}$ (Dividing throughout by m_2)

$$= \frac{-8 \times \frac{2}{7} + 10}{\frac{2}{7} + 1} = \frac{\frac{-16}{7} + 10}{\frac{9}{7}} = \frac{-16 + 70}{9} = \frac{54}{9} = 6$$

Therefore, the point (–4, 6) divides the line segment joining the points A(–6, 10) and B (3, –8) in the ratio 2 : 7.

Think - Discuss

The line joining points A(6, 9) and B(–6, –4) are given

(a) In which ratio does origin divide $\overline{AB}$? And what it is called for $\overline{AB}$?

(b) In which ratio does the point P(2, 3) divide $\overline{AB}$?

(c) In which ratio does the point Q(–2, –3) divide $\overline{AB}$?

(d) In how many equal parts is $\overline{AB}$ divided by P and Q?

(e) What do we call P and Q for $\overline{AB}$?

7.7 Trisectional Points of a Line

Example-14. Find the coordinates of the points of trisection (*The points which divide a line segment into 3 equal parts are said to be the sectional points*) of the line segment joining the points A(2,–2) and B(–7, 4).

Solution : Let P and Q be the points of trisection of AB i.e., AP=PQ=QB (see figure below).

Therefore, P divides AB internally in the ratio 1 : 2.

Therefore, the coordinates of P are (by applying the section formula)

$$P(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right)$$

$$\left(\frac{1(-7)+2(2)}{1+2}, \frac{1(4)+2(-2)}{1+2}\right)$$

$$\text{i.e., } \left(\frac{-7+4}{3}, \frac{4-4}{3}\right) = \left(\frac{-3}{3}, \frac{0}{3}\right) = (-1, 0)$$

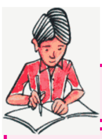
Now, Q also divides AB internally in the ratio 2:1.

So,the coordinates of Q are

$$= \left(\frac{2(-7)+1(2)}{2+1}, \frac{2(4)+1(-2)}{2+1}\right)$$

$$\text{i.e., } \left(\frac{-14+2}{3}, \frac{8-2}{3}\right) = \left(\frac{-12}{3}, \frac{6}{3}\right) = (-4, 2)$$

Therefore, the coordinates of the points of trisection of the line segment are P(-1, 0) and Q(-4, 2)



Do This

1. Find the trisectional points of line joining (2, 6) and (–4, 8).
2. Find the trisectional points of line joining (–3, –5) and (–6, –8).

Example-15. Find the ratio in which the y-axis divides the line segment joining the points (5, –6) and (1, –4). Also find the point of intersection.

Solution : Let the ratio be K : 1. Then by the section formula, the coordinates of the point which divides AB in the ratio K : 1 are

$$\left(\frac{K(-1)+1(5)}{K+1}, \frac{K(-4)+1(-6)}{K+1}\right)$$

i.e., $\left(\frac{-K+5}{K+1}, \frac{-4K-6}{K+1}\right)$

This point lies on the y-axis, and we know that on the y-axis the abscissa is 0.

Therefore, $\frac{-K+5}{K+1} = 0$

$-K + 5 = 0 \Rightarrow K = 5.$

So, the ratio is K : 1 = 5 : 1

Putting the value of K = 5, we get the point of intersection as

$$= \left(\frac{-5+5}{5+1}, \frac{-4(5)-6}{5+1}\right) = \left(0, \frac{-20-6}{6}\right) = \left(0, \frac{-26}{6}\right) = \left(0, \frac{-13}{3}\right)$$

Example-16. Show that the points A(7, 3), B(6, 1), C(8, 2) and D(9, 4) taken in that order are vertices of a parallelogram.

Solution : Let the points A(7, 3), B(6, 1), C(8, 2) and D(9, 4) are vertices of a parallelogram.

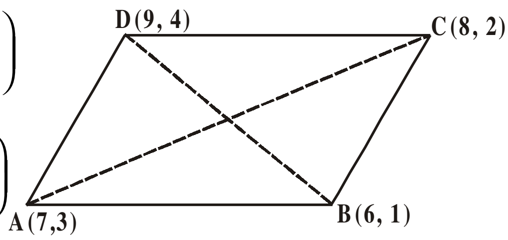
We know that the diagonals of a parallelogram bisect each other.

$\therefore$ So the midpoints of the diagonals AC and DB should be equal.

Now, we find the mid points of AC and DB by using $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ formula.

midpoint of $AC = \left(\frac{7+8}{2}, \frac{3+2}{2}\right) = \left(\frac{15}{2}, \frac{5}{2}\right)$

midpoint of $DB = \left(\frac{9+6}{2}, \frac{4+1}{2}\right) = \left(\frac{15}{2}, \frac{5}{2}\right)$



Hence, midpoint of AC = midpoint of DB.

Therefore, the points A, B, C, D are vertices of a parallelogram.

Example-17. If the points A(6, 1), B(8, 2), C(9, 4) and D(p, 3) are the vertices of a parallelogram, taken inorder, find the value of P.

