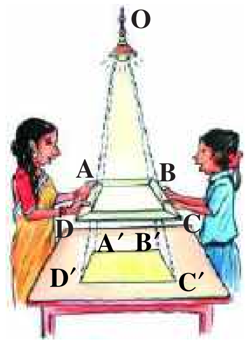
To understand the similarity of figures more clearly, let us perform the following activity.

Activity



Place a table directly under a lighted bulb, fitted in the ceiling in your classroom. Cut a polygon, say ABCD, from a plane cardboard and place it parallel to the ground between the bulb and the table. Then, a shadow of quadrilateral ABCD is cast on the table. Mark the outline of the shadow as quadrilateral $A'B'C'D'$.

Now this quadrilateral $A'B'C'D'$ is enlargement or magnification of quadrilateral ABCD. Further, A' lies on ray OA where 'O' is the bulb, B' on $\overrightarrow{OB}$, C' on $\overrightarrow{OC}$ and D' on $\overrightarrow{OD}$. Quadrilaterals ABCD and $A'B'C'D'$ are of the same shape but of different sizes.

A' corresponds to vertex A and we denote it symbotically as $A' \leftrightarrow A$. Similarly $B' \leftrightarrow B$, $C' \leftrightarrow C$ and $D' \leftrightarrow D$.

By actually measuring angles and sides, you can verify

(i) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle D = \angle D'$ and

(ii) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \dfrac{DA}{D'A'}$.

This emphasises that two polygons with the same number of sides are similar if

(i) All the corresponding angles are equal and

(ii) All the corresponding sides are in the same ratio (or in proportion)

Is a square similar to a rectangle? In both the figures, corresponding angles are equal but their corresponding sides are not in the same ratio. Hence, they are not similar. For similarity of polygons only one of the above two conditions is not sufficient, both have to be satisfied.

Think - Dissuss

Can you say that a square and a rhombus are similar? Discuss with your friends. Write why the conditions are not sufficient.

Do This

1. Fill in the blanks with similar / not similar.
   (i) All squares are ....
   (ii) All equilateral triangles are ....
   (iii) All isosceles triangles are ....
   (iv) Two polygons with same number of sides are .... if their corresponding angles are equal and corresponding sides are equal.
   (v) Reduced and Enlarged photographs of an object are ....
   (vi) Rhombus and squares are .... to each other.
2. Write the True / False for the following statements.
   (i) Any two similar figures are congruent.
   (ii) Any two congruent figures are similar.
   (iii) Two polygons are similar if their corresponding angles are equal.
3. Give two different examples of pair of
   (i) Similar fgures (ii) Non similar figures

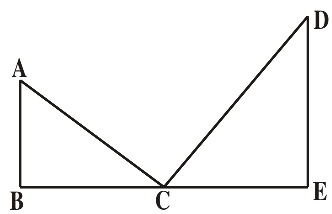
8.3 Similarity of Triangles

In the example we had drawn two triangles, those two triangles showed the property of similarity. We know that, two triangles are similar if their

(i) Corresponding Angles are equal and

(ii) Corresponding sides are in the same ratio (in proportion)

In ΔABC and ΔDEC in the introduction,



$\angle A = \angle D$, $\angle B = \angle E$, $\angle ACB = \angle DCE$

Also $\dfrac{DE}{AB} = \dfrac{EC}{BC} = \dfrac{DC}{AC} = K$ (scale factor)

then ΔABC is similar to ΔDEC

Symbolically we write $\Delta ABC \sim \Delta DEC$

(Symbol '~' is read as "Is similar to")

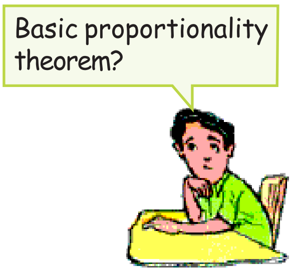
As we have stated K is a scale factor, So

if $K > 1$ we get enlarged figures,

$K = 1$ We get congruent figures and

$K < 1$ gives reduced (or diminished) figures

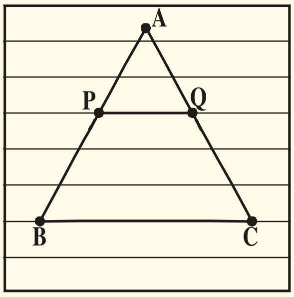
Further, in triangles ABC and DEC, corresponding angles are equal. So they are called equiangular triangles. The ratio of any two corresponding sides in two equiangular triangles is always the same. For proving this, Basic Proportionality theorem is used. This is also known as Thales Theorem.



To understand Basic proportionality theorem or Thales theorem, let us do the following activity.

ACTIVITY

Take any ruled paper and draw a triangle on that with base on one of the lines. Several lines will cut the triangle ABC. Select any one line among them and name the points where it meets the sides AB and AC as P and Q.



Find the ratio of $\frac{AP}{PB}$ and $\frac{AQ}{QC}$. What do you observe?

The ratios will be equal. Why ? Is it always true? Try for different lines intersecting the triangle. We know that all the lines on a ruled paper are parallel and we observe that every time the ratios are equal.

So in ΔABC, if $PQ \parallel BC$ then $\frac{AP}{PB} = \frac{AQ}{QC}$.

This is known as the result of basic proportionality theorem.

8.3.1 Basic Proportionality Theorem (Thales Theorem)

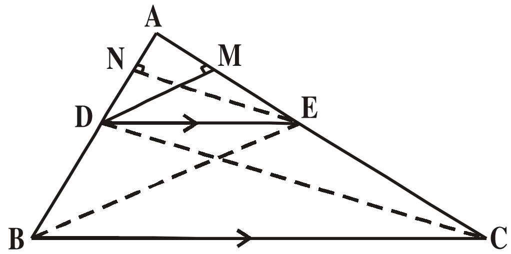
Theorem-8.1 : If a line is drawn parallel to one side of a triangle to intersect the other two sides in distinct points, then the other two sides are divided in the same ratio.

Given : In ΔABC, $DE \parallel BC$ which intersects sides AB and AC at D and E respectively.

RTP: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction : Join B, E and C, D and then draw

$DM \perp AC$ and $EN \perp AB$.



Proof : Area of $\Delta ADE = \frac{1}{2} \times AD \times EN$

Area of $\Delta BDE = \frac{1}{2} \times BD \times EN$

So $$\frac{ar(\Delta ADE)}{ar(\Delta BDE)} = \frac{\frac{1}{2}\times AD\times EN}{\frac{1}{2}\times BD\times EN} = \frac{AD}{BD} \qquad ...(1)$$

Again Area of $\Delta ADE = \frac{1}{2}\times AE\times DM$

Area of $\Delta CDE = \frac{1}{2}\times EC\times DM$

$$\frac{ar(\Delta ADE)}{ar(\Delta CDE)} = \frac{\frac{1}{2}\times AE\times DM}{\frac{1}{2}\times EC\times DM} = \frac{AE}{EC} \qquad ...(2)$$

Observe that ΔBDE and ΔCDE are on the same base DE and between same parallels BC and DE.

So $ar(\Delta BDE) = ar(\Delta CDE)$ ...(3)

From (1) (2) and (3), we have

$$\frac{AD}{DB} = \frac{AE}{EC}$$

Hence proved.

Is the converse of the above theorem also true? To examine this, let us perform the following activity.

Activity

Draw an angle XAY on your note book and on ray AX, mark points B_1, B_2, B_3, B_4 and B such that

$AB_1 = B_1B_2 = B_2B_3 = B_3B_4 = B_4B = 1$cm (say)

Similarly on ray AY, mark points C_1, C_2, C_3, C_4 and C such that

$AC_1 = C_1C_2 = C_2C_3 = C_3C_4 = C_4C = 2$ cm (say)

Join B_1, C_1 and B, C.

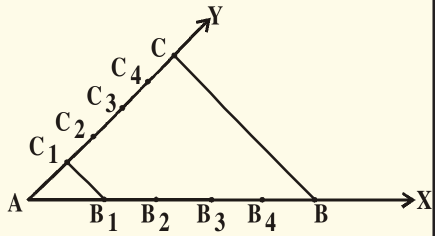
Observe that $\frac{AB_1}{B_1B} = \frac{AC_1}{C_1C} = \frac{1}{4}$ and $B_1C_1 \parallel BC$

Similarly, joining B_2C_2, B_3C_3 and B_4C_4, you see that

$$\frac{AB_2}{B_2B} = \frac{AC_2}{C_2C} = \frac{2}{3} \text{ and } B_2C_2 \parallel BC$$

$$\frac{AB_3}{B_3B} = \frac{AC_3}{C_3C} = \frac{3}{2} \text{ and } B_3C_3 \parallel BC$$

$$\frac{AB_4}{B_4B} = \frac{AC_4}{C_4C} = \frac{4}{1} \text{ and } B_4C_4 \parallel BC$$



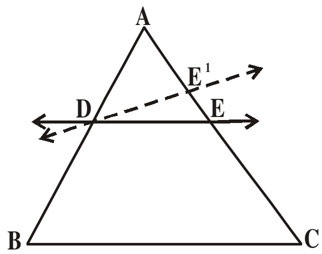
From this we obtain the following theorem called converse of the Thales theorem

Theorem-8.2 : If a line divides two sides of a triangle in the same ratio, then the line is parallel to the third side.

Given : In ΔABC, a line DE is drawn such that $\frac{AD}{DB} = \frac{AE}{EC}$

RTP : $DE \parallel BC$

Proof : Assume that DE is not parallel to BC then draw the line DE^1 parallel to BC



So $\frac{AD}{DB} = \frac{AE'}{E'C}$ (why ?)

$\therefore \frac{AE}{EC} = \frac{AE'}{E'C}$ (why ?)

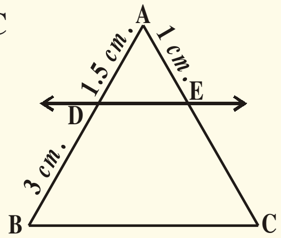
Adding 1 to both sides of the above, you can see that E and E' must coincide (why ?)

Try This

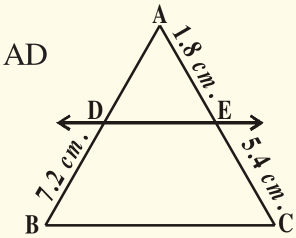
1. E and F are points on the sides PQ and PR respectively of ΔPQR. For each of the following, state whether. $EF \parallel QR$ or not?
   (i) PE = 3.9 cm EQ = 3 cm PF = 3.6 cm and FR = 2.4 cm
   (ii) PE = 4 cm, QE = 4.5 cm, PF = 8 cm and RF = 9 cm.
   (iii) PQ = 1.28 cm PR = 2.56 cm PE = 1.8 cm and PF = 3.6 cm

2. In the following figures DE || BC.

(i) Find EC

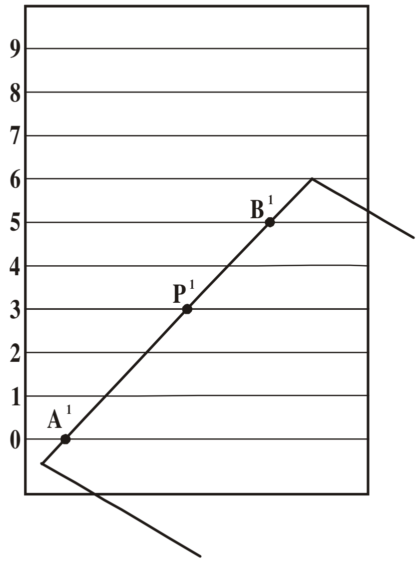


(ii) Find AD



Construction : Division of a line segment (using Thales theorem)

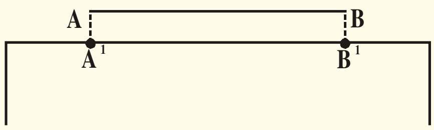
Madhuri drew a line segment. She wants to divide it in the ratio of 3 : 2. She measured it by using a scale and divided it in the required ratio. Meanwhile her elder sister came. She saw this and suggested Madhuri to divide the line segment in the given ratio without measuring it Madhuri was puzzled and asked her sister for help to do it. Then her sister explained. You may also do it by the following activity.



ACTIVITY

Take a sheet of paper from a lined note book. Number the lines by 1, 2, 3, ... starting with the bottom line numbered '0'.

Take a thick cardboard paper (or file card or chart strip) and place it against the given line segment AB and transfer its length to the card. Let A^1 and B^1 denote the points on the file card corresponding to A and B.



Now place A^1 on the zeroeth line of the lined paper and rotate the card about A^1 unitl point B^1 falls on the 5^{th} line (3 + 2).

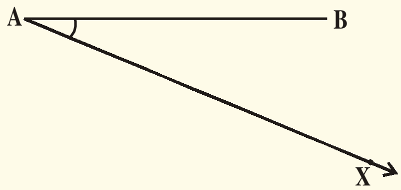
Mark the point where the third line touches the file card, by P^1.

Again place this card along the given line segment and transfer this point P^1 and denote it with 'P'.

So P is required point which divides the given line segment in the ratio 3:2.

Now let us learn how this construction can be done.

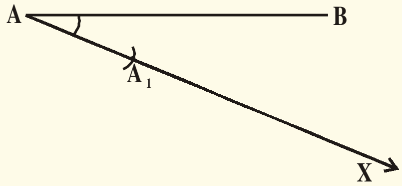
Given a line segment AB. We want to divide it in the ratio m : n where m and n are both positive integers. Let us take m = 3 and n = 2.

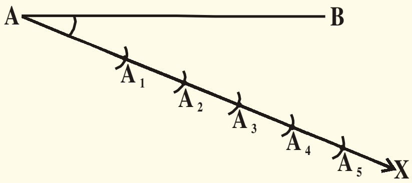


Steps :

1. Draw a ray AX through A making an acute angle with AB.

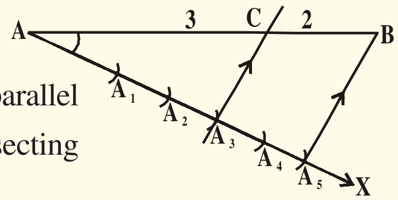
2. With 'A' as centre and with any length draw an arc on ray AX and label the point A_1.





3. Using the same compass setting and with A_1 as centre draw another arc and locate A_2.

4. Like this locate 5 points $(= m + n)$ A_1, A_2, A_3, A_4, A_5 such that $AA_1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5$

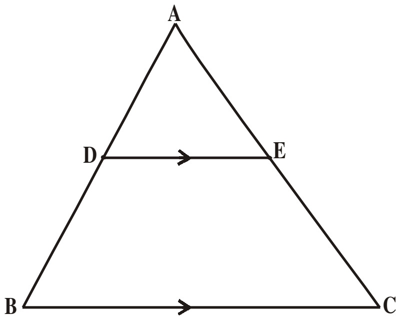


5. Join A_5B. Now through point $A_3 (m = 3)$ draw a line parallel to A_5B (by making an angle equal to $\angle A\ A_5\ B$) intersecting AB at C and obeserve that AC : CB = 3 : 2.

Now let us solve some examples on Thales theorem and its converse.

Example-1. In ΔABC, $DE \parallel BC$ and $\frac{AD}{DB} = \frac{3}{5}$.

AC = 5.6. Find AE.



Solution : In ΔABC, $DE \parallel BC$

$$\Rightarrow \quad \frac{AD}{DB} = \frac{AE}{EC} \text{ (by B.P.T)}$$

$$\text{but} \quad \frac{AD}{DB} = \frac{3}{5} \quad \text{So} \quad \frac{AE}{EC} = \frac{3}{5}$$

Given AC = 5.6 and AE : EC = 3 : 5.

$$\frac{AE}{AC - AE} = \frac{3}{5}$$

$$\frac{AE}{5.6 - AE} \quad \frac{3}{5} \text{ (cross multiplication)}$$

$$5AE = (3 \times 5.6) - 3AE$$

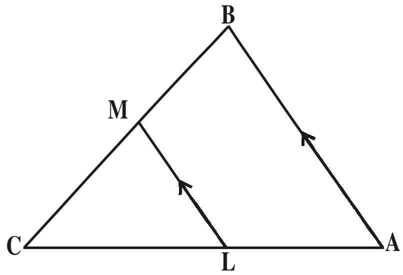
$$8AE = 16.8$$

$$AE = \frac{16.8}{8} = 2.1\text{cm.}$$

Example-2. In the given figure LM || AB

$AL = x - 3$, $AC = 2x$, $BM = x - 2$

and $BC = 2x + 3$ find the value of x



Solution : In ΔABC, LM || AB

$$\Rightarrow \frac{AL}{LC} = \frac{BM}{MC} \text{ (by B.P.T)}$$

$$\frac{x-3}{2x-(x-3)} = \frac{x-2}{(2x+3)-(x-2)}$$

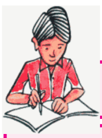
$$\frac{x-3}{x+3} \qquad \frac{x-2}{x+5} \text{ (cross multiplication)}$$

$$(x-3)(x+5) = (x-2)(x+3)$$

$$x^2 + 2x - 15 = x^2 + x - 6$$

$$\Rightarrow \quad 2x - x = -6 + 15$$

$$x = 9$$

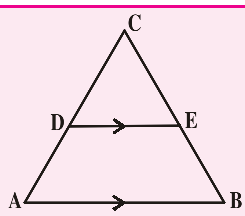


Do This

1. What value(s) of x will make DE || AB, in the given figure ?

   $AD = 8x + 9$, $CD = x + 3$

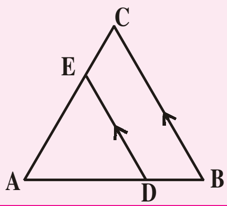
   $BE = 3x + 4$, $CE = x$.



2. In ΔABC, DE || BC. $AD = x$, $DB = x - 2$,

   $AE = x + 2$ and $EC = x - 1$.

   Find the value of x.



Example-3. The diagonals of a quadrilateral ABCD intersect each other at point 'O' such that $\frac{AO}{BO} = \frac{CO}{DO}$. Prove that ABCD is a trapezium.

Solution : Given : In quadrilateral ABCD , $\frac{AO}{BO} = \frac{CO}{DO}$.

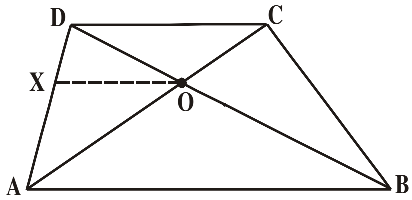
RTP : ABCD is a trapezium.

Construction : Through 'O' draw a line parallel to AB which meets DA at X.

Proof : In ΔDAB, XO || AB (by construction)

$$\Rightarrow \frac{DX}{XA} = \frac{DO}{OB} \qquad \text{(by basic proportionality theorem)}$$

$$\frac{AX}{XD} = \frac{BO}{OD} \qquad \text{..... (1)}$$



again $$\frac{AO}{BO} = \frac{CO}{DO} \qquad \text{(given)}$$

$$\frac{AO}{CO} = \frac{BO}{OD} \qquad \text{..... (2)}$$

From (1) and (2)

$$\frac{AX}{XD} = \frac{AO}{CO}$$

In ΔADC, XO is a line such that $\frac{AX}{XD} = \frac{AO}{OC}$

$\Rightarrow$ XO || DC (by converse of the basic the proportionality theorem)

$\Rightarrow$ AB || DC

In quadrilateral ABCD, AB || DC

$\Rightarrow$ ABCD is a trapezium (by definition)

Hence proved.

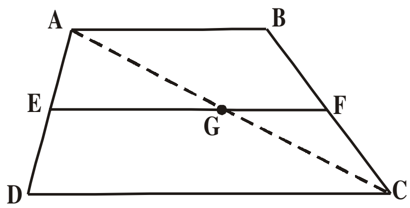
Example-4. In trapezium ABCD, AB || DC. E and F are points on non-parallel sides AD and BC respectively such that EF || AB. Show that $\frac{AE}{ED} = \frac{BF}{FC}$.

Solution : Let us join AC to intersect EF at G.

AB || DC and EF || AB (given)

$\Rightarrow$ EF || DC (Lines parallel to the same line are parallel to each other)

In ΔADC, EG || DC



So $\frac{AE}{ED} = \frac{AG}{GC}$ (by BPT) ...(1)

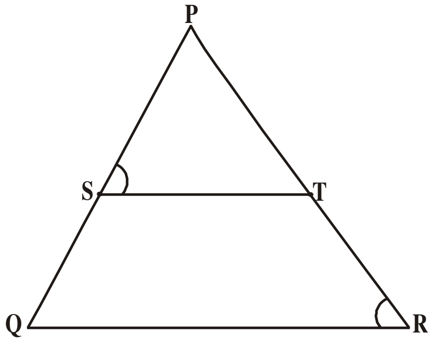
Similarly, In ΔCAB, GF || AB

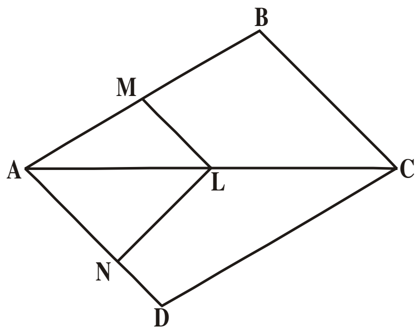
$\frac{CG}{GA} = \frac{CF}{FB}$ (by BPT) i.e., $\frac{AG}{GC} = \frac{BF}{FC}$ ...(2)

From (1) & (2) $\frac{AE}{ED} = \frac{BF}{FC}$.

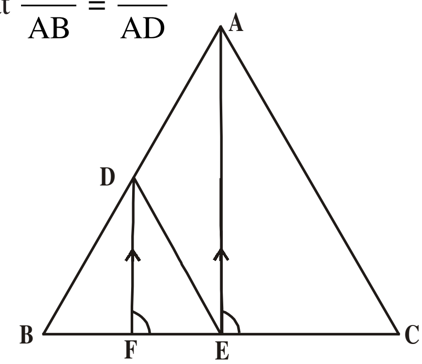
Exercise - 8.1

1. In ΔPQR, ST is a line such that $\frac{PS}{SQ} = \frac{PT}{TR}$ and also $\angle PST = \angle PRQ$.
   Prove that ΔPQR is an isosceles triangle.

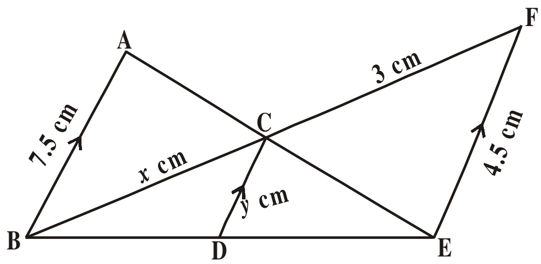




2. In the given figure, $LM \parallel CB$ and $LN \parallel CD$
   Prove that $\frac{AM}{AB} = \frac{AN}{AD}$



3. In the given figure, $DE \parallel AC$ and $DF \parallel AE$
   Prove that $\frac{BF}{FE} = \frac{BE}{EC}$.

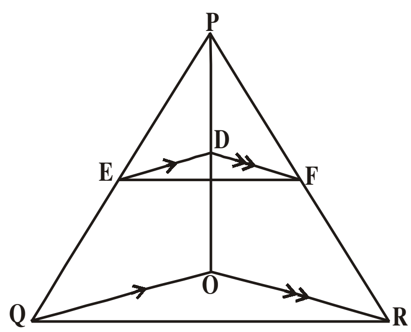


4. In the given figure, $AB \parallel CD \parallel EF$.
   given $AB = 7.5$ cm $DC = y$cm
   $EF = 4.5$ cm, $BC = x$ cm.
   Calculate the values of x and y.

5. Prove that a line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side (Using basic proportionality theorem).

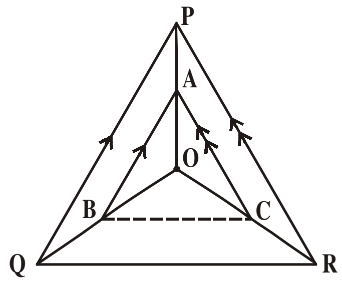
6. Prove that a line joining the midpoints of any two sides of a triangle is parallel to the third side. (Using converse of basic proportionality theorem)

7. In the given figure, $DE \parallel OQ$ and $DF \parallel OR$. Show that $EF \parallel QR$.



8. In the adjacent figure, A, B and C are points on OP, OQ and OR respectively such that AB || PQ and AC||PR.

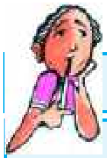
   Show that BC || QR.



9. ABCD is a trapezium in which AB||DC and its diagonals intersect each other at point 'O'.

   Show that $\frac{AO}{BO} = \frac{CO}{DO}$.

10. Draw a line segment of length 7.2 cm and divide it in the ratio 5 : 3. Measure the two parts.



Think - Discuss and Write

Discuss with your friends that in what way similarity of triangles is different from similarity of other polygons?

8.4 Criteria for Similarity of Triangles

We know that two triangles are similar if corresponding angles are equal and corresponding sides are proportional. For checking the similarity of two triangles, we should check for the equality of corresponding angles and equality of ratios of their corresponding sides. Let us make an attempt to arrive at certain criteria for similarity of two triangles. Let us perform the following activity.

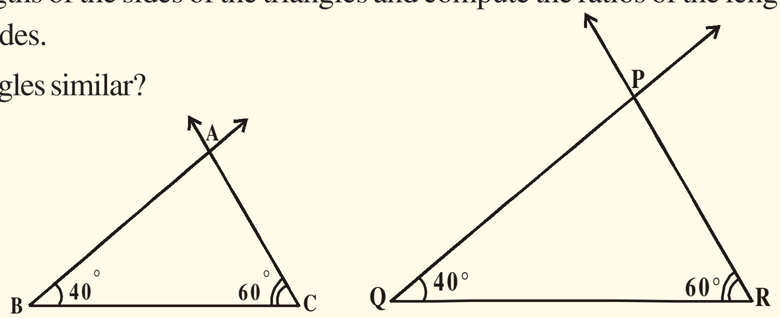
Activity

Use a protractor and ruler to draw two non congruent triangles so that each triangle has a 40° and 60° angle. Check the figures made by you by measuring the third angles of two triangles.

It should be each 80° (why?)

Measure the lengths of the sides of the triangles and compute the ratios of the lengths of the corresponding sides.

Are the triangles similar?



This activity leads us to the following criterion for similarity of two triangles.

8.4.1 AAA Criterion for Similarity of Triangles

Theorem-8.3 : In two triangles, if the angles are equal, then the sides opposite to the equal angles are in the same ratio (or proportional) and hence the two triangles are similar.

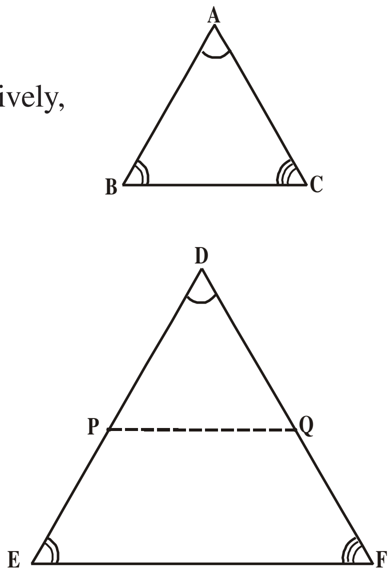
Given : In triangles ABC and DEF,

$\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$

RTP : $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

Construction : Locate points P and Q on DE and DF respectively,

such that AB = DP and AC = DQ. Join PQ.



Proof : $\Delta ABC \cong \Delta DPQ$ (why ?)

This gives $\angle B = \angle P = \angle E$ and PQ || EF (How ?)

$\therefore \frac{DP}{PE} = \frac{DQ}{QF}$ (why ?)

i.e., $\frac{AB}{DE} = \frac{AC}{DF}$ (why ?)

Similarly $\frac{AB}{DE} = \frac{BC}{EF}$ and So $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

Hence proved.

Note : If two angles of a triangle are respectively equal to the two angles of another triangle, then by the angle sum property of a triangle, third angles will also be equal.

So AA similarity criterion is stated as if two angles of one triangle are respectively equal to the two angles of anther triangle, then the two triangles are similar.

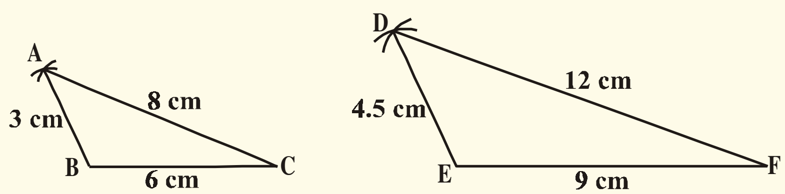
What about the converse of the above statement ?

If the sides of a triangle are respectively proportional to the sides of another triangle, is it true that their corresponding angles are equal ?

Let us exercise it through an activity.

