01 Real Numbers

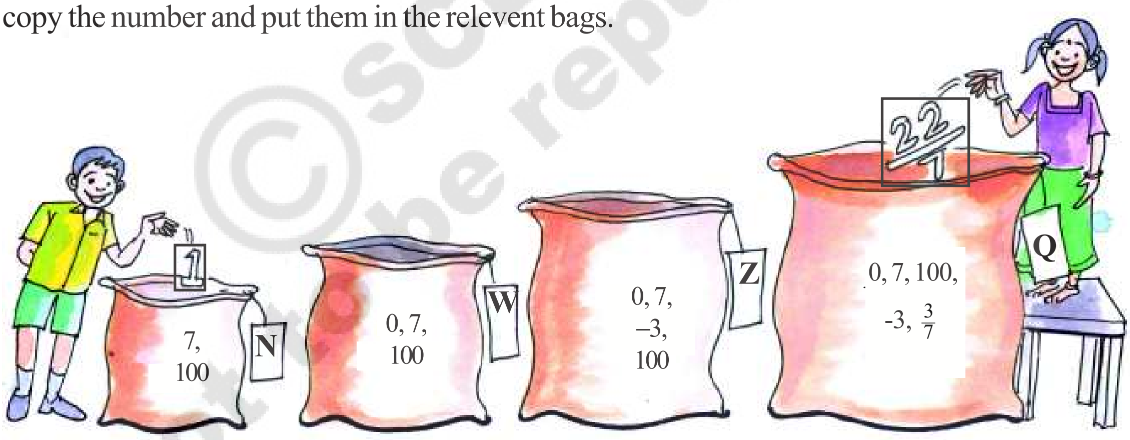
1.1 Introduction

Let us have a brief review of various types of numbers.

Consider the following numbers.

$$7, 100, 9, 11, -3, 0, -\frac{1}{4}, 5, 1, \frac{3}{7}, -1, 0.12, -\frac{13}{17}, 13.222 ..., 19, \frac{-5}{3}, \frac{213}{4}, \frac{-69}{1}, \frac{22}{7}, 5.\overline{6}$$

John and Sneha want to label the above numbers and put them in the bags they belong to. Some of the numbers are there in their respective bags..... Now you pick up rest of the numbers and put them into the bags which they belong. If one number can go in more than one bag then copy the number and put them in the relevent bags.



You have observed bag N contains natural numbers. Bag W contains whole numbers. Bag Z contains integers and bag Q contains the rational numbers.

The bag Z contains integers which is the collection of negative numbers and whole numbers. It is denoted by I or Z and we write,

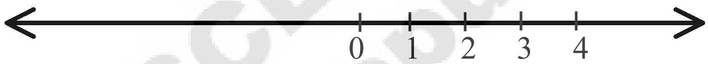
$$Z = \{... -3, -2, -1, 0, 1, 2, 3, ...\}$$

Similarly the bag Q contains all numbers that are of the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

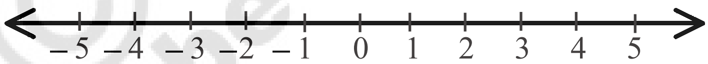
You may noticed that natural numbers, whole numbers, integers and rational numbers can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

For example, -15 can be written as $\frac{-15}{1}$; here p = -15 and q = 1. Look at the Example $\frac{1}{2} = \frac{2}{4} = \frac{10}{20} = \frac{50}{100}$ ... and so on. These are equivalent rational numbers (or fractions). It means that the rational numbers do not have a unique representation in the form p/q, where p and q are integers and $q \neq 0$. However, when we say p/q is a rational number or when we represent p/q on number line, we assume that $q \neq 0$ and that p and q have no common factors other than the universal factor '1' (i.e., p and q are co-primes.) So, on the number line, among the infinitely many fractions equivalent to $\frac{1}{2}$, we will choose $\frac{1}{2}$ i.e., the simplest form to represent all of them. To understand this lets make a number line.

You know that to represent whole numbers on number line, we draw a line and mark a point '0' on it. Then we can set off equal distances on the right side of the point '0' and label the points of division as 1, 2, 3, 4, …



The integer number line is made like this,



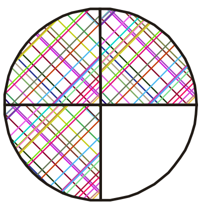
Do you remember how to represent the rational numbers on the number line?

To recall this let's first take the fraction $\frac{3}{4}$ and represent it pictorically as well as on number line.

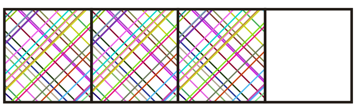
We know that in $\frac{3}{4}$, 3 is the numerator and 4 is the denominator.

Which means that 3 parts taken out of 4 equal parts from a given unit.

Here are few representations of $\frac{3}{4}$.

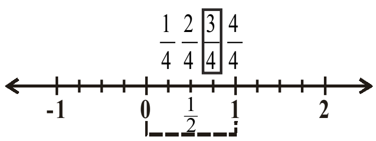


$\frac{3}{4}$



$\frac{3}{4}$

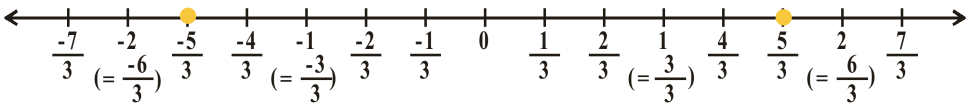
(Pictorically)



(Number line)

Example-1. Represent $\frac{5}{3}$ and $-\frac{5}{3}$ on the number line.

Solution : Draw an integer line representing $-2, -1, 0, 1, 2$.



Divide each unit into three equal parts to the right and left sides of zero respectively. Take five parts out of these. The fifth point on the right of zero represents $\frac{5}{3}$ and the fifth one to the left of zero represents $\frac{-5}{3}$.

DO THIS

1. Represent $\frac{-3}{4}$ on number line.
2. Write 0, 7, 10, -4 in p/q form.
3. **Guess my number** : Your friend chooses an integer between 0 and 100. You have to find out that number by asking questions, but your friend can only answer 'yes' or 'no'. What strategy would you use?

Example-2. Are the following statements True? Give reasons for your answers with an example.

i. Every rational number is an integer.

ii. Every integer is a rational number

iii. Zero is a rational number

Solution : i. False: For example, $\frac{7}{8}$ is a rational number but not an integer.

ii. True: Because any integer can be expressed in the form $\frac{p}{q}$ $(q \neq 0)$ for example

$-2 = \frac{-2}{1} = \frac{-4}{2}$. Thus it is a rational number.