Solution : We know that diagonals of parallelogram bisect each other.

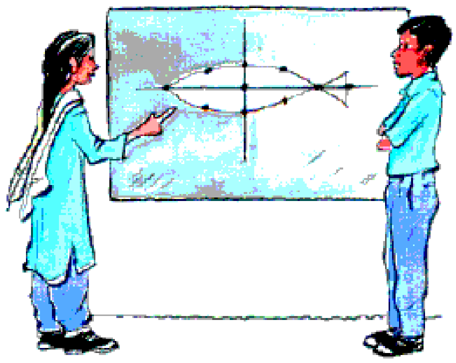
So, the coordinates of the midpoint of AC = Coordinates of the midpoint of BD.

i.e., $$\left(\frac{6+9}{2}, \frac{1+4}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

$$\left(\frac{15}{2}, \frac{5}{2}\right) = \left(\frac{8+p}{2}, \frac{5}{2}\right)$$

$$\frac{15}{2} = \frac{8+p}{2}$$

$$15 = 8 + p \Rightarrow p = 7.$$



Exercise - 7.2

1. Find the coordinates of the point which divides the join of (–1, 7) and (4, –3) in the ratio 2 : 3.
2. Find the coordinates of the points of trisection of the line segment joining (4, –1) and (–2, –3).
3. Find the ratio in which the line segment joining the points (–3, 10) and (6, –8)is divided by (–1, 6).
4. If (1, 2), (4, y), (x, 6) and (3, 5) are the vertices of a parallelogram taken in order, find x and y.
5. Find the coordinates of a point A, where AB is the diameter of a circle whose centre is (2, –3) and B is (1, 4).
6. If A and B are (–2, –2) and (2, –4) respectively. Find the coordinates of P such that $AP = \frac{3}{7}$ AB and P lies on the segment AB.
7. Find the coordinates of points which divide the line segment joining A(–4, 0) and B(0, 6) into four equal parts.

8. Find the coordinates of the points which divides the line segment joining A(–2, 2) and B(2, 8) into four equal parts.

9. Find the coordinates of the point which divides the line segment joining the points $(a+b, a-b)$ and $(a-b, a+b)$ in the ratio 3 : 2 internally.

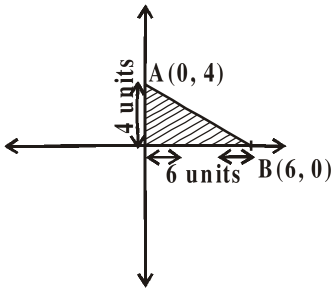
10. Find the coordinates of centroid of the following:

    i. (-1, 3), (6, -3) and (-3, 6) ii. (6, 2), (0, 0) and (4, -7)

    iii. (1, -1), (0, 6) and (-3, 0)

7.8 Area of the Triangle

Consider the points A(0, 4) and B(6, 0) which form a triangle with origin O on a plane as shown in figure.



What is the area of the ΔAOB?

ΔAOB is right angle triangle and the base is 6 units (i.e., x coordinate) and height is 4 units (i.e., y coordinate).

$$\therefore \text{Area of } \Delta AOB = \frac{1}{2}\times \text{base}\times \text{height}$$

$$= \frac{1}{2}\times 6\times 4 = 12 \text{ square units.}$$

Try This

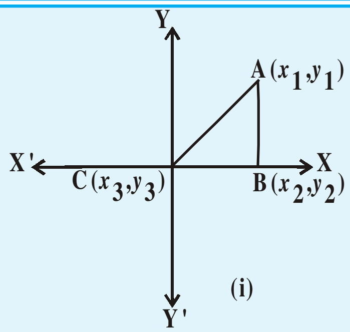
Take a point A on X-axis and B on Y-axis and find area of the triangle AOB. Discuss with your friends what did they do?

Think - Discuss

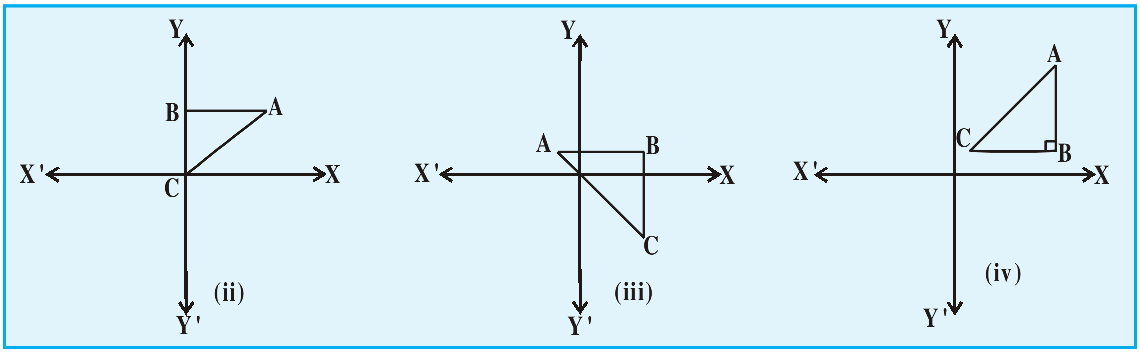
Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

Then find the area of the following triangles in a plane.

