03

The Elements of Geometry

3.1 Introduction

You may have seen large structures like bridges, dams, school buildings, hostels, hospitals etc. The construction of these structures a big task for the engineers.

Do you know how we estimate the cost of the construction? Besides wages of the labour, cost of cement and concrete it depends upon the size and shape of the structure.

The size and shape of a structure include the foundation, plinth area, size of the walls, elevation, roof etc. To understand the geometric principles involved in these constructions, we should know the basic elements of geometry and its application.

We also know that geometry is widely used in daily life activities such as paintings, handicrafts, laying of floor designs, ploughing and sowing of seeds in fields. So in other words, we can say that the life without geometry is unimaginable.

The great construction like the Pyramids in Egypt, the Great wall of China, Temples, Mosques, Cathedral, Tajmahal, Charminar and altars in India, Eifel tower of France etc. are some of the best examples of application of geometry.

In this chapter, we will look into the history to understand the roots of geometry and the different schools of thought that have developed the geometry and its comparison with modern geometry.

3.2 History

The domains of mathematics which study the shapes and sizes of structures are described under geometry. The word *geometry* is derived from the Greek 'geo' means earth and 'metrein' means measure.

The earliest recorded beginnings of geometry can be traced to early people, who discovered obtuse angled triangles in the ancient Indus valley and ancient Babylonia. The 'Bakshali manuscript' employs a handful of geometric problems including problems about volumes of irregular solids. Remnants of geometrical knowledge of the Indus Valley civilization can be found in excavations at Harappa and Mohenjo-Daro where there is evidence of circle-drawing instruments from as early as 2500 B.C.

The 'Sulba Sutras' in Vedic Sanskrit lists the rules and geometric principles involved in the construction of ritual fire altars. The amazing idea behind the construction of fire altars is that they occupy same area although differ in their shapes. Boudhayana (8th century B.C.) composed the Boudhayana Sulba Sutra, the best-known Sulba Sutra which contains examples of simple Pythagorean triples such as (3,4,5), (5,12,13), (8,15,17)… etc. as well as a statement of Pythagorean theorem for the sides of a rectangle.

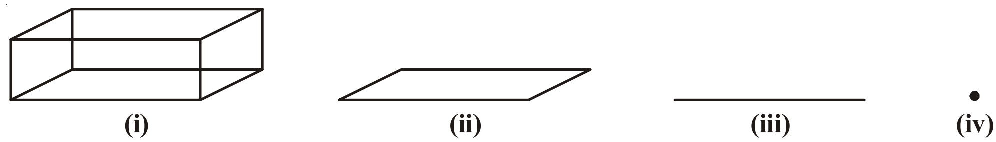
Ancient Greek mathematicians conceived ***geometry*** as the crown jewel of their sciences. They expanded the range of geometry to many new kinds of figures, curves, surfaces and solids. They found the need of establishing a proposed statement as universal truth with the help of logic. This idea led the Greek mathematician Thales to think of deductive proof.

Pythagoras of Ionia might have been a student of Thales and the theorem that was named after him might not have been his discovery, but he was probably one of the mathematicians who had given a deductive proof of it. Euclid (325-265B.C) of Alexandria in Egypt wrote 13 books called 'The Elements'. Thus Euclid created the first system of thought based on fundamental definitions, axioms, propositions and rules of inference or logic.

3.3 EUCLID'S ELEMENTS

Euclid thought geometry as an abstract model of the world in which they lived. The notions of point, line, plane (or surface) and so on were derived from what was seen around them. From studies of the space and solids in the space around them, an abstract geometrical notion of a solid object was developed. A solid has shape, size, position and can be moved from one place to another. Its boundaries are called **surfaces**. They separate one part of the solid from another, and are said to have no thickness. The boundaries of the surfaces are **curves** or straight **lines**. These lines end in **points.** Consider the steps from solids to points (solids-surfaces-lines-point)

Observe the figure given in the next page. This figure is a cuboid (a solid) [fig.(i)]. It has three dimensions namely length, breadth and height. If it loses one dimension i.e. height then it will have only two dimensions which is rectangle. You know that a rectangle has two dimensions length and breadth [fig.(ii)]. If it further loses another dimension i.e. breadth then it will leave with only line segment [fig.(iii)] and if it has to lose one more dimension, there remain only the points [fig.(iv)]. We may recall that a point has no dimensions. Similarly when we see the edge of a table or a book, we can visualise it as a line. The end point of a line or the point where two lines meet is a point.



solids →	surfaces/curves →	lines →	points
3-D	2-D	1-D	no dimension

These are the fundamental terms of geometry. With the use of these terms other terms like line segment, angle, triangle etc. are defined.