(i.e. any integer 'b' can be represented as $\frac{b}{1}$)

iii. True: Because 0 can be expressed as $\frac{0}{2}, \frac{0}{7}, \frac{0}{13}$ ($\frac{p}{q}$ form, where p, q are integers and $q \neq 0$)

('0' can be represented as $\frac{0}{x}$ where 'x' is an integer and $x \neq 0$)

Example-3. Find two rational numbers between 3 and 4 by mean method.

Solution :

Method-I : We know that the rational number lie between two rational numbers a and b is $\frac{a+b}{2}$.

Here $a = 3$ and $b = 4$, (we know that $\frac{a+b}{2}$ is the mean of two integers 'a', 'b' and it lie between 'a' and 'b')

So, $\frac{(3+4)}{2} = \frac{7}{2}$ which is in between 3 and 4. $3 < \frac{7}{2} < 4$

If we continue the above process, we can find many more rational number between 3 and $\frac{7}{2}$

$$\frac{3+\frac{7}{2}}{2} = \frac{\frac{6+7}{2}}{2} = \frac{\frac{13}{2}}{2} = \frac{13}{2\times 2} = \frac{13}{4}$$

$$3 < \frac{13}{4} < \frac{7}{2} < 4$$

Method-II : The other option to find two rational numbers in single step.

Since we want two numbers, we write 3 and 4 as rational numbers with denominator $2 + 1 = 3$

i.e., $3 = \frac{3}{1} = \frac{6}{2} = \frac{9}{3}$ and $4 = \frac{4}{1} = \frac{8}{2} = \frac{12}{3} = \frac{16}{4}$

Then you can see that $\frac{10}{3}, \frac{11}{3}$ are rational numbers between 3 and 4.

$$3 = \frac{9}{3} < \left(\frac{10}{3} < \frac{11}{3}\right) < \frac{12}{3} = 4$$

Now if you want to find 5 rational numbers between 3 and 4, then we write 3 and 4 as rational number with denominator $5 + 1 = 6$.

i.e. $3 = \frac{18}{6}$ and $4 = \frac{24}{6}$ $\qquad 3 = \frac{18}{6} < \left(\frac{19}{6}, \frac{20}{6}, \frac{21}{6}, \frac{22}{6}, \frac{23}{6}\right) < \frac{24}{6} = 4$

From this, you might have realised the fact that there are infinitely many rational numbers between 3 and 4. Check, whether this holds good for any other two rational numbers? Thus we can say that , there exist infinite number of rational numbers between any two given rational numbers.

Do This

i. Find five rational numbers between 2 and 3 by mean method.

ii. Find 10 rational numbers between $-\frac{3}{11}$ and $\frac{8}{11}$.

Example-4. Express $\frac{7}{16}$, $\frac{2}{3}$ and $\frac{10}{7}$ in decimal form.

Solution :

$16\overline{)7.00000}$ = 0.4375

0
70
64
60
48
120
112
80
80
0

$\therefore \frac{7}{16} = 0.4375$
is a terminating decimal

$7\overline{)10}$ = 1.428571

7
30
28
20
14
60
56
40
35
50
49
10
7
3

$\therefore \frac{10}{7} = 1.\overline{428571}$
is a non-terminating recurring decimal

$3\overline{)2.0000}$ = 0.666

18
20
18
20
18
2

$\therefore \frac{2}{3} = 0.666 = 0.\overline{6}$
is a non-terminating recurring decimal

DO THIS

Find the decimal form of (i) $\frac{1}{17}$ (ii) $\frac{1}{19}$

Example-5. Express 3.28 in the form of $\frac{p}{q}$ (where p and q are integers, $q \neq 0$).

Solution :

$$3.28 = \frac{328}{100}$$

$$= \frac{328 \div 2}{100 \div 2} = \frac{164}{50}$$

$$= \frac{164 \div 2}{50 \div 2} = \frac{82}{25}$$ (Numerator and denominator are co-primes)

$$\therefore 3.28 = \frac{82}{25}$$

Example-6. Express $1.\overline{62}$ in the $\frac{p}{q}$ form where $q \neq 0$; p, q are integers.

Solutions : Let $x = 1.626262.....$ (1)

multiplying both sides of equation (1) by 100, we get

$100x = 162.6262...$ (2)

Subtracting (2) from (1) we get

$$\begin{array}{rl} 100x = & 162.6262... \\ x = & 1.6262... \\ - & - \\ \hline 99x = & 161 \\ \hline \end{array}$$

$$x = \frac{161}{99}$$

$$\therefore 1.\overline{62} = \frac{161}{99}$$

TRY THESE

I. Find the decimal values of the following:

i. $\frac{1}{2}$	ii. $\frac{1}{2^2}$	iii. $\frac{1}{5}$	iv. $\frac{1}{5\times 2}$
v. $\frac{3}{10}$	vi. $\frac{27}{25}$	vii. $\frac{1}{3}$	viii. $\frac{7}{6}$
ix. $\frac{5}{12}$	x. $\frac{1}{7}$		

Observe the following decimals

$\frac{1}{2} = 0.5$ $\quad \frac{1}{10} = 0.1$ $\quad \frac{32}{5} = 6.4$ $\quad \frac{1}{3} = 0.333...$ $\quad \frac{4}{15} = 0.2\overline{6}$

Can you tell the special character of denominator which makes the fraction as terminating or non-terminating recurring decimals?

Write prime factors of denominator of each rational number.

What did you observe from the results?

EXERCISE - 1.1

1. (a) Write any three rational numbers

   (b) Explain rational number is in your own words.

2. Give one example each to the following statements.

   i. A number which is rational but not an integer

   ii. A whole number which is not a natural number

   iii. An integer which is not a whole number

   iv. A number which is natural number, whole number, integer and rational number.

   v. A number which is an integer but not a natural number.

3. Find five rational numbers between 1 and 2.

4. Insert three rational numbers between $\frac{2}{3}$ and $\frac{3}{5}$

5. Represent $\frac{8}{5}$ and $\frac{-8}{5}$ on a numberline

6. Express the following rational numbers as decimals numbers

   I. i) $\frac{242}{1000}$ ii) $\frac{354}{500}$ iii) $\frac{2}{5}$ iv) $\frac{115}{4}$

   II. i) $\frac{2}{3}$ ii) $-\frac{25}{36}$ iii) $\frac{22}{7}$ iv) $\frac{11}{9}$

7. Express each of the following decimals in $\frac{p}{q}$ form where $q \neq 0$ and p, q are integers

   i) 0.36 ii) 15.4 iii) 10.25 iv) 3.25

8. Express each of the following decimal number in the $\frac{p}{q}$ form

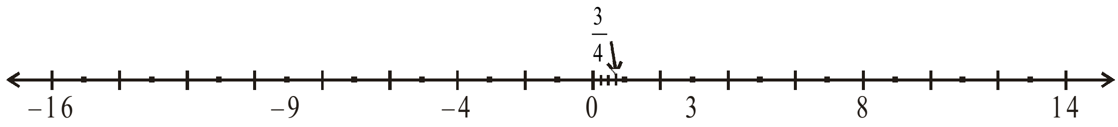
   i) $0.\overline{5}$ ii) $3.\overline{8}$ iii) $0.\overline{36}$ iv) $3.12\overline{7}$

9. Without actually dividing find which of the following are terminating decimals.

   (i) $\frac{3}{25}$ (ii) $\frac{11}{18}$ (iii) $\frac{13}{20}$ (iv) $\frac{41}{42}$

1.2 IRRATIONAL NUMBERS

Let us take a look at the number line again. Are we able to represent all the numbers on the number line? The fact is that there are infinite numbers left on the number line.