And discuss with your friends in groups about the area of that triangle.

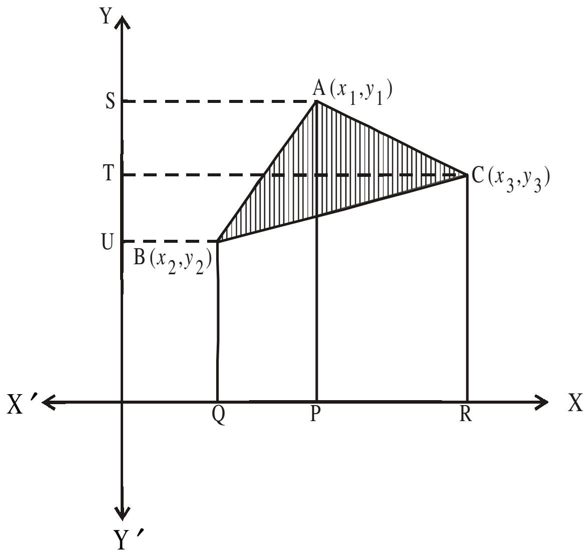


(i)



Area of the triangle

Let ABC be any triangle whose vertices are A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) .



Draw AP, BQ and CR perpendiculars from A, B and C respectively to the x-axis.

Clearly ABQP, APRC and BQRC are all trapezia as shown in figure.

Now from figure, it is clear that

Area of ΔABC = area of trapezium ABQP + area of trapezium APRC – area of trapezium BQRC

$\because$ Area of trapezium $= \frac{1}{2}$ (sum of the parallel sides) (distance between them)

$$\text{Area of } \Delta ABC = \frac{1}{2}(BQ+AP)QP + \frac{1}{2}(AP+CR)PR - \frac{1}{2}(BQ+CR)QR \qquad \text{.... (1)}$$

Here from the figure

$BQ = y_2, \ AP = y_1, \ QP = OP - OQ = x_1 - x_2$

$CR = y_3, \ PR = OR - OP = x_3 - x_1$

$QR = OR - OQ = x_3 - x_2$

Therefore, Area of ΔABC [from (1)]

$$= \frac{1}{2}\left|(y_2 + y_1)\,(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)\,(x_3 - x_1) - \frac{1}{2}(y_3 + y_3)\,(x_3 - x_2)\right|$$

$$= \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

Thus, the area of ΔABC is the numerical value of the expression

$$\frac{1}{2}\left|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right|$$

Let us try some examples.

Example-18. Find the area of a triangle whose vertices are (1, –1), (–4, 6) and (–3, –5).

Solution : The area of the triangle formed by the vertices A(1, –1), B(–4, 6) and C(–3, –5), by using the formula above

$$=\frac{1}{2}\left|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\right|$$

is given by

$$=\frac{1}{2}\left|1(6+5)+(-4)(-5+1)+(-3)(-1-6)\right|$$

$$=\frac{1}{2}\left|11+16+21\right|=24$$

So the area of the triangle is 24 square units.

Example-19. Find the area of a triangle formed by the points A(5, 2), B(4, 7) and C(7, –4).

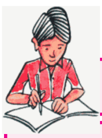
Solution : The area of the triangle formed by the vertices A(5, 2), B(4, 7) and C(7, –4) is given by

$$\frac{1}{2}\left|5(7+4)+4(-4-2)+7(2-7)\right|$$

$$=\frac{1}{2}\left|55-24-35\right|=\left|\frac{-4}{2}\right|=\left|-2\right|$$

Since area is a measure, which cannot be negative, we will take the numberical value of -2 or absolute value i.e., $|-2|=2$.

Therefore, the area of the triangle = 2 square units.



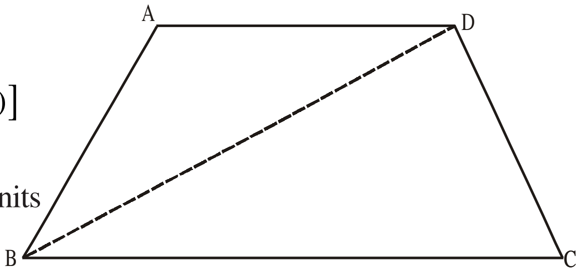
Do This

1. Find the area of the triangle whose vertices are
2. (5, 2) (3, –5) and (–5, –1)
3. (6, –6), (3, –7) and (3, 3)

Example-20. If A(–5, 7), B(–4,–5), C(–1, –6) and D(4,5) are the vertices of a quadrilateral. Then, find the area of the quadrilateral ABCD.

Solution : By joining B to D, you willl get two triangles ABD, and BCD.