Activity

Draw two triangles ABC and DEF such that AB = 3 cm, BC = 6 cm, CA = 8 cm, DE = 4.5 cm, EF = 9 cm and FD = 12 cm.



So you have $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = \frac{2}{3}$.

Now measure the angles of both the triangles. What do you observe? What can you say about the corresponding angles? They are equal, so the triangles are similar. You can verify it for different triangles.

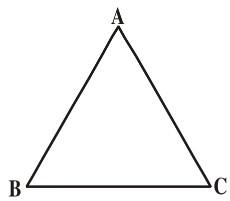
From the above activity, we can give the following criterion for similarity of two triangles.

8.4.2. SSS Criterion for Similarity of Triangles

Theorem-8.4 : If in two triangles, the sides of one triangle are proportional to the sides of the other triangle, then their corresponding angles are equal and hence the triangles are similar.

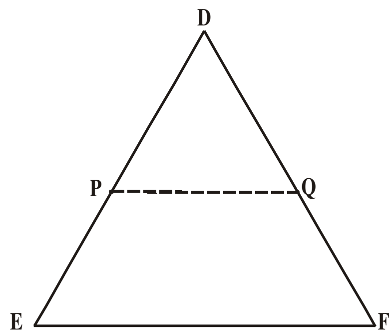
Given : ΔABC and ΔDEF are such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} \quad (<1)$$



RTP : $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

Construction : Locate points P and Q on DE and DF respectively such that AB = DP and AC = DQ. Join PQ.



Proof : $\frac{DP}{PE} = \frac{DQ}{QF}$ and PQ || EF (why ?)

So $\angle P = \angle E$ and $\angle Q = \angle F$ (why ?)

$\therefore \frac{DP}{DE} = \frac{DQ}{DF} = \frac{PQ}{EF}$

So $\frac{DP}{DE} = \frac{DQ}{DF} = \frac{BC}{EF}$ (why ?)

So BC = PQ (Why ?)

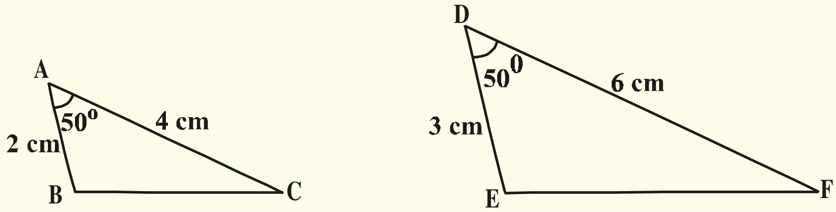
$\Delta ABC \cong \Delta DPQ$ (why ?)

So $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$ (How ?)

We studied that for similarity of two polygons any one condition is not sufficient. But for the similarity of triangles, there is no need for fulfillment of both the conditions as one automatically implies the other. Now let us look for SAS similarity criterion. For this, let us perform the following activity.

Activity

Draw two triangles ABC and DEF such that AB = 2 cm, $\angle A = 50^0$ AC = 4cm, DE = 3 cm, $\angle D = 50^0$ and DF = 6cm.



Observe that $\frac{AB}{DE} = \frac{AC}{DF} = \frac{2}{3}$ and $\angle A = \angle D = 50^0$.

Now measure $\angle B$, $\angle C$, $\angle E$, $\angle F$ also measure BC and EF.

Observe that $\angle B = \angle E$ and $\angle C = \angle F$ also $\frac{BC}{EF} = \frac{2}{3}$.

So, the two triangles are similar. Repeat the same for triangles with different measurements, which gives the following criterion for similarity of triangles.

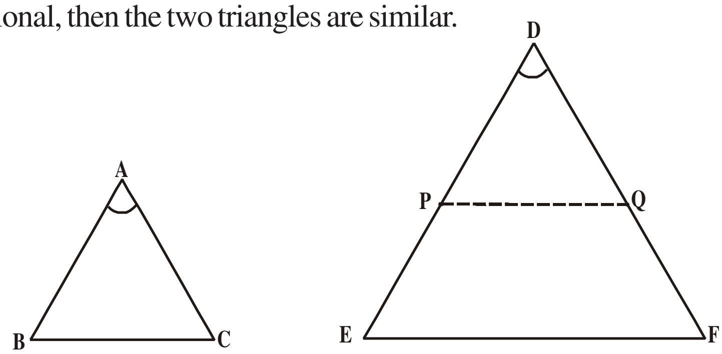
8.4.3 SAS Criterion for Similarity of Triangles

Theorem-8.5 : If one angle of a triangle is equal to one angle of the other triangle and the sides including these angles are proportional, then the two triangles are similar.

Given : In ΔABC and ΔDEF

$\frac{AB}{DE} = \frac{AC}{DF}$ (<1) and

$\angle A = \angle D$



RTP : $\Delta ABC \sim \Delta DEF$

Construction : Locate points P and Q on DE and DF respectively such that AB = DP and AC = DQ. Join PQ.

Proof : PQ || EF and $\Delta ABC \cong \Delta DPQ$ (How ?)

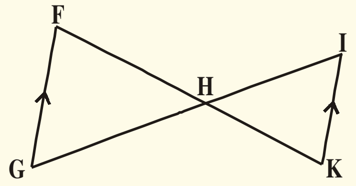
So $\angle A = \angle D$, $\angle B = \angle P$, $\angle C = \angle Q$

$\therefore$ $\Delta ABC \sim \Delta DEF$ (why ?)

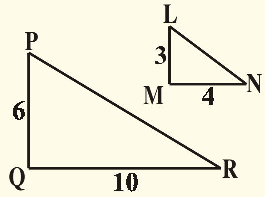
Try This

1. Are the triangles similar? If so, name the criterion of similarity. Write the similarity relation is symbolic form.

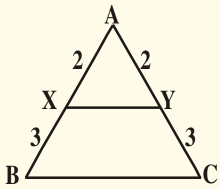
(i)



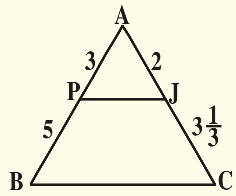
(ii)



(iii)



(iv)



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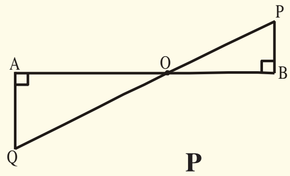

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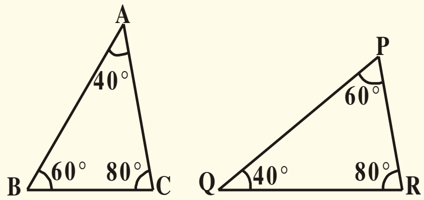

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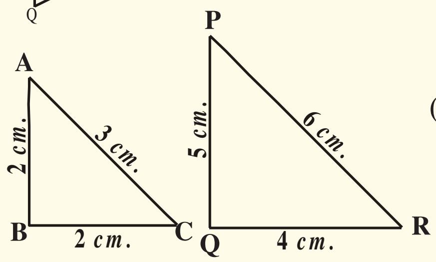

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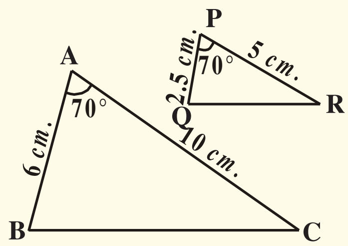
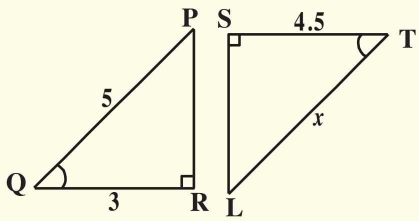
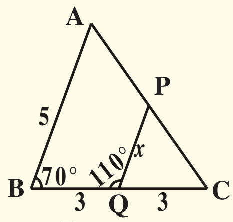


2. Explain why the triangles are similar and then find the value of x.

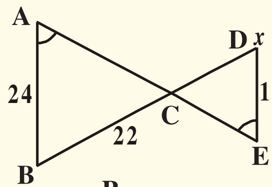
(i)



(ii)



(iii)



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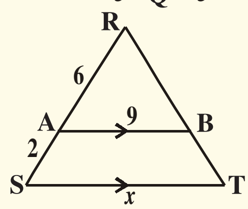

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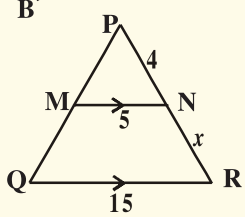

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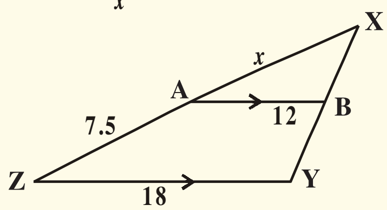
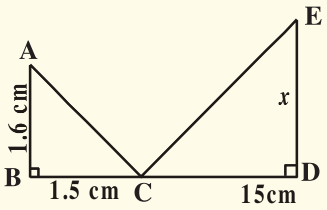
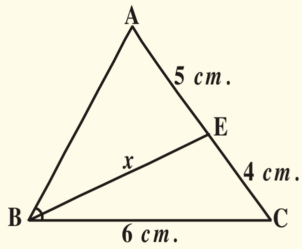
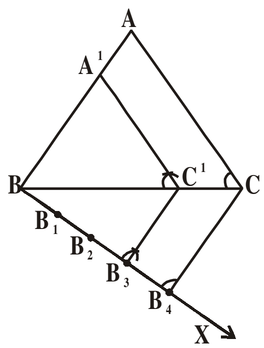


(vii) 

(viii) 

Construction : To construct a triangle similar to a given triangle as per given scale factor.

a) Construct a triangle similar to a given triangle ABC with its sides equal to $\frac{3}{4}$ of corresponding sides of ΔABC (scale factor $\frac{3}{4}$)



Steps : 1. Draw a ray BX, making an acute angle with BC on the side opposite to vertex A.

2. Locate 4 points B_1, B_2, B_3 and B_4 on BX so that $BB_1 = B_1B_2 = B_2B_3 = B_3B_4$.

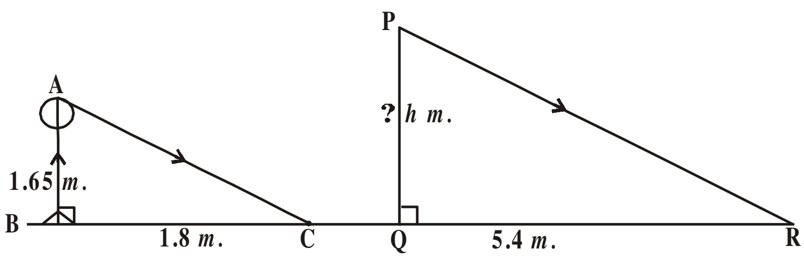
3. Join B_4C and draw a line through B_3 parallel to B_4C intersecting BC at C'.

4. Draw a line through C' parallel to CA to intersect AB at A'.

So $\Delta A'BC'$ is the required triangle.

Let us take some examples to illustrate the use of these criteria.

Example-5. A person 1.65m tall casts 1.8m shadow. At the same instance, a lamp-posts casts a shadow of 5.4 m. Find the height of the lamppost.



Solution: In ΔABC and ΔPQR

$\angle B = \angle Q = 90^0$.

$\angle C = \angle R$ (AC || PR, all sun's rays are parallel at any instance)

$\Delta ABC \sim \Delta PQR$ (by AA similarity)

$\frac{AB}{PQ} = \frac{BC}{QR}$ (cpst, corresponding parts of Similar triangles)

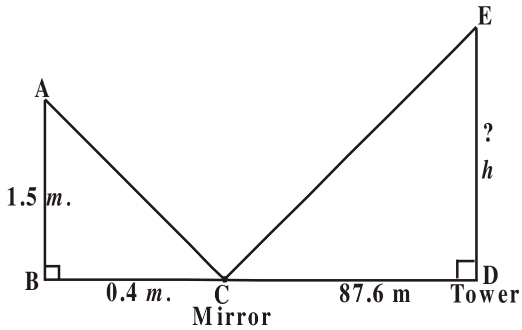
$$\frac{1.65}{PQ} = \frac{1.8}{5.4}$$

$$PQ = \frac{1.65 \times 5.4}{1.8} = 4.95m$$

The height of the lamp post is 4.95m.

Example-6. A man sees the top of a tower in a mirror which is at a distance of 87.6m from the tower. The mirror is on the ground facing upwards. The man is 0.4m away from the mirror and his height is 1.5m. How tall is the tower?

Solution : In ΔABC & ΔEDC



$\angle ABC = \angle EDC = 90°$

$\angle BCA = \angle DCE$ (angle of incidence and angle of reflection are same)

$\Delta ABC \sim \Delta EDC$ (by AA similarity)

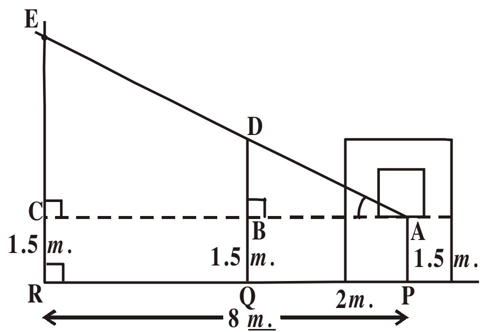
$$\frac{AB}{ED} = \frac{BC}{CD} \Rightarrow \frac{1.5}{h} = \frac{0.4}{87.6}$$

$$h = \frac{1.5 \times 87.6}{0.4} = 328.5m$$

Hence, the height of the towers is 328.5m.

Example7. Gopal is worrying that his neighbour can see into his living room from the top floor of his house. He has decided to build a fence that is high enough to block the view from their top floor window. What should be the height of the fence? The measurements are given in the figure.

Solution : In ΔABD & ΔACE



$\angle B = \angle C = 90°$

$\angle A = \angle A$ (common angle)

$\Delta ABD \sim \Delta ACE$ (by AA similarity)

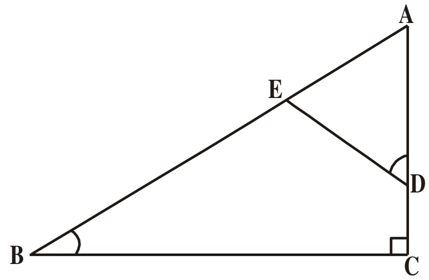
$$\frac{AB}{AC} = \frac{BD}{CE} \Rightarrow \frac{2}{8} = \frac{BD}{1.2}$$

$$BD = \frac{2 \times 1.2}{8} = \frac{2.4}{8} = 0.3m$$

Total height of the fence required is 1.5 m. + 0.3 m. = 1.8m to block the neightbour's view.

Exercise - 8.2

1. In the given figure, $\angle ADE = \angle B$



   (i) Show that $\Delta ABC \sim \Delta ADE$

   (ii) If AD = 3.8 cm, AE = 3.6cm

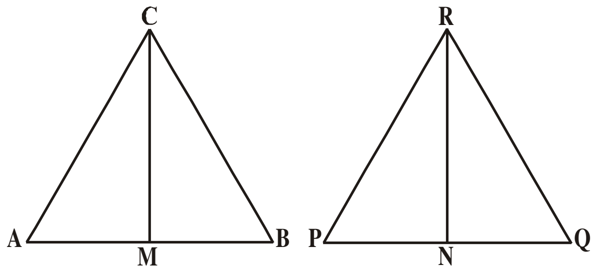
   BE = 2.1 cm BC = 4.2 cm

   find DE.

2. The perimeters of two similar triangles are 30 cm and 20 cm respectively. If one side of the first triangle is 12 cm, determine the corresponding side of the second triangle.

3. A girl of height 90 cm is walking away from the base of a lamp post at a speed of 1.2 m/sec. If the lamp post is 3.6m above the ground, find the length of her shadow after 4 seconds.

4. CM and RN are respectively the medians of ΔABC and ΔPQR. Prove that



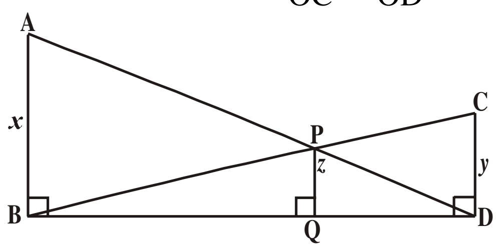
   (i) $\Delta AMC \sim \Delta PNR$

   (ii) $\dfrac{CM}{RN} = \dfrac{AB}{PQ}$

   (iii) $\Delta CMB \sim \Delta RNQ$

5. Diagonals AC and BD of a trapezium ABCD with AB || DC intersect each other at the point 'O'. Using the criterion of similarity for two triangles, show that $\dfrac{OA}{OC} = \dfrac{OB}{OD}$.

6. AB, CD, PQ are perpendicular to BD.



   AB = x, CD = y amd PQ = Z

   prove that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$.

7. A flag pole 4m tall casts a 6 *m*., shadow. At the same time, a nearby building casts a shadow of 24m. How tall is the building ?

8. CD and GH are respectively the bisectors of $\angle ACB$ and $\angle EGF$ such that D and H lie on sides AB and FE of ΔABC and ΔFEG respectively. If $\Delta ABC \sim \Delta FEG$ then show thatG

   (i) $\dfrac{CD}{GH} = \dfrac{AC}{FG}$ (ii) $\Delta DCB \sim \Delta HGE$ (iii) $\Delta DCA \sim \Delta HGF$

9. AX and DY are altitudes of two similar triangles ΔABC and ΔDEF. Prove that AX : DY = AB : DE.

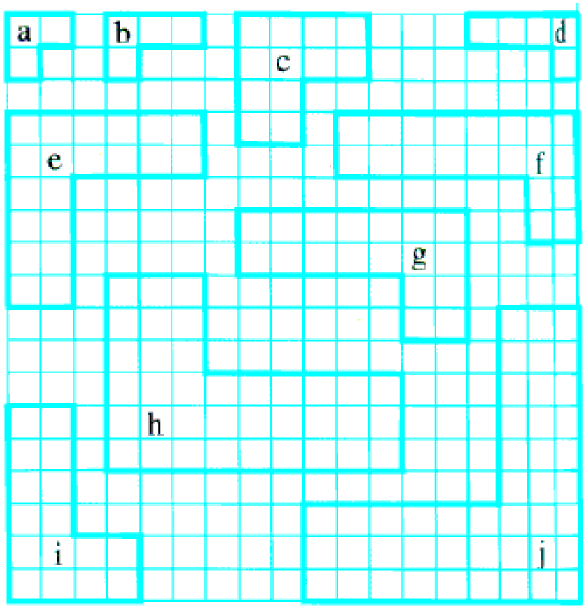
10. Construct a triangle shadow similar to the given ΔABC, with its sides equal to $\frac{5}{3}$ of the corresponding sides of the triangle ABC.

11. Construct a triangle of sides 4cm, 5 cm and 6 cm. Then, construct a triangle similar to it, whose sides are $\frac{2}{3}$ of the corresponding sides of the first triangle.

12. Construct an Isosceles triangle whose base is 8cm and altitude is 4 *cm*. Then, draw another triangle whose sides are $1\frac{1}{2}$ times the corresponding sides of the isosceles triangle.

8.5 Areas of Similar Triangles

For two similar triangles, ratio of their corresponding sides is the same. Do you think there is any relationship between the ratio of their areas and the ratio of their corresponding sides ? Let us do the following activity to understand this.



Activity

Make a list of pairs of similar polygons in this figure.

Find

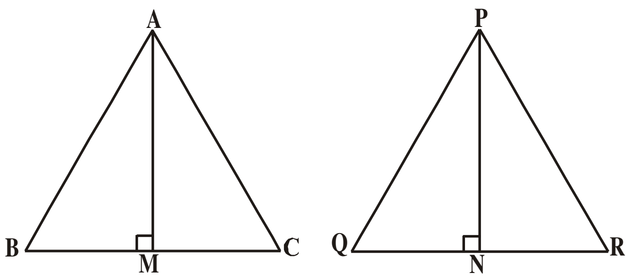
(i) The ratio of similarity and

(ii) The ratio of areas.

You will observe that ratio of areas is the square of the ratoio of their corresponding sides.

Let us prove it like a theorem.

Theorem-8.6 : The ratio of the areas of two similar triangles is equal to the ratio of the squares of their corresponding sides.



Given : $\Delta ABC \sim \Delta PQR$

RTP : $\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{CA}{RP}\right)^2$.

Construction : Draw $AM \perp BC$ and $PN \perp QR$.

Proof : $\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \dfrac{\frac{1}{2} \times BC \times AM}{\frac{1}{2} \times QR \times PN} = \dfrac{BC \times AM}{QR \times PN}$ ...(1)

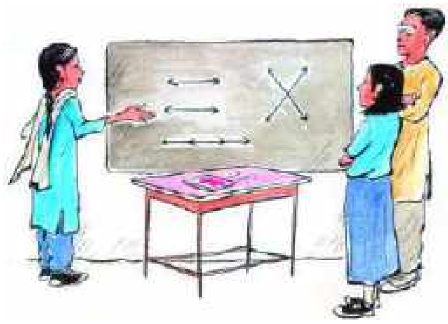
In ΔABM & ΔPQN

$\angle B = \angle Q$ ($\because \Delta ABC \sim \Delta PQR$)

$\angle M = \angle N = 90^0$

$\therefore \Delta ABM \sim \Delta PQN$ (by AA similarity)

$\dfrac{AM}{PN} = \dfrac{AB}{PQ}$ ...(2)



Also $\Delta ABC \sim \Delta PQR$ (given)

$\boxed{\dfrac{AB}{PQ} = \dfrac{BC}{QR}} = \dfrac{AC}{PR}$ ...(3)

$\therefore \dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \dfrac{AB}{PQ} \times \dfrac{AB}{PQ}$ from (1), (2) and (3)

$= \left(\dfrac{AB}{PQ}\right)^2$.

Now by using (3), we get

$\dfrac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\dfrac{AB}{PQ}\right)^2 = \left(\dfrac{BC}{QR}\right)^2 = \left(\dfrac{AC}{PR}\right)^2$

Hence proved.

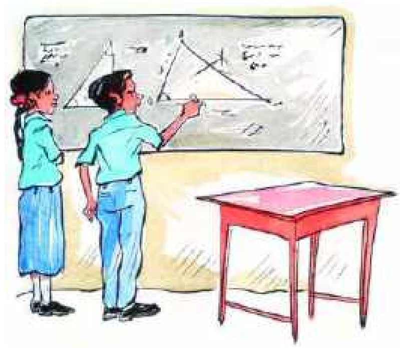
Now let us see some examples.

Example-8. Prove that if the areas of two similar triangles are equal, then they are congruent.

Solution : $\Delta ABC \sim \Delta PQR$

$$\text{So } \frac{ar(\Delta ABC)}{ar(\Delta PQR)} = \left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{AC}{PR}\right)^2$$

$$\text{But } \frac{ar(\Delta ABC)}{ar(\Delta PQR)} = 1 \quad (\because \text{ areas are equal})$$



$$\left(\frac{AB}{PQ}\right)^2 = \left(\frac{BC}{QR}\right)^2 = \left(\frac{AC}{PR}\right)^2 = 1$$

So $AB^2 = PQ^2$

$BC^2 = QR^2$

$AC^2 = PR^2$

From which we get $AB = PQ$

$BC = QR$

$AC = PR$

$\therefore \quad \Delta ABC \cong \Delta PQR$ (by SSS congruency)

Example-9. $\Delta ABC \sim \Delta DEF$ and their areas are respectively $64 cm^2$ and $121\ cm^2$.

If EF = 15.4 cm., then find BC.

Solution : $\frac{ar(\Delta ABC)}{ar(\Delta DEF)} = \left(\frac{BC}{EF}\right)^2$

$$\frac{64}{121} = \left(\frac{BC}{15.4}\right)^2$$

$$\frac{8}{11} = \frac{BC}{15.4} \Rightarrow BC = \frac{8 \times 15.4}{11} = 11.2cm.$$

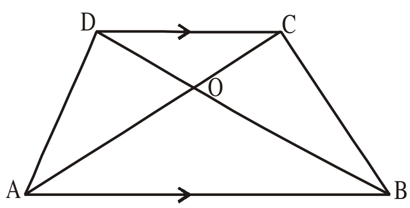
Example-10. Diagonals of a trapezium ABCD with AB || DC, intersect each other at the point 'O'. If AB = 2CD, find the ratio of areas of triangles AOB and COD.

Solution : In trapezium ABCD, AB || DC also AB = 2CD.

In ΔAOB and ΔCOD

$\angle AOB = \angle COD$ (vertically opposite angles)

$\angle OAB = \angle OCD$ (alternate interior angles)



$\Delta AOB \sim \Delta COD$ (by AA similarity)

$$\frac{ar(\Delta AOB)}{ar(\Delta COD)} = \frac{AB^2}{DC^2}$$

$$= \frac{(2DC)^2}{(DC)^2} = \frac{4}{1}$$

$\therefore$ $ar(\Delta AOB) : ar(\Delta COD) = 4 : 1$.

Exercise - 8.3

1. Equilateral triangles are drawn on the three sides of a right angled triangle. Show that the area of the triangle on the hypotenuse is equal to the sum of the areas of triangles on the other two sides.

2. Prove that the area of the equilateral triangle described on the side of a square is half the area of the equilateral triangles described on its diagonal.

3. D, E, F are mid points of sides BC, CA, AB of ΔABC. Find the ratio of areas of ΔDEF and ΔABC.

4. In ΔABC, XY || AC and XY divides the triangle into two parts of equal area. Find the ratio of $\frac{AX}{XB}$.

5. Prove that the ratio of areas of two similar triangles is equal to the square of the ratio of their corresponding medians.

6. $\Delta ABC \sim \Delta DEF$. BC = 3cm EF = 4cm and area of $\Delta ABC = 54\ cm^2$. Determine the area of ΔDEF.

7. ABC is a triangle and PQ is a straight line meeting AB in P and AC in Q. If AP = 1 cm. and BP = 3cm., AQ = 1.5 cm., CQ = 4.5 cm.

   Prove that (area of ΔAPQ) = $\frac{1}{16}$ (area of ΔABC).

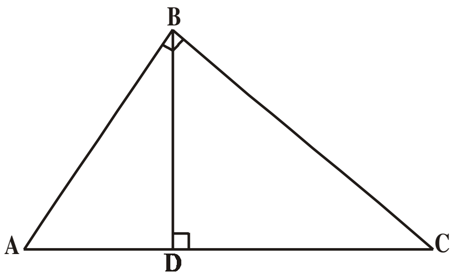
8. The areas of two similar triangles are $81cm^2$ and $49\ cm^2$ respectively. If the attitude of the bigger triangle is 4.5 cm. Find the corresponding attitude fo the smaller triangle.

8.6 Pythagoras Theorem

You are familar with the Pythagoras theorem, you had verified this theorem through some activities. Now we shall prove this theorem using the concept of similarity of triangles. For this, we make use of the following result.

Theorem-8.7 : If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, then the triangles on both sides of the perpendicular are similar to the whole triangle and to each other.

Proof: ABC is a right triangle, right angled at B. Let BD be the perpendicular to hypotenuse AC.



In ΔADB and ΔABC

$\angle A = \angle A$

And $\angle ADB = \angle ABC$ (why ?)

So $\Delta ADB \sim \Delta ABC$ (How ?) ...(1)

Similarly, $\Delta BDC \sim \Delta ABC$ (How ?) ...(2)

So from (1) and (2), triangles on both sides of the perpendicular BD are similar to the whole triangle ABC.

Also since $\Delta ADB \sim \Delta ABC$

$\Delta BDC \sim \Delta ABC$

So $\Delta ADB \sim \Delta BDC$

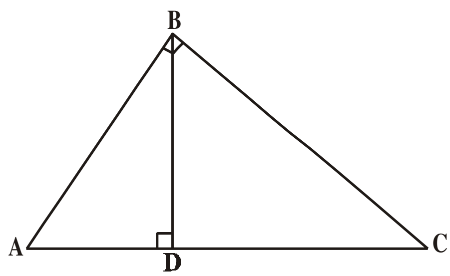
This leads to the following theorem.

Think - Discuss

For a right angled triangle with integer sides atleast one of its measurements must be an even number. Why? Discuss this with your friends and teachers.

8.6.1 Pythagoras Theorem (Baudhayan Theorem)

Theorem-8.8 : In a right triangle, the square of hypotenuse is equal to the sum of the squares of the other two sides.



Given: ΔABC is a right triangle right angled at B.

RTP : $AC^2 = AB^2 + BC^2$

Construction : Draw $BD \perp AC$.