To understand this, consider these equations

(i) $x^2 = 4$ (ii) $3x = 4$ (iii) $x^2 = 2$

For equation (i) we know that value of x for this equation is 2 and -2. We can plot 2 and -2 on the number line.

For equation (ii) $3x = 4$ on dividing both sides by, 3 we get $\frac{3x}{3} = \frac{4}{3} \Rightarrow x = \frac{4}{3}$. We can plot this on the number line.

When we solve the equation (iii) $x^2 = 2$, taking square root for both the sides of the equation $\sqrt{x^2} = \sqrt{2} \Rightarrow x = \sqrt{2}$

Can we represent $\sqrt{2}$ on number line ?

What is the value of $\sqrt{2}$? To which numbers $\sqrt{2}$ belongs?

Let us find the value of $\sqrt{2}$ by long division method.

	1.4142135
1	$2.\overline{00}\ \overline{00}\ \overline{00}\ \overline{00}\ \overline{00}\ \overline{00}\ \overline{00}$
	1
24	100
	96
281	400
	281
2824	11900
	11296
28282	60400
	56564
282841	383600
	282841
2828423	10075900
	8485269
28284265	159063100
	141421325
28284270	17611775

Step 1 : After 2, place decimal points.

Step 2 : After decimal points write 0's.

Step 3 : Group '0' is pairs and put a bar over them.

Step 4 : Then after follow the method of find the square root of perfect square.

$\therefore \sqrt{2} = 1.4142135 \ldots$

If you go on finding the value of $\sqrt{2}$, you observe that $\sqrt{2}$ =1.4142135623731..... is neither terminating nor repeating decimal.

So far we have observed that the decimal number is either terminating or non-terminating repeating decimal, which can be expressed in $\frac{p}{q}$ form. These are known as rational numbers.

But decimal number for $\sqrt{2}$ is non-terminating and non-recurring decimal. Can you represent this using bar? No we can't. These type of numbers are called irrational numbers and they can't be represented in p/q form. That is $\sqrt{2} \neq p/q$ (for any integers p and q, $q \neq 0$).

Similarly $\sqrt{3} = 1.7320508075689.....$

$\sqrt{5} = 2.2360679774998.....$

These are non-terminating, non-recuring decimals. These are known as irrational numbers and are denoted by 'S' or 'Q''.

Examples of irrational numbers

(1) 2.1356217528...,   (2) $\sqrt{2}$, $\sqrt{3}$, π, etc.

The 5th Century BC the Pythagorean in Greece, the follower of the famous mathematician and philosopher Pythagoras, were the first to discover the numbers which were not rationals. These numbers are called irrational numbers. The Pythagoreans proved that $\sqrt{2}$ is irrational number. Later Theodorus of Cyrene showed that $\sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$ and $\sqrt{17}$ are also irrational numbers. There is a reference of irrationals in calculation of square roots in Sulba Sutra (800 BC).

Observe the following table

$\sqrt{1}$	=	1
$\sqrt{2}$	=	1.414213.....
$\sqrt{3}$	=	1.7320508.....
$\sqrt{4}$	=	2
$\sqrt{5}$	=	2.2360679.....
$\sqrt{6}$	=	
$\sqrt{7}$	=	
$\sqrt{8}$	=	
$\sqrt{9}$	=	3

If 'n' is a natural number other than a perfect square then $\sqrt{n}$ is an irrational number

Now can you identify which are rational and which are irrational?

$\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$ - are rational numbers.

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$ - are irrational numbers.

Think Discuss and Write

Kruthi said $\sqrt{2}$ can be written as $\frac{\sqrt{2}}{1}$ which is in $\frac{p}{q}$ form. So $\sqrt{2}$ is a rational number.

Do you agree with her argument ?

Know About π

π is defined as the ratio of the circumfrence (C) of a circle to its diameter (d). i.e. $\pi = \frac{c}{d}$

As π is in the form of ratio, this seems to contradict the fact that π is irrational. The circumference (C) and the diameter (d) of a circle are incommesurable. i.e. there does not exist a common unit to measure that allows us to measure the both. If you measure accurately then atleast either **C** or **d** is irrational. So π is regarded as irrational.

The Greek genius Archimedes was the first to compute the value of π. He showed the value of π lie between 3.140845 and 3.142857. (i.e., $3.140845 < \pi < 3.142857$) Aryabhatta (476-550 AD), the great Indian mathematician and astronomer, found the value of π correctly to four decimal places 3.1416. Using high speed computers and advanced algorithams, π has been computed to over 1.24 trillion decimal places .

π =3.14159265358979323846264338327950 ….. The decimal expansion of π is non-terminating non-recurring. So π is an irrational number. Note that, we often take $\frac{22}{7}$ as an approximate value of π, but $\pi \neq \frac{22}{7}$.

We celebrate March 14th as π day since it is 3.14 and t 1 : 59 (as π = 3.14159 ....). What a coincidence, Albert Einstein was born on March 14th, 1879!

Try These

Find the value of $\sqrt{3}$ upto six decimals.

1.3 Representing irrational numbers on Number line

We have learnt that there exist a rational number between any two rational numbers. Therefore, when two rational numbers are represented by points on number line, we can use a point to represent a rational number between them. So there are infinitely many points representing rational numbers. It seems that the number line is consisting of points which represent rational numbers only. Is it true? Can't you represent $\sqrt{2}$ on number line? Let us discuss and locate irrational numbers such as $\sqrt{2}$, $\sqrt{3}$ on number line.

Example-7. Locate $\sqrt{2}$ on number line

Solution : At O draw a unit square OABC on number line with each side 1 unit in length.