The area of ΔABD



$$= \frac{1}{2}\left[-5(-5-5)+(-4)(5-7)+4(7+5)\right]$$

$$= \frac{1}{2}(50+8+48) = \frac{106}{2} = 53 \text{ square units}$$

Also, The area of ΔBCD $= \frac{1}{2}\left[-4(-6-5)-1(5+5)+4(-5+6)\right]$

$$= \frac{1}{2}\left[44-10+4\right] = 19 \text{ Square units}$$

Area of ΔABD + area of ΔBCD

So, the area of quadrilateral ABCD = 53+19 = 72 square units.

Try This

Find the area of the square formed by (0, –1), (2, 1) (0, 3) and (–2, 1) taken inorder are as vertices.

Think - Discuss

Find the area of the triangle formed by the following points

(i) (2, 0), (1, 2), (1, 6)

(ii) (3, 1), (5, 0), (1, 2)

(iii) (–1.5, 3), (6, 2), (–3, 4)

What do you observe?

Plot these points three different graphs. What do you observe?

Can we have a triangle is 0 square units ?

What does it mean?

7.8.1. Collinearity

Suppose the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are lying on a line. Then, they can not form a triangle. i.e. area of ΔABC is zero.

When the area of a triangle is zero then the three points said to be collinear points.

Example-21. The points (3, –2) (–2, 8) and (0, 4) are three points in a plane. Show that these points are collinear.

Solution : By using area of the triangle formula

$$\Delta = \frac{1}{2}\left|3(8-4)+(-2)(4-(-2))+0((-2)-8)\right|$$

$$= \frac{1}{2}\left|12-12\right| = 0$$

The area of the triangle is 0. Hence the three points are collinear or the lie on the same line.

Do This

Verify wheather the following points are

(i) (1, –1), (4, 1), (–2, –3)

(ii) (1, –1), (2, 3), (2, 0)

(iii) (1, –6), (3, –4), (4, –3)

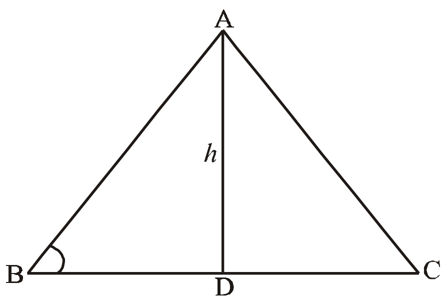
7.8.2. Area of a Triangle- 'Heron's Formula'

We know the formula for area of the triangle is $\frac{1}{2}\times\text{base}\times\text{height}$.

Any given triangle is may be a right angle triangle, equillateral triangle and isosceles triangle. Can we calculate the area of the triangle ?

If we know the base and height directly we apply the above formula to find the area of a triangle.

The height (h) is not known, how can we find its area?

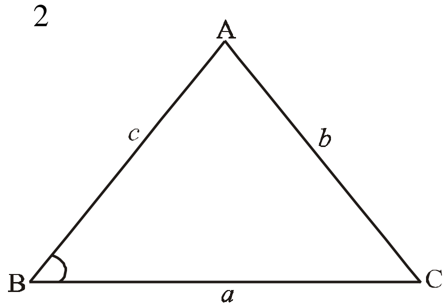


For this Heron, a Ancient Greek mathematician, derived a formula for a triangle whose lengths of sides are a, b and c.

$$A = \sqrt{S(S-a)(S-b)(S-c)}\,, \text{ where } \quad S = \frac{a+b+c}{2}$$

For example, we find the area of the triangle whose lengths of sides are 12m, 9m, 15m by using Heron's formula we get

$$A = \sqrt{S(S-a)(S-b)(S-c)}\,, \text{ where } \quad S = \frac{a+b+c}{2}$$



$$S = \frac{12+9+15}{2} = \frac{36}{2} = 18m$$

Then $S - a = 18 - 12 = 6$m

$S - b = 18 - 9 = 9$m

$S - c = 18 - 15 = 3$m

$$A = \sqrt{18(6)(9)(3)} = \sqrt{2916} = 54 \text{ square meter.}$$

Do This

(i) Find the area of the triangle whose lengths of sides are 15m, 17m, 21m (use Heron's Formula) and verify your answer by using the formula $A = \frac{1}{2}bh$.

(ii) Find the area of the triangle formed by the points (0, 0), (4, 0), (4, 3) by using Heron's formula.

Example-22. Find the value of 'b' for which the points are collinear.

Solution : Let given points A(1, 2), B(−1, b), C(−3, −4)

Then $x_1 = 1, y_1 = 2$; $\quad x_2 = -1, y_2 = b$; $\quad x_3 = -3, \ y_3 = -4$

We know, area of $\Delta ABC = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$

$$\therefore \ \frac{1}{2}\left|1(b+4) + (-1)(-4,-2) + (-3)(2-b)\right| = 0 \quad (\because \text{The given points are collinear})$$

$|b + 4 + 6 - 6 + 36| = 0$

$|4b + 4| = 0$

$4b + 4 = 0$

$\therefore b = -1$

Exercise - 7.3

1. Find the area of the triangle whose vertices are
   (i) (2, 3) (–1, 0), (2, –4) (ii) (–5, –1), (3, –5), (5, 2)
   (iii) (0, 0), (3, 0) and (0, 2)
2. Find the value of 'K' for which the points are collinear.
   (i) (7, –2) (5, 1) (3, K) (ii) (8, 1), (K, –4), (2, –5)
   (iii) (K, K) (2, 3) and (4, –1).
3. Find the area of the triangle formed by joining the mid-points of the sides of the triangle whose vertices are (0, –1), (2, 1) and (0, 3). Find the ratio of this area to the area of the given triangle.
4. Find the area of the quadrilateral whose vertices, taken inorder, are (–4, –2), (–3, –5), (3, –2) and (2, 3).
5. Find the area of the triangle formed by the points (8, –5), (–2, –7) and (5, 1) by using Herones formula.