Proof : $\Delta ADB \sim \Delta ABC$

$$\Rightarrow \frac{AD}{AB} = \frac{AB}{AC} \quad \text{(sides are proportional)}$$

$AD \cdot AC = AB^2$ ...(1)

Also, $\Delta BDC \sim \Delta ABC$

$$\Rightarrow \frac{CD}{BC} = \frac{BC}{AC}$$

$CD \,.\, AC = BC^2$ ...(2)

On adding (1) & (2)

$AD \,.\, AC + CD \,.\, AC = AB^2 + BC^2$

$AC\,(AD + CD) = AB^2 + BC^2$

$AC \,.\, AC = AB^2 + BC^2$

$$\boxed{AC^2 = AB^2 + BC^2}$$

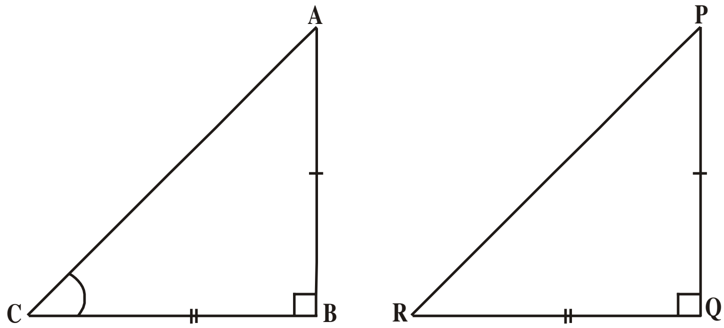
The above theorem was earlier given by an ancient Indian mathematician **Baudhayan** (about 800 BC) in the following form.

"The diagonal of a rectangle produces by itself the same area as produced by its both sides (i.e. length and breadth)." So sometimes, this theorem is also referred to as the Baudhayan theorem.

What about the converse of the above theorem ?

We prove it like a theorem, as done earlier also.

Theorem-8.9 : In a triangle if square of one side is equal to the sum of squares of the other two sides, then the angle opposite to the first side is a right angle and the triangle is a right angled triangle.



Given : In ΔABC,

$AC^2 = AB^2 + BC^2$

RTP : $\angle B = 90^0$.

Construction : Construct a right angled triangle ΔPQR right angled at Q such that PQ = AB and QR = BC.

Proof : In ΔPQR, $PR^2 = PQ^2 + QR^2$ (Pythagores theorem as $\angle Q = 90^0$)

$PR^2 = AB^2 + BC^2$ (by construction) ...(1)

but $AC^2 = AB^2 + BC^2$ (given) ...(2)

$\therefore$ AC = PR from (1) & (2)

Now In ΔABC and ΔPQR

AB = PQ (by construction)

BC = QE (by construction)

AC = PR (proved)

$\therefore \quad \Delta ABC \cong \Delta PQR$ (by SSS congruency)

$\therefore \quad \angle B = \angle Q$ (by cpct)

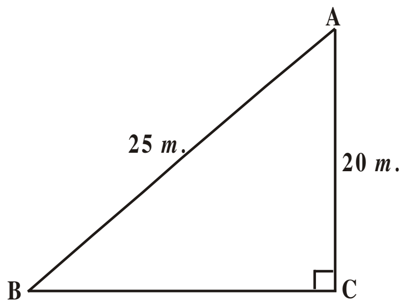
but $\angle Q = 90°$ (by construction)

$\therefore \quad \angle B = 90°$.

Hence proved.

Now let us take some examples.

Example-11. A ladder 25m long reaches a window of building 20m above the ground. Determine the distance of the foot of the ladder from the building.



Solution : In ΔABC, $\angle C = 90°$

$\Rightarrow AB^2 = AC^2 + BC^2$ (by Pythogorous theorem)

$25^2 = 20^2 + BC^2$

$BC^2 = 625 - 400 = 225$

$BC = \sqrt{225} = 15m$

Hence, the foot of the ladder is at a distance of 15m from the building.

Example-12. BL and CM are medians of a triangle ABC right angled at A.

Prove that $4(BL^2 + CM^2) = 5BC^2$.

Solution : BL and CM are medians of ΔABC in which $\angle A = 90°$.

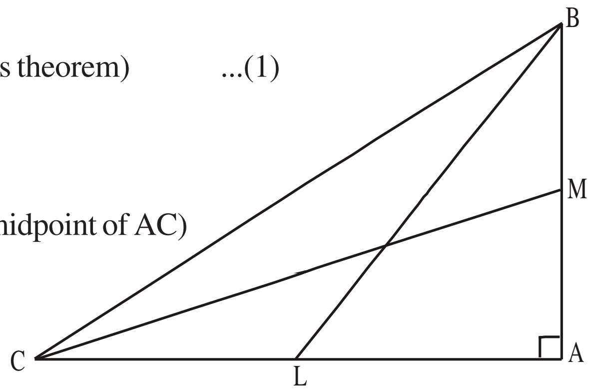
In ΔABC

$BC^2 = AB^2 + AC^2$ (Pythagores theorem) ...(1)

In ΔABL, $BL^2 = AL^2 + AB^2$

So $BL^2 = \left(\dfrac{AC}{2}\right)^2 + AB^2$ ($\because$ L is the midpoint of AC)

$BL^2 = \dfrac{AC^2}{4} + AB^2$



$\therefore \quad 4BL^2 = AC^2 + 4AB^2$ ...(2)

In ΔCMA, $CM^2 = AC^2 + AM^2$

$CM^2 = AC^2 + \left(\dfrac{AB}{2}\right)^2$ ($\because$ M is the mid point of AB)

$$CM^2 = AC^2 + \frac{AB^2}{4}$$

$$4CM^2 = 4AC^2 + AB^2 \qquad ...(3)$$

On adding (2) and (3), we get

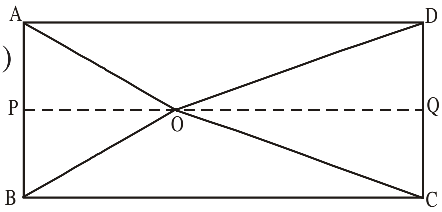
$$4(BL^2 + CM^2) = 5(AC^2 + AB^2)$$

$$\therefore\ 4(BL^2 + CM^2) = 5BC^2 \quad \text{from (1).}$$

Example-13. 'O' is any point inside a rectangle ABCD.

Prove that $OB^2 + OD^2 = OA^2 + OC^2$

Solution : Through 'O' draw PQ || BC so that P lies on AB and Q lies on DC.



Now PQ || BC

$\therefore$ PQ $\perp$ AB & PQ $\perp$ DC ($\because \angle B = \angle C = 90°$)

So, $\angle BPQ = 90°$ & $\angle CQP = 90°$

$\therefore$ BPQC and APQD are both rectangles.

Now from ΔOPB. $OB^2 = BP^2 + OP^2 \quad ...(1)$

Similarly from ΔOQD, we have $OD^2 = OQ^2 + DQ^2 \quad ...(2)$

From ΔOQC, we have $OC^2 = OQ^2 + CQ^2 \quad ...(3)$

And from ΔOAP, $OA^2 = AP^2 + OP^2$

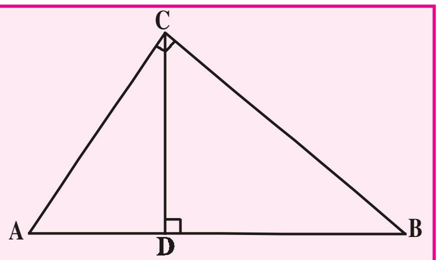
Adding (1) & (2)

$$\begin{aligned} OB^2 + OD^2 &= BP^2 + OP^2 + OQ^2 + DQ^2 \\ &= CQ^2 + OP^2 + OQ^2 + AP^2 \qquad (\because BP = CQ \text{ and } DQ = AP) \\ &= CQ^2 + OQ^2 + OP^2 + AP^2 \\ &= OC^2 + OA^2 \text{ (from (3) \& (4))} \end{aligned}$$

Do This

1. In ΔACB, $\angle C = 90^0$ and CD $\perp$ AB

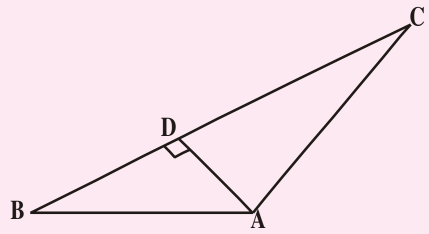
   Prove that $\dfrac{BC^2}{AC^2} = \dfrac{BD}{AD}$.



2. A ladder 15m long reaches a window which is 9 m above the ground on one side of a street. Keeping its foot at the same point, the ladder is turned to other side of the street to reach a window 12m high. Find the width of the street.

3. In the given fig. if AD $\perp$ BC

   Prove that $AB^2 + CD^2 = BD^2 + AC^2$.



Example-14. The hypotenuse of a right triangle is 6m more than twice of the shortest side. If the third side is 2 *m*., less than the hypotenuse, find the sides of the triangle.

Solution : Let the shortest side be x m.

Then htypotenuse = (2x + 6)m and third side = (2x + 4)m.

by Pythagores theorem, we have

$(2x + 6)^2 = x^2 + (2x + 4)^2$

$4x^2 + 24x + 36 = x^2 + 4x^2 + 16x + 16$

$x^2 - 8x - 20 = 0$

$(x - 10)(x + 2) = 0$

$x = 10$ or $x = -2$

but x can't be negative as side of a triangle.

$\therefore x = 10$

Hence, the sides of the triangle are 10m, 26m and 24m.

Example-15. ABC is a right triangle right angled at C. Let BC = a, CA = b, AB = c and let p be the length of perpendicular from C on AB. Prove that (i) pc = ab (ii) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Solution :

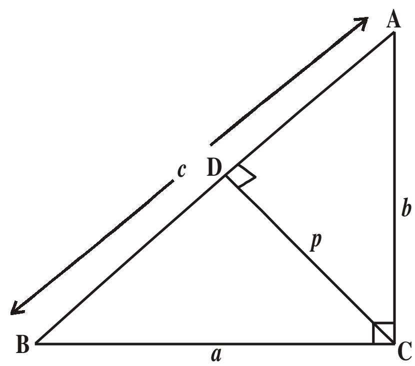
(i) CD $\perp$ AB and CD = p.

Area of $\Delta ABC = \frac{1}{2} \times AB \times CD$

$= \frac{1}{2} cp$.

also area of $\Delta ABC = \frac{1}{2} \times BC \times AC$

$= \frac{1}{2} ab$



$$\frac{1}{2}cp = \frac{1}{2}ab$$

$$cp = ab \qquad ...(1)$$

(ii) Since ΔABC is a right triangle right angled at C.

$$AB^2 = BC^2 + AC^2$$

$$c^2 = a^2 + b^2$$

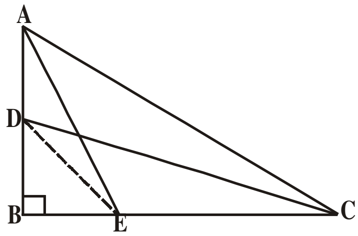
$$\left(\frac{ab}{p}\right)^2 = a^2 + b^2$$

$$\frac{1}{p^2} = \frac{a^2+b^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Exercise - 8.4

1. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.

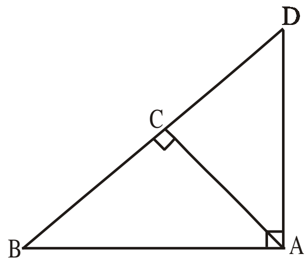
2. ABC is a right triangle right angled at B. Let D and E be any points on AB and BC respectively.
   Prove that $AE^2 + CD^2 = AC^2 + DE^2$.



3. Prove that three times the square of any side of an equilateral triangle is equal to four times the square of the altitude.

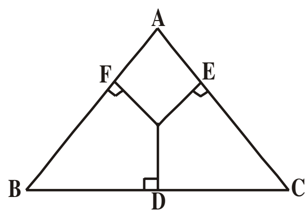
4. PQR is a triangle right angled at P and M is a point on QR such that $PM \perp QR$.
   Show that $PM^2 = QM \cdot MR$.

5. ABD is a triangle right angled at A and $AC \perp BD$
   Show that (i) $AB^2 = BC.BD$.
   (ii) $AC^2 = BC.DC$
   (iii) $AD^2 = BD.CD$.



6. ABC is an isosceles triangle right angled at C. Prove that $AB^2 = 2AC^2$.

7. 'O' is any point in the interior of a triangle ABC.
   $OD \perp BC$, $OE \perp AC$ and $OF \perp AB$, show that
   (i) $OA^2+OB^2+OC^2-OD^2-OE^2-OF^2 = AF^2+BD^2+CE^2$
   (ii) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$.

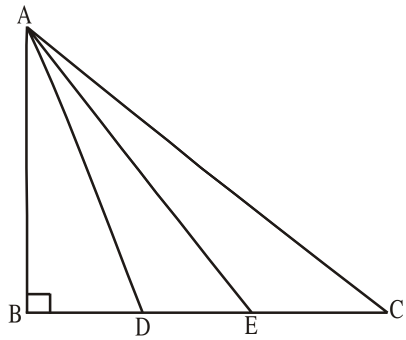


8. A wire attached to veritical pole of height 18m is 24m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?

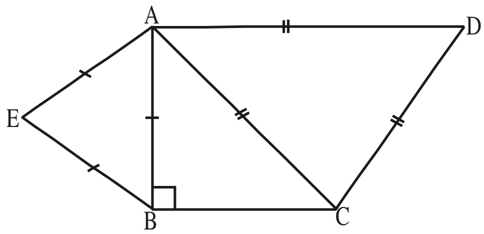
9. Two poles of heights 6m and 11m stand on a plane ground. If the distance between the feet of the poles is 12m find the distance between their tops.

10. In an equilateral triangle ABC, D is a point on side BC such that BD = $\frac{1}{3}$ BC. Prove that $9AD^2 = 7AB^2$.

11. In the given figure, ABC is a triangle right angled at B. D and E are ponts on BC trisect it. Prove that $8AE^2 = 3AC^2 + 5AD^2$.



12. ABC is an isosceles triangle right angled at B. Similar triangles ACD and ABE are constructed on sides AC and AB. Find the ratio between the areas of ΔABE and ΔACD.



8.7 Different forms of theoritical statements

1. Negation of a statement :

We have a statement and if we add "Not" after the statement, we will get a new statement; which is called negation of the statement.

For example take a statement "ΔABC is a equilateral". If we denote it by "*p*", we can write like this.

p : Triangle ABC is equilateral and its negation will be "Triangle ABC is not equilateral". Negation of statement *p* is denoted by $\sim p$; and read as negotiation of *p*. the statement $\sim p$ negates the assertion that the statement *p* makes.

When we write the negation of the statements we would be careful that there should no confusion; in understanding the statement.

Observe this example carefully

P : All irrational numbers are real numbers. We can write negation of *p* like these ways.

i) $\sim p$: All irrational numbers are not real numbers.

ii) $\sim p$ Not all the irrational are real numbers.

How do we decide which negation is correct? We use the following criterion "Let p be a statement and $\sim p$ its negation. Then $\sim p$ is false whenever p is true and $\sim p$ is true whenever p is false.

For example s : $2 + 2 = 4$ is True

$\sim s$: $2 + 2 \neq 4$ is False

2. Converse of a statement :

A sentence which is either true or false is called a simple statement. If we combine two simple statements then we will get a compound statement. Connecting two simple statements with the use of the words "If and then" will give a compound statement which is called implication (or) conditional.

Combining two simple statements p & q using if and then, we get p implies q which can be denoted by $p \Rightarrow q$. In this $p \Rightarrow q$, suppose we interchange p and q we get $q \Rightarrow p$. This is called its converse.

Example : $p \Rightarrow q$: In ΔABC, if AB = AC then $\angle C = \angle B$

Converse $q \Rightarrow p$: In ΔABC, if $\angle C = \angle B$ then AB = AC

3. Proof by contradiction :

In this proof by contradiction, we assume the negation of the statement as true; which we have to prove. In the process of proving we get contradiction somewhere. Then, we realize that this contradiction occur because of our wrong assumption which is negation is true. Therefore we conclude that the original statement is true.

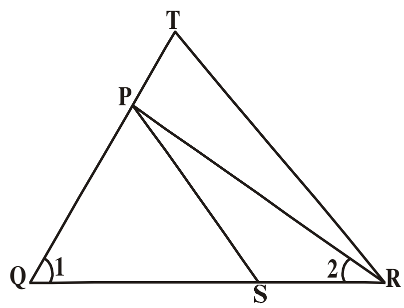
Optional Exercise

[This exercise is not meant for examination]

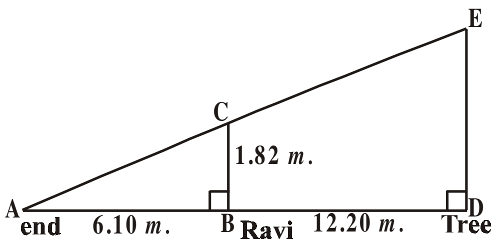
1. In the given figure,

$$\frac{QT}{PR} = \frac{QR}{QS} \text{ and } \angle 1 = \angle 2$$

prove that $\Delta PQS \sim \Delta TQR$.



2. Ravi is 1.82m tall. He wants to find the height of a tree in his backyard. From the tree's base he walked 12.20 *m*. along the tree's shadow to a position where the end of his shadow exactly overlaps the end of the tree's shadow. He is now 6.10m from the end of the shadow. How tall is the tree ?

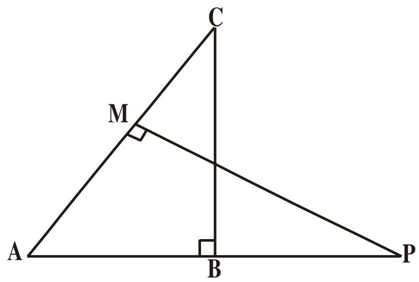


3. The diagonal AC of a parallelogram ABCD intersects DP at the point Q, where 'P' is any point on side AB. Prove that $CQ \times PQ = QA \times QD$.

4. ΔABC and ΔAMP are two right triangles right angled at B and M respectively.

   Prove that (i) $\Delta ABC \sim \Delta AMP$

   (ii) $\dfrac{CA}{PA} = \dfrac{BC}{MP}$.



5. An aeroplane leeves an airport and flies due north at a speed of 1000 kmph. At the same time another aeroplane leeves the same airport and flies due west at a speed of 1200 kmph. How far apart will the two planes be after $1\frac{1}{2}$ hour?

6. In a right triangle ABC right angled at C. P and Q are points on sides AC and CB respectively which divide these sides in the ratio of 2 : 1.

   Prove that (i) $9AQ^2 = 9AC^2 + 4BC^2$

   (ii) $9BP^2 = 9BC^2 + 4AC^2$

   (iii) $9(AQ^2 + BP^2) = 13AB^2$

What We Have Discussed

1. Two figures having the same shape but not necessarily of the same size are called similar figures.
2. All the congruent figures are similar but the converse is not true.
3. Two polygons of the same number of sides are similar

   If(i) their corresponding angles are equal and

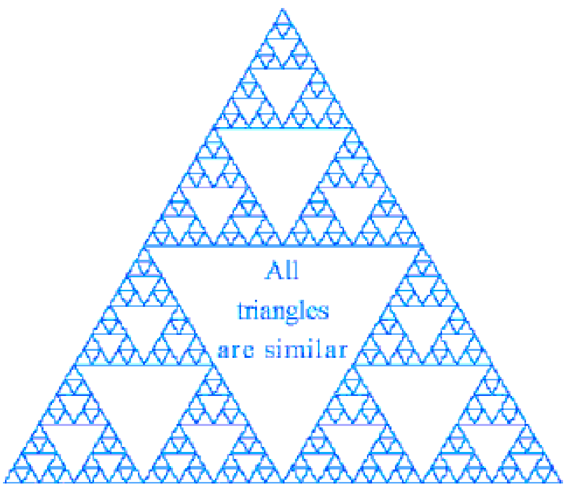
   (ii) Their corresponding sides are in the same ratio (ie propostion)

   For similarity of polygons either of the above two condition is not sufficient.
4. If a line is drawn parallel to one side of a triangle to intersect the other two sides at distinct points then the other two sides are divided in the same ratio.

5. If a line divides any two sides of a triangle in the same ratio, then the line is parllel to the third side.
6. In in two triangles, angles are equal, then their corresponding sides are in the same ratio and hence the two triangles are similar (AAA similarity)
7. If two angles of a triangle are equal to the two angles of another triangle, then third angles of both triangles are equal by angle sum property of at triangle.
8. In in two triangles, corresponding sides are in the serve ratio, then their corresponding angles are equal and hence the triangles are similar. (SSS similar)
9. If one angle of triangle is equal to one angle of another triangle and the sides including these angles are in the same ratio then the triangle are similar. (SAS similarity)
10. The ratio of areas of two similar triangles is equal to the square of the ratio of their corresponding sides.
11. If a perpendicular is drawn from the vertex of a right the triangles on both sides of the perpendicular are similar to the whole triangle and also to each other.
12. In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides (Pythogores Theorem).
13. In a triangle, square of one side is equal to the sum of the squares of the other two sides, then the angle opposite to the first side is a right angle.

Puzzle

Draw a triangle. Join the mid-point of the sides of the triangle. You get 4 triangles Again join the mid-points of these triangles. Repeat this process. All the triangles drawn are similar triangles. Why ? think and discuss with your friends.

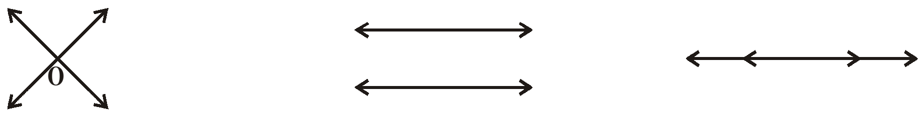


CHAPTER

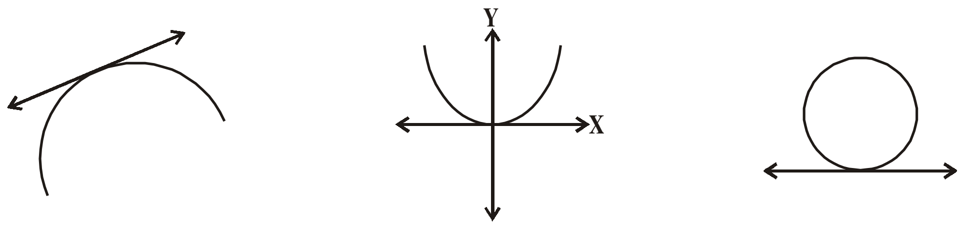
9 Tangents and Secants to a Circle

9.1 Introduction

We have seen two lines mostly intersect at a point or do not intersect in a plane. In some situations they coincide with each other.



Similarly, what happens when a curve and a line is given in a plane? You know a curve may be a parabola as you have seen in polynomials or a simple closed curve "circle" which is a collection of all those points on a plane that are at a constant distance from a fixed point.



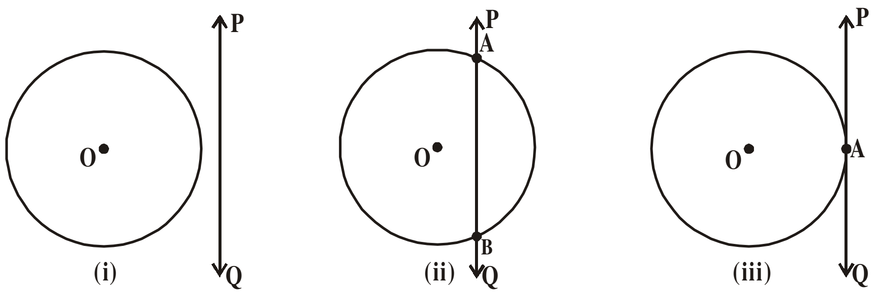
You might have seen circular objects rolling on a plane creating a path. For example; riding a bicycle, wheels of train on the track etc., where it seems to be a circle and a line. Does there a relation exist between them?

Let us see what happens, if a circle and a line are given in a plane.

9.1.1 A Line and A Circle

You are given a circle and a line drawn on a paper. Salman argues that there can only be 3 possible ways of presenting them.

Consider a circle 'O' and a line PQ, the three possibilities are given in figure below:



In Fig.(i), the line PQ and the circle have no common point. In this case PQ is a non-intersecting line with respect to the circle.

In Fig.(ii), the line PQ intersects the circle at two points A and B. It forms a **chord** on the circle AB with two common points. In this case the line PQ is a **secant** of the circle.

In Fig.(iii), There is only one point A, common to the line PQ and the circle. This line is called a **tangent** to the circle.

You can see that there cannot be any other position of the line with respect to the circle. We will study the existence of tangents to a circle and also study their properties and constructions.

Do yo know?

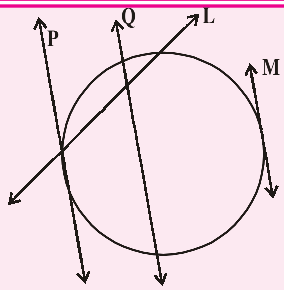
The word 'tangent' comes from the latin word 'tangere', which means to touch and was introduced by Danish mathematician Thomas Fineke in 1583.

Do This

i. Draw a circle with any radius. Draw four tangents at different points. How many tangents can you draw to this circle ?

ii. How many tangents you can draw to circle from a point away from it.

iii. Which of the following are tangents to the circles



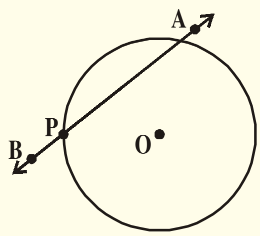
9.2 Tangents of a Circle

We can see that tangent can be drawn at any point lying on the circle. Can you say how many tangents can be drawn at any point on the surface of the circle.

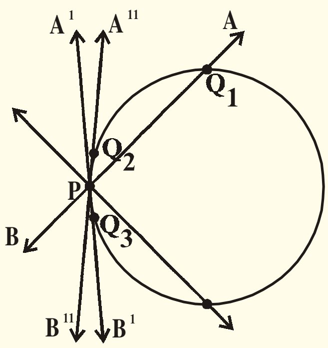
To understand this let us consider the following activity.

Activity

Take a circular wire and attach a straight wire AB at a point P of the circular wire, so that the system rotate about the point P in a plane.



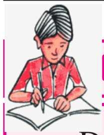
The circular wire represents a circle and the straight wire AB represents a line intersects the circle at point P.



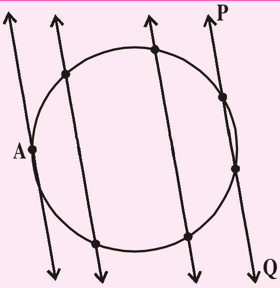
Put the system on a table and gently rotate the wire AB about the point P to get different positions of the straight wire as shown in the figure. The wire intersects the circular wire at P and at one more point through the points Q_1, Q_2 or Q_3 etc. So while it generally intersects circular wire at two points one of which is P in one particular position, it intersects the circle only at the point P (See position $A'B'$ of AB). This is the position of a tangent at the point P of the circle. You can check that in all other positions of AB it will intersect the circle at P and at another point, $A'B'$ is a tangent to the circle at P.

We see that there is only one tangent to the circle at point P.

Moving wire AB in either direction from this position makes it cut the circular wire in two points. All these are therefore secants. Tangent is a special case of a secant where the two points of intersection of a line with a circle coincide.



Do This



Draw a circle and a secant PQ of the circle on a paper as shown below. Draw various lines parallel to the secant on both sides of it.

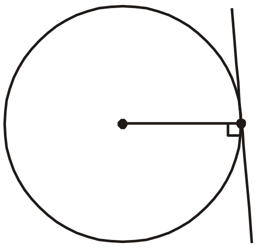
What happens to the length of chord coming closer and closer to the centre of the circle?

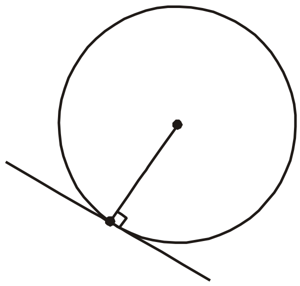
What is the longest chord?

How many tangents can you draw to a circle, which are parallel to each other ?

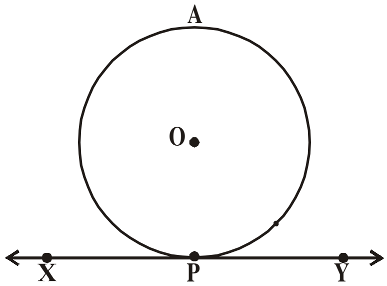
The common point of the tangent and the circle is called the point of contact and the tangent is said to touch the circle at the common point.

Observe the tangents to the circle in the figures given below:





How many tangents can you draw to a circle at a point? How many tangents can you obtain to the circle in all? See the points of contact. Draw radii from the points of contact. Do you see anything special about the angle between the tangents and the radii at the points of contact. All appear to be perpendicular to the corresponding tangents. We can also prove it. Let us see how.

