Marks 100

Duration 3 hours

Section A: Each question carries 1 mark

1. There is a 2 x 2 matrix A such that $A \times adj(A) = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$. Find the determinant value of matrix A.
2. A piece wise function is given below. Find a value of k such that the given function is continuous at x = 2.
$$f(x) = \begin{cases} \dfrac{(x+2)^2 - 16}{x-3} & ,x \neq 2 \\ k, & ,x = 2 \end{cases}$$
3. Evaluate the following indefinite integral in a suitable domain.
$$\int \frac{1 + \cos(x)}{(x + \sin(x))^7} dx$$
4. We have two planes represented by the equations: $3x - 2y + 3z = \ 8$ and $6x - 4y + 6z = 32$. Find the distance between these two planes.

Section B: Each question carries 2 marks

5. Given that M is a 3rd order skewed matrix, write a step wise proof for determinant value of M to be equal to 0, |M| = 0.
6. Apply the Rolle's theorem to the function $f(x) = 8x + e^{-3x}$ over the interval $[-2, 3]$ and thus find the value of c that satisfies $f'(c) = 0$.
7. Find out how fast is the surface area of a cube increasing when the length of an edge of the cube is 15 cm and its volume is increasing at a steady rate of 6 cu cm/ sec.
8. You are given the function $f(x) = x^3 - 5x^2 + 2x - 80$. You are told that it is increasing over the set of real numbers R. Prove it.
9. The point P lies on the line joining the points A(2,-1,3) and B(1,4,-3). The x co ordinate of the point P is 7. Find the y coordinate of the point.
10. A six sided fair die has three faces green and three faces blue. The green faces are numbered 1, 2 and 3 and the blue faces are numbered 4, 5 and 6. The die is rolled and two events are described as follows:
    Event M = Even number shows up
    Event N = Blue face shows up.
    Are these two events independent? Justify showing work.
11. A firm manufacturing bicycles makes two types of bicycles: gearless bicycles and geared bicycles. In the long term, the business projections indicate that the demand of gearless bicycles is at least 100 per day and that of geared bicycles is at least 80 per day. However the production capacity of the firm is limited. They cannot produce more than 200 gearless and 170 geared bicycles per day. To fulfil a particular whole seller's order they need to send at least 200 bicycles each day. If the loss incurred on each gearless bicycle is $2 and the profit made on each

geared bicycle is \$5, then find out using a linear programming model how many bicycles of each type must be manufactured and shipped each day to maximize the net profits.

12. Evaluate the following indefinite integral in suitable domain.

$$\int \frac{1}{21-6x-x^2}\,dx$$

Section C: Each question carries 4 marks

13. Solve the following inverse trigonometric equation in suitable domain.

$$\tan^{-1}\left(\frac{(x+1)}{(x-1)}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7)$$

14. Use the theorems and properties of determinants that you have learned about to prove that the value of the following determinant is $-(x-y)(y-z)(z-x)$

$$\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

OR

Find a matrix X such that the product of the following matrices is equal to what is given in there.

$$\begin{bmatrix} -1 & 2 \\ 5 & 0 \\ 3 & -4 \end{bmatrix} \times X = \begin{bmatrix} 15 & -5 & 5 \\ 15 & 35 & -25 \\ -27 & 17 & -15 \end{bmatrix}$$

15. Use implicit differentiation to find the first derivative of y with respect to x for the following equation.

$$x^{2y}+y^{2x}=a^{2b}$$

OR

Show that if $y=ax+a^2$ where a is an arbitrary constant, then $\left(\frac{dy}{dx}\right)^2+x\left(\frac{dy}{dx}\right)=y$.