Based on the above observatioin, Euclid defind point, line, plane.

In Book 1 of his Elements, Euclid listed 23 definitions. Some of them are given below.

Euclid 300 B.C
Father of Geometry

- A **point** is that which has no part
- A **line** is breadthless length
- The ends of a line are points
- A **straight line** is a line which lies evenly with the points on itself
- A **surface** is that which has length and breadth only
- The edges of surface are lines
- A **plane surface** is a surface which lies evenly with the straight lines on itself

In defining terms like point, line and plane, Euclid used words or phrases like 'part', 'breadth', 'evenly' which need **defining** or further explanation for the sake of clarity. In defining terms like plane, if we say 'a plane' occupies some area then 'area' is again to be clarified. So to define one term you need to define more than one term resulting in a chain of definitions without an end. So, mathematicians agreed to leave such terms as undefined. However we do have a intuitive feeling for the geometric concepts of a point than what the "definition" above gives us. So, we represent a point as a dot, even though a dot has some dimension. The Mohist philosophers in ancient China said "the line is divided into parts and that part which has no remaining part is a point.

A similar problem arises in definition 2 above, since it refers to breadth and length, neither of which has been defined. Because of this, a few terms are kept undefined while developing any course of study. So, in geometry, we *take a point, a line and a plane* (*in Euclid's words a plane surface*) *as undefined terms.* The only thing is that we can represent them intuitively, or explain them with the help of 'physical models.'

Euclid then used his definitions in assuming some geometric properties which need no proofs. These assumptions are self-evident truths. He divided them into two types: axioms and postulates.

3.3.1 Axioms and Postulates

Axioms are statements which are self evident or assumed to be true with in context of a particular mathematical system. For example when we say "The whole is always greater than the parts." It is a self evident fact and does not require any proof. This axiom gives us the definition of 'greater than'. For example, if a quantity P is a part of another quantity C, then C can be written as the sum of P and some third quantity R. Symbolically, $C > P$ means that there is some R such that $C = P + R$.

Euclid used this common notion or axiom throughout the mathematics not particularly in geometry but the term postulate was used for the assumptions made in geometry. The axioms are the foundation stones on which the structure of geometry is developed. These axioms arise in different situations.

Some of the Euclid's axioms are given below.

- Things which are equal to the same things are equal to one another
- If equals are added to equals, the wholes are equal
- If equals are subtracted from equals, the remainders are also equal.
- Things which coincide with one another are equal to one another.
- Things which are double of the same things are equal to one another
- Things which are halves of the same things are equal to one another

These 'common notions' refer to magnitudes of some kind. The first common notion could be applied to plane figures. For example, if the area of an object say A equals the area of another object B and the area of the object B equals that of a square, then the area of the object A is also equals to the area of the square.

Magnitudes of the same kind can be compared and added, but magnitudes of different kinds cannot be compared. For example, a line cannot be added to the area of objects nor can an angle be compared to a pentagon.

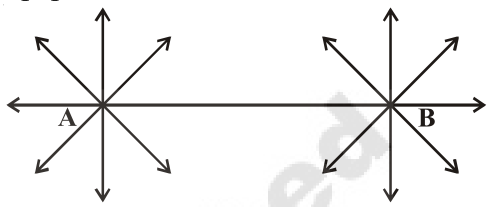
Try This

Can you give any two axioms from your daily life.

Now let's discuss Euclids five postulates:

1. Mark two distinct points A and B on a sheet of paper.

Draw a straight line passing through the points A and B. How many such lines can be drawn through point A and B? We can not draw more than one distinct line through two given points.



Euclid first postulate gives the above concept. Postulate is as follows-

Postulate-1 : There is a unique line that passes through the given two distinct points.