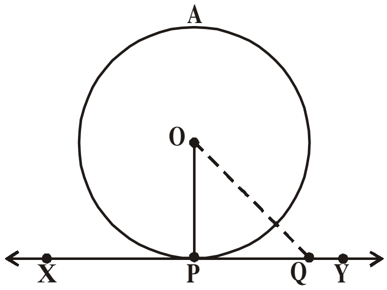


Theorem-9.1 : The tangent at any point of a circle is perpendicular to the radius through the point of contact.

Given : A circle with centre 'O' and a tangent XY to the circle at a point P.

To prove : OP is perpendicular to XY. (i.e OP $\perp$ XY)

Proof : Here, we will use the method that assumes that the statement is wrong and shows that such an assumption leads to a fallacy. So we will suppose OP is not perpendicular to XY. Take a point Q on XY other than P and join OQ.



The point Q must lie outside the circle (why?) (Note that if Q lies inside the circle, XY becomes a secant and not a tangent to the circle)

Therefore, OQ is longer than the radius OP of the circle [Why?]

i.e., $OQ > OP$.

This must happen for all points on the line XY. It is therefore true that OP is the shortest of all the distances of the point O to the points of XY.

So our assumption that OP is not perpendicular to XY is false. Therefore, OP is perpendicular to XY.

Hence proved.

Note : The line containing the radius through the point of contact is also called the 'normal to the circle at the point'.

Try This

How can you prove the converse of the above theorem.

"If a line in the plane of a circle is perpendicular to the radius at its endpoint on the circle, then the line is tangent to the circle".

We can find some more results using the above theorem

(i) Since there can be only one perpendicular OP at the point P, it follows that one and only one tangent can be drawn to a circle at a given point on the circumference.

(ii) Since there can be only one perpendicular to XY at the point P, it follows that the perpendicular to a tangent at its point of contact passes through the centre.

Think about these. Discuss these among your friends and with your teachers.

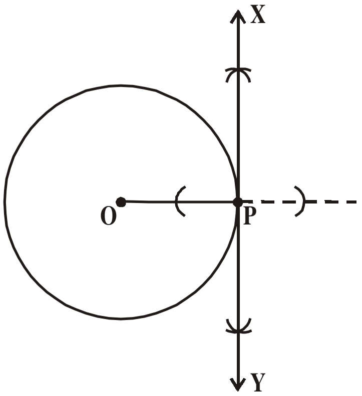
9.2.1 Construction of Tangent to a Circle

How can we construct a line that would be tangent to a circle at a given point on it? We use what we just found the tangent has to be perpendicular to the radius at the point of contact. To draw a tangent through the point of contact we need to draw a line prependicular to the radius at that point. To draw this radius we need to know the center of the circle. Let us see the steps for this construction.

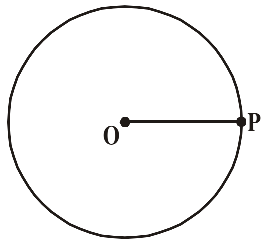
Construction : Construct a tangent to a circle at a given point when the centre of the circle is known.

We have a circle with centre 'O' and a point P anywhere on its circumference. Then we have to construct a tangent through P.

Steps of Construction :



1. Draw a circle with centre 'O' and mark a point 'P' anywhere on it. Join OP.



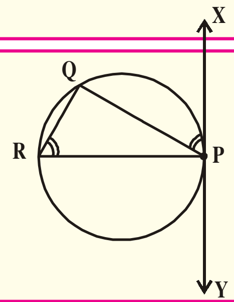
2. Draw a perpendicular line through the point P and name it as XY, as shown in the figure.
3. XY is the required tangent to the given circle passing through P.

Can you draw one more tangent through P ? give reason.

Try This

How can you draw the tangent to a circle at a given point when the centre of the circle is not known?

Hint : Draw equal angles $\angle QPX$ and $\angle PRQ$. Explain the construction.



9.2.2 Finding Length of the Tangent

Can we find the length of the tangent to a circle from a given point? Is the length of tangents from a given point to the circle the same? Let us examine this.

Example : Find the length of the tangent to a circle with centre 'O' and radius = 6 *cm*. from a point P such that OP = 10 *cm*.

Solution : Tangent is perpendicular to the radius at the point of contact (Theorem 9.1)

Here PA is tangent segment and OA is radius of circle

$\therefore$ $OA \perp PA \Rightarrow \angle OAP = 90°$

Now in ΔOAP, $OP^2 = OA^2 + PA^2$ (pythagoras theorem)

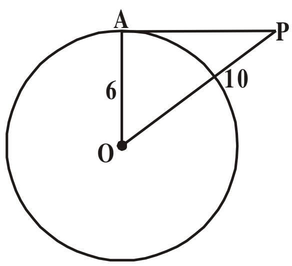
$$10^2 = 6^2 + PA^2$$

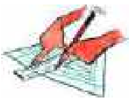
$$100 = 36 + PA^2$$

$$PA^2 = 100 - 36$$

$$= 64$$

$\therefore$ $PA = \sqrt{64} = 8$ *cm*.





EXERCISE - 9.1

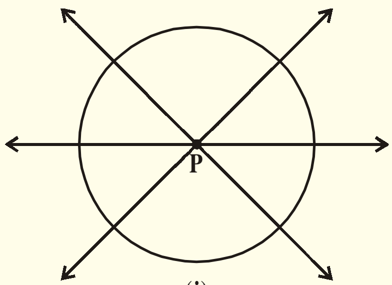
1. Fill in the blanks
   (i) A tangent to a circle intersects it in .... point (s).
   (ii) A line intersecting a circle in two points is called a ....
   (iii) A circle can have .... parallel tangents at the most.
   (iv) The common point of a tangent to a circle and the circle is called ....
   (v) We can draw .... tangents to a given circle.
2. A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q so that OQ = 12 cm. Find length of PQ.
3. Draw a circle and two lines parallel to a given line such that one is a tangent and the other, a secant to the circle.
4. Calculate the length of tangent from a point 15 *cm*. away from the centre of a circle of radius 9 *cm*.
5. Prove that the tangents to a circle at the end points of a diameter are parallel.

9.3 NUMBER OF TANGENT TO A CIRCLE FROM ANY POINT

To get an idea of the number of tangents from a point on a circle, Let us perform the following activity.

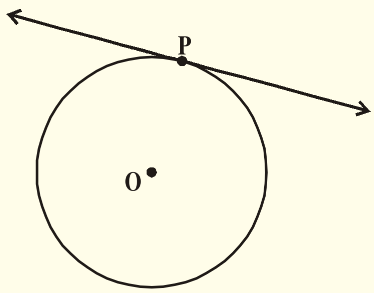
ACTIVITY

(i) Draw a circle on a paper. Take a point P inside it. Can you draw a tangent to the circle through this point ? You will find that all the lines through this point intersect the circle in two points. What are these ? These are all secants of a circle. So, it is not possible to draw any tangent to a circle through a point inside it. (See the adjacent figure)



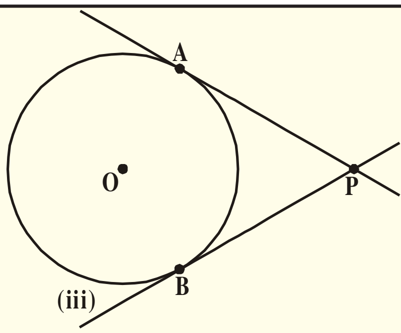
(i)

(ii) Next, take a point P on the circle and draw tangents through this point. You have observed that there is only one tangent to the circle at a such a point. (See the adjacent figure)



(ii)

(iii) Now, take a point P outside the circle and try to draw tangents to the circle from this point. What do you observe? You will find that you can draw exactly two tangents to the circle through this point (See the adjacent figure)



(iii)

Now, we can summarise these facts as follows :

Case (i) : There is no tangent to a circle passing through a point lying inside the circle.

Case(ii) : There is one and only one tangent to a circle passing through a point lying on the circle.

Case(iii) : There are exactly two tangents to a circle through a point lying outside the circle in this case, A and B are the points of contacts of the tangents PA and PB respectively.

The length of the segment from the external point P and the point of contact with the circle is called the length of the tangent from the point P to the circle.

Note that in the above figure (iii), PA and PB are the length of the tangents from P to the circle. What is the relation between lengths PA and PB.

Theorem-9.2 : The lengths of tangents drawn from an external point to a circle are equal.

Given : A circle with centre O, P is a point lying outside the circle and PA and PB are two tangents to the circle from P. (See figure)

To prove : PA = PB

Proof : Join OA, OB and OP.

$\angle OAP = \angle OBP = 90^0$ (Angle between radii and tangents according to theorem 9.1)

Now in the two right triangles ΔOAP and ΔOBP,

OA = OB (radii of same circle)

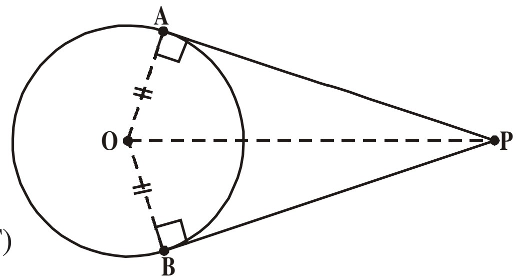
OP = OP (Common)

Therfore, By R.H.S. Congruency axiom,

$\Delta OAP \cong \Delta OBP$

This gives PA = PB (CPCT)

Hence Proved.



TRY THIS

Use pythagoras theorem and write proof of above theorem.

9.3.1. Construction of Tangents to a Circle from an External point

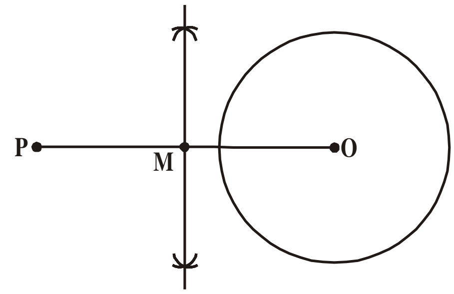
You saw that if a point lies outside the circle, there will be exactly two tangents to the circle from this point. We shall now see how to draw these tangents.

Construction : To construct the tangents to a circle from a point outside it.

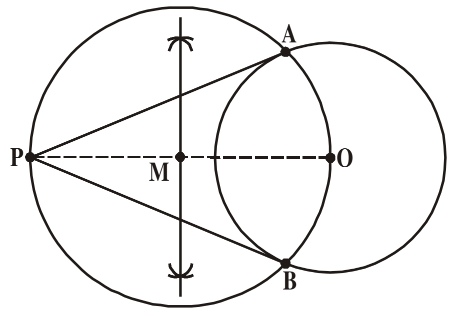
Given : We are given a circle with centre 'O' and a point P outside it. We have to construct two tangents from P to the circle.

Steps of construction :

Step(i) : Join PO and draw a perpendicular bisector of it. Let M be the midpoint of PO.



Step (ii) : Taking M as centre and PM or MO as radius, draw a circle. Let it intersect the given circle at the points A and B.



Step (iii) : Join PA and PB. Then PA and PB are the required two tangents.

Proof : Now, Let us see how this construction is justified.

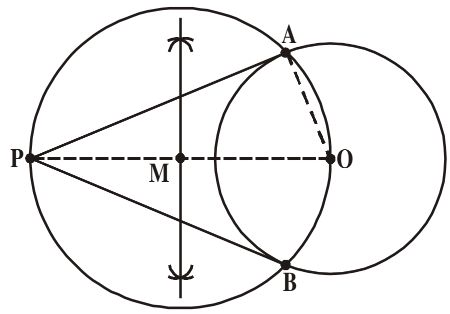
Join OA. Then $\angle PAO$ is an angle in the semicircle and,

therefore, $\angle PAO = 90°$.

Can we say that $PA \perp OA$?

Since, OA is a radius of the given circle, PA has to be a tangent to the circle (By converse theorem of 9.1)

Similarly, PB is also a tangent to the circle.



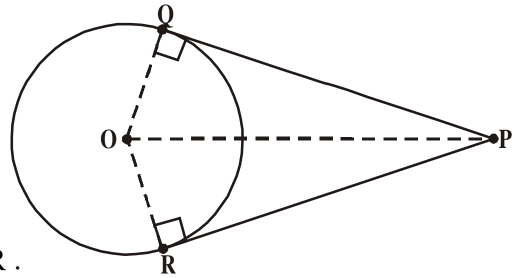
Hence proved.

Some interesting statements about tangents and secants and their proof:

Statement-1 : The centre of a circle lies on the bisector of the angle between two tangents drawn from a point outside it. Can you think how we can prove it?

Proof : Let PQ and PR be two tangents drawn from a point P outside of the circle with centre O

Join OQ and OR, triangles OQP and ORP are congruent because we know that,



$\angle OQP = \angle ORP = 90^o$ (Theorem 9.1)

OQ = OR (Radii)

OP is common.

This means $\angle OPQ = \angle OPR$ (CPCT)

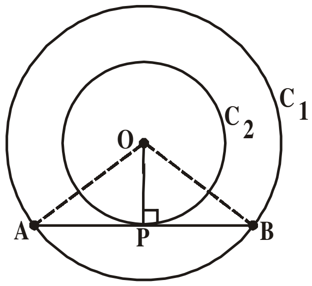
Therefore, OP is the angle bisector of $\angle QPR$.

Hence, the centre lies on the bisector of the angle between the two tangents.

Statement-2 : In two concentric circles, such that a chord of the bigger circle, that touches the smaller circle is bisected at the point of contact with the smaller circle.

Can you see how is this?

Proof : We are given two concentric circles C_1 and C_2 with centre O and a chord AB of the larger circle C_1, touching the smaller circle C_2 at the point P (See figure) we need to prove that AP = PB.



Join OP.

Then AB is a tangent to the circle C_2 at P and OP is its radius.

Therefore, by Theorem 9.1

$$OP \perp AB$$

Now, ΔOAP and ΔOBP are congruent. (Why?) This means AP = PB. Therefore, OP is the bisector of the chord AB, as the perpendicular from the centre bisects the chord.

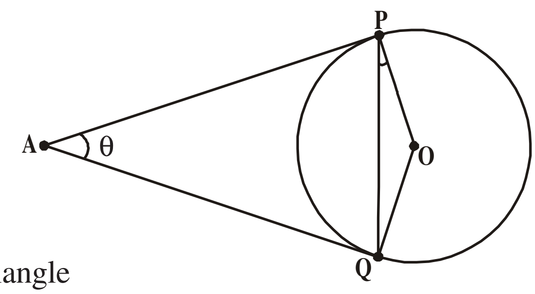
Statement-3 : If two tangents AP and AQ are drawn to a circle with centre O from an external point A then $\angle PAQ = 2\angle OPQ = 2\angle OQP$.

Can you see?

Proof : We are given a circle with centre O, an external point A and two tangents AP and AQ to the circle, where P, Q are the points of contact (See figure).

We need to prove that

$\angle PAQ = 2\angle OPQ$



Let $\angle PAQ = \theta$

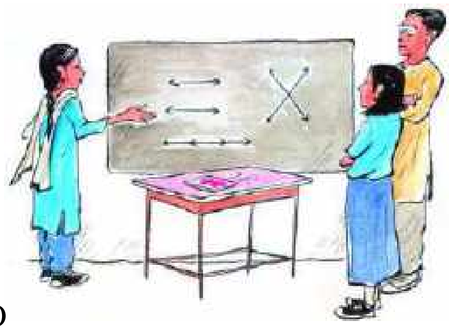
Now, by Theorem 9.2,

AP = AQ, So ΔAPQ is an isosceles triangle

Therefore, $\angle APQ + \angle AQP + \angle PAQ = 180°$ (Sum of three angles)

$$\angle APQ = \angle AQP = \frac{1}{2}(180° - \theta)$$

$$= 90° - \frac{1}{2}\theta$$



Also, by Theorem 9.1,

$\angle OPA = 90°$

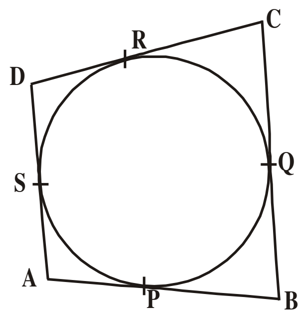
So, $\angle OPQ = \angle OPA - \angle APQ$

$$= 90° - \left[90 - \frac{1}{2}\theta\right] = \frac{1}{2}\theta = \frac{1}{2}\angle PAQ$$

This gives $\angle OPQ = \frac{1}{2}\angle PAQ$.

$\therefore$ $\angle PAQ = 2\angle OPQ$. Similarly $\angle PAQ = 2\angle OQP$

Statement-4 : If a circle touches all the four sides of a quadrilateral ABCD at points PQRS. Then AB+CD = BC + DA



Can you think how do we proceed? AB, CD, BC, DA are all chords to a circle.

For the circle to touch all the four sides of the quadrilateral at points P, Q, R, S, it has to be inside the quadrilateral. (See figure)

How do we proceed further?

Proof : The circle touched the sides AB, BC, CD and DA of Quadrilateral ABCD at the points P, Q, R and S respectively as shown

Since by theorem 9.2, the two tangents to a circle drawn from a point outside it, are equal,

AP = AS

BP = BQ

DR = DS

and CR = CQ

On adding, We get

AP + BP + DR + CR = AS + BQ + DS + CQ

or (AP + PB) + (CR + DR) = (BQ + QC) + (DS + SA)

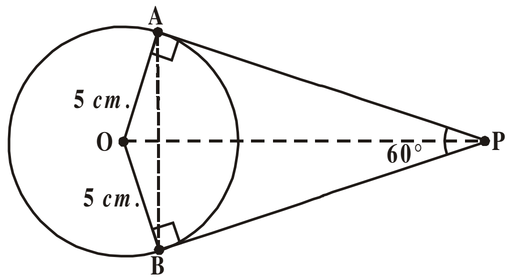
or AB + CD = BC + DA.

Let us do an example of analysing a situation and know how we would construct something.

Example-1. Draw a pair of tangents to a circle of radius 5cm which are inclined to each other at an angle 60°.

Solution : To draw the circle and the two tangents we need to see how we proceed. We only have the radius of the circle and the angle between the tangents. We do not know the distance of the point from where the tangents are drawn to the circle and we do not know the length of the tangents either. We know only the angle between the tangents. Using this, we need to find out the distance of the point outside the circle from which we have to draw the tangents.

To begin let us consider a circle with centre 'O' and radius 5cm. Let PA and PB are two tangents draw from a point 'P' outside the circle and the angle between them is 60^o. In this $\angle APB = 60^o$. Join OP.



As we know,

OP is the bisector of $\angle APB$,

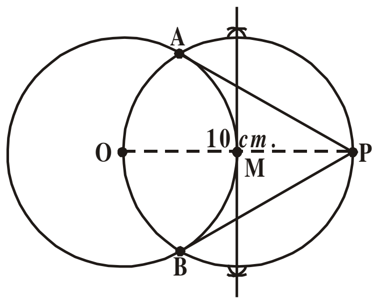
$$\angle OAP = \angle OPB = \frac{60^o}{2} = 30^o \quad (\because \ \Delta OAP \cong \Delta OBP)$$

Now In ΔOAP,

$$\sin 30^o = \frac{\text{Opp. side}}{\text{Hyp}} = \frac{OA}{OP}$$

$$\frac{1}{2} = \frac{5}{OP} \text{ (From trigonometric ratio)}$$

$OP = 10$ cm.

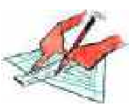


Now we can draw a circle of radius 5 cm with centre 'O'. We then mark a point at a distance of 10 cm from the centre of the circle. Join OP and complete the construction as given in construction 9.2. Hence PA and PB are the required pair of tangents to the given circle.

You can also try this construction without using trigonometric ratio.

Try This

Draw a pair of radii OA and OB such that $\angle BOA = 120^o$. Draw the bisector of $\angle BOA$ and draw lines perpendiculars to OA and OB at A and B. These lines meet on the bisector of $\angle BOA$ at a point which is the external point and the perpendicular lines are the required tangents. Construct and Justify.



Exercise - 9.2

1. Choose the correct answer and give justification for each.

(i) The angle between a tangent to a circle and the radius drawn at the point of contact is

(a) 60° (b) 30° (c) 45° (d) 90°

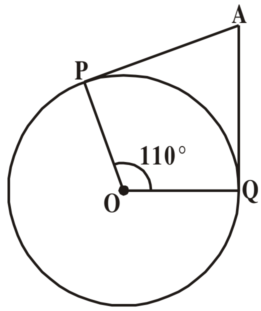
(ii) From a point Q, the length of the tangent to a circle is 24 *cm*. and the distance of Q from the centre is 25 cm. The radius of the circle is

(a) 7cm (b) 12 cm (c) 15cm (d) 24.5cm

(iii) If AP and AQ are the two tangents a circle with centre O so that $\angle POQ = 110°$, then $\angle PAQ$ is equal to

(a) 60° (b) 70°

(c) 80° (d) 90°



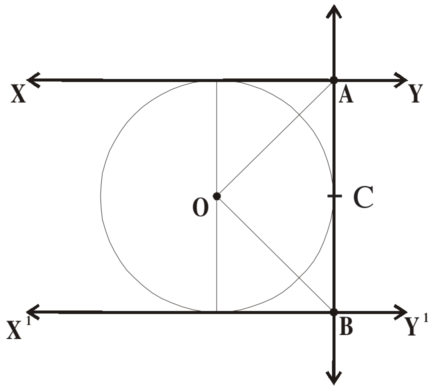
(iv) If tangents PA and PB from a point P to a circle with centre O are inclined to each other at angle of 80°, then $\angle POA$ is equal to

(a) 50° (b) 60° (c) 70° (d) 80°

(v) In the figure XY and X^1Y^1 are two parallel tangents to a circle with centre O and another tangent AB with point of contact C intersecting XY at A and X^1Y^1 at B then $\angle AOB$ =

(a) 80° (b) 100°

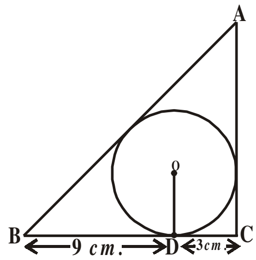
(c) 90° (d) 60°



2. Two concentric circles are radii 5 cm and 3cm are drawn. Find the length of the chord of the larger circle which touches the smaller circle.

3. Prove that the parallelogram circumscribing a circle is a rhombus.

4. A triangle ABC is drawn to circumscribe a circle of radius 3 *cm*. such that the segments BD and DC into which BC is divided by the point of contact D are of length 9 *cm*. and 3 *cm*. respectively (See adjacent figure). Find the sides AB and AC.

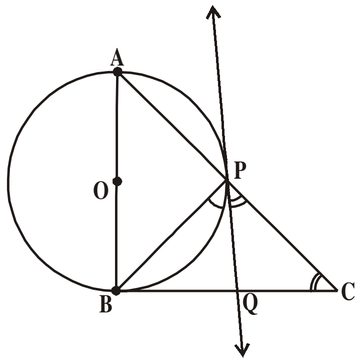


5. Draw a circle of radius 6cm. From a point 10 cm away from its centre, construct the pair of tangents to the circle and measure their lengths. Verify by using Pythogoras Theorem.

6. Construct a tangent to a circle of radius 4cm from a point on the concentric circle of radius 6cm and measure its length. Also verify the measurement by actual calculation.

7. Draw a circle with the help of a bangle, Take a point outside the circle. Construct the pair of tangents from this point to the circle measure them. Write conclusion.

8. In a right triangle ABC, a circle with a side AB as diameter is drawn to intersect the hypotenuse AC in P. Prove that the tangent to the circle at P bisects the side BC.



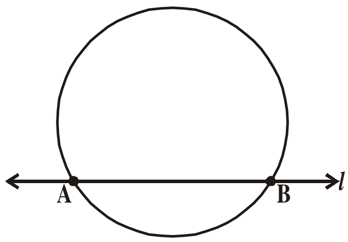
9. Draw a tangent to a given circle with center O from a point 'R' outside the circle. How many tangents can be drawn to the circle from that point?

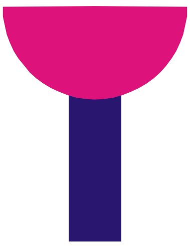
   Hint : The distance of two points to the point of contact is the same.

9.4 Segment of a circle formed by a secant

We have seen a line and a circle. When a line meets a circle in only one point, it is a tangent. A secant is a line which intersects the circle at two distint points represented in the chord.

Here '*l*' is the secant and AB is the chord.

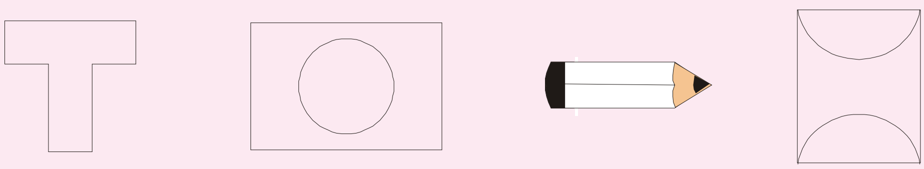




Shankar is making a picture by sticking pink and blue paper. He makes many pictures. One picture he makes is of washbasin. How much paper does he need to make this picture? This picture can be seen in two parts. A rectangle is there, but what is the remaining part? It is the segment of the circle. We know how to find the area of rectangle. How do we find the area of the segment? In the following discussion we will try to find this area.

Do This

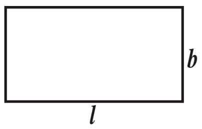
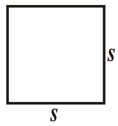
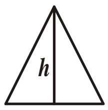
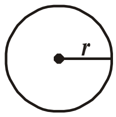
Shankar made the following pictures also along with washbasin.



What shapes can they be broken into that we can find area easily?

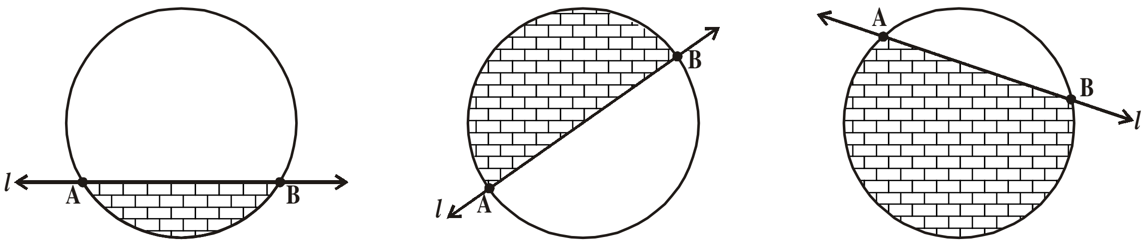
Make some more pictures and think of the shapes they can be divided into different parts.

Lets us recall how to find the area of the following geometrical figures as given in the table.

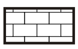
S.No.	Figure	Dimentions	Area
1.		length = l breadth = b	$A = lb$
2.		Side = s	$A = s^2$
3.		base = b	$A = \frac{1}{2} bh$
4.		radius = r	$A = \pi r^2$

9.4.1. Finding the Area of Segment of a Circle

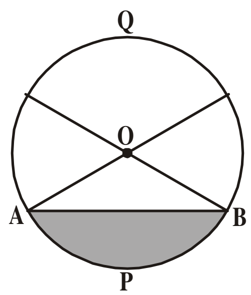
To estimate the area of segment of a circle, Swetha made the segments by drawing secants to the circle.



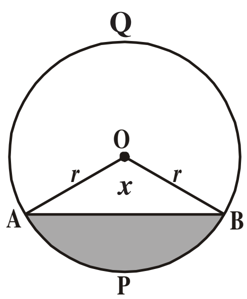
(i) (ii) (iii)

As you know a segment is a region, bounded by the arc and a chord, we can see the area that is shaded () in fig.(i) is a minor segment, semicircle in fig.(ii) and major segment in fig.(iii).

How do we find the area of the segment? Do the following activity.



Take a circular shaped paper and fold it along with a chord less than the diameter and shade the smaller part as shown in in the figure. What do we call this smaller part? It is a minor segment (APB). What do we call the unshaded portion of the circle? Obviously it is a major segment (AQB).



You have already come across the sector and segment in earlier classes. The portion of some unshaded part and shaded part (minor segment) is a sector which is the combination of a triangle and a segment.

Let OAPB be a sector of a circle with centre O and radius 'r' as shown in the figure. Let the angle measure of $\angle$AOB be 'x'.

You know that area of circle when the angle measure at the centre is 360° is πr^{2}.

So, when the degree measure of the angle at the centre is 1°, then area of sector is $\frac{1°}{360°} \times \pi r^2$.

Therefore, when the degree measure of the angle at the centre is $x°$, the area of sector is $\frac{x°}{360°} \times \pi r^2$.

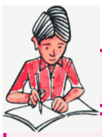
Now let us take the case of the area of the segment APB of a circle with centre 'O' and radius 'r', you can see that

Area of the segment APB = Area of the sector OAPB - Area of ΔOAB

$$= \frac{x°}{360°} \times \pi r^2 - \text{area of } \Delta OAB$$

Try This

How can you find the area of major segment using area of minor segment?