16. Evaluate the following indefinite integral in suitable domain.

$$\int \frac{\sin(x)}{(9+\cos^2(x))\,(10-9\cos^2(x))}\,dx$$

17. Evaluate any one of the following definite integrals.

a. $\int_0^{\frac{1}{2}} \frac{\sin^{-1}(x)}{(1-x^2)^{\frac{3}{2}}}\,dx$

b. $\int_{-1}^{2} |2x - 1|\, dx$

18. Solve the following differential equation:

$$dy + ydx = (\cos x - \sin x)dx$$

19. Prove the following statement: For an equilateral triangle, the centroid and the in-centre are the same point. Use the result to find the co-ordinates of the in-centre of the triangle having vertices P(6, 4, 6), Q(12, 4, 0) and R(4, 2, -2).

20. Four points have the following position vectors:

$$2\hat{\imath} + 5\hat{\jmath} + \hat{k}$$
$$-\hat{\jmath} - 4\,\hat{k}$$
$$3\hat{\imath} - \lambda\hat{\jmath} + 8\hat{k}$$
$$-4\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$$

Find a value of the λ that makes those points coplanar.

21. There are four balls numbered 2, 5, 8 and 11 in a lotto jar. Michel draws two balls at random from the jar without replacement. If we define X as a random variable such that it denotes the sum of the numbers appearing on the two balls, find the mean and standard deviation of the distribution of X.

22. Of the people in a small village it is known that 30% of the have been given 100% vaccination towards a particular disease. 70% of them have been given partial vaccination towards that disease. From previous results it is known that 70% of those who have been given 100% vaccination are susceptible to contacting that disease and 10% of those given partial vaccination are also susceptible to contacting the disease. At the end of the incubation period of the vaccination program a person is chosen at random. It is found that his person has contacted the disease. What is the probability that this person was given 100% vaccination? Is the vaccination really effective? Justify your answer.

23. Maximize the objective function given by:

$$Z = x + y - 50$$

When it is subject to the following constraints:

$$50x + 24y \leq 2400$$
$$30x + 33y \leq 2100$$
$$x \geq 45$$
$$y \geq 5$$

Note that the solution has to be found graphically.

Section C: Each question carries 6 marks

24. Find the product of the two matrices given below:

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$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & 0 \\ 3 & -5 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

Use the result to solve the system of equations given below:

$$x + y - z = 4$$
$$2x - y + 3z = 9$$
$$4y + 5z = 1$$

25. Given the function $f(x) = \frac{8x+9}{9x+8}$ such that it is a function from $R - \left\{-\frac{8}{9}\right\}$ to $R - \left\{\frac{8}{9}\right\}$. Prove that this function is a bijection. Then find its inverse.

OR

Given that set B is a Cartesian product set $Q \times Q$. Also that, # is a binary operation that is defined on B such that $(b_1, b_2)\#(b_3, b_4) = (b_1 b_3, b_2 + b_1 b_4)$ for all (b_1, b_2) and (b_3, b_4) belonging to set B. First determine whether the described binary operation is commutative and associative. Also find its identity element in set B. Also find the invertible elements of set B.

26. A closed cuboidal box with a square base has a specific volume. Prove that for minimum volume, the height of the box should be equal to the side of the square base.
27. Find the area of a triangle having vertices (4,5), (-3, 2) and (8,-4) using integrals.

    OR

    Region R is enclosed between the parabola $4y = 5x^2$ and the line $5y + 6x = 37$. Find the area of this region using integrals.

28. A radioactive body decays at a rate that is proportional to its mass at a particular instant of time. The mass of the body after decaying for a day is 100 gms. After two days the mass further reduces to 80 gms. What was the initial mass of the body before it started to decay?
29. The line passing through the points (1, 2, 3) and (2, -3, 1) cuts the plane that contains the points (4, 2, -3), (3, -4, 5) and (0, 4, 3). Find the co-ordinates of the point of intersection of the line with the plane.

    OR

    A plane shifts such that it remains at a distance of 3p from the origin. This plane cuts the co-ordinate axes in points X, Y and Z. Prove that the equation of the path of the centroid of the triangle XYZ can be given by the equation:

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$