By Pythagoras theorem OB = $\sqrt{1^2+1^2} = \sqrt{2}$

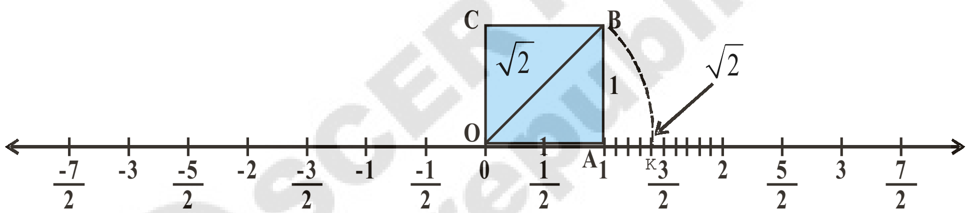


Fig. (i)

We have seen that OB = $\sqrt{2}$. Using a compass with centre O and radius OB, draw an arc on the right side to O intersecting the number line at the point K. Now K corresponds to $\sqrt{2}$ on the number line.

Example-8. Locate $\sqrt{3}$ on number line.

Solution : Let us return to fig. (i)

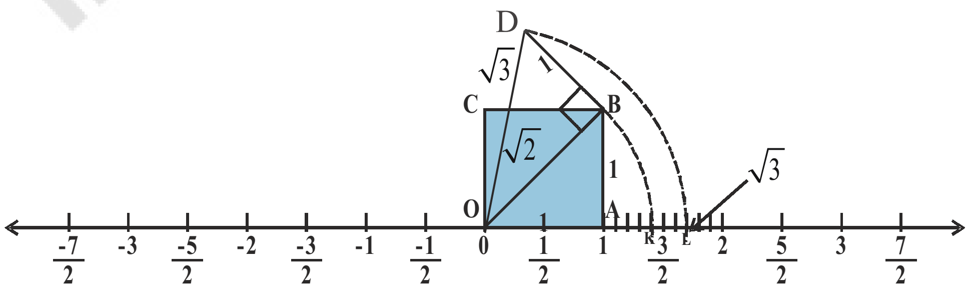


Fig. (ii)

Now construct BD of 1 unit length perpendicular to OB as in Fig. (ii). Join OD

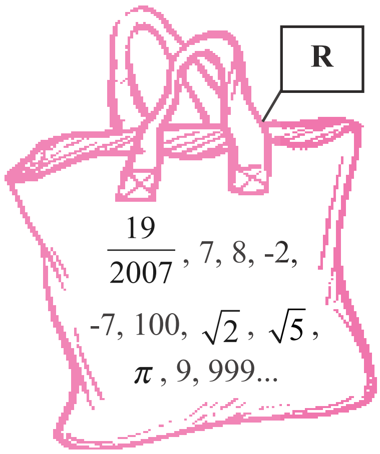
By Pythagoras theorem, OD = $\sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{2+1} = \sqrt{3}$

Using a compass, with centre O and radius OD, draw an arc which intersects the number line at the point L right side to 0. Then 'L' corresponds to $\sqrt{3}$. From this we can conclude that many points on the number line can be represented by irrational numbers also. In the same way, we can locate $\sqrt{n}$ for any positive integers n, after $\sqrt{n-1}$ has been located.

TRY THESE

Locate $\sqrt{5}$ and $-\sqrt{5}$ on number line. [**Hint** : $5^2 = (2)^2 + (1)^2$]

1.3 REAL NUMBERS



All rational numbers can be written in the form $\frac{p}{q}$, where p and q are integers and q $\neq$ 0. There are also other numbers that cannot be written in the form $\frac{p}{q}$, where p and q are integers and are called irrational numbers. If we represent all rational numbers and all irrational numbers and put these on the number line, would there be any point on the number line that is not covered?

The answer is no! The collection of all rational and irrational numbers completely covers the line. This combination makes a new collection and called Real Numbers, denoted by R. Real numbers cover all the points on the number line. We can say that every real number is represented by a unique point on the number line. Also, every point on the number line represents a unique real number. So we call this as the real number line.

Here are some examples of Real numbers

$-5.6, \sqrt{21}, -2, 0, 1, \frac{1}{5}, \frac{22}{7}, \pi, \sqrt{2}, \sqrt{7}, \sqrt{9}$, 12.5, 12.5123..... etc. Find which of these real numbers are also rational numbers.

Example-9. Find any two irrational numbers between $\frac{1}{5}$ and $\frac{2}{7}$.

Solution : We know that $\frac{1}{5} = 0.20$

$$\frac{2}{7} = 0.\overline{285714}$$

To find two irrational numbers between $\frac{1}{5}$ and $\frac{2}{7}$, we need to look at the decimal form of the two numbers and then proceed. We can find infinitely many such irrational numbers.

Examples of two such irrational numbers are

0.201201120111..., 0.24114111411114…, 0.25231617181912..., 0.267812147512 …

Can you find four more irrational numbers between $\frac{1}{5}$ and $\frac{2}{7}$?

Example-10. Find an irrational number between 3 and 4.

Solution :

> If a and b are two positive rational numbers such that ab is not a perfect square of a rational number, then $\sqrt{ab}$ is an irrational number lying between *a* and *b*.

$\therefore$ An irrational number between 3 and 4 is $\sqrt{3 \times 4} = \sqrt{3} \times \sqrt{4}$

$= \sqrt{3} \times 2 = 2\sqrt{3}$

Example-11. Examine, whether the following numbers are rational or irrational :

(i) $\left(3+\sqrt{3}\right)+\left(3-\sqrt{3}\right)$ (ii) $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

(iii) $\frac{10}{2\sqrt{5}}$ (iv) $\left(\sqrt{2}+2\right)^2$

Solution :

(i) $\left(3+\sqrt{3}\right)+\left(3-\sqrt{3}\right) = 3+\sqrt{3}+3-\sqrt{3}$

$= 6$, which is a rational number.

(ii) $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right)$

We know that $(a+b)(a-b) \equiv a^2 - b^2$ is an identity.

Thus $\left(3+\sqrt{3}\right)\left(3-\sqrt{3}\right) = 3^2-\left(\sqrt{3}\right)^2 = 9 - 3 = 6$ which is a rational number.

(iii) $\dfrac{10}{2\sqrt{5}} = \dfrac{10 \div 2}{2\sqrt{5} \div 2} = \dfrac{5}{\sqrt{5}} = \dfrac{\sqrt{5}\times\sqrt{5}}{\sqrt{5}} = \sqrt{5}$, which is an irrational number.

(iv) $\left(\sqrt{2}+2\right)^2 = \left(\sqrt{2}\right)^2 + 2.\sqrt{2}\ .2 + 2^2 = 2+4\sqrt{2}+4$

$= 6+4\sqrt{2}$, which is an irrational number.

EXERCISE - 1.2

1. Classify the following numbers as rational or irrational.

   (i) $\sqrt{27}$ (ii) $\sqrt{441}$ (iii) 30.232342345…

   (iv) 7.484848… (v) 11.2132435465 (vi) 0.3030030003.....