Do This

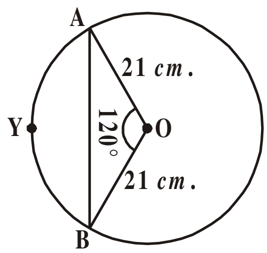
1. Find the area of sector, whose radius is 7 *cm*. with the given angle:

   i. 60° ii. 30° iii. 72° iv. 90° v. 120°

2. The length of the minute hand of a clock is 14 *cm*. Find the area swept by the minute hand in 10 minutes.

Now, we will see an example to find area of segment of a circle.

Example-1. Find the area of the segment AYB showing in the adjacent figure. If radius of the circle is 21 cm and $\angle AOB = 120^0$ (Use $\pi = \frac{22}{7}$ and $\sqrt{3} = 1.732$)



Solution : Area of the segment AYB

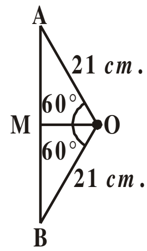
= Area of sector OAYB – Area of ΔOAB

Now, area of the sector OAYB = $\frac{120^0}{360^0} \times \frac{22}{7} \times 21 \times 21 \text{ cm}^2$

$= 462 \text{ cm}^2$ ...(1)

For finding the area of ΔOAB, draw OM $\perp$ AB as shown in the figure:-

Note OA = OB. Therefore, by RHS congruence, $\Delta AMO \cong \Delta BMO$



So, M is the midpoint of AB and $\angle AOM = \angle BOM = \frac{1}{2}\times 120^0 = 60^0$

Let, $OM = x$ cm

So, from ΔOMA, $\frac{OM}{OA} = \cos 60^0$.

or, $\frac{x}{21} = \frac{1}{2}$ $\left(\because \cos 60^0 = \frac{1}{2}\right)$

or, $x = \frac{21}{2}$

So, $OM = \frac{21}{2}$ cm

Also, $\frac{AM}{OA} = \sin 60°$

$\frac{AM}{21} = \frac{\sqrt{3}}{2}$ $\left(\because \sin 60^0 = \frac{\sqrt{3}}{2}\right)$

So, $AM = \frac{21\sqrt{3}}{2}$ cm.

Therefore $AB = 2AM = \frac{2\times 21\sqrt{3}}{2} cm. = 21\sqrt{3}$ cm

So, Area of $\Delta OAB = \frac{1}{2}\times AB \times OM$

$= \frac{1}{2}\times 21\sqrt{3}\times\frac{21}{2}$ cm^2.

$= \frac{441}{4}\sqrt{3}$ cm^2. ...(2)

Therefore, area of the segment AYB $= \left(462 - \frac{441}{4}\sqrt{3}\right)$cm^2.

($\because$ from (1), (2)]

$= \frac{21}{4}\left(88 - 21\sqrt{3}\right)$ cm^2

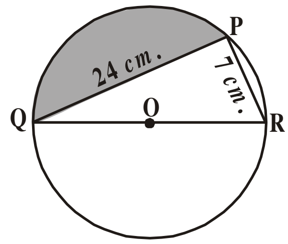
$= 271.047\ cm^2$

Example-2. Find the area of the segments shaded in figure, if PQ = 24 *cm.*, PR = 7 *cm.* and QR is the diameter of the circle with centre O (Take $\pi = \frac{22}{7}$)

Solution : Area of the segments shaded = Area of sector OQPR - Area of triangle PQR.

Since QR is diameter, $\angle QPR = 90°$ (Angle in a semicircle)

So, using pythagoras Theorem



In ΔQPR, $QR^2 = PQ^2 + PR^2$

$= 24^2 + 7^2$

$= 576 + 49$

$= 625$

$QR = \sqrt{625} = 25\ cm.$

Then radius ofthe circle $= \frac{1}{2}\ QR$

$= \frac{1}{2}\ (25) = \frac{25}{2}\ cm.$

Now, area of semicircle OQPR $= \frac{1}{2}\ \pi r^2$

$= \frac{1}{2} \times \frac{22}{7} \times \frac{25}{2} \times \frac{25}{2}$

$= 327.38\ cm^2$ ..... (1)

Area of right angled triangle QPR $= \frac{1}{2} \times PR \times PQ$

$= \frac{1}{2} \times 7 \times 24$

$= 84\ cm^2$ ..... (2)

From (1) and (2),

Area of the shaded segments $= 327.38 - 84$

$= 243.38\ cm^2$

Example-3. A round table top has six equal designs as shown in the figure. If the radius of the table top is 14 *cm.*, find the cost of making the designs with paint at the rate of ₹5 per cm^2. (use $\sqrt{3} = 1.732$)

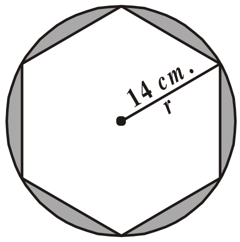
Solution : We know that the radius of circumscribing circle of a regular hexagon is equal to the length of its side.

$\therefore$ Each side of regular hexagon = 14 *cm*.

Therefore, Area of six design segments = Area of circle - Area of the regular hexagon.

Now, Area of circle $= \pi r^2$

$$= \frac{22}{7} \times 14 \times 14 = 616\ cm^2 \quad ..... (1)$$



Area of regular hexagon $= 6 \times \frac{\sqrt{3}}{4}\ a^2$

$$= 6 \times \frac{\sqrt{3}}{4} \times 14 \times 14$$

$$= 509.2\ cm^2 \quad ..... (2)$$

Hence, area of six designs $= 616 - 509.21$ (from (1), (2)

$$= 106.79\ cm^2.$$

Therefore, cost of painting the design at the rate of ₹5 per cm^2

$$= ₹106.79 \times 5$$

$$= ₹533.95$$

Exercise - 9.3

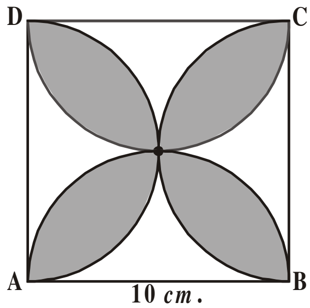
1. A chord of a circle of radius 10 *cm*. subtends a right angle at the centre. Find the area of the corresponding: (use $\pi = 3.14$)

   i. Minor segment ii. Major segment

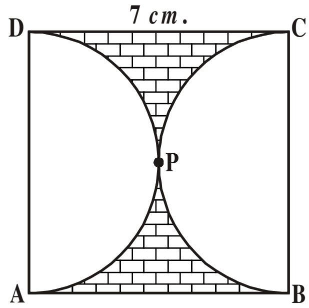
2. A chord of a circle of radius 12 *cm*. subtends an angle of 120° at the centre. Find the area of the corresponding minor segment of the circle (use $\pi = 3.14$ and $\sqrt{3} = 1.732$)

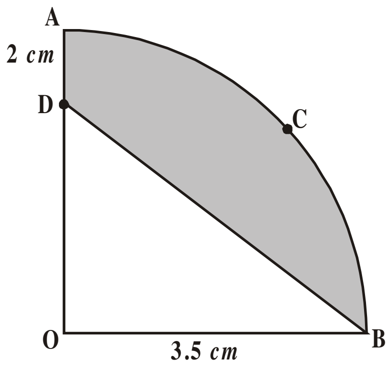
3. A car has two wipers which do not overlap. Each wiper has a blade of length 25 *cm*. sweeping through an angle of 115°. Find the total area cleaned at each sweep of the blades. (use $\pi = \frac{22}{7}$)

4. Find the area of the shaded region in figure, where ABCD is a square of side 10 *cm*. and semicircles are drawn with each side of the square as diameter (use $\pi = 3.14$)



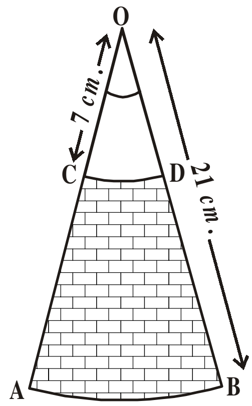
5. Find the area of the shaded region in figure, if ABCD is a square of side 7 *cm*. and APD and BPC are semicircles. (use $\pi = \frac{22}{7}$)

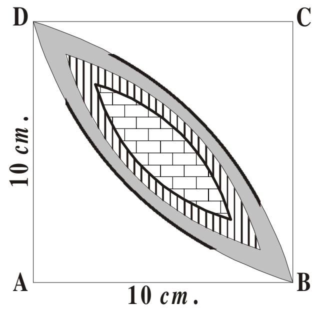




6. In figure, OACB is a quadrant of a circle with centre O and radius 3.5 *cm*. If OD = 2 *cm*., find the area of the shaded region. (use $\pi = \frac{22}{7}$)

7. AB and CD are respectively arcs of two concentric circles of radii 21 *cm*. and 7 *cm*. with centre O (See figure). If $\angle AOB = 30°$, find the area of the shaded region. (use $\pi = \frac{22}{7}$)



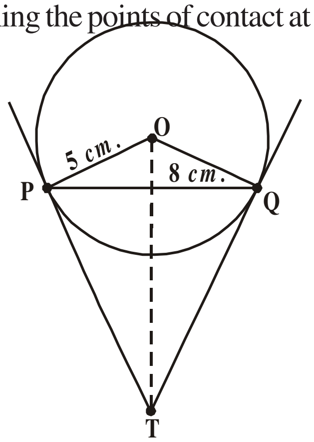


8. Calculate the area of the designed region in figure, common between the two quadrants of the circles of radius 10 *cm*. each. (use $\pi = 3.14$)

Optional Exercise

[This exercise is not meant for examination]

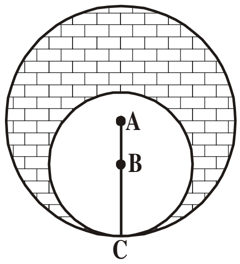
1. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line - segment joining the points of contact at the centre.
2. PQ is a chord of length 8cm of a circle of radius 5cm. The tangents at P and Q intersect at a point T (See figure). Find the length of TP.

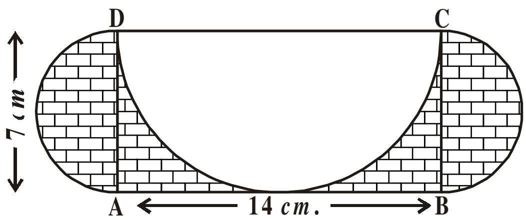


3. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.
4. Draw a line segment AB of length 8cm. Taking A as centre, draw a circle of radius 4cm and taking B as centre, draw another circle of radius 3cm. Construct tangents to each circle from the centre of the other circle.

5. Let ABC be a right triangle in which AB = 6cm, BC = 8 cm and $\angle B = 90^0$. BD is the perpendicular from B on AC. The circle through B, C, D is drawn. Construct the tangents from A to this circle.

6. Find the area of the shaded region in the figure, given in which two circles with centres A and B touch each other at the point C. If AC = 8 *cm*. and AB = 3 *cm*.





7. ABCD is a rectangel with AB = 14 *cm*. and BC = 7 *cm*. Taking DC, BC and AD as diameters, three semicircles are drawn as shown in the figure. Find the area of the shaded region.

What We Have Discussed

In this chapter, we have studied the following points.

1. The meaning of a tangent to a circle and a secant. We have used the idea of the chord of a circle.
2. We used the ideas of different kinds of triangles particularly right angled triangles and isosceles traingles.
3. We learn the following:

   a) The tangents to a circle is prependicular to the radius through the point of contact.

   b) The lengths of the two tangents from an external point to a circle are equal.
4. We learnt to do the following:

   a) To construct a tangent to a circle at a given point when the centre of the circle is known.

   b) To construct the pair of tangents from an external point to a circle.
5. We learnt to understand how to prove some statements about circles and tangents. In this process learnt to use previous results and build on them logically to from new results.
6. We have learnt

   Area of segment of a circle = Area of the corresponding sector - Area of the corresponding triangle.

CHAPTER 10

Mensuration

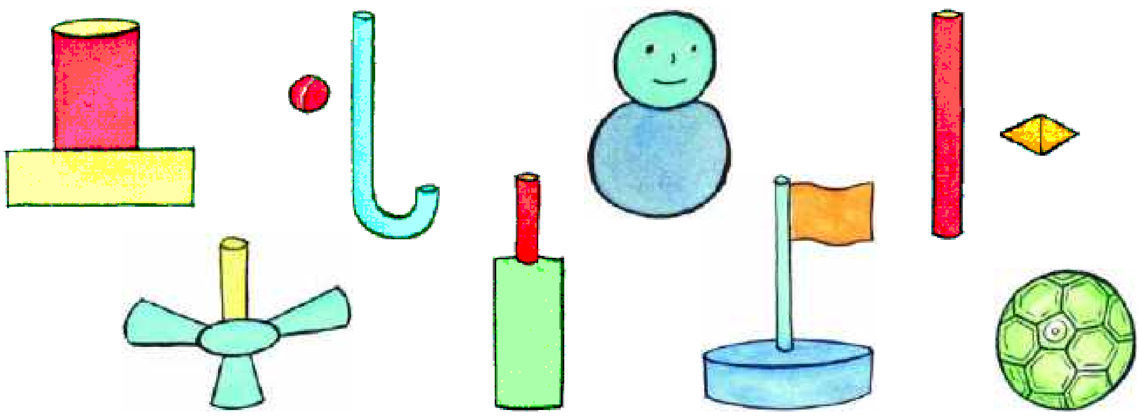
10.1 INTRODUCTION

In classes VIII and IX, we have learnt about area, surface area and volume of solid shapes. We did many exercises to understand what they mean. We used them in real life situations and identified what we needed and what was to be measured or estimated. For example, to find the quantity of paint required to white wash a room, we need the surface area and not the volume. To find the number of boxes that would contains a quantity of grain, we need the volume and not the area.

TRY THIS

1. Consider the following situations. In each find out whether you need volume or area and why?
   - i. Quantitiy of water inside a bottle.
   - ii. Canvas needed for making a tent.
   - iii. Number of bags inside the lorry.
   - iv. Gas filled in a cylinder.
   - v. Number of match sticks that can be put in the match box.
2. Compute 5 more such examples and ask your friends to choose what they need?

We see so many things of different shapes (combination of two or more) around us. Houses stand on pillars, storage water tanks are cylindrical and are placed on cuboidal foundations, a cricket bat has a cylindrical handle and a flat main body, etc. Think of different things around you. Some of these are shown below:

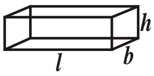
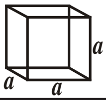
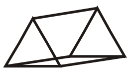
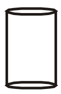
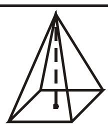
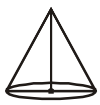
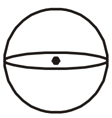
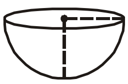


Of these objects like football have shapes where we know that the surface area and volume. We can however see that other objects can be seen as combinations of the solid shapes. So, their surface area and volume we now have to find. The table of the solid shapes, their areas and volumes are given later.

TRY THIS

1. Break the pictures in the previous figure into solids of known shapes.
2. Think of 5 more things around you that can be seen as a combination of shapes. Name the shapes that combine to make them.

Let us recall the surface areas and volumes of different solid shapes.

S. No.	Name of the solid	Figure	Lateral / Curved surface area	Total surface area	Volume	Nomen-clature
1.	Cuboid		$2h(l+b)$	$2(lb+bh+hl)$	lbh	l:length b:breadth h:height
2.	Cube		$4a^2$	$6a^2$	a^3	a:side of the cube
3.	Right prism		Perimeter of base $\times$ height	Lateral surface area+2(area of the end surface)	area of base $\times$height	-
4.	Regular circular Cylinder		$2\pi rh$	$2\pi r(r+h)$	πr^2h	r:radius of the base h:height
5.	Right pyramid		$\frac{1}{2}$ (perimeter of base)$\times$ slant height	Lateral surfaces area+area of the base	$\frac{1}{3}$ area of the base $\times$height	-
6.	Right circular cone		πrl	$\pi r(l+r)$	$\frac{1}{3}\pi r^2h$	r:radius of the base h:height l:slant height
7.	Sphere		$4\pi r^2$	$4\pi r^2$	$\frac{4}{3}\pi r^3$	r:radius
8.	Hemisphere		$2\pi r^2$	$3\pi r^2$	$\frac{2}{3}\pi r^3$	r:radius

Now, let us see some examples to illustrate the shapes in the table.

Example-1. The radius of a conical tent is 7 meters and its height is 10 meters. Calculate the length of canvas used in making the tent if width of canvas is 2m. $\left[\text{Use } \pi = \frac{22}{7}\right]$

Solution : If the radius of conical tent is given $(r) = 7$ metres

$$\text{Height}(h) = 10\, m.$$

$\therefore$ So, the slant height of the cone $l^2 = r^2 + h^2 \Rightarrow l = \sqrt{r^2 + h^2}$

$$= \sqrt{49 + 100}$$

$$= \sqrt{149} = 12.2\, m.$$

Now, Surface area of the tent $= \pi r l$

$$= \frac{22}{7} \times 7 \times 12.2\ m^2$$

$$= 268.4\ m^2.$$

Area of canvas used $= 268.4\text{m}^2$

It is given the width of the canvas = 2m

Length of canvas used $= \frac{\text{Area}}{\text{width}} = \frac{268.4}{2} = 134.2m$

Example-2. An oil drum is in the shape of a cylinder having the following dimensions: diameter is 2 m. and height is 7 meters. The painter charges ₹3 per m^2 to paint the drum. Find the total charges to be paid to the painter for 10 drums ?

Solution : It is given that diameter of the (oil drum) cylinder = 2 m.

Radius of cylinder $= \frac{d}{2} = \frac{2}{2} = 1\ m$

Total surface area of a cylinderical drum $= 2 \times \pi r(r + h)$

$$= 2 \times \frac{22}{7} \times 1(1 + 7)$$

$$= 2 \times \frac{22}{7} \times 8$$

$$= \frac{352}{7} m^2. = 50.28\ m^2$$

So, the total surface area of a drum $= 50.28\ m^2$

Painting change per $1m^2$ $=$ ₹3.

Cost of painting of 10 drums $= 50.28 \times 3 \times 10$

$=$ ₹1508.40

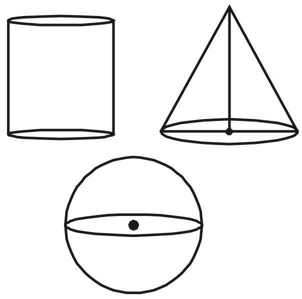
Example-3. A sphere, a cylinder and a cone are of the same radius and same height. Find the ratio of their curved surface areas?

Solution : Let r be the common radius of a sphere, a cone and cylinder.

Height of sphere = its diameter = $2r$.

Then, the height of the cone = height of cylinder = height of sphere.

$= 2r$.



Let l be the slant height of cone $= \sqrt{r^2 + h^2}$

$$= \sqrt{r^2 + (2r)^2} = \sqrt{5}r$$

$\because$ S_1 = Curved surface area of sphere $= 4\pi r^2$

S_2 = Curved surface area of cylinder, $2\pi rh = 2\pi r \times 2r = 4\pi r^2$

S_3 = Curved surface area of cone $= \pi rl = \pi r \times \sqrt{5}\, r = \sqrt{5}\, \pi r^2$

Ratio of curved surface area as

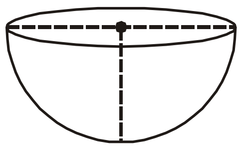
$$\therefore\ S_1 : S_2 : S_3 = 4\pi r^2 : 4\pi r^2 : \sqrt{5}\, \pi r^2$$

$$= 4 : 4 : \sqrt{5}$$

Example-4. A company wanted to manufacture 1000 hemispherical basins from a thin steel sheet. If the radius of hemisperical basin is 21 *cm*., find the required area of steel sheet to manufacture the above hemispherical basins?

Solution : Radius of the hemisphesical basin (r) = 21 cm

Surface area of a hemispherical basin



$$= 2\pi r^2$$

$$= 2 \times \frac{22}{7} \times 21 \times 21$$

$$= 2772\ \text{cm}^2.$$

So, surface area of a hemispherical basin

$= 2772\ cm^2$.

Hence, the steel sheet required for one basin $= 2772\ cm^2$

Total area of steel sheet required for 1000 basins $= 2772 \times 1000$

$= 2772000\ cm^2$

$= 277.2\ m^2$

Example-5. A right circular cylinder has base radius 14cm and height 21cm.

Find: (i) Area of base or area of each end (ii) Curved surface area

(iii) Total surface area and (iv) Volume of the right circular cylinder.

Solution : Radius of the cylinder (r) = 14cm

Height of the cylinder (h) = 21cm

Now (i) Area of base(area of each end) $\pi r^2 = \frac{22}{7}(14)^2 = 616\ cm^2$

(ii) Curved surface area $= 2\pi rh = 2\times\frac{22}{7}\times 14\times 21 = 1848 cm^2$.

(iii) Total surface area $= 2\times$area of the base + curved surface area

$= 2 \times 616 + 1848 = 3080\ cm^2$.

(iv) Volume of cylinder $= \pi r^2 h =$ area of the base $\times$ height

$= 616 \times 21 = 12936\ cm^3$.

Example-6. Find the volume and surface area of a sphere of radius 2.1cm ($\pi = \frac{22}{7}$)

Solution : Radius of sphere(r) = 2.1 cm

Surface area of sphere $= 4\pi r^2$

$$= 4\times\frac{22}{7}\times(2.1)^2 = 4\times\frac{22}{7}\times\frac{21}{10}\times\frac{21}{10}$$

$$= \frac{1386}{25} = 55.44\ cm^2$$

Volume of sphere $= \frac{4}{3}\pi r^3 = \frac{4}{3}\times\frac{22}{7}\times(2.1)^3$

$$= \frac{4}{3} \times \frac{22}{7} \times 2.1 \times 2.1 \times 2.1 = 38.808 \text{ cm}^3.$$

Example-7. Find the volume and the total surface area of a hemisphere of radius 3.5 *cm*. $\left(\pi = \frac{22}{7}\right)$

Solution : Radius of sphere (r) is 3.5 cm $= \frac{7}{2}$ cm

Volume of hemisphere $= \frac{2}{3}\pi r^3$

$$= \frac{2}{3} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times \frac{7}{2} = \frac{539}{6} = 89.83 \text{ cm}^3$$

Total surface area $= 3\pi r^2$

$$= 3 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} = \frac{231}{2} = 115.5 \ cm^2$$

Exercise - 10.1

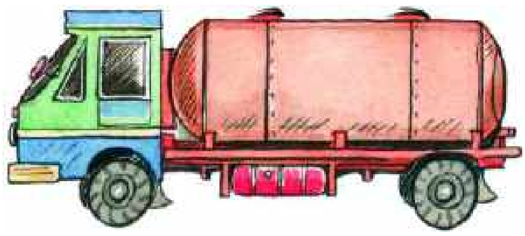
1. A joker's cap is in the form of right circular cone whose base radius is 7cm and height is 24 cm. Find the area of the sheet required to make 10 such caps.
2. A sports company was ordered to prepare 100 paper cylinders for shuttle cocks. The required dimensions of the cylinder are 35 cm length /height and its radius is 7 cm. Find the required area of thin paper sheet needed to make 100 cylinders?
3. Find the volume of right circular cone with radius 6 *cm*. and height 7cm.
4. The lateral surface area of a cylinder is equal to the curved surface area of a cone. If the radius be the same, find the ratio of the height of the cylinder and slant height of the cone.
5. Aself help group wants to manufacture joker's caps (conical caps) of 3*cm*. radius and 4 *cm*. height. If the available colour paper sheet is 1000 cm^2, than how many caps can be manufactured from that paper sheet?
6. A cylinder and cone have bases of equal radii and are of equal heights. Show that their volumes are in the ratio of 3:1.
7. A solid iron rod has a cylinderical shape. Its height is 11 *cm*. and base diameter is 7cm. Then find the total volume of 50 rods?

8. A heap of rice is in the form of a cone of diameter 12 m. and height 8 m. Find its volume? How much canvas cloth is required to cover the heap ? (Use $\pi = 3.14$)

9. The curved surface area of a cone is 4070 cm^2 and its diameter is 70 cm. What is its slant height?

10.2 Surface Area of the Combination of Solids

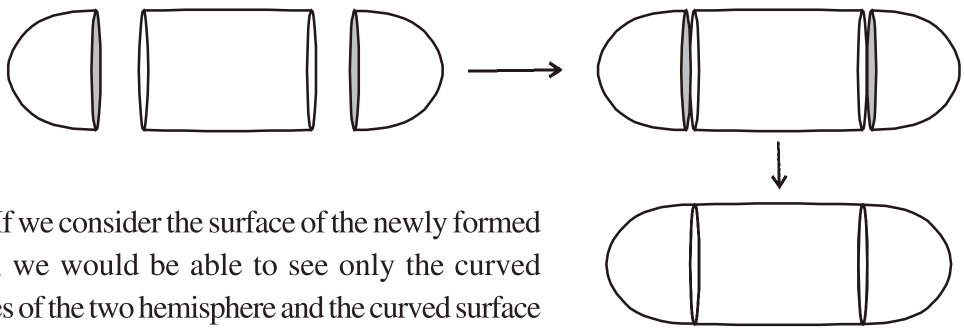
We have seen solids which are made up of combination of solids known like sphere cylinder and cone. We can observe in our real life also like wooden things, house items, medicine capsules, bottles, oil-tankers etc., We eat ice-cream in our daily life. Can you tell how many solid figures are there in it? It is usually made up of cone and hemisphere.



Lets take another example of an oil-tanker/ water-tanker. Is it a single shaped object? You may guess that it is made up of a cylinder with two hemisphere at it ends.

If, for some reason you wanted to find the surface areas or volumes or capacities of such objects, how would you do it? We cannot classify these shapes under any of the solids you have already studied.

As we have seen, the oil-tanker was made up of a cylinder with two hemispheres stuck at either end. It will look like the following figure:



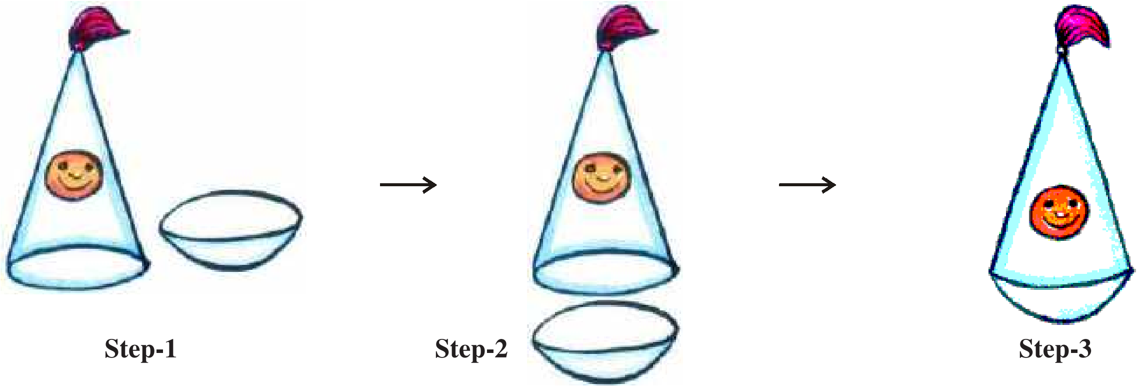
If we consider the surface of the newly formed object, we would be able to see only the curved surfaces of the two hemisphere and the curved surface of the cylinder.

TSA of new solid = CSA of one hemisphere + CSA of cylinder + CSA of other hemisphere

here TSA and CSA stand for 'total surface area' and 'curved surface area' respectively. Now look at another example.

Devarsha wants to make a toy by putting together a hemisphere and a cone. Let us see the steps that he should be going through.

First, he should take a cone and hemisphere and bring their flat faces together. Here, of course, he should take the base radius of the cone equal to the radius of the hemisphere, for the toy is to have a smooth surface. So, the steps would be as shown below:



Step-1 Step-2 Step-3

At the end, he got a nice round-bottomed toy. Now, if he wants to find how much paint he should be required to colour the surface of the toy, what should he know? He needs to know the surface area of the toy, which consists of the CSA of the hemisphere and the CSA of the cone.

So, we can say that

TSA of the toy = CSA of Hemisphere + CSA of cone

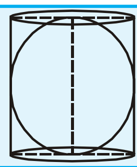
Try This

- Use known solid shapes and make as many objects (by combining more than two) as possible that you come across in your daily life.

[**Hint** : Use clay, or balls, pipes, paper cones, boxes like cube, cuboid etc]

Think - Discuss

A sphere is inscribed in a cylinder. Is the surface of the sphere equal to the curved surface of the cylinder? If yes, explain how?