7.9 Straight Lines

Bharadwaj and Meena are discussing to find solutions for a linear equation in two variable.

Bharadwaj : Can you find solutions for $2x + 3y = 12$

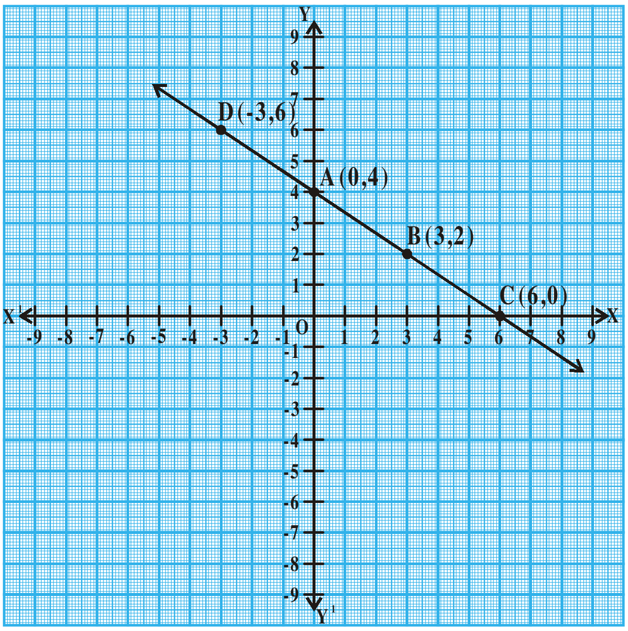
Meena : Yes, I have done this, see

x	0	3	6	–3
y	4	2	0	6

$$2x+3y = 12$$

$$3y = 12 - 2x$$

$$y = \frac{12-2x}{3}$$



Meena : Can you write these solutions in order pairs

Bharadwaj : Yes, (0, 4), (3, 2), (6, 0), (–3, 6)

Meena, can you plot these points on the coordinate plane.

Meena : I have done case like this case

Bharadwaj : What do you observe?

What does this line represent?

Meena : It is a straight line.

Bharadwaj : Can you identify some more points on this line?

Can you help Meena to find some more points on this line?

.................., ..., ..., ...

And In this line, what is $\overline{AB}$ called ?

$\overline{AB}$ is a line segment.

Do This

Plot these points on the coordinates axis and join Them:

1. A(1, 2), B(–3, 4), C(7, –1)
2. P(3, –5) Q(5, –1), R(2, 1), S(1, 2)

Which gives a straight line? Which is not? why?

Think - Discuss

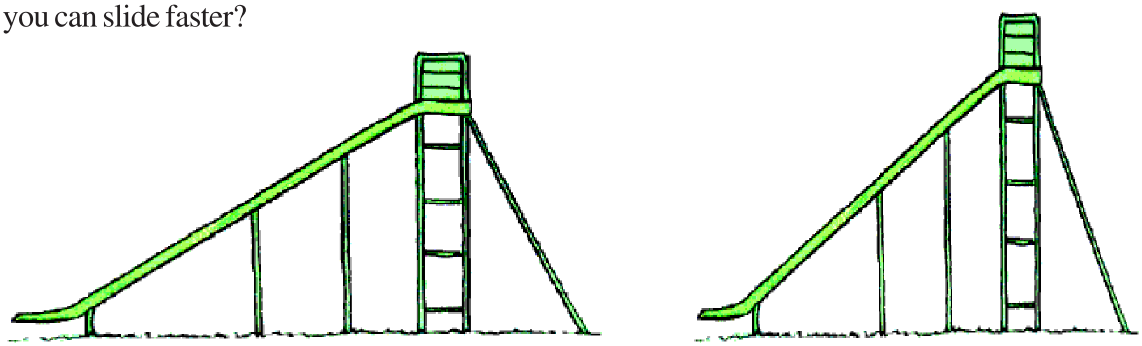
Is $y = x + 7$ represent a straight line? draw the line on the coordinate plane.

At which point does this line intersect Y-axis?