In Euclid's terms, "To draw a straight line from any point to any point".

2. Draw a line segment PQ on a sheet of paper.

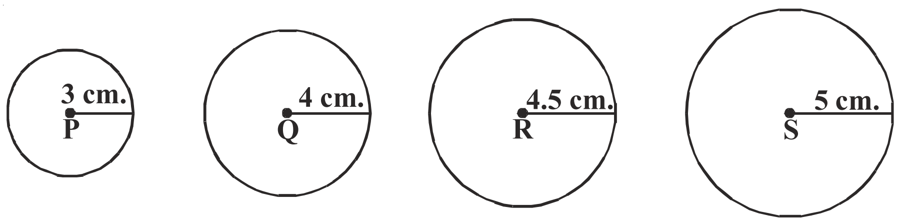
Extend the line segment both sides .

How far the line segment PQ can be extended both sides? Does it have any end points? We see that the line segment PQ can be extended on both sides and the line PQ has no ends points. Euclid conceived this idea in his second axiom.

Postulate-2 : A line segment can be extended on either side to form a straight line.

In Euclid's terms 'To produce a finite straight line continuously in a straight line' Euclid used the term 'terminated line' for 'a line segment'.

3. Radii of four circles are given as 3 cm, 4 cm, 4.5 cm and 5 cm. Using a compass, draw circles with these radii taking P, Q, R and S as their centres.



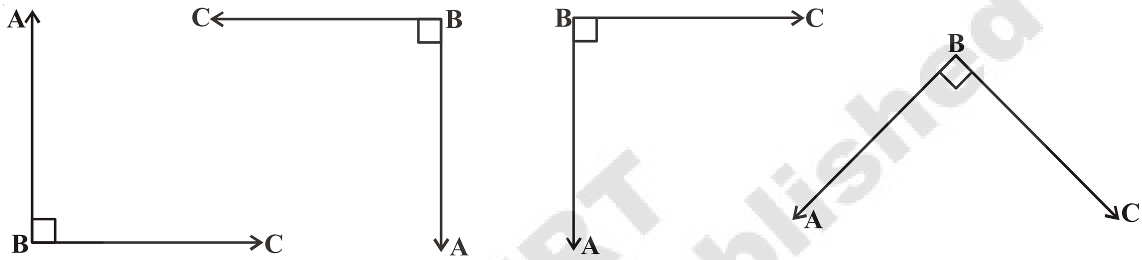
If the centre and radius of a circle are given can you draw the circle? We can draw a circle with any centre and any radius. (See chapter-12 Circle)

Euclid's third postulate states the above idea.

(To describe a circle with any centre and distance)

Postulate-3 : We can describe a circle with any centre and radius.

4. Take a grid paper. Draw different figures which represent a right angle. Cut them along their arms and place all angles one above other. What do you observe?



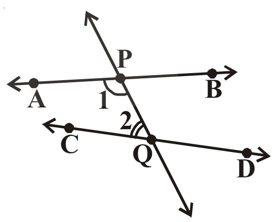
You observe that both the arms of each angle fall on one above the other, (i.e.) all right angles are equal. This is nothing but Euclids fourth axiom. Can you say this for any angle? Euclid take right angle as a reference angle for all the other angles and situation which he stated further.

Postulate-4 : All right angles are equal to one another.

Now we shall look at the Euclid's fifth postulate and its equivalent version.

Postulate-5 : If a straight line falling on two straight lines makes the interior angles on the same side of it taken together is less than two right angles, then the two straight lines, if produced infinitely, meet on that side on which the sum of the angles is less than two right angles.

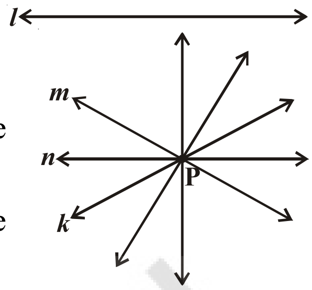
Note : For example, the line PQ in figure falls on lines AB and CD such that the sum of the interior angles 1 and 2 is less than 180° on the left side of PQ. Therefore, the lines AB and CD will eventually intersect on the left side of PQ.