Example-8. A right triangle, whose base and height are 15 *cm*. and 20 *cm*. respectively is made to revolve about its hypotenuse. Find the volume and surface area of the double cone so formed (Use π=3.14).

Solution : Let ABC be the right angled triangle such that

AB = 15cm and AC = 20 cm

Using Pythagoras theorem in ΔABC we have

$BC^2 = AB^2 + AC^2$

$BC^2 = 15^2 + 20^2$

$BC^2 = 225 + 400 = 625$

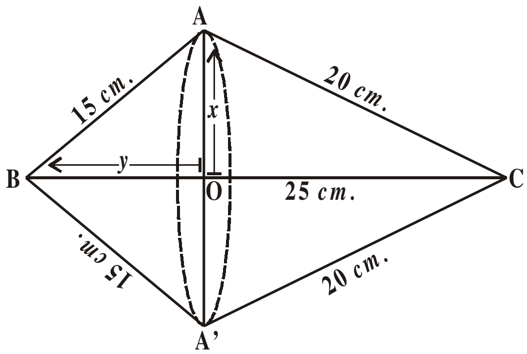
$BC = \sqrt{625} = 25$ cm.

Let OA = x and OB = y.

In triangles ABO and ABC, we have $\angle BOA = \angle BAC$ and $\angle ABO = \angle ABC$

So , by angle - angle - criterion of similarity, we have $\Delta BOA \sim \Delta BAC$

Therefore, $\frac{BO}{BA} = \frac{OA}{AC} = \frac{BA}{BC}$



$\Rightarrow \frac{y}{15} = \frac{x}{20} = \frac{15}{25}$

$\Rightarrow \frac{y}{15} = \frac{x}{20} = \frac{3}{5}$

$\Rightarrow \frac{y}{15} = \frac{3}{5}$ and $\frac{x}{20} = \frac{3}{5}$

$\Rightarrow y = \frac{3}{5} \times 15$ and $x = \frac{3}{5} \times 20$

$\Rightarrow y = 9$ and $x = 12$.

Thus, we have

OA = 12 cm and OB = 9cm

When the ABC is revolved about the hypotenuse. we get a double cone as shown in figure.

Volume of the double cone = volume of the cone CAA' + volume of the cone BAA'

$$= \frac{1}{3}\pi(OA)^2 \times OC + \frac{1}{3}\pi(OA)^2 \times OB$$

$$= \frac{1}{3}\pi \times 12^2 \times 16 + \frac{1}{3}\pi \times 12^2 \times 9$$

$$= \frac{1}{3}\pi \times 144(16+9)$$

$$= \frac{1}{3} \times 3.14 \times 144 \times 25 \text{ cm}^3$$

$= 3768$ cm^3.

Surface area of the doubled cone = (Curved surface area of cone CAA')

+ (Curved surface area of cone BAA')

$= (\pi \times OA \times AC) + (\pi \times OA \times AB)$

$= (\pi \times 12 \times 20) + (\pi \times 12 \times 15)\ cm^2$

$= 420\ \pi\ cm^2$

$= 420 \times 3.14\ cm^2$

$= 1318.8\ cm^2$.

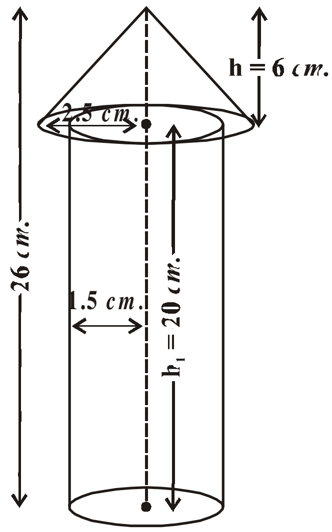
Example-9. A wooden toy rocket is in the shape of a cone mounted on a cylinder as shown in the adjacent figure. The height of the entire rocked is 26 cm, while the height of the conical part is 6cm. The base of the conical position has a diameter of 5cm, while the base diameter of the cylindrical portion is 3cm. If the conical portion is to be painted orange and the cylindrical portion is to be painted yellow, find the area of the rocket painted with each of these color (Take $\pi = 3.14$)

Solution : Let 'r' be the radius of the base of the cone and its slant height be 'l'. Further, let r_1 be the radius of cylinder and h_1 be its height

We have,

$r = 2.5\ cm.,\ \ h = 6\ cm.$

$r_1 = 1.5\ cm.\ \ h_1 = 20\ cm.$



Now, $l = \sqrt{r^2 + h^2}$

$\Rightarrow \quad l = \sqrt{(2.5)^2 + 6^2}$

$l = \sqrt{6.25 + 36} = \sqrt{42.25} = 6.5$

Now, area to be painted orange

= Curved surface area of the cone

$= \pi r l$

$= 3.14\ \{2.5 \times 6.5\}$

$= 51.025\ cm^2$

Area to be painted yellow

= Curved surface area of the cylinder + Area of the base of the cylinder

$= 2\pi r_1 h_1 + \pi r_1^2$

$= \pi r_1\ (2h_1 + r_1)$

$= 3.14 \times 1.5\ (2 \times 20 + 1.5)\ cm^2$

$= 3.14 \times 1.5 \times 41.5\ \text{cm}^2$

$= 4.71 \times 41.5\ \text{cm}^2$

$= 195.465\ \text{cm}^2$.

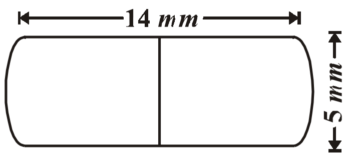
Therefore, area to be painted yellow = $195.465\ cm^2$

Exercise - 10.2

1. A toy is in the form of a cone mounted on a hemisphere. The diameter of the base and the height of the cone are 6 cm and 4 cm respectively. Determine the surface area of the toy. [use $\pi = 3.14$]

2. A solid is in the form of a right circular cylinder with a hemisphere at one end and a cone at the other end. The radius of the common base is 8 *cm*. and the heights of the cylindrical and conical portions are 10 cm and 6 cm respectivly. Find the total surface area of the solid. [use $\pi = 3.14$]

3. A medicine capsule is in the shape of a cylinder with two hemispheres stuck to each of its ends. The length of the capsule is 14 *mm*. and the width is 5 *mm*. Find its surface area.



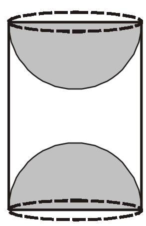
4. Two cubes each of volume 64 cm^3 are joined end to end together. Find the surface area of the resulting cuboid.

5. A storage tank consists of a circular cylinder with a hemisphere stuck on either end. If the external diameter of the cylinder be 1.4 *m*. and its length be 8 *m*. find the cost of painting it on the outside at rate of ₹20 per m^2.

6. A sphere, a cylinder and a cone have the same radius. Find the ratio of their curved surface areas.

7. A hemisphere is cut out from one face of a cubical wooden block such that the diameter of the hemisphere is equal to the length of the cube. Determine the surface area of the remaining solid.

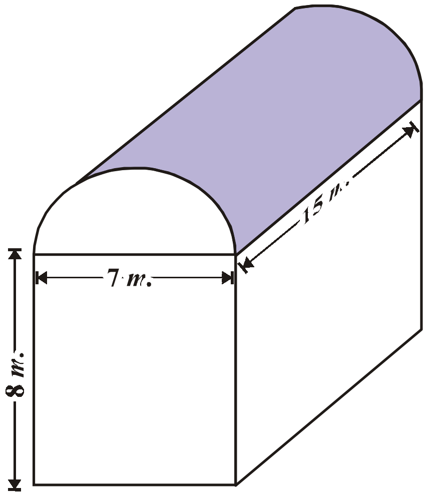
8. A wooden article was made by scooping out a hemiphere from each end of a solid cylinder, as shown in the figure. If the height of the cylinder is 10 *cm*. and its base is of 3.5 *cm*, find the total surface area of the article.



10.3 Volume of Combination of Solids

Let us understand volume through an example.

Suresh runs an industry in a shed which is in the shape of a cuboid surmounted by a half cylinder. The base of the shed is of dimensions 7 $m.$ $\times$ 15 $m.$ and the height of the cuboidal portion is 8 m. Find the volume of air that the shed can hold? Further suppose the machinery in the shed occupies a total space of 300 m^3 and there are 20 workers, each of whom occupies about 0.08 m^3 space on an average. Then how much air is in the shed ?



The volume of air inside the shed (when there are no people or machinery) is given by the volume of air inside the cuboid and inside the half cylinder taken together. The length, breadth and height of the cuboid are 15 $m.$, 7 $m.$ and 8 $m.$ respectively. Also the diameter of the half cylinder is 7 $m.$ and its height is 15 $m.$

So the required volume = volume of the cuboid + $\frac{1}{2}$ volume of the cylinder.

$$= \left[15\times7\times8+\frac{1}{2}\times\frac{22}{7}\times\frac{7}{2}\times\frac{7}{2}\times15\right]m^3$$

$$= 1128.75\text{m}^3.$$

Next, the toal space occupied by the machinery

$$= 300 \text{ m}^3.$$

And the total space occupied by the workers

$$= 20\times0.08\ m^3$$

$$= 1.6\text{m}^3$$

Therefore, the volume of the air, when there are machinery and workers

$$= 1128.75 – (300.00 + 1.60)$$

$$= 1128.75 - 301.60 = 827.15\ m^3$$

Note **:** In calculating the surface area of combination of solids, we can not add the surface areas of the two solids because some part of the surface areas disappears in the process of joining them. However, this will not be the case when we calculate the volume. The volume of the solid formed by joining two basic solids will acutally be the sum of the volumes of the constituents as we seen in the example above.

TRY THIS

1. If the diameter of the cross - section of a wire is decreased by 5%, by what percentage should the length be increased so that the volume remains the same ?
2. Surface area of a sphere and cube are equal. Then find the ratio of their volumes.

Let us see some more examples.

Example-10. A solid toy is in the form of a right circular cylinder with hemispherical shape at one end and a cone at the other end. Their common diameter is 4.2 cm and the height of the cylindrical and conical portions are 12cm and 7cm respectively. Find the volume of the solid toy. $\left(\text{Use } \pi = \frac{22}{7}\right)$.

Solution : Let height of the conical portion $h_1 = 7\text{cm}$

The height of cylindrical portion $h_2 = 12$ cm

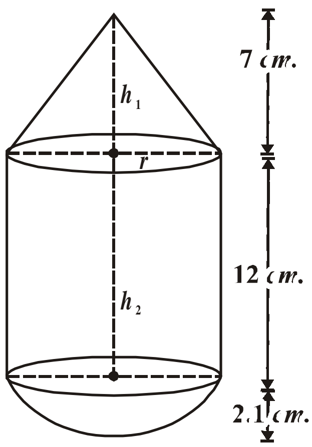
Radius $(r) = \frac{4.2}{2} = 2.1 = \frac{21}{10}$ cm

Volume of the solid toy

= Volume of the Cone + Volume of the Cylinder + Volume of the Hemisphere.

$$= \frac{1}{3}\pi r^2 h_1 + \pi r^2 h_2 + \frac{2}{3}\pi r^3$$

$$= \pi r^2 \left[\frac{1}{3}h_1 + h_2 + \frac{2}{3}r\right]$$



$$= \frac{22}{7} \times \left(\frac{21}{10}\right)^2 \times \left[\frac{1}{3}\times 7 + 12 + \frac{2}{3}\times\frac{21}{10}\right]$$

$$= \frac{22}{7} \times \frac{441}{100} \times \left[\frac{7}{3} + \frac{12}{1} + \frac{7}{5}\right]$$

$$= \frac{22}{7} \times \frac{441}{100} \times \left[\frac{35+180+21}{15}\right]$$

$$= \frac{22}{7} \times \frac{441}{100} \times \frac{236}{15} = \frac{27258}{125} = 218.064 \text{ cm}^3.$$

Example-11. A cylindrical container is filled with ice-cream whose diameter is 12 *cm*. and height is 15 *cm*. The whole ice-cream is distributed to 10 children in equal cones having hemispherical tops. If the height of the conical portion is twice the diameter of its base, find the diameter of the ice-cream cone.

Solution : Let the radius of the base of conical ice cream = x cm

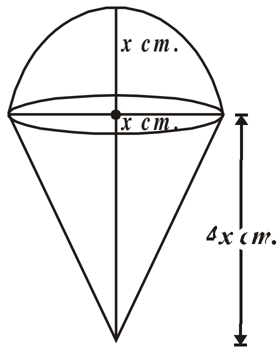
$\therefore$ diameter = $2x$ cm

Then, the height of the conical ice-cream

= 2 (diameter) = 2(2x) = 4x cm

Volume of ice - cream cone

= Volume of conical porton + Volume of hemispherical portion



$$= \frac{1}{3}\pi r^2 h + \frac{2}{3}\pi r^3$$

$$= \frac{1}{3}\pi x^2 (4x) + \frac{2}{3}\pi x^3$$

$$= \frac{4\pi x^3 + 2\pi x^3}{3} = \frac{6\pi x^3}{3}$$

$$= 2\pi x^3 \text{ cm}^3$$

Diameter of cylindrical container = 12 cm

Its height (h) = 15 cm

$\therefore$ Volume of cylindrical container = $\pi r^2 h$

$$= \pi(6)^2\ 15$$

$$= 540\pi \text{ cm}^3$$

Number of children to whom ice-cream is given = 1

$$\frac{\text{Volume of cylindrical container}}{\text{Volume of one ice-cream cone}} = 10$$

$$\Rightarrow \quad \frac{540\pi}{2\pi x^3} = 10$$

$$2\pi x^3 \times 10 = 540\pi$$

$$\Rightarrow \quad x^3 = \frac{540}{2\times 10} = 27$$

$$\Rightarrow \quad x^3 = 27$$

$\Rightarrow \quad x^3 = 3^3$

$\Rightarrow \quad x = 3$

$\therefore$ Diameter of ice-cream cone $2x = 2(3) = 6$cm

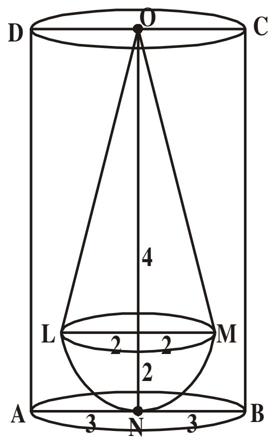
Example-12. A solid consisting of a right circular cone standing on a hemisphere, is placed upright in a right circular cylinder full of water and touches the bottom. Find the volume of water left in the cylinder, given that the radius of the cylinder is 3 *cm*. and its height is 6cm. The radius of the hemisphere is 2 *cm*. and the height of the cone is 4 *cm*.

$\left(\text{Take } \pi = \frac{22}{7}\right)$.

Solution : In the figure drawn here,

ABCD is a cylinder and LMN is a Hemisphere

OLM is a cone. We know that where a solid consisting of a cone and hemisphere is immersed in the cylinder full of water, then some water equal to the volume of the solid, is displaced.



Volume of Cylinder $= \pi r^2 h = \pi \times 3^2 \times 6 = 54\,\pi$ cm^3

Volume of Hemisphere $= \frac{2}{3}\pi r^3 = \frac{2}{3} \times \pi \times 2^3 = \frac{16}{3}\pi \text{cm}^3$

Volume of Cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3} \times \pi \times 2^2 \times 4 = \frac{16}{3}\pi \text{cm}^3$

Volume of cone and hemisphere $= \frac{16}{3}\pi + \frac{16}{3}\pi$

$= \frac{32}{3}\pi$

Volume of water left in cylinder

= Volume of Cylinder - Volume of Cone and Hemisphere

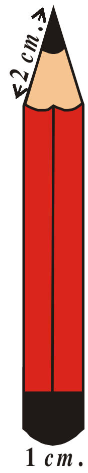
$= \text{Volume of cylinder} - \frac{32\pi}{3}$

$= 54\pi - \frac{32\pi}{3}$

$= \frac{162\pi - 32\pi}{3} = \frac{130\pi}{3}$

$$= \frac{130}{3} \times \frac{22}{7} = \frac{2860}{21} = 136.19\,\text{cm}^3$$

Example-13. A cylindrical pencil is sharpened to produce a perfect cone at one end with no over all loss of its length. The diameter of the pencil is 1cm and the length of the conical portion is 2cm. Calculate the volume of the shavings. Give your answer correct to two places if it is in decimal $\left[\text{use } \pi = \frac{355}{113}\right]$.



Solution : Diameter of the pencil = 1cm

So, radius of the pencil $(r) = 0.5$ cm

Length of the conical portion $= h = 2$cm

Volume of showings = Volume of cylinder of length 2 cm and base radius 0.5 *cm*.

– volume of the cone formed by this cylinder

$$= \pi r^2 h - \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^2 h$$

$$= \frac{2}{3} \times \frac{355}{113} \times (0.5)^2 \times 2\ \text{cm}^3 = 1.05\ \text{cm}^3$$

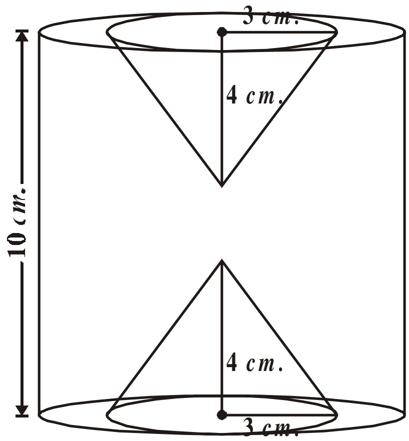
Exercise-10.3

1. An iron pillar consists of a cylindrical portion of 2.8 *m*. height and 20 *cm*. in diameter and a cone of 42 *cm*. height surmounting it. Find the weight of the pillar if 1 cm^3 of iron weighs 7.5 *g*.

2. A toy is made in the form of hemisphere surmounted by a right cone whose circular base is joined with the plane surface of the hemisphere. The radius of the base of the cone is 7 *cm*. and its volume is $\frac{3}{2}$ of the hemisphere. Calculate the height of the cone and the surface area of the toy correct to 2 places of decimal $\left(\text{Take } \pi = 3\frac{1}{7}\right)$.

3. Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 7 *cm*.

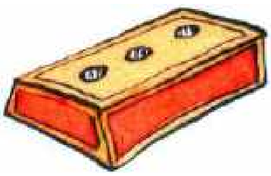
4. A cylinderical tub of radius 5cm and length 9.8 cm is full of water. A solid in the form of right circular cone mounted on a hemisphere is immersed into the tub. The radius of the hemisphere is 3.5 cm and height of cone outside the hemisphere is 5cm. Find the volume of water left in the tub $\left(\text{Take } \pi = \dfrac{22}{7}\right)$.

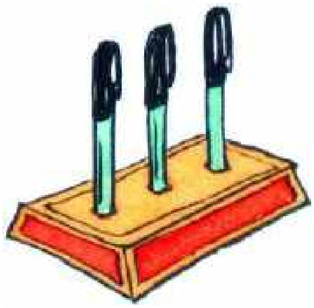


5. In the adjacent figure, the height of a solid cylinder is 10 cm and diameter is 7cm. Two equal conical holes of radius 3cm and height 4 cm are cut off as shown the figure. Find the volume of the remaining solid.

6. Spherical Marbles of diameter 1.4 *cm*. are dropped into a cylindrical beaker of diameter 7 *cm*., which contains some water. Find the number of marbles that should be dropped in to the beaker, so that water level rises by 5.6 *cm*.

7. A pen stand is made of wood in the shape of cuboid with three conical depressions to hold the pens. The dimensions of the cuboid are 15cm by 10 cm by 3.5 cm. The radius of each of the depression is 0.5 cm and the depth is 1.4cm. Find the volume of wood in the entire stand.



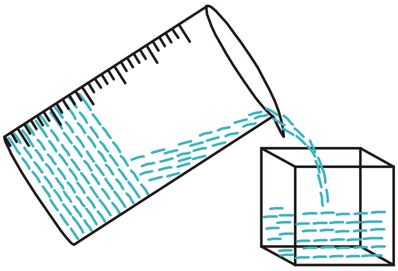


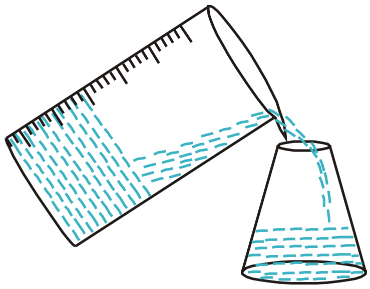
10.4 Conversion of Solid from One Shape to Another

A women self help group (DWACRA) prepares candles by melting down cuboid shape wax. In gun factories spherical bullets are made by melting solid cube of lead, goldsmith prepares various ornaments by melting cubiod gold biscuts. In all these cases, the shapes of solids are converted into another shape. In this process, the volume always remains the same.

How does this happen? If you want a candle of any special shape, you have to give heat to the wax in metal container till it is completely melted into liquid. Then you pour it into another container which has the special shape that you want.

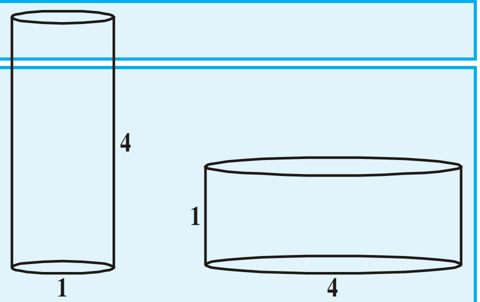
For example, lets us take a candle in the shape of solid cylinder, melt it and pour whole of the molton wax into another container shaped like a sphere. On cooling, you will obtain a candle in the shape of sphere. The volume of the new candle will be the same as the volume of the earlier candle. This is what we have to remember when we come across objects which are converted from one shape to another, or when a liquid which originally filled a container of a particular shape is poured into another container of a different shape or size as you observe in the following figures.





Think- Discuss

Which barrel shown n the adjacent figure can hold more water? Discuss with your friends.



To understand what has been discussed, let us consider some examples.

Example-14. A cone of height 24cm and radius of base 6cm is made up of modelling clay. A child reshapes it in the form of a sphere. Find the radius of the sphere.

Solution : Volume of cone = $\frac{1}{3} \times \pi \times 6 \times 6 \times 24 \text{ cm}^3$

If r is the radius of the sphere, then its volume is $\frac{4}{3}\pi r^3$

Since the volume of clay in the form of the cone and the sphere remains the same, we have

$$\frac{4}{3}\pi r^3 = \frac{1}{3}\pi \times 6 \times 6 \times 24$$

$$r^3 = 3 \times 3 \times 24 = 3 \times 3 \times 3 \times 8$$

$$r^3 = 3^3 \times 2^3$$

$$r = 3 \times 2 = 6$$

Therefore the radius of the sphere is 6cm.

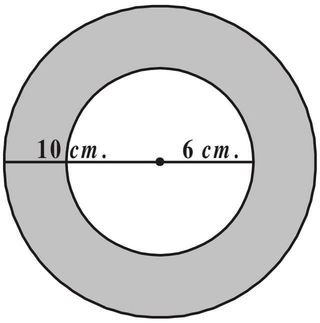
Do This

1. A copper rod of diameter 1 *cm.* and length 8 *cm.* is drawn into a wire of length 18m of uniform thickness. Find the thickness of the wire.

2. Pravali house has a water tank in the shape of a cylinder on the roof. This is filled by pumping water from a sump (an under ground tank) which is in the shape of a cuboid. The sump has dimensions 1.57 *m.* $\times$1.44 *m.* $\times$ 9.5 *cm.* The water tank has radius 60 *cm.* and height 95 *cm.* Find the height of the water left in the sump after the water tank has been completely filled with water from the sump which had been full of water. Compare the capacity of the tank with that of the sump. $(\pi = 3.14)$

Example-15. The diameter of the internal and external surfaces of a hollow hemispherical shell are 6 *cm.* and 10 *cm.* respectively. It is melted and recast into a solid cylinder of diameter 14 *cm.* Find the height of the cylinder.

Solution : Radius of Hollow hemispherical shell $= \frac{10}{2} = 5\ cm. = R$



Internal radius of hollow hemispherical shell $= \frac{6}{2} = 3\ cm. = r$

Volume of hollow hemispherical shell

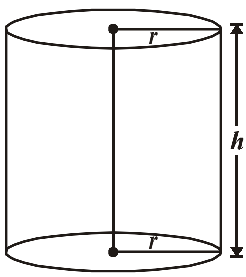
= External volume - Internal volume

$$= \frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3$$

$$= \frac{2}{3}\pi (R^3 - r^3)$$

$$= \frac{2}{3}\pi (5^3 - 3^3)$$

$$= \frac{2}{3}\pi (125 - 27)$$



$$= \frac{2}{3}\pi \times 98\,cm^3 = \frac{196\pi}{3} cm^3 \qquad ...(1)$$

Since, this hollow hemispherical shell is melted and recast into a solid cylinder. So their volumes must be equal

Diameter of cylinder = 14 *cm.* (Given)

So, radius of cylinder = 7 *cm.*

Let the height of cylinder $= h$

$\therefore$ volume of cylinder $= \pi r^2 h$

$$= \pi \times 7 \times 7 \times h \text{ cm}^3 = 49\pi h \text{ cm}^3 \qquad ...(2)$$

According to given condition

volume of Hollow hemispherical shell = volume of solid cylinder

$$\frac{196}{3}\pi = 49\pi h \qquad \text{[From equatiion (1) and (2)]}$$

$$\Rightarrow \quad h = \frac{196}{3 \times 49} = \frac{4}{3} \text{ cm.}$$

Hence, height of the cylinder = 1.33 cm.

Example-16. A hemispherical bowl of internal radius 15 *cm*. contains a liquid. The liquid is to be filled into cylindrical bottles of diameter 5 *cm*. and height 6 *cm*. How many bottles are necessary to empty the bowl ?

Solution : Volume of hemisphere $= \frac{2}{3}\pi r^3$

Internal radius of hemisphere $r = 15$ *cm*.

$\therefore$ volume of liquid contained in hemispherical bowl

$$= \frac{2}{3}\pi(15)^3 \text{ cm}^3$$

$$= 2250 \, \pi \text{ cm}^3.$$

This liquid is to be filled in cylindrical bottles and the height of each bottle $(h) = 6$ *cm*.

Radius of cylindrical bottle (R) $= \frac{5}{2}$ *cm*.

$\therefore$ Volume of 1 cylindrical bottle $= \pi R^2 h$

$$= \pi \times \left(\frac{5}{2}\right)^2 \times 6$$

$$= \pi \times \frac{25}{4} \times 6 \text{cm}^3 = \frac{75}{2} \pi \text{ cm}^3$$

$$\text{Number of cylindrical bottles required} = \frac{\text{Volume of hemispherical bowl}}{\text{Volume of 1 cylindrical bottle}}$$

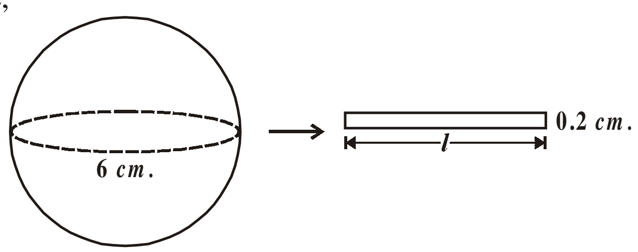
$$= \frac{2250\pi}{\frac{75}{2}\pi} = \frac{2\times 2250}{75} = 60.$$

Example-17. The diameter of a metallic sphere is 6cm. It is melted and drawn into a wire having diameter of the cross section as 0.2 cm. Find the length of the wire.

Solution : We have, diameter of metallic sphere = 6cm

$\therefore$ Radius of metallic shpere = 3cm

Also, we have,



Diameter of cross - section of cylindrical wire = 0.2 *cm*.

Radius of cross section of cylinder wire = 0.1 *cm*.

Let the length of wire be *l cm*.

Since the metallic sphere is converted into a cylidrical shaped wire of length *h* cm.

$\therefore$ Volume of the metal used in wire = Volume of the sphere

$$\pi\times(0.1)^2\times h = \frac{4}{3}\times\pi\times 3^3$$

$$\pi\times\left(\frac{1}{10}\right)^2\times h = \frac{4}{3}\times\pi\times 27$$

$$\pi\times\frac{1}{100}\times h = 36\pi$$

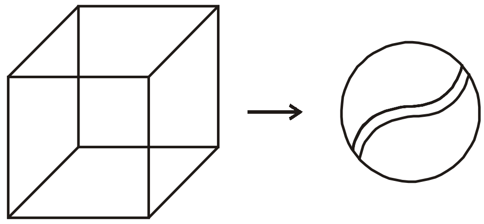
$$h = \frac{36\pi\times 100}{\pi}\text{ cm}$$

$$= 3600\ cm. = 36\ m.$$

Therefore, the length of the wire is 36 *m*.