How much angle does it make with X-axis? Discuss with your friends

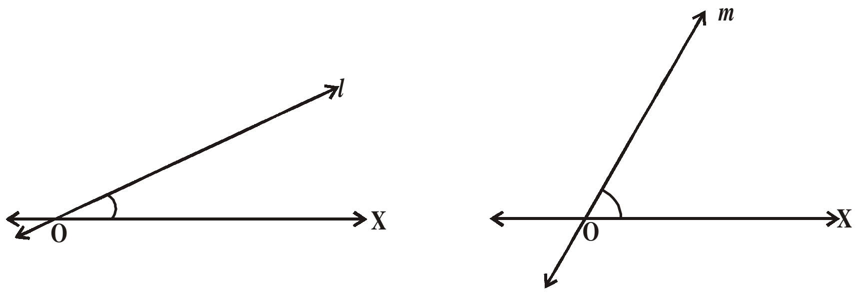
7.9.1 Slope of the straight line

You might have seen a slider in a park. Two sliders have been given here. On which slider you can slide faster?



Obviousely your answer will be second "Why"?

Observe these lines.



Which line makes more angle with OX ?

Since the line "*m*" makes a greater angle with OX than line '*l*'.

line '*m*' has a greater "slope" than line '*l*'. We may also term the "Steepness" of a line as its slope.

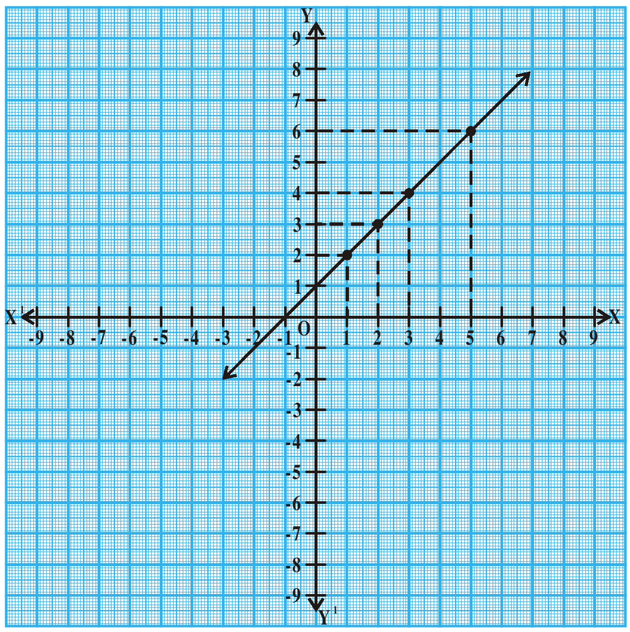
How we find the slope of a line?

Activity

Consider the line given in the figure indentify the points on the line and fill the table below.

x coordinate	1	-	-	4	-
y coordinate	2	3	4	-	6

We can observe that y coordinates change when x coordinates change.



When y coordinate increases from $y_1 = 2$ to $y_2 = 3$,

So the change in y is = ...

Then corresponding change in 'x' is = ...

$$\therefore \frac{\text{change in } y}{\text{change in } x} = \text{..................}$$

When y coordinate increases from $y_1 = 2$, $y_3 = 4$

So, the change in y is = ...

The corresponding change in x is ....

$$\text{So, } \frac{\text{change in } y}{\text{change in } x} = \text{...............}$$

Then can you try other points on the line choose any two points and fill in the table.

y value		Change in y	x	Change in x		$\frac{\text{change in } y}{\text{change in } x}$
2	4	-	1	2	1	-
-	-	-	-	-	-	-
-	-	-	-	-	-	-

What can you conclude from above activity?

Therefore, there is a relation between the ratio of change in y to change in x on a line has relation with angle made by it with X-axis.

You will learn the concept of $\tan\theta$ from trigonometry

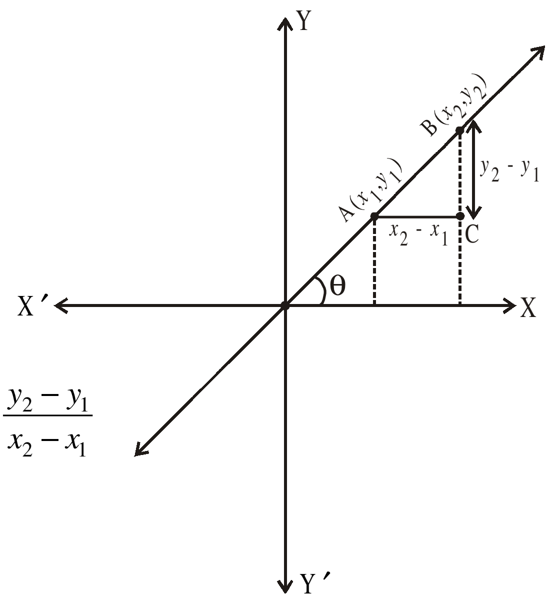
i.e., $$\tan\theta = \frac{\text{Opposite side of angle }\theta}{\text{adjecent side of angle }\theta} = \frac{\text{Change in } y}{\text{Change in } x}$$

7.9.2 Slope of a line joining Two Points

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on a line 'l' as shown in figure

The slope of a line $= \dfrac{\text{change in } y}{\text{change in } x}$

Slope of $\overline{AB} = m = \dfrac{y_2 - y_1}{x_2 - x_1}$



Slope will be denoted by 'm' and the line 'l' makes the angle θ with X-axis.