2. Explain with an example how irrational numbers differ from rational numbers?

3. Find an irrational number between $\dfrac{5}{7}$ and $\dfrac{7}{9}$. How many more there may be?

4. Find two irrational numbers between 0.7 and 0.77

5. Find the value of $\sqrt{5}$ upto 3 decimal places.

6. Find the value of $\sqrt{7}$ up to six decimal places by long division method.

7. Locate $\sqrt{10}$ on number line.

8. Find atleast two irrational numbers between 2 and 3.

9. State whether the following statements are true or false. Justify your answers.

   (i) Every irrational number is a real number.

   (ii) Every rational number is a real number.

   (iii) Every real number need not be a rational number

   (iv) $\sqrt{n}$ is not irrational if n is a perfect square.

   (v) $\sqrt{n}$ is irrational if n is not a perfect square.

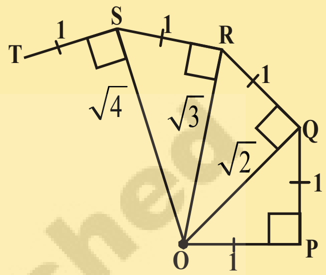
   (vi) All real numbers are irrational.

ACTIVITY

Constructing the 'Square root spiral'.

Take a large sheet of paper and construct the 'Square root spiral' in the following manner.

Step 1 : Start with point 'O' and draw a line segment $\overline{OP}$ of 1 unit length.



Step 2 : Draw a line segment $\overline{PQ}$ perpendicular to $\overline{OP}$ of unit length (where OP = PQ = 1) (see Fig)

Step 3 : Join O, Q. (OQ = $\sqrt{2}$)

Step 4 : Draw a line segment $\overline{QR}$ of unit length perpendicular to $\overline{OQ}$.

Step 5 : Join O, R. (OR = $\sqrt{3}$)

Step 6 : Draw a line segment RS of unit length perpendicular to $\overline{OR}$.

Step 7 : Continue in this manner for some more number of steps, you will create a beautiful spiral made of line segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, $\overline{ST}$, $\overline{TU}$ ... etc. Note that the line segments $\overline{OQ}$, $\overline{OR}$, $\overline{OS}$, $\overline{OT}$, $\overline{OU}$ ... etc. denote the lengths $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}$ respectively.

1.4 Representing Real numbers on the Number line through Successive magnification

In the previous section, we have seen that any real number has a decimal expansion.

Now first let us see how to represent terminating decimal on the number line.

Suppose we want to locate 2.776 on the number line. We know that this is a terminating decimal and this lies between 2 and 3.

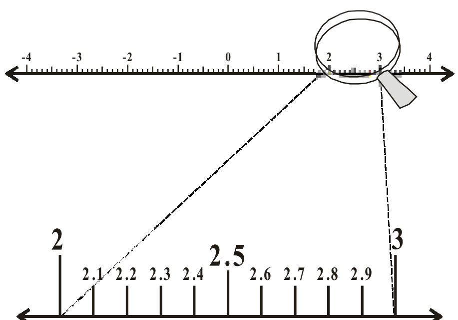


Fig.(i)

So, let us look closely at the portion of the number line between 2 and 3. Suppose we divide this into 10 equal parts as in Fig.(i). Then the markings will be like 2.1, 2.2, 2.3 and so on. To have a clear view, let us assume that we have a magnifying glass in our hand and look at the portion between 2 and 3. It will look like what you see in figure (i).

Now, 2.776 lies between 2.7 and 2.8. So, let us focus on the portion between 2.7 and 2.8 (See Fig. (ii). We imagine to divide this again into ten equal parts. The first mark will represent 2.71, the second is 2.72, and so on. To see this clearly, we magnify this as shown in Fig(ii).

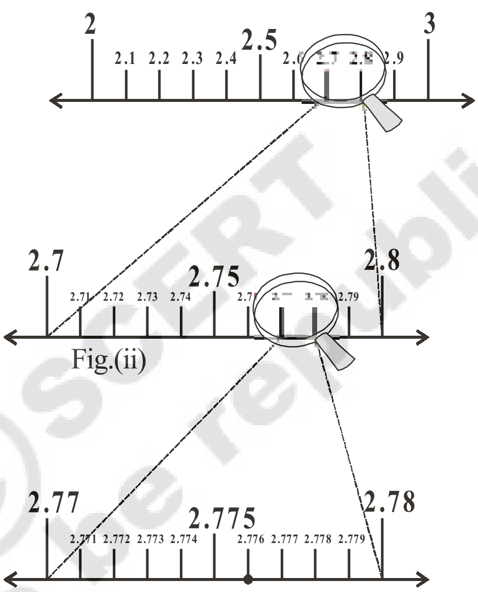


Fig.(ii)
Fig.(iii)

Again 2.776 lies between 2.77 and 2.78. So, let us focus on this portion of the number line see Fig. (iii) and imagine to divide it again into ten equal parts. We magnify it to see it better, as in Fig.(iii).

The first mark represents 2.771, second mark 2.772 and so on, 2.776 is the 6^{th} mark in these subdivisions.

We call this process of visualization of presentation of numbers on the number line through a magnifying glass, as the process of successive magnification.

Now let us try and visualize the position of a real number with a non-terminating recurring decimal expansion on the number line by the process of successive magnification with the following example.

Example-11. Visualise the representation of $3.5\overline{8}$ on the number line through successive magnification upto 4 decimal places.

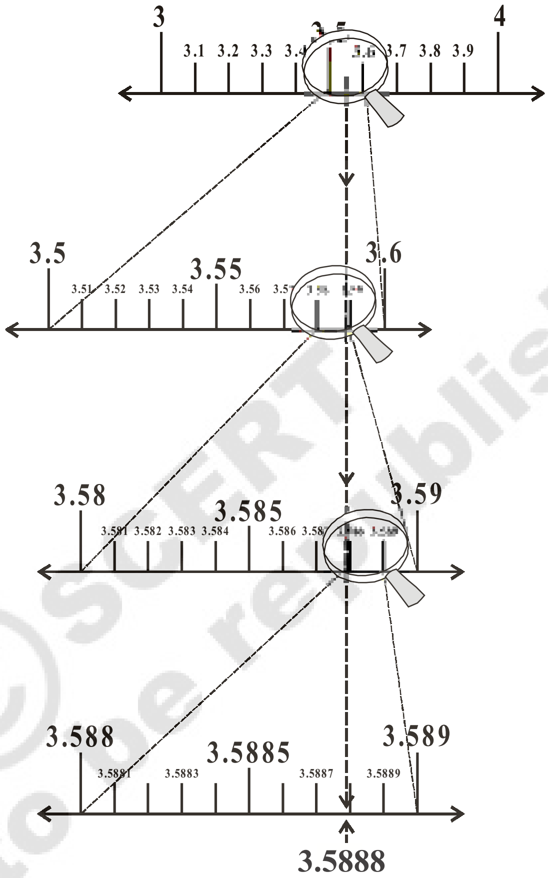
Solution: Once again we proceed the method of successive magnification to represent 3.5888 on number line.