Example-18. How many spherical balls can be made out of a solid cube of lead whose edge measures 44 cm and each ball being 4 *cm*. in diameter.

Solution : Side of lead cube = 44 *cm*.



Radius of spherical ball $= \frac{4}{2}\ cm. = 2\ cm.$

Now volume of a spherical bullet $= \frac{4}{3}\pi r^3$

$$= \frac{4}{3}\times\frac{22}{7}\times 2^3\ \text{cm}^3$$

$$= \frac{4}{3}\times\frac{22}{7}\times 8\ \text{cm}^3$$

Volume of x sperical bullet $= \frac{4}{3}\times\frac{22}{7}\times 8\times x\ \text{cm}^3$

It is clear that volume of x sperical bullets = Volume of lead cube

$$\Rightarrow \frac{4}{3}\times\frac{22}{7}\times 8\times x = (44)^3$$

$$\Rightarrow \frac{4}{3}\times\frac{22}{7}\times 8\times x = 44\times 44\times 44$$

$$\Rightarrow x = \frac{44\times 44\times 44\times 3\times 7}{4\times 22\times 8}$$

$$x = 2541$$

Hence, total number of sperical bullets = 2541.

Example-19. A women self help group (DWACRA) is supplied a rectangular solid (cuboid shape) of wax with diameters 66 *cm*., 42 *cm*., 21 *cm*., to prepare cylindrical candles each 4.2 *cm*. in diameter and 2.8 *cm*. of height. Find the number of candles.

Solution : Volume of wax in the rectangular solid = lbh

$$= (66\times 42\times 21)\ cm^3.$$

Radius of cylindrical candle $= \frac{4.2}{2}\ cm. = 2.1\ cm.$

Height of cylindrical candle = 2.8 *cm*.

Volume of candle $= \pi r^2 h$

$$= \frac{22}{7} \times (2.1)^2 \times 2.8$$

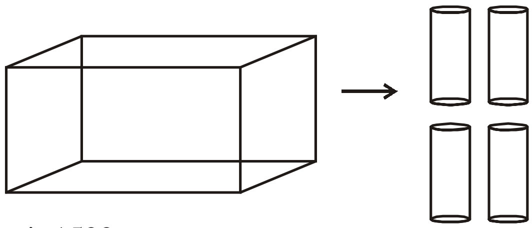
Volume of x cylindrical wax candles $= \frac{22}{7} \times 2.1 \times 2.1 \times 2.8 \times x$

$\because$ Volume of x cylindrical candles = volume of wax in rectangular shape

$$\therefore \frac{22}{7} \times 2.1 \times 2.1 \times 2.8 \times x = 66 \times 42 \times 21$$

$$x = \frac{66 \times 42 \times 21 \times 7}{22 \times 2.1 \times 2.1 \times 2.8}$$

$$= 1500$$



Hence, the number of cylindrical wax candles is 1500.

Exercise - 10.4

1. A metallic sphere of radius 4.2 *cm.* is melted and recast into the shape of a cylinder of radius 6cm. Find the height of the cylinder.
2. Metallic spheres of radius 6 *cm.*, 8 *cm.* and 10 *cm.* respectively are melted to form a single solid sphere. Find the radius of the resulting sphere.
3. A 20m deep well with diameter 7 *m.* is dug and the earth from digging is evenly spread out to form a platform 22 *m.* by 14 *m.* Find the height of the platform.
4. A well of diameter 14 *m.* is dug 15 *m.* deep. The earth taken out of it has been spread evenly all around it in the shape of a circular ring of width 7 *m.* to form an embankment. Find the height of the embankment.
5. A container shaped like a right circular cylinder having diameter 12 *cm.* and height 15 *cm.* is full of ice cream. The icecream is to be filled into cones of height 12 *cm.* and diameter 6 *cm.*, having a hemispherical shape on the top. Find the number of such cones which can be filled with ice cream.
6. How many silver coins, 1.75 *cm.* in diameter and thickness 2 *mm.*, need to be melted to form a cuboid of dimensions 5.5 *cm.* × 10 *cm.* × 3.5 *cm.*?
7. A vessel is in the form of an inverted cone. Its height is 8 *cm.* and the radius of its top is 5 *cm.* It is filled with water up to the rim. When lead shots, each of which is a sphere of

radius 0.5cm are dropped into the vessel, $\frac{1}{4}$ of the water flows out. Find the number of lead shots dropped into the vessel.

8. A solid metallic sphere of diameter 28 cm is melted and recast into a number of smaller cones, each of diameter $4\frac{2}{3}$ cm and height 3cm. Find the number of cones so formed.

Optional Exercise

[This exercise is not meant for examination purpose]

1. A golf ball has diameter equal to 4.1 *cm*. Its surface has 150 dimples each of radius 2 *mm*. Calculate total surface area which is exposed to the surroundings. (Assume that the dimples are all hemispherical) $\left[\pi = \frac{22}{7}\right]$

2. A cylinder of radius 12 *cm*. contains water to a depth of 20 *cm*. A spherical iron ball is dropped in to the cylinder and thus the level of water is raised by 6.75 *cm*. Find the radius of the ball. $\left[\pi = \frac{22}{7}\right]$

3. A solid toy is in the form of a right circular cylinder with a hemispherical shape at one end and a cone at the other end. Their common diameter is 4.2 *cm*. and height of the cylindrical and concial portion are 12 *cm*. and 7 *cm*. respectively. Find the volume of the solid toy. $\left[\pi = \frac{22}{7}\right]$

4. Three metal cubes with edges 15 *cm*., 12 *cm*. and 9 *cm*. respectively are melted together and formed into a simple cube. Find the diagonal of this cube.

5. A hemispherical bowl of internal diameter 36 *cm*. contains a liquid. This liquid is to be filled in cylindrical bottles of radius 3 *cm*. and height 6 *cm*. How many bottles are required to empty the bowl?

What We Have Discussed.

1. The volume of the solid formed by joining two basic solids is the sum of the volumes of the constituents.

2. In calculating the surface area of a combination of solids, we can not add the surface area of the two constituents, because some part of the surface area disappears in the process of joining them.

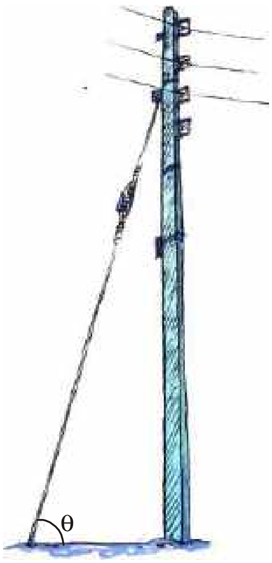
CHAPTER

11 Trigonometry

11.1 Introduction

We have seen triangles and their properties in previous classes. There, we observed different daily life situations where we were using triangles.

Let's again look at some of the daily life examples.

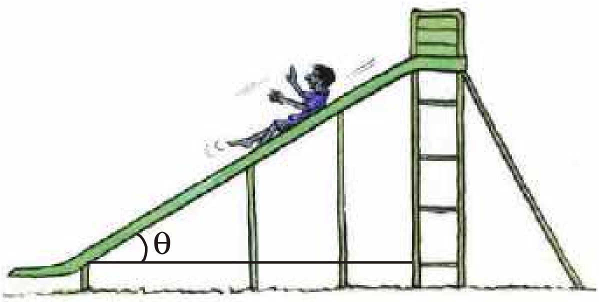


- Electric poles are present everywhere. They are usually erected by using a metal wire. The pole, wire and the ground form a triangle. But, if the length of the wire decreases, what will be the shape of the triangle and what will be the angle of the wire with the ground ?

- A person is whitewashing a wall with the help of a ladder which is kept as shown in the adjacent figure on left. If the person wants to paint at a higher position, what will the person do? What will be the change in angle of the ladder with the ground ?

- In the temple at Jainath in Adilabad district, which was built in 13th century, the first rays of the Sun fall at the feet of the Idol of Suryanarayana Swami in the month of December. There is a relation between distance of Idol from the door, height of the hole on the door from which Sun rays are entering and angle of sun rays in that month. Is there any triangle forming in this context?

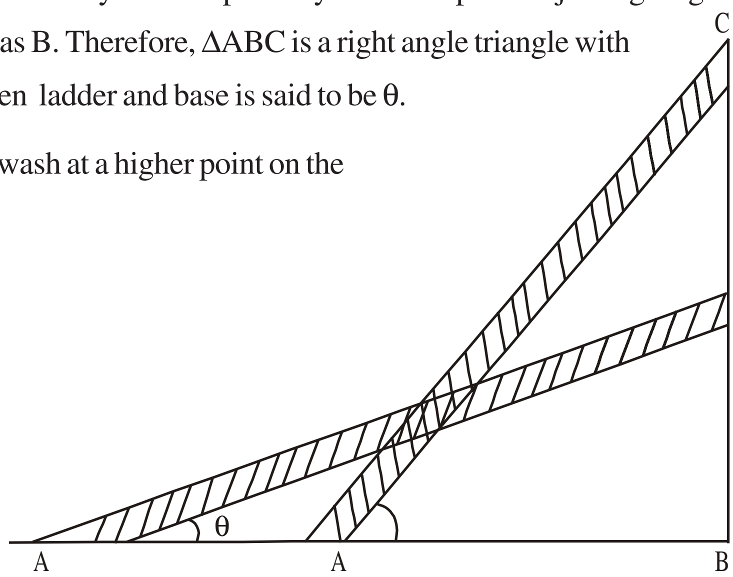
- In a play ground, children like to slide on slider and slider is on a defined angle from earth. What will happen to the slider if we change the angle? Will children still be able to play on it?



The above examples are geometrically showing the application part of triangles in our daily life and we can measure the heights, distances and slopes by using the properties of triangles. These types of problems are part of 'trigonometry' which is a branch of mathematics.

Now look at the example of a person who is white washing the wall with the help of a ladder as shown in the previous figure. Let us observe the following conditions.

We denote the foot of the ladder by A and top of it by C and the point of joining height of the wall and base of the ladder as B. Therefore, ΔABC is a right angle triangle with right angle at B. The angle between ladder and base is said to be θ.



1. If the person wants to white wash at a higher point on the wall-
   - What happens to the angle made by the ladder with the ground?
   - What will be the change in the distance AB?
2. If the person wants to white wash at a lower point on the wall-
   - What happens to the angle made by the ladder with the ground?
   - What will be the change in the distance AB?

We have observed in the above example of a person who was white washing. When he wants to paint at higher or lower points, he should change the position of ladder. So, when 'θ' is increased, the height also increases and the base decreases. But, when θ is decreased, the height also decreases and the base increases. Do you agree with this statement?

Here, we have seen a right angle triangle ABC and have given ordinary names to all sides and angles. Now let's name the sides again because trigonometric ratios of angles are based on sides only.

11.1.1 Naming the Sides in a Right Triangle

Let's take a right triangle ABC as show in the figure.

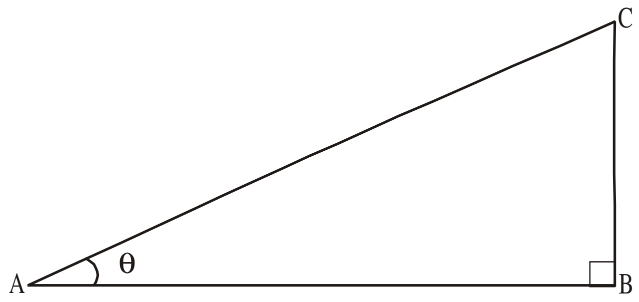
In triangle ABC, we can consider $\angle CAB$ as A where angle A is an acute angle. Since AC is the longest side, it is called "hypotenuse".

Here you observe the position of side BC with respect to angle A. It is opposite to angle A and we can call it as "**opposite side of angle A**". And the remaining side AB can be called as "**Adjacent side of angle A**"

AC = Hypotenuse

BC = Opposite side of angle A

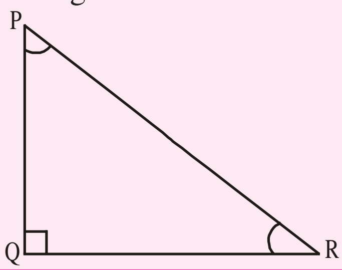
AB = Adjacent side of angle A



Do This

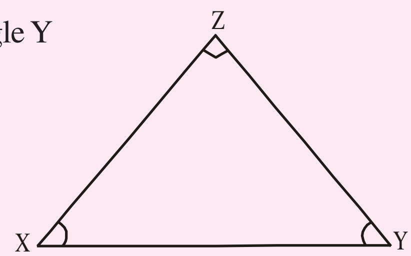
Identify "Hypotenuse", "Opposite side" and "Adjacent side" for the given angles in the given triangles.

1. For angle R



2. (i) For angle X

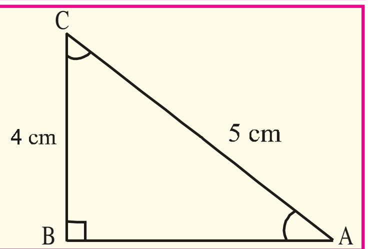
(ii) For angle Y



Try This

Write lengths of "Hypotenuse", "Opposite side" and "Adjacent side" for the given angles in the given triangles.

1. For angle C
2. For angle A

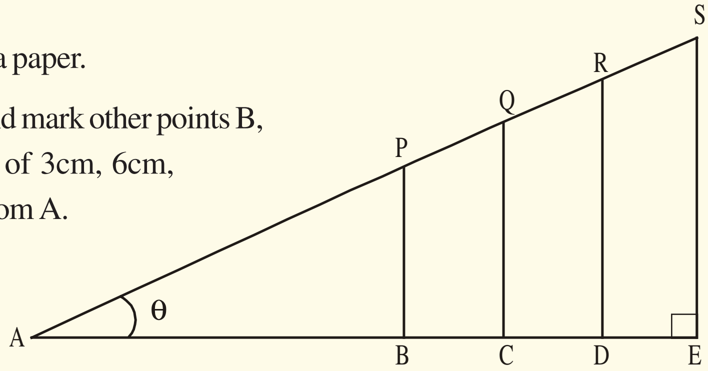


What do you observe? Is there any relation between the opposite side of the angle A and adjacent side of angle C? Like this, suppose you are erecting a pole by giving support of strong ropes. Is there any relationship between the length of the rope and the length of the pole? Here, we have to understand the relationship between the sides and angles we will study this under the section called trigonometric ratios.

11.2 Trigonometric Ratios

We have seen the example problems in the beginning of the chapter which are related to our daily life situations. Let's know about the trigonometric ratios and how they are defined.

Activity



1. Draw a horizontal line on a paper.
2. Let the initial point be A and mark other points B, C, D and E at a distance of 3cm, 6cm, 9cm, 15cm respectively from A.
3. Draw the perpendiculars BP, CQ, DR and ES of lengths 4cm, 8cm, 12cm, 16cm from the points B, C, D and E respectively.
4. Then join AP, PQ, QR and RS.
5. Find lengths of AP, AQ, AR and AS.

Length of hypotenuse	Length of opposite side	Length of adjacent side	$\dfrac{\text{Opposite side}}{\text{Hypotenuse}}$	$\dfrac{\text{Adjacent side}}{\text{Hypotenuse}}$

Then find the ratios of $\dfrac{BP}{AP}, \dfrac{CQ}{AQ}, \dfrac{DR}{AR}$ and $\dfrac{ES}{AS}$.

Did you get the same ratio as $\dfrac{4}{5}$?

Similarly try to find the ratios $\dfrac{AB}{AP}, \dfrac{AC}{AQ}, \dfrac{AD}{AR}$ and $\dfrac{AE}{AS}$? What do you observe?

11.2.1 Defining Trigonometeric Ratios

In the above activity, when we observe right angle triangles ABP, ACQ, ADR and AES, $\angle A$ is common, $\angle B$, $\angle C$, $\angle D$ and $\angle E$ are right angles and $\angle P$, $\angle Q$, $\angle R$ and $\angle S$ are also equal. Hence, we can say that triangles ABP, ACQ, ADR and AES are similar triangles. When we observe the ratio of opposite side of angle A and hypotenuse in a right angle triangle and the ratio of similar sides in another triangle, it is found to be constant in all the above right angle triangles ABP, ACQ, ADR and AES. And the ratios $\frac{BP}{AP}, \frac{CQ}{AQ}, \frac{DR}{AR}$ and $\frac{ES}{AS}$ can be named as "**sine A**" or simply "**sin A**" in those triangles. If the value of angle A is "x" when it was measured, then the ratio would be "sin x".

Hence, we can conclude that the ratio of opposite side of an angle (measure of the angle) and length of the hypotenuse is constant in all similar right angle triangles. This ratio will be named as "sine" of that angle.

Similarly, when we observe the ratios $\frac{AB}{AP}, \frac{AC}{AQ}, \frac{AD}{AR}$ and $\frac{AE}{AS}$, it is also found to be constant. And these are the ratios of the adjacent sides of the angle A and hypotenuses in right angle triangles ABP, ACQ, ADR and AES. So, the ratios $\frac{AB}{AP}, \frac{AC}{AQ}, \frac{AD}{AR}$ and $\frac{AE}{AS}$ will be named as "**cosine A**" or simply "**cos A**" in those triangles. If the value of the angle A is "x", then the ratio would be "cos x"

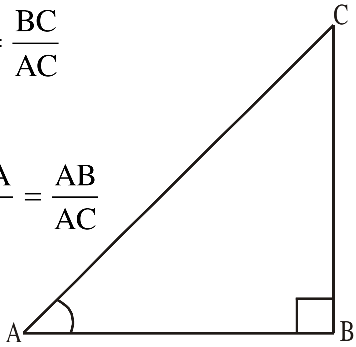
Hence, we can also conclude that the ratio of the adjacent side of an angle (measure of the angle) and length of the hypotenuse is constant in all similar right triangles. This ratio will be named as "cosine" of that angle.

Similarly, the ratio of opposite side and adjacent side of an angle is constant and it can be named as "tangent" of that angle.

Let's Define Ratios in a Right Angle Triangle

Consider a right angle triangle ABC having right angle at B as shown in the following figure.

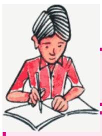
Then, trigonometric ratios of the angle A in right angle triangle ABC are defined as follows :



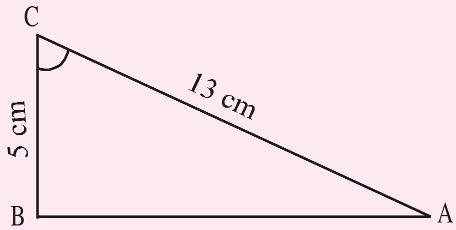
$$\text{sine of } \angle A = \sin A = \frac{\text{Length of the side opposite to angle A}}{\text{Length of hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \cos A = \frac{\text{Length of the side adjacent to angle A}}{\text{Length of hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \tan A = \frac{\text{Length of the side opposite to angle A}}{\text{Length of the side adjacent to angle A}} = \frac{BC}{AB}$$



Do This



1. Find (i) sin C (ii) cos C and (iii) tan C in the adjacent triangle.
2. In a triangle XYZ, $\angle Y$ is right angle, XZ = 17 m and YZ = 15 cm, then find (i) sin X (ii) cos Z (iii) tan X
3. In a triangle PQR with right angle at Q, the value of $\angle P$ is x, PQ = 7 cm and QR = 24 cm, then find sin x and cos x.

Try This

In a right angle triangle ABC, right angle is at C. BC + CA = 23 cm and BC – CA = 7cm, then find sin A and tan B.

Think - Discuss

Discuss between your friends that

(i) $\sin x = \frac{4}{3}$ does exist for some value of angle x?

(ii) The value of sin A and cos A is always less than 1. Why?

(iii) tan A is product of tan and A.

There are three more ratios defined in trigonometry which are considered as multiplicative inverse of the above three ratios.

Multiplicative inverse of "sine A" is "cosecant A". Simply written as "cosec A"

i.e., $\text{cosec } A = \dfrac{1}{\sin A}$

Similarly, multiplicative inverses of "cos A" is secant A" (simply written as "sec A") and that of "tan A" is "cotangent A (simply written as cot A)

i.e., $\sec A = \dfrac{1}{\cos A}$ and $\cot A = \dfrac{1}{\tan A}$

How can you define 'cosec' in terms of sides?

If $\sin A = \dfrac{\text{Opposite side of the angle A}}{\text{Hypotenuse}}$,

then $\text{cosec } A = \dfrac{\text{Hypotenuse}}{\text{Opposite side of the angle A}}$

Try This

What will be the ratios of sides for sec A and cot A?

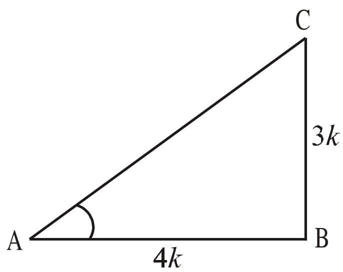
Think - Discuss

- Is $\dfrac{\sin A}{\cos A}$ equal to tan A ?
- Is $\dfrac{\cos A}{\sin A}$ equal to cot A ?

Let us see some examples

Example-1. If $\tan A = \dfrac{3}{4}$, then find the other trigonometric ratio of angle A.

Solution : Given $\tan A = \dfrac{3}{4}$



Hence $\tan A = \dfrac{\text{Opposite side}}{\text{Adjacent side}} = \dfrac{3}{4}$

Therefore, opposite side : adjacent side = 3:4

For angle A, opposite side = BC = $3k$

Adjacent side = AB = $4k$ (where k is any positive number)

Now, we have in triangle ABC (by Pythagoras theorem)

$$AC^2 = AB^2 + BC^2$$
$$= (3k)^2 + (4k)^2 = 25k^2$$
$$AC = \sqrt{25k^2}$$
$$= 5k = \text{Hypotenuse}$$

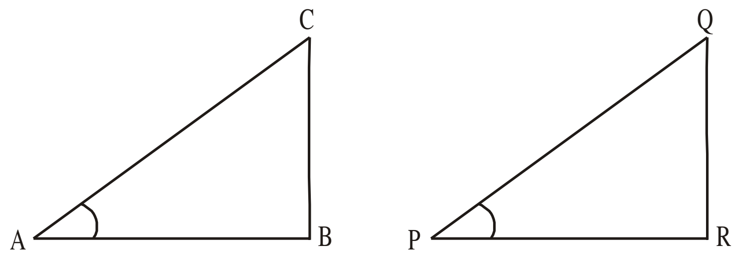
Now, we can easily write the other ratios of trigonometry

$$\sin A = \frac{3k}{5k} = \frac{3}{5} \quad \text{and} \quad \cos A = \frac{4k}{5k} = \frac{4}{5}$$

And also $\text{cosec } A = \frac{1}{\sin A} = \frac{5}{3}$, $\sec A = \frac{1}{\cos A} = \frac{5}{4}$, $\cot A = \frac{1}{\tan A} = \frac{4}{3}$.

Example-2. If $\angle A$ and $\angle P$ are acute angles such that $\sin A = \sin P$ then prove that $\angle A = \angle P$

Solution : Given $\sin A = \sin P$



we have $\sin A = \frac{BC}{AC}$

and $\sin P = \frac{QR}{PQ}$

Then $\frac{BC}{AC} = \frac{QR}{PQ}$

Therefore, $\frac{BC}{AC} = \frac{QR}{PQ} = k$

By using Pythagoras theorem

$$\frac{AB}{PR} = \frac{\sqrt{AC^2 - BC^2}}{\sqrt{PQ^2 - QR^2}} = \frac{\sqrt{AC^2 - k^2BC^2}}{\sqrt{PQ^2 - k^2QR^2}} = \frac{AC}{PQ} \quad \text{(From (1))}$$

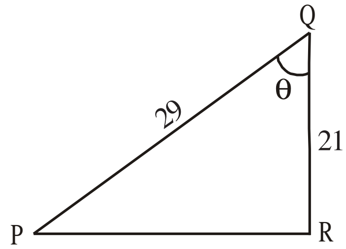
Hence, $\frac{AC}{PQ} = \frac{AB}{PR} = \frac{BC}{QR}$ then $\Delta ABC \sim \Delta PQR$

Therefore, $\angle A = \angle P$

Example-3. Consider a triangle PQR, right angled at P, in which PQ = 29 units, QR = 21 units and $\angle PQR = \theta$, then find the values of

(i) $\cos^2\theta + \sin^2\theta$ and (ii) $\cos^2\theta - \sin^2\theta$

Solution : In PQR, we have



$$PR = \sqrt{PQ^2 - QR^2} = \sqrt{(29)^2 - (21)^2}$$

$$= \sqrt{400} = 20 \text{ units}$$

$$\sin\theta = \frac{PR}{PQ} = \frac{20}{29}$$

$$\cos\theta = \frac{QR}{PQ} = \frac{21}{29}$$

Now (i) $\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{441+400}{841} = 1$

(ii) $\cos^2\theta - \sin^2\theta = \left(\frac{20}{29}\right)^2 - \left(\frac{21}{29}\right)^2 = \frac{-41}{841}$

Exercise - 11.1

1. In right angle triangle ABC, 8 cm, 15 cm and 17 cm are the lengths of AB, BC and CA respectively. Then, find out sin A, cos A and tan A.

2. The sides of a right angle triangle PQR are PQ = 7 cm, QR = 25cm and $\angle Q = 90^o$ respectively. Then find, tan Q − tan R.

3. In a right angle triangle ABC with right angle at B, in which $a = 24$ units, $b = 25$ units and $\angle BAC = \theta$. Then, find $\cos\theta$ and $\tan\theta$.

4. If $\cos A = \frac{12}{13}$, then find $\sin A$ and $\tan A$.

5. If $3\tan A = 4$, then find $\sin A$ and $\cos A$.

6. If $\angle A$ and $\angle X$ are acute angles such that cos A = cos X then show that $\angle A = \angle X$.

7. Given $\cot\theta = \frac{7}{8}$, then evaluate (i) $\frac{(1+\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(1-\cos\theta)}$ (ii) $\frac{(1+\sin\theta)}{\cos\theta}$

8. In a right angle triangle ABC, right angle is at B, if $\tan A = \sqrt{3}$ then find the value of

   (i) $\sin A \cos C + \cos A \sin C$ (ii) $\cos A \cos C - \sin A \sin C$

11.3 Trigonometric Ratios of some specific angles

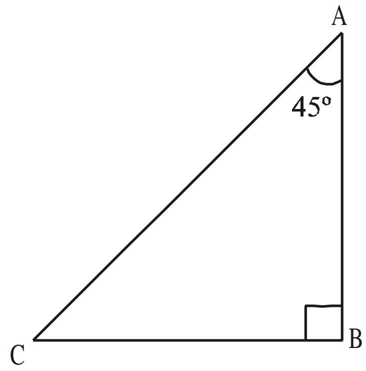
We already know about isosceles right angle triangle and right angle triangle with angles 30º, 60º and 90º.

Can we find sin 30° or tan 60° or cos 45° etc. with the help of these triangles?

Does sin 0° or cos 0° exist?

11.3.1 Trigonometric Ratios of 45°

In isosceles right angle triangle ABC right angled at B



$\angle A = \angle C = 45^{\circ}$ (why ?) and $BC = AB$ (why ?)

Let's assume the length of $BC = AB = a$

Then, $AC^2 = AB^2 + BC^2$ (by Pythagoras theorem)

$= a^2 + a^2 = 2a^2$,

Therefore, $AC = a\sqrt{2}$

Using the definitions of trigonometric ratios,

$$\sin 45^{\circ} = \frac{\text{Length of the opposite side to angle } 45^{\circ}}{\text{Length of hypotenuse}} = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

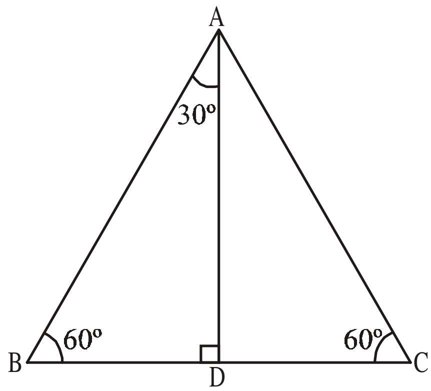
$$\cos 45^{\circ} = \frac{\text{Length of the adjacent side to angle } 45^{\circ}}{\text{Length of hypotenuse}} = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45^{\circ} = \frac{\text{Length of the opposite side to angle } 45^{\circ}}{\text{Length of the adjacent side to angle } 45^{\circ}} = \frac{BC}{AC} = \frac{a}{a} = 1$$

Like this you can determine the values of cosec 45°, sec 45° and cot 45°.

11.3.2 Trigonometric Ratios of 30° and 60°

Let us now calculate the trigonometric ratios of 30° and 60°. To calculate them, we will take an equilateral triangle, draw a perpendicular which can divide the triangle into two equal right angle triangles having angles 30°, 60° and 90° in each.