This postulate has acquired much importance as many mathematicians including Euclid were convinced that the fifth postulate is a theorem. Consequently for two thousand years mathematicians tried to prove that the V postulate was a consequence of Euclid's nine other axioms. They tried by assuming other proposition (John Play Fair) which are equivalent to it.

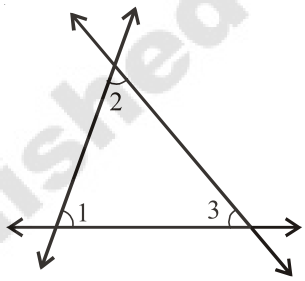
3.3.2 Equivalent version of fifth postulate or equivalents of fifth postulate

There are some noteworthy alternatives proposed by later mathematicians.



- Through a point not on a given line, exactly one parallel line may be drawn to the given line. (John Play Fair – 1748-1819)

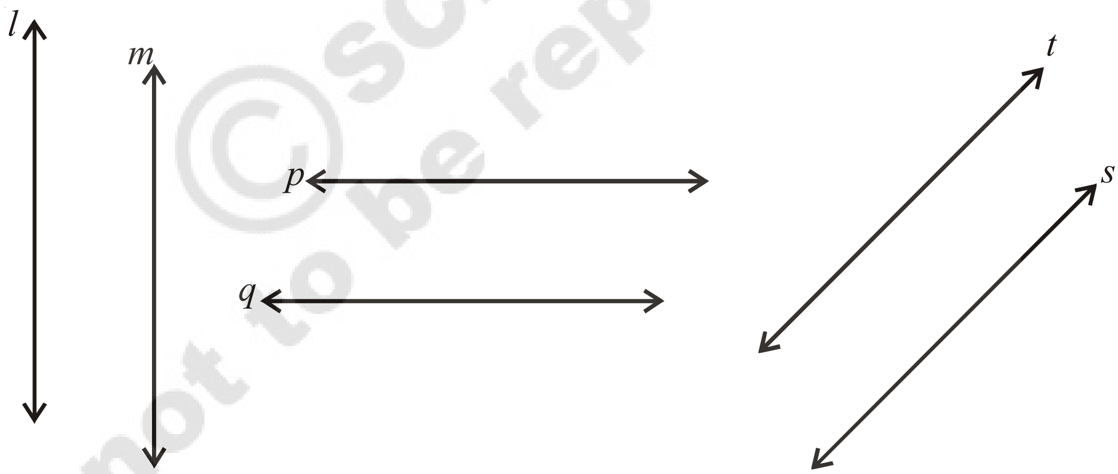
Let l be a line and P be a point, not on l. So through P, there exists only one line parallel to l. This is called Play Fair's axiom.



- The sum of angles of any triangle is a constant and is equal to two right angles. (Legendre)

$\angle 1 + \angle 2 + \angle 3 = 180^0$

- There exists a pair of lines everywhere equidistant from one another. (Posidominus)



- If a straight line intersects any one of two parallel lines, then it will intersect the other also.(Proclus)

- Straight lines parallel to the same straight line are parallel to one another.(Proclus)

If any one of these statements is substituted for Euclid's fifth postulate leaving the first four the same, the same geometry is obtained.

So after stating these five postulates, Euclid used them to prove many more results by applying deductive reasoning and the statements that were proved are called propositions or theorems.

Sometimes a certain statement that you think is to be true but that is an educated guess based on observations. Such statements which are neither proved nor disproved are called conjectures (hypothesis). Mathematical discoveries often start out as conjectures (hypothesis). "Every even number greater than 4 can be written as sum of two primes" is a conjecture (hypothesis) stated by Gold Bach.

A conjecture (hypothesis) that is proved to be true is called a theorem. A theorem is proved by a logical chain of steps. A proof of a theorem is an argument that establishes the truth of the theorem beyond doubt.

Euclid deducted as many as 465 propositions in a logical chain using defined terms, axioms, postulates and theorems already proven in that chain.

Let us study how Euclid axioms and postulates can be used in proving the results.

Example-1. If A,B,C are three points on a line and B lies between A and C, then prove that AC - AB = BC.

Solution : In the figure, AC coincides with AB+BC

Euclid's 4th axiom says that things which coincide with one another are equal to one another. Therefore it can be deducted that

$AB + BC = AC,$