Step 1 :

Step 2 :

Step 3 :

Step 4 :



EXERCISE - 1.3

1. Visualise 2.874 on the number line, using successive magnification.
2. Visualilse $5.\overline{28}$ on the number line, upto 3 decimal places.

1.5 OPERATIONS ON REAL NUMBERS

We have learnt, in previous class, that rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication. And also, we learnt that rational numbers are closed with respect to addition, subtraction, multiplication. Can you say irrational numbers are also closed under four fundamental operations?

Look at the following examples

$\left(\sqrt{3}\right)+\left(-\sqrt{3}\right)=0$. Here 0 is a rational number.

$\left(\sqrt{5}\right)-\left(\sqrt{5}\right)=0$. Here 0 is a rational number.

$\left(\sqrt{2}\right).\left(\sqrt{2}\right)=2$. Here 2 is a rational number.

$\frac{\sqrt{7}}{\sqrt{7}}=1$. Here 1 is a rational number.

What did you observe? The sum, difference, quotients and products of irrational numbers need not be irrational numbers.

So we can say irrational numbers are not closed with respect to addition, subtraction, multiplication and divisioin.

Let us see some problems on irrational numbers.

Example-12. Check whether (i) $5\sqrt{2}$ (ii) $\frac{5}{\sqrt{2}}$ (iii) $21+\sqrt{3}$ (iv) $\pi+3$ are irrational numbers or not?

Solution : We know $\sqrt{2}=1.414...$, $\sqrt{3}=1.732...$, $\pi=3.1415...$

(i) $5\sqrt{2} = 5(1.414\ldots) = 7.070\ldots$

(ii) $\frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{5\sqrt{2}}{2}=\frac{7.070}{2}=3.535\ldots$ (from i)

(iii) $21+\sqrt{3} = 21+1.732\ldots = 22.732\ldots$

(iv) $\pi+3 = 3.1415\ldots + 3 = 6.1415\ldots$

All these are non-terminating, non-recurring decimals.

Thus they are irrational numbers.

> If q is rational and s is irrational then q + s, q - s, qs and $\frac{q}{s}$ $(s\neq 0)$ are irrational numbers

Example-13. Subtract $5\sqrt{3}+7\sqrt{5}$ from $3\sqrt{5}-7\sqrt{3}$

Solution : $(3\sqrt{5}-7\sqrt{3}) - (5\sqrt{3}+7\sqrt{5})$

$= 3\sqrt{5}-7\sqrt{3} - 5\sqrt{3}-7\sqrt{5}$

$= -4\sqrt{5}-12\sqrt{3}$

$= -(4\sqrt{5}+12\sqrt{3})$

Example-14. Multiply $6\sqrt{3}$ with $13\sqrt{3}$

Solution : $6\sqrt{3}\times 13\sqrt{3} = 6\times 13\times\sqrt{3}\times\sqrt{3} = 78\times 3 = 234$

We now list some properties relating to square roots, which are useful in various ways.

Let a and b be positive real numbers. Then

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(ii) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$; if $b \neq 0$

(iii) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a - b$

(iv) $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right) = a^2 - b$

(v) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{c}+\sqrt{d}\right) = \sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}$

(vi) $\left(\sqrt{a}+\sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$

Let us look at some particular cases of these properties.

Example-15. Simplify the following expressions:

(i) $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right)$ (ii) $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$

(iii) $\left(\sqrt{5}+\sqrt{2}\right)^2$ (iv) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)$

Solution :

(i) $\left(3+\sqrt{3}\right)\left(2+\sqrt{2}\right) = 6+3\sqrt{2}+2\sqrt{3}+\sqrt{6}$

(ii) $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) = 2^2-\left(\sqrt{3}\right)^2 = 4 - 3 = 1$

(iii) $\left(\sqrt{5}+\sqrt{2}\right)^2 = \left(\sqrt{5}\right)^2+2\sqrt{5}\sqrt{2}+\left(\sqrt{2}\right)^2 = 5+2\sqrt{10}+2 = 7+2\sqrt{10}$

(iv) $\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right) = \left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2 = 5-2 = 3$

1.5.1 Rationalising the Denominator

Can we locate $\frac{1}{\sqrt{2}}$ on the number line ?

What is the value of $\frac{1}{\sqrt{2}}$?

How do we find the value? As $\sqrt{2}$ = 1.4142135..... which is neither terminating nor repeating. Can you divide 1 with this?

It does not seem to be easy to find $\frac{1}{\sqrt{2}}$.

Let us try to change the denominator into a rational form.

To rationalise the denominator of $\frac{1}{\sqrt{2}}$, multiply the numerator and the denominator of $\frac{1}{\sqrt{2}}$ by $\sqrt{2}$, we get

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$. Yes, it is half of $\sqrt{2}$.

Now can we plot $\frac{\sqrt{2}}{2}$ on the number line ? It lies between 0(zero) and $\sqrt{2}$.

Observe that $\sqrt{2} \times \sqrt{2} = 2$. Thus we say $\sqrt{2}$ is the rationalising factor (R.F) of $\sqrt{2}$

Similarly $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$. Then $\sqrt{2}$ and $\sqrt{8}$ are rationalising factors of each other $\sqrt{2} \times \sqrt{18} = \sqrt{36} = 6$, etc. Among these $\sqrt{2}$ is the simplest rationalising factor of $\sqrt{2}$.

Note that if the product of two irrational numbers is a rational number then each of the two is the rationalising factor (R.F) of the other. Also notice that the R.F. of a given irrational number is not unique. It is convenient to use the simplest of all R.F.s of given irrational number.

DO THIS

Find rationalising factors of the denominators of (i) $\frac{1}{2\sqrt{3}}$ (ii) $\frac{3}{\sqrt{5}}$ (iii) $\frac{1}{\sqrt{8}}$.