So AB line segment makes the same angle θ with AC also.

$$\therefore \tan\theta = \frac{\text{Opposite side of angle }\theta}{\text{adjecent wide of angle }\theta} = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = m$$

Hence $\therefore m = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1}$

It is the formula to find slope of line segment $\overline{AB}$ which is having end points are (x_1, y_1), (x_2, y_2).

If θ is angle made by the line with X-axis, then $m = \tan\theta$.

Example-22. The end points of a line are (2, 3), (4, 5). Find the slope of the line.

Solution : Points of a line are (2, 3), (4, 5) then slope of the line

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{4-2} = \frac{2}{2} = 1$$

Slope of the given line is 1.

Do This

Find the slope of $\overleftrightarrow{AB}$ with the given end points.

1. A(4, –6) B(7, 2)
2. A(8, –4), B(–4, 8)
3. A(–2, –5), B(1, –7)

Try This

Find the slope of $\overleftrightarrow{AB}$ with the points lying on

1. A(2, 1), B(2, 6)
2. A(–4, 2), B(–4, –2)
3. A(–2, 8), B(–2, –2)
4. Justify that the line $\overleftrightarrow{AB}$ line segment formed by given points is parallel to Y-axis. What can you say about their slope? Why?

Think - Discuss

Find the slope $\overleftrightarrow{AB}$ with the points lying on A(3, 2), (–8, 2)

When the line $\overleftrightarrow{AB}$ parallel to X-axis ? Why ?

Think and discuss with your friends in groups.

Example-23. Determine x so that 2 is the slope of the line through P(2, 5) and Q(x, 3).

Solution : Slope of the line passing through P(2, 5) and Q(x, 3) is 2.

Here, $x_1 = 5$, $y_1 = 5$, $x_2 = x$, $y_2 = 3$

$$\text{Slope of a } \overline{PQ} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-5}{x-2} = \frac{-2}{x-2} \Rightarrow \frac{-2}{x-2} = 2$$

$$\Rightarrow -2 = 2x - 4 \qquad \Rightarrow 2x = 2 \qquad \Rightarrow x = 1$$

Exercise - 7.4

1. Find the slope of the line joining the two given points

   (i) (4, –8) and (5, –2)

   (ii) (0, 0) and $(\sqrt{3}, 3)$

   (iii) (2a, 3b) and (a, –b)

   (iv) (a, 0) and (0, b)

   (v) A(–1.4, –3.7), B(–2.4, 1.3)

   (vi) A(3, –2), B(–6, –2)

   (vii) $A\left(-3\frac{1}{2}, 3\right)$, $B\left(-7, 2\frac{1}{2}\right)$

   (viii) A(0, 4), B(4, 0)

Optional Exercise

[This exercise is not meant for examination]

1. Centre of the circle Q is on the Y-axis. And the circle passes through the points (0, 7) and (0, –1). Circle intersects the positive X-axis at (P, 0). What is the value of 'P'.

2. A triangle ΔABC is formed by the points A(2, 3), B(–2, –3), C(4, –3). What is the point of intersection of side BC and angular bisector of A.

3. The side of BC of an equilateral triangle ΔABC is parallel to X-axis. Find the slopes of line along sides BC, CA and AB.

4. A right triangle has sides 'a' and 'b' where $a > b$. If the right angle is bisected then find the distance between orthocentres of the smaller triangles using coordinate geometry.

5. Find the centroid of the triangle formed by the line $2x + 3y - 6 = 0$. With the coordinate axes.

What We Have Discussed

1. The distance between two points P(x_1, y_1) and Q(x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. The distance of a point P(x, y) from the origin is $\sqrt{x^2 + y^2}$.

3. The distance between two points (x_1, y_1) and (x_1, y_2) on a line parallel to Y-axis is $|y_2 - y_1|$.

4. The distance between two point (x_1, y_1) and (x_2, y_2) on a line parallel to X-axis is $|x_2 - x_1|$.

5. The coordinates of the point P(x, y) which divides the line segment joining the points A (x_1, y_1) and B(x_2, y_2) internally in the ratio $m_1 : m_2$ are $\left[\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right]$.