Consider an equilateral triangle ABC. Since each angle is 60^o in an equilateral triangle, we have $\angle A = \angle B = \angle C = 60^o$ and the sides of equilateral triangle is AB = BC = CA = $2a$ units.

Draw the perpendicular line AD from vertex A to BC as shown in the adjacent figure.

Perpendicular AD acts as "angle bisector of angle A" and "bisector of the side BC" in the equilateral triangle ABC.

Therefore, $\angle BAD = \angle CAD = 30^o$.

Since point D divides the side BC into equal halves,

$$BD = \frac{1}{2}BC = \frac{2a}{2} = a \text{ units.}$$

Consider right angle triangle ABD in the above given figure.

We have AB = $2a$ and BD = a

Then $AD^2 = AB^2 - BD^2$ by (Pythagoras theorem)

$= (2a)^2 - (a)^2 = 3a^2$.

Therefore, $AD = a\sqrt{3}$

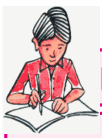
From definitions of trigonometric ratios,

$$\sin 60^o = \frac{AD}{AB} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^o = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

So, similarly $\tan 60^o = \sqrt{3}$ (why?)

Like the above, you can also determine the reciprocals, cosec 60^o, sec 60^o and cot 60^o by using the ratio concepts.



Do This

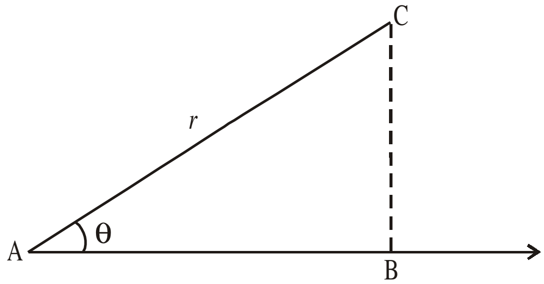
Find cosec 60^o, sec 60^o and cot 60^o.

Try This

Find sin 30^o, cos30^o, tan 30^o, cosec 30^o, sec30^o and cot 30^o by using the ratio concepts.

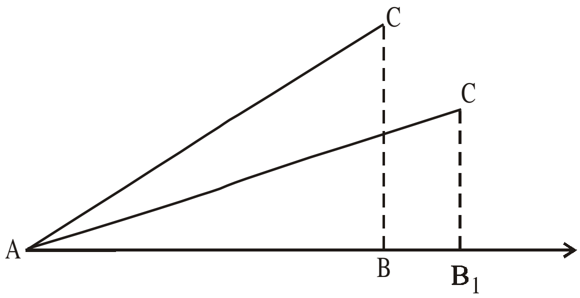
11.3.3 Trigonometric Ratios of 0^o and 90^o

Till now, we have discussed trigonometric ratios of 30^o, 45^o and 60^o. Now let us determine the trigonometric ratios of angles 0^o and 90^o.

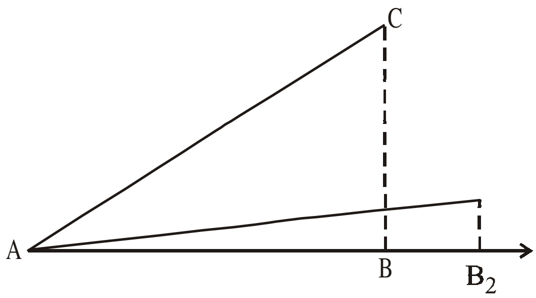


Suppose a segment AC of length r is making an acute angle with ray AB. Height of C from B is BC. When AC leans more on AB so that the angle made by it decreases, then what happens to the lengths of BC and AB ?

As the angle A decreases, the height of C from AB ray decreases and foot B is shifted from B to B_1 and B_2 and gradually when the angle becomes zero, height (i.e. opposite side of the angle) will also become zero (0) and adjacent side would be equal to AC i.e. length equal to *r*.



Step (i)



Step (ii)

Let us look at the trigonometric ratios

$$\sin A = \frac{BC}{AC} \text{ and } \cos A = \frac{AB}{AC}$$

If $A = 0^o$ then $BC = 0$ and $AC = AB = r$

$$\text{then } \sin 0^o = \frac{0}{r} = 0 \text{ and } \cos 0^o = \frac{r}{r} = 1$$

$$\text{we know that } \tan A = \frac{\sin A}{\cos A}$$

$$\text{So, } \tan 0^o = \frac{\sin 0^o}{\cos 0^o} = \frac{0}{1} = 0$$

Think - Discuss

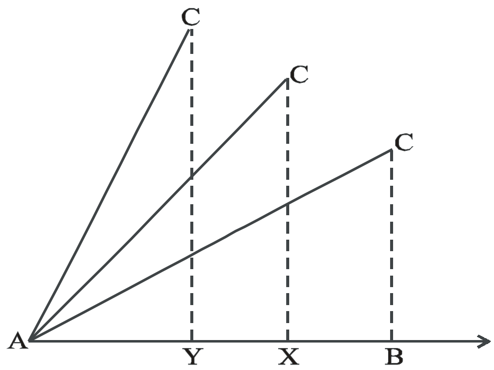
Discuss between your friend about the following conditions:

1. What can you say about $\operatorname{cosec} 0^o = \dfrac{1}{\sin 0^o}$? Is it defined? Why ?

2. What can you say about $\cot 0^o = \frac{1}{\tan 0^o}$. Is it defined? Why ?

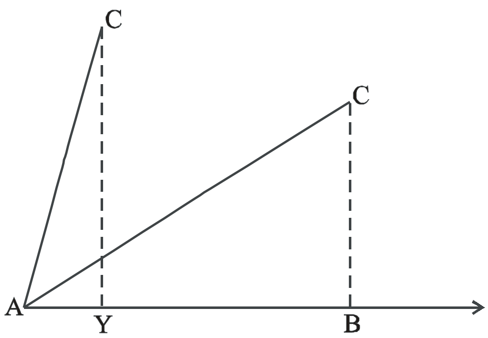
3. $\sec 0^o = 1$. Why ?

Now let us see what happens when angle made by AC with ray AB increases. When angle A is increased, height of point C increases and the foot of the perpendicular shifts from B to X and then to Y and so on. In other words, we can say that the height BC increases gradually, the angle on C gets continuous increment and at one stage the angle reaches 90^o. At that time, point B reaches A and AC equal to BC.

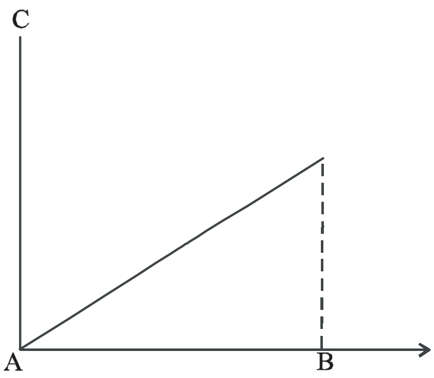


Step (i)

So, when the angle becomes 90^o, base (i.e. adjacent side of the angle) would become zero (0), the height of C from AB ray increases and it would be equal to AC and that is the length equal to r.



Step (ii)



Step (iii)

Now let us see trigonometric ratios

$$\sin A = \frac{BC}{AC} \text{ and } \cos A = \frac{AB}{AC}$$

If $A = 90^o$ then $AB = 0$ and $AC = BC = r$

then $\sin 90^o = \frac{r}{r} = 1$ and $\cos 90^o = \frac{0}{r} = 0$

TRY THIS

Find the ratios for $\tan 90^o$, $\text{cosec } 90^o$, $\sec 90^o$ and $\cot 90^o$.

Now, let us see the values of trigonometric ratios of all the above discussed angles in the form of a table.

Table 11.1

$\angle A$	0^o	30^o	45^o	60^o	90^o
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cot A	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cosec A	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

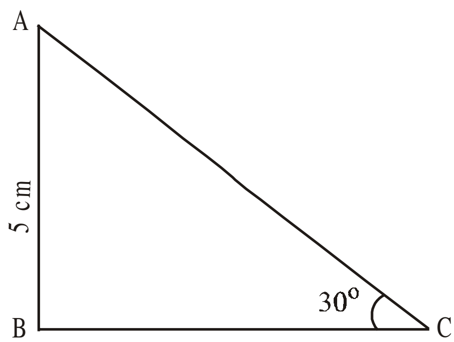
Think - Discuss

What can you say about the values of sin A and cos A, as the value of angle A increases from 0^o to 90^o? (observe the above table)

If $A \geq B$, then $\sin A \geq \sin B$. Is it true ?

If $A \geq B$, then $\cos A \geq \cos B$. Is it true ? Discuss.

Example-4. In ΔABC, right angle is at B, AB = 5 cm and $\angle ACB = 30^o$. Determine the lengths of the sides BC and AC.

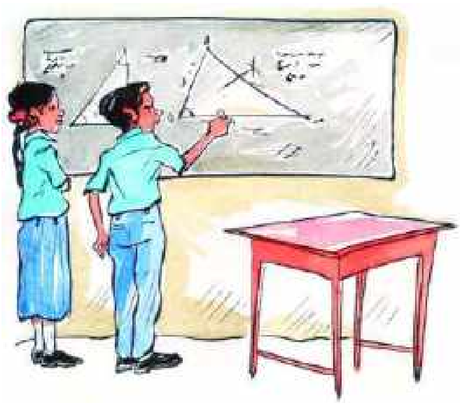


Solution : Given AB=5 cm and $\angle ACB=30^o$. To find the length of side BC, we will choose the trignometric ratio involving BC and the given side AB. Since BC is the side adjacent to angle C and AB is the side opposite to angle C.

Therefore,

$$\frac{AB}{BC} = \tan C$$

i.e. $\dfrac{5}{BC} = \tan 30^o = \dfrac{1}{\sqrt{3}}$



which gives $BC = 5\sqrt{3}$ cm

Now, by using the Pythagoras theorem

$$AC^2 = AB^2 + BC^2$$

$$AC^2 = 5^2 + 5\sqrt{3}^{\,2}$$

$$AC^2 = 25 + 75$$

$$AC = \sqrt{100} \quad = 10 \text{ cm}$$

Example-5. A chord of a circle of radius 6cm is making an angle 60^o at the centre. Find the length of the chord.

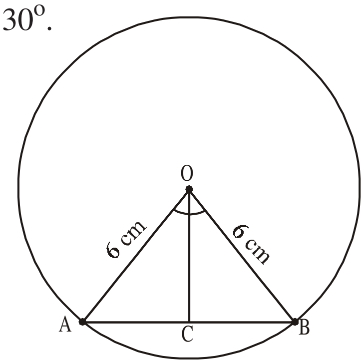
Solution : Given the radius of the circle $OA = OB = 6$cm

$\angle AOB = 60^o$

OC is height from 'O' upon AB and it is a angle bisector.

then, $\angle COB = 30^o$.

Consider ΔCOB



$\sin 30^o = \dfrac{BC}{OB}$

$\dfrac{1}{2} = \dfrac{BC}{6}$

$BC = \dfrac{6}{2} = 3$.

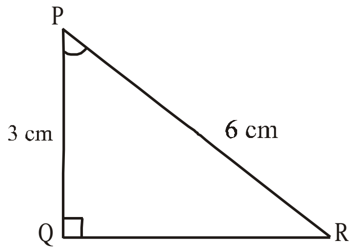
But, length of the chord $AB = 2BC$

$= 2 \times 3 = 6$ cm

$\therefore$ Therefore, length of the chord = 6 cm

The first use of the idea of **'sine'** in the way we use it today was in the book *Aryabhatiyam* by Aryabhatta, in A.D. 500. Aryabhatta used the word ***ardha-jya*** for the half-chord, which was shortened to ***jya*** or ***jiva*** in due course. When the *Aryabhatiyam* was translated into Arabic, the word ***jiva*** was retained as it is. The word ***jiva*** was translated into ***sinus,*** which means curve, when the Arabic version was translated into Latin. Soon the word ***sinus***, also used as ***sine***, became common in mathematical texts throughout Europe. An English Professor of astronomy Edmund Gunter (1581–1626), first used the abbreviated notation **'*sin*'.**

Example-6. In ΔPQR, right angle is at Q, PQ = 3 cm and PR = 6 cm. Determine $\angle QPR$ and $\angle PRQ$.



Solution : Given PQ = 3 cm and PR = 6 cm

Therefore, $\frac{PQ}{PR} = \sin R$

or $\sin R = \frac{3}{6} = \frac{1}{2}$

So, $\angle PRQ = 30^o$

and therefore, $\angle QPR = 60^o$ (why?)

Note **:** If one of the sides and any other part (either an acute angle or any side) of a right angle triangle is known, the remaining sides and angles of the triangle can be determined.

Example-7. If $\sin(A-B) = \frac{1}{2}$, $\cos(A+B) = \frac{1}{2}$, $0^o < A+B \leq 90^o$, $A > B$, find A and B.

Solution : Since $\sin(A-B) = \frac{1}{2}$, therefore, $A - B = 30^o$ (why?)

Also, since $\cos(A+B) = \frac{1}{2}$, therefore, $A + B = 60^o$ (why?)

Solving the above equations, we get : $A = 45^o$ and $B = 15^o$. (How?)

EXERCISE - 11.2

1. Evaluate the following.

(i) $\sin 45^o + \cos 45^o$

(ii) $\dfrac{\cos 45^o}{\sec 30^o + \operatorname{cosec} 60^o}$

(iii) $\dfrac{\sin 30^o + \tan 45^o - \operatorname{cosec} 60^o}{\cot 45^o + \cos 60^o - \sec 30^o}$

(iv) $2\tan^2 45^o + \cos^2 30^o - \sin^2 60^o$

(v) $\dfrac{\sec^2 60^o - \tan^2 60^o}{\sin^2 30^o + \cos^2 30^o}$

2. Choose the right option and justify your choice-

(i) $\dfrac{2\tan 30^o}{1+\tan^2 45^o}$

(a) $\sin 60^o$ (b) $\cos 60^o$ (c) $\tan 30^o$ (d) $\sin 30^o$

(ii) $\dfrac{1-\tan^2 45^o}{1+\tan^2 45^o}$

(a) $\tan 90^o$ (b) 1 (c) $\sin 45^o$ (d) 0

(iii) $\dfrac{2\tan 30^o}{1-\tan^2 30^o}$

(a) $\cos 60^o$ (b) $\sin 60^o$ (c) $\tan 60^o$ (d) $\sin 30^o$

3. Evaluate $\sin 60^o \cos 30^o + \sin 30^o \cos 60^o$. What is the value of $\sin(60^o + 30^o)$. What can you conclude ?

4. Is it right to say $\cos(60^o + 30^o) = \cos 60^o \cos 30^o - \sin 60^o \sin 30^o$.

5. In right angle triangle ΔPQR, right angle is at Q and PQ = 6cms $\angle RPQ = 60^o$. Determine the lengths of QR and PR.

6. In ΔXYZ, right angle is at Y, YZ = x, and XY = $2x$ then determine $\angle YXZ$ and $\angle YZX$.

7. Is it right to say that $\sin(A+B) = \sin A + \sin B$? Justify your answer.

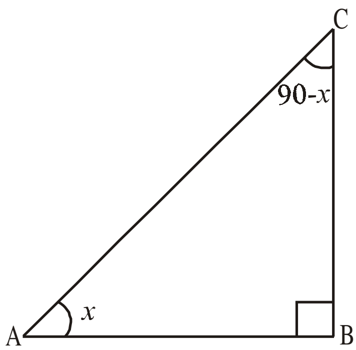
Think - Discuss

For which value of acute angle (i) $\dfrac{\cos\theta}{1-\sin\theta} + \dfrac{\cos\theta}{1+\sin\theta} = 4$ is true?

For which value of $0^o \le \theta \le 90^o$, above equation is not defined?

11.4 Trigonometric Ratios of Complementary Angles

We already know that two angles are said to be complementary, if their sum is equal to 90^o. Consider a right angle triangle ABC with right angle at B. Are there any complementary angles in this triangle?



Since angle B is 90^o, sum of other two angles must be 90^o. ($\because$ Sum of angles in a triangle 180^o)

Therefore, $\angle A + \angle C = 90^o$. Hence $\angle A$ and $\angle C$ are said to be complementary angles.

Let us assume that $\angle A = x$, then for angle x, BC is opposite side and AB is adjacent side.

$\sin x = \frac{BC}{AC}$ $\cos x = \frac{AB}{AC}$ $\tan x = \frac{BC}{AB}$

$\operatorname{cosec} x = \frac{AC}{BC}$ $\sec x = \frac{AC}{AB}$ $\cot x = \frac{AB}{BC}$

If $\angle A + \angle C = 90^o$, then we have $\angle C = 90^o - \angle A$

And we have that $\angle A = x$, then $\angle C = 90^o - x$

Let us look at what would be "Opposite side" and "Adjacent side" of the angle $(90^o - x)$in the triangle ABC.

$\sin(90^o - x) = \frac{AB}{AC}$ $\cos(90^o - x) = \frac{BC}{AC}$ $\tan(90^o - x) = \frac{AB}{BC}$

$\operatorname{Cosec}(90^o - x) = \frac{AC}{AB}$ $\sec(90^o - x) = \frac{AC}{BC}$ $\cot(90^o - x) = \frac{BC}{AB}$

Now, if we compare the ratios of angles x and $(90^o - x)$ from the above values of different triginometric terms.

There can be three possibilities in above figure.

$\sin(90^o - x) = \frac{AB}{AC} = \cos x$ and $\cos(90^o - x) = \frac{BC}{AC} = \sin x$

$\tan(90^o - x) = \frac{AB}{BC} = \cot x$ and $\cot(90^o - x) = \frac{BC}{AB} = \tan x$

$\operatorname{cosec}(90^o - x) = \frac{AC}{AB} = \sec x$ and $\sec(90^o - x) = \frac{AC}{BC} = \operatorname{cosec} x$

Think - Discuss

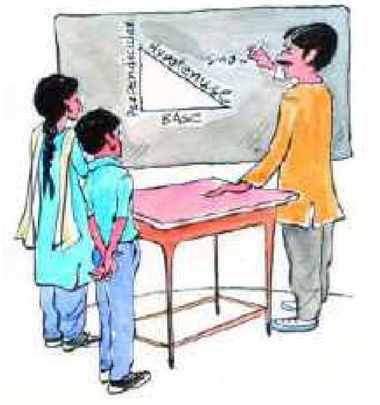
Check and discuss the above relations in the case of angles between 0º and 90º, whether they hold for these angles or not?

So, $\sin(90^o - A) = \cos A$ $\cos(90^o - A) = \sin A$

$\tan(90^o - A) = \cot A$ and $\cot(90^o - A) = \tan A$

$\sec(90^o - A) = \operatorname{cosec} A$ $\operatorname{cosec}(90^o - A) = \sec A$

Now, let us consider some examples



Example-8. Evaluate $\dfrac{\sec 35^o}{\operatorname{cosec} 55^o}$

Solution : $\operatorname{cosec} A = \sec (90^o - A)$

$\operatorname{cosec} 55^o = \sec (90^o - 35^o)$

$\operatorname{cosec} 55^o = \sec 35^o$

Now $\dfrac{\sec 35^o}{\operatorname{cosec} 55^o} = \dfrac{\sec 35^o}{\sec 35^o} = 1$

Example-9. If $\cos 7A = \sin(A - 6^o)$, where 7A is an acute angle, find the value of A.

Solution : Given $\cos 7A = \sin(A - 6^o)$ ...(1)

$\sin (90 - 7A) = \sin (A - 6^o)$

since $(90 - 7A)$ & $(A - 6^o)$ are both acute angles,

therefore

$90^o - 7A = A - 6^o$

$8A = 96^o$

which gives $A = 12^o$.

Example-10. If $\sin A = \cos B$, then prove that $A + B = 90^o$.

Solution : Given that $\sin A = \cos B$ ...(1)

We know $\cos B = \sin (90^o - B)$, we can write (1) as

$\sin A = \sin (90^o - B)$

If A, B are acute angles, then $A = 90^o - B$

$\Rightarrow A + B = 90^o$.

Example-11. Express $\sin 81^o + \tan 81^o$ in terms of trigonometric ratios of angles between 0^o and 45^o

Solution : We can write $\sin 81^o = \cos(90^o - 81^o) = \cos 9^o$

$\tan 81^o = \tan(90^o - 81^o) = \cot 9^o$

Then, $\sin 81^o + \tan 81^o = \cos 9^o + \cot 9^o$

Example-12. If A, B and C are interior angles of triangle ABC, then show that $\sin\frac{B+C}{2} = \cos\frac{A}{2}$

Solution : Given A, B and C are interior angles of right angle triangle ABC then $A + B + C = 180^o$.

On dividing the above equation by 2 on both sides, we get

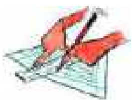
$$\frac{A}{2} + \frac{B+C}{2} = 90^o$$

$$\frac{B+C}{2} = 90^o - \frac{A}{2}$$

On taking sin ratio on both sides

$$\sin\left(\frac{B+C}{2}\right) = \sin\left(90^o - \frac{A}{2}\right)$$

$$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2} \text{ ; hence proved.}$$



EXERCISE 11.3

1. Evaluate

   (i) $\frac{\tan 36^o}{\cot 54^o}$ (ii) $\cos 12^o - \sin 78^o$ (iii) $\text{cosec } 31^o - \sec 59^o$

   (iv) $\sin 15^o \sec 75^o$ (vi) $\tan 26^o \tan 64^o$

2. Show that

   (i) $\tan 48^o \tan 16^o \tan 42^o \tan 74^o = 1$

   (ii) $\cos 36^o \cos 54^o - \sin 36^0 \sin 54^o = 0.$

3. If $\tan 2A = \cot(A - 18^o)$, where 2A is an acute angle. Find the value of A.

4. If $\tan A = \cot B$ where A and B are acute angles, prove that $A + B = 90^o$.

5. If A, B and C are interior angles of a triangle ABC, then show that $\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$

6. Express $\sin 75^o + \cos 65^o$ in terms of trigonometric ratios of angles between 0^o and 45^o.

11.5 Trigonometric identities

We know that an identity is that mathematical equation which is true for all the values of the variables in the equation.

For example $(a+b)^2 = a^2 + b^2 + 2ab$ is an identity.

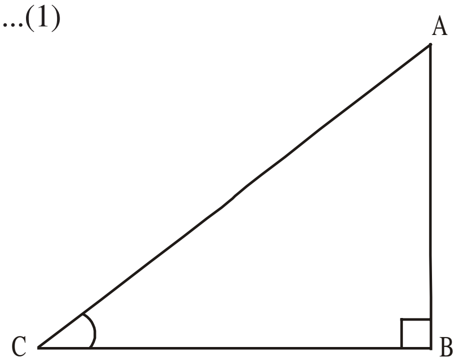
In the same way, an identity equation having trigonometric ratios of an angle is called trigonometric idenitity. And it is true for all the values of the angles involved in it.

Here, we will derive a trigonometric identity and remaining would be based on that.

Consider a right angle triangle ABC with right angle is at B, so

From Pythagoras theorem

We have $AB^2 + BC^2 = AC^2$ ....(1)



Dividing each term by AC^2, we get

$$\Rightarrow \frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

i.e., $\left[\frac{AB}{AC}\right]^2 + \left[\frac{BC}{AC}\right]^2 = \left[\frac{AC}{AC}\right]^2$

i.e., $(\cos A)^2 + (\sin A)^2 = 1$

Here, we generally write $\cos^2 A$ in the place of $(\cos A)^2$

i.e., $(\cos A)^2 = \cos^2 A$ (Do not write $\cos A^2$)

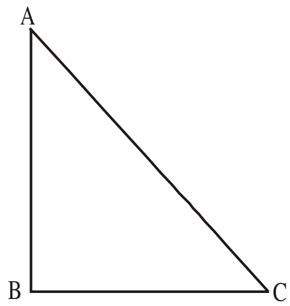
$\therefore$ above equation is $\cos^2 A + \sin^2 A = 1$

We have given an equation having a variable A(angle) and above equation is true for all the value of A. Hence the above equation is a trigonometric identity.

Therefore, we have trigonometric idenity

$$\mathbf{\cos^2 A + \sin^2 A = 1}$$

Let us look at another trigonometric idenity



From equation (1) we have

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$ (Dividing each term by AB^2)

$$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e., $1 + \tan^2 A = \sec^2 A$

Similarly, on dividing (1) by BC^2, we get $\cot^2 A + 1 = \text{cosec}^2 A$.

By using above identities, we can express each trigonometric ratio in terms of another ratio. If we know the value of a ratio, we can find all other ratios by using these identities.

Think - Discuss

Are these identities true for $0^0 \le A \le 90^0$? If not, for which values of A they are true?

- $\sec^2 A - \tan^2 A = 1$
- $\text{cosec}^2 A - \cot^2 A = 1$

Do This

(i) If $\sin C = \frac{15}{17}$, then find cos A.

(ii) If $\tan x = \frac{5}{12}$, then find sec x.

(iii) If $\text{cosec}\,\theta = \frac{25}{7}$, then find cot x.

Try This

Evaluate the following and justify your answer.

(i) $\dfrac{\sin^2 15^o + \sin^2 75^o}{\cos^2 36^o + \cos^2 54^o}$

(ii) $\sin 5^o \cos 85^o + \cos 5^o \sin 85^o$

(iii) $\sec 16^o \ \text{cosec}\, 74^o - \cot 74^o \tan 16^o$.

Example-13. Show that $\cot\theta + \tan\theta = \sec\theta \ \text{cosec}\,\theta$.

Solution : LHS $= \cot\theta + \tan\theta$

$$= \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta} \qquad \text{(why ?)}$$

$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1}{\sin\theta \ \cos\theta} \qquad \text{(why ?)}$$

$$= \frac{1}{\sin\theta} \ \frac{1}{\cos\theta} = \operatorname{cosec}\theta \ \sec\theta$$

Example-14. Show that $\tan^2\theta + \tan^4\theta = \sec^4\theta - \sec^2\theta$

Solution : L.H.S. $= \tan^2\theta + \tan^4\theta$

$= \tan^2\theta\,(1 + \tan^2\theta)$

$= \tan^2\theta \,.\, \sec^2\theta$ (Why ?)

$= (\sec^2\theta - 1)\sec^2\theta$ (Why ?)

$= \sec^4\theta - \sec^2\theta = \text{R.H.S}$

Example-15. Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$

Solution : LHS $= \sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}}$ (multiply and divide by $1 + \cos\theta$)

$$= \sqrt{\frac{1+\cos\theta}{1-\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}} \qquad \text{(Why ?)}$$

$$= \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \operatorname{cosec}\theta + \cot\theta = \text{R.H.S.}$$

EXERCISE 11.4

1. Evaluate the following :

   (i) $(1 + \tan\theta + \sec\theta)\,(1 + \cot\theta - \operatorname{cosec}\theta)$

   (ii) $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2$

   (iii) $(\sec^2\theta - 1)\,(\operatorname{cosec}^2\theta - 1)$

2. Show that $(\text{cosec}\,\theta - \cot\theta)^2 = \dfrac{1-\cos\theta}{1+\cos\theta}$

3. Show that $\sqrt{\dfrac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

4. Show that $\dfrac{1-\tan^2 A}{\cot^2 A - 1} = \tan^2 A$

5. Show that $\dfrac{1}{\cos\theta} - \cos\theta = \tan\theta.\sin\theta$

6. Simplify $\sec A\,(1-\sin A)\,(\sec A + \tan A)$

7. Prove that $(\sin A + \text{cosec}\,A)^2 + (\cos A + \sec A)^2 = 7 + \tan^2 A + \cot^2 A$

8. Simplify $(1 - \cos\theta)\,(1 + \cos\theta)\,(1 + \cot^2\theta)$

9. If $\sec\theta + \tan\theta = p$, then what is the value of $\sec\theta - \tan\theta$?

10. If $\text{cosec}\,\theta + \cot\theta = k$ then prove that $\cos\theta = \dfrac{k^2-1}{k^2+1}$

Optional Exercise

[This exercise is not meant for examination]

1. Prove that $\dfrac{\cot\theta - \cos\theta}{\cot\theta + \cos\theta} = \dfrac{\text{cosec}\,\theta - 1}{\text{cosec}\,\theta + 1}$

2. Prove that $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$ using the identity $\sec^2\theta = 1 + \tan^2\theta$.

3. Prove that $(\text{cosec}\,A - \sin A)(\sec A - \cos A) = \dfrac{1}{\tan A + \cot A}$

4. Prove that $\dfrac{1+\sec A}{\sec A} = \dfrac{\sin^2 A}{1-\cos A}$.

5. Show that $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right) = \left(\dfrac{1+\tan A}{1-\cot A}\right)^2 = \tan^2 A$

6. Prove that $\left(\dfrac{(\sec A - 1)}{(\sec A + 1)}\right) = \left(\dfrac{(1-\cos A)}{(1+\cos A)}\right)$