Example-16. Rationalise the denominator of $\frac{1}{4+\sqrt{5}}$

Solution : We know that $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right)=a^2-b$

Multiplying the numerator and denominator of $\frac{1}{4+\sqrt{5}}$ by $4-\sqrt{5}$

$$\frac{1}{4+\sqrt{5}}\times\frac{4-\sqrt{5}}{4-\sqrt{5}}=\frac{4-\sqrt{5}}{4^2-\left(\sqrt{5}\right)^2}=\frac{4-\sqrt{5}}{16-5}=\frac{4-\sqrt{5}}{11}$$

Example-17. Rationalise the denominator of $\frac{1}{7+4\sqrt{3}}$

Solution : $\frac{1}{7+4\sqrt{3}}\times\frac{7-4\sqrt{3}}{7-4\sqrt{3}}=\frac{7-4\sqrt{3}}{7^2-\left(4\sqrt{3}\right)^2}=\frac{7-4\sqrt{3}}{49-16\times3}$

$$=\frac{7-4\sqrt{3}}{49-48}=7-4\sqrt{3}$$

Example-18. Simplify $\frac{1}{7+4\sqrt{3}}+\frac{1}{2+\sqrt{5}}$

Solution : The rationalising factor of $7+4\sqrt{3}$ is $7-4\sqrt{3}$ and the rationalising factor of $2+\sqrt{5}$ is $2-\sqrt{5}$.

$$=\frac{1}{7+4\sqrt{3}}+\frac{1}{2+\sqrt{5}}$$

$$=\frac{1}{7+4\sqrt{3}}\times\frac{7-4\sqrt{3}}{7-4\sqrt{3}}+\frac{1}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}$$

$$=\frac{7-4\sqrt{3}}{7^2-(4\sqrt{3})^2}+\frac{2-\sqrt{5}}{2^2-(\sqrt{5})^2}$$

$$=\frac{7-4\sqrt{3}}{49-48}+\frac{2-\sqrt{5}}{(4-5)}$$

$$=\frac{7-4\sqrt{3}}{1}+\frac{2-\sqrt{5}}{(-1)}$$

$$=7-4\sqrt{3}-2+\sqrt{5}=5-4\sqrt{3}+\sqrt{5}$$

1.5.2 Law of Exponents for real numbers

Let us recall the laws of exponents.

i) $a^m . a^n = a^{m+n}$ ii) $(a^m)^n = a^{mn}$ iii) $\frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n \\ 1 & \text{if } m = n \\ \frac{1}{a^{n-m}} & \text{if } m < n \end{cases}$

iv) $a^m b^m = (ab)^m$ v) $\frac{1}{a^n} = a^{-n}$ vi) $a^0 = 1$

Here 'a', 'm' and 'n' are integers and a $\neq$ 0. 'a' is called the base and m, n are the exponents.

For example

i) $7^3.7^{-3} = 7^{3+(-3)} = 7^0 = 1$ ii) $(2^3)^{-7} = 2^{-21} = \frac{1}{2^{21}}$

iii) $\frac{23^{-7}}{23^4} = 23^{-7-4} = 23^{-11}$ iv) $(7)^{-13}.(3)^{-13} = (7\times3)^{-13} = (21)^{-13}$

Suppose we want to do the following computations

i) $2^{\frac{2}{3}}.2^{\frac{1}{3}}$ ii) $\left(5^{\frac{1}{7}}\right)^4$ iii) $\frac{3^{\frac{1}{5}}}{3^{\frac{1}{3}}}$ iv) $7^{\frac{1}{17}}.11^{\frac{1}{17}}$

How do we go about it? The exponents and bases in the above examples are rational numbers. Thus there is a need to extend the laws of exponents to bases of positive real numbers and to the exponents as rational numbers. Before we state these laws, we need first to understand what is n^{th} root of a real number.

We know if $3^2 = 9$ then $\sqrt{9} = 3$ (square root of 9 is 3)

i.e., $\sqrt[2]{9} = 3$

If $5^2 = 25$ then $\sqrt{25} = 5$ i.e., $\sqrt[2]{25} = 5$ moreover $\sqrt[2]{25} = (25)^{\frac{1}{2}} = \left(5^2\right)^{\frac{1}{2}} = 5^{2\times\frac{1}{2}} = 5$

Observe the following

If $2^3 = 8$ then $\sqrt[3]{8} = 2$ (cube root of 8 is 2); $\sqrt[3]{8} = 8^{\frac{1}{3}} = \left(2^3\right)^{\frac{1}{3}} = 2$

$2^4 = 16$ then $\sqrt[4]{16} = 2$ (4^{th} root of 16 is 2); $\sqrt[4]{16} = (16)^{\frac{1}{4}} = \left(2^4\right)^{\frac{1}{4}} = 2$

$2^5 = 32$ then $\sqrt[5]{32} = 2$ (5^{th} root of 32 is 2); $\sqrt[5]{32} = (32)^{\frac{1}{5}} = (2^5)^{\frac{1}{5}} = 2$

$2^6 = 64$ then $\sqrt[6]{64} = 2$ (6^{th} root of 64 is 2); $\sqrt[6]{64} = (64)^{\frac{1}{6}} = (2^6)^{\frac{1}{6}} = 2$

....

Similarly if $\mathbf{a^n = b}$ then $\sqrt[n]{\mathbf{b}} = \mathbf{a}$ (n^{th} root of b is a); $\sqrt[n]{b} = (b)^{\frac{1}{n}} = (a^n)^{\frac{1}{n}} = a$

Let $\mathbf{a} > 0$ be a real number and '**n**' be a positive integer.

If $\mathbf{b^n = a}$, for some positive real number '**b**', then **b** is called nth root of '**a**' and we write $\sqrt[n]{\mathbf{a}} = \mathbf{b}$. In the earlier discussion laws of exponents were defined for integers. Let us extend the laws of exponents to the bases of positive real numbers and rational exponents.

Let $a > 0$ be a real number and p and q be rational numbers then, we have

i) $a^p . a^q = a^{p+q}$ ii) $(a^p)^q = a^{pq}$

iii) $\dfrac{a^p}{a^q} = a^{p-q}$ iv) $a^p . b^q = (ab)^p$ v) $\sqrt[n]{a} = a^{\frac{1}{n}}$

Now we can use these laws to answer the questions asked earlier.

Example-19. Simplify

i) $2^{\frac{2}{3}}.2^{\frac{1}{3}}$ ii) $\left(5^{\frac{1}{7}}\right)^4$ iii) $\dfrac{3^{\frac{1}{5}}}{3^{\frac{1}{3}}}$ iv) $7^{\frac{1}{17}}.11^{\frac{1}{17}}$

Solution : i) $2^{\frac{2}{3}}.2^{\frac{1}{3}} = 2^{\left(\frac{2}{3}+\frac{1}{3}\right)} = 2^{\frac{3}{3}} = 2^1 = 2$

ii) $\left(5^{\frac{1}{7}}\right)^4 = 5^{\frac{4}{7}}$

iii) $\dfrac{3^{\frac{1}{5}}}{3^{\frac{1}{3}}} = 3^{\left(\frac{1}{5}-\frac{1}{3}\right)} = 3^{\frac{3-5}{15}} = 3^{\frac{-2}{15}} = \dfrac{1}{3^{2/15}}$

iv) $7^{\frac{1}{17}}.11^{\frac{1}{17}} = (7\times11)^{\frac{1}{17}} = 77^{\frac{1}{17}}$

DO THIS

Simplify:

i. $(16)^{\frac{1}{2}}$ ii. $(128)^{\frac{1}{7}}$

iii. $(343)^{\frac{1}{5}}$

Surd

If '**n**' is a positive integer greater than 1 and '**a**' is a positive rational number but not $\mathbf{n}^{th}$ power of any rational number then $\sqrt[n]{\mathbf{a}}$ (or) $\mathbf{a}^{1/n}$ is called a surd of $\mathbf{n}^{th}$ order. In general we say the positive $\mathbf{n}^{th}$ root of **a** is called a surd or a radical. Here a is called radicand, $\sqrt[n]{\ }$ is called radical sign. and **n** is called the degree of radical.

Forms of Surd

Exponential form $a^{\frac{1}{n}}$

Radical form $\sqrt[n]{a}$

Here are some examples for surds.

$\sqrt{2}, \sqrt{3}, \sqrt[3]{9}$, ... etc

Consider the real number $\sqrt{7}$. It may also be written as $7^{\frac{1}{2}}$. Since 7 is not a square of any rational number, $\sqrt{7}$ is a surd.

Consider the real number $\sqrt[3]{8}$. Since 8 is a cube of a rational number 2, $\sqrt[3]{8}$ is not a surd.

Consider the real number $\sqrt{\sqrt{2}}$. It may be written as $\left(2^{\frac{1}{2}}\right)^{\frac{1}{2}} = 2^{\frac{1}{4}} = \sqrt[4]{2}$. So it is a surd.

DO THIS

1. Write the following surds in exponential form

   i. $\sqrt{2}$ ii. $\sqrt[3]{9}$ iii. $\sqrt[5]{20}$ iv $\sqrt[17]{19}$

2. Write the surds in radical form:

   i. $5^{\frac{1}{7}}$ ii. $17^{\frac{1}{6}}$ iii. $5^{\frac{2}{5}}$ iv $142^{\frac{1}{2}}$

EXERCISE - 1.4

1. Simplify the following expressions.

   i) $\left(5+\sqrt{7}\right)\left(2+\sqrt{5}\right)$ ii) $\left(5+\sqrt{5}\right)\left(5-\sqrt{5}\right)$

   iii) $\left(\sqrt{3}+\sqrt{7}\right)^2$ iv) $\left(\sqrt{11}-\sqrt{7}\right)\left(\sqrt{11}+\sqrt{7}\right)$

2. Classify the following numbers as rational or irrational.

   i) $5-\sqrt{3}$ ii) $\sqrt{3}+\sqrt{2}$ iii) $\left(\sqrt{2}-2\right)^2$

iv) $\frac{2\sqrt{7}}{7\sqrt{7}}$ v) 2π vi) $\frac{1}{\sqrt{3}}$ vii) $(2+\sqrt{2})(2-\sqrt{2})$

3. In the following equations, find whether variables x, y, z etc. represent rational or irrational numbers

i) $x^2 = 7$ ii) $y^2 = 16$ iii) $z^2 = 0.02$

iv) $u^2 = \frac{17}{4}$ v) $w^2 = 27$ vi) $t^4 = 256$

4. The ratio of circumference to the diameter of a circle $\frac{c}{d}$ is represented by π. But we say that π is an irrational number. Why?

5. Rationalise the denominators of the following:

i) $\frac{1}{3+\sqrt{2}}$ ii) $\frac{1}{\sqrt{7}-\sqrt{6}}$ iii) $\frac{1}{\sqrt{7}}$ iv) $\frac{\sqrt{6}}{\sqrt{3}-\sqrt{2}}$

6. Simplify each of the following by rationalising the denominator:

i) $\frac{6-4\sqrt{2}}{6+4\sqrt{2}}$ ii) $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$ iii) $\frac{1}{3\sqrt{2}-2\sqrt{3}}$ iv) $\frac{3\sqrt{5}-\sqrt{7}}{3\sqrt{3}+\sqrt{2}}$

7. Find the value of $\frac{\sqrt{10}-\sqrt{15}}{2\sqrt{2}}$ upto three decimal places. (take $\sqrt{2} = 1.414$ and $\sqrt{5} = 2.236$)

8. Find:

i) $64^{\frac{1}{6}}$ ii) $32^{\frac{1}{5}}$ iii) $625^{\frac{1}{4}}$

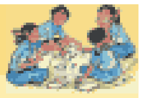
iv) $16^{\frac{3}{2}}$ v) $243^{\frac{2}{5}}$ vi) $(46656)^{\frac{-1}{6}}$

9. Simplify: $\sqrt[4]{81} - 8\sqrt[3]{343} + 15\sqrt[5]{32} + \sqrt{225}$

10. If '*a*' and '*b*' are rational numbers, find the value of *a* and *b* in each of the following equations.

i) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = a + b\sqrt{6}$ ii) $\frac{\sqrt{5}+\sqrt{3}}{2\sqrt{5}-3\sqrt{3}} = a - b\sqrt{15}$

WHAT WE HAVE DISCUSSED



In this chapter we have discussed the following points:

1. A number which can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ is called a rational number.
2. A number which cannot be written in the form $\frac{p}{q}$, for any integers p, q and $q \neq 0$ is called an irrational number.
3. The decimal expansion of a rational number is either terminating or non-terminating recurring.
4. The decimal expansion of an irrational number is non-terminating and non-recurring.
5. The collection of all rational and irrational numbers is called Real numbers.
6. There is a unique real number corresponding to every point on the number line. Also corresponding to each real number, there is a unique point on the number line.
7. If q is rational and s is irrational, then q+s, q-s, qs and $\frac{q}{s}$ are irrational numbers.
8. If n is a natural number other than a perfect square, then $\sqrt{n}$ is an irrational number.
9. The following identities hold for positive real numbers a and b

   i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$ ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ $(b \neq 0)$

   iii) $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a - b$ iv) $\left(a+\sqrt{b}\right)\left(a-\sqrt{b}\right) = a^2 - b$

   v) $\left(\sqrt{a}+\sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$
10. To rationalise the denominator of $\frac{1}{\sqrt{a}+b}$, we multiply this by $\frac{\sqrt{a}-b}{\sqrt{a}-b}$, where a, b are integers.
11. Let $a > 0$ be a real number and p and q be rational numbers. Then

    i) $a^p . a^q = a^{p+q}$ ii) $(a^p)^q = a^{pq}$ iii) $\frac{a^p}{a^q} = a^{p-q}$

    iv) $a^p . b^p = (ab)^p$
12. If 'n' is a positive integer > 1 and 'a' is a positive rational number but not n^{th} power of any rational number then $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$ is called a surd of nth order.