6. The mid-point of the line segment joining the points P(x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

7. The point that divides each median in the ratio 2 : 1 is the centroid of a triangle.

8. The centroid of a triangle is the point of intersection of its medians. Hence the coordinates of the centroid are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

9. The area of the triangle formed by the points (x_1, y_1) (x_2, y_2) and (x_3, y_3) is the numerical value of the expression $\frac{1}{2}\ |x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)|$

10. Area of a triangle formula 'Heron's Formula'

$$A = \sqrt{S(S-a)\ (S-b)\ (S-c)} \qquad \because\ S = \frac{a+b+c}{2}$$

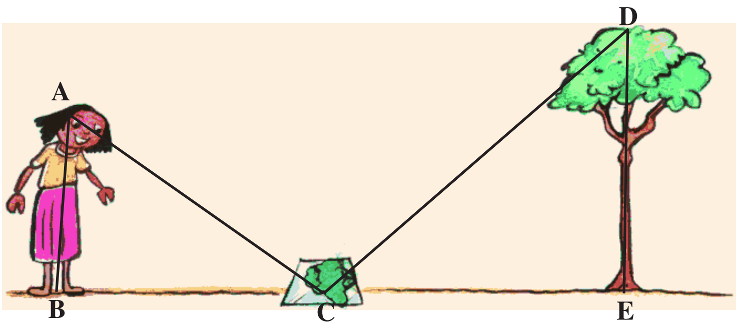
(a, b, c are three sides of ΔABC)

11. Slope of the line containing the points (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$

CHAPTER 8

Similar Triangles

8.1 Introduction

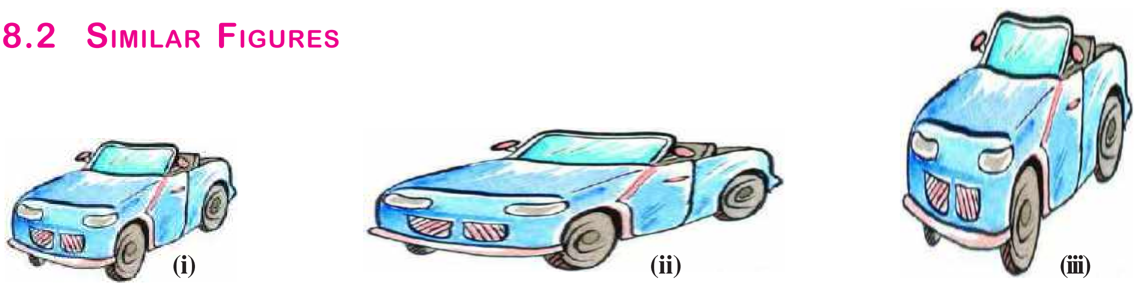


There is a tall tree in the backyard of Snigdha's house. She wants to find out the height of that tree but she is not sure about how to find it. Meanwhile, her uncle arrives at home. Snigdha requests her uncle to help her with the height. He thinks for a while and then ask her to bring a mirror. He places it on the ground at a certain distance from the base of the tree. He then asked Snigdha to stand on the otherside of the mirror at such a position from where she is able to see the top of the tree in that mirror.

When we draw the figure from (AB) girl to the mirror (C) and mirror to the tree (DE) as above, we observe triangles ABC and DEC. Now, what can you say about these two triangles? Are they congruent? No, because although they have the same shape but their sizes are different. Do you know what we call the geometrical figures which have the same shape, but are not necessarily of the same size? They are called **similar figures**.

Can you guess how the heights of trees, mountains or distances of far-away, objects such as the Sun have been found out? Do you think these can be measured directly with the help of a measuring tape? The fact is that all these heights and distances have been found out using the idea of indirect measurements which is based on the principle of similarity of figures.

8.2 Similar Figures



Observe the object (car) in the previous figure.

If its breadth is kept the same and the length is doubled, it appears as in fig.(ii).

If the length in fig.(i) is kept the same and its breadth is doubled, it appears as in fig.(iii).

Now, what can you say about fig.(ii) and (iii)? Do they resemble fig.(i)? We find that the figure is distorted. Can you say that they are similar? No, they have same shape, yet they are not similar.

Think what a photographer does when she prints photographs of different sizes from the same film (negative) ? You might have heard about stamp size, passport size and post card size photographs. She generally takes a photograph on a small size film, say 35 *mm*., and then enlarges it into a bigger size, say 45 mm (or 55 mm). We observe that every line segment of the smaller photograph is enlarged in the ratio of 35 : 45 (or 35 : 55). Further, in the two photographs of different sizes, we can see that the angles are equal. So, the photographs are similar.


(i)


(ii)


(iii)

Similarly in geometry, two polygons of the same number of sides are similar if their corresponding angles are equal and their corresponding sides are in the same ratio or proportion.

A polygon in which all sides and angles are equal is called a regular polygon.

The ratio of the corresponding sides is referred to as scale factor (or representative factor). In real life, blue prints for the construction of a building are prepared using a suitable scale factor.

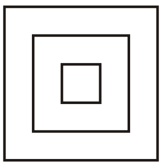
Think, Discuss and Write

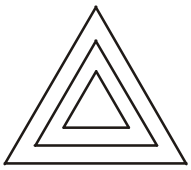
Can you give some more examples from your daily life where scale factor is used.

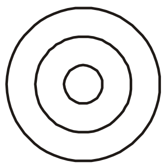
All regular polygons having the same number of sides are always similar. For example, all squares are similar, all equalateral triangles are similar and so on.

Circles with same radius are congruent and those with different radii are not congruent. But, as all circles have same shape, they are all similar.

We can say that all congruent figures are similar but all similar figures need not be congruent.


Similar Squares


Similar equilateral triangles


Similar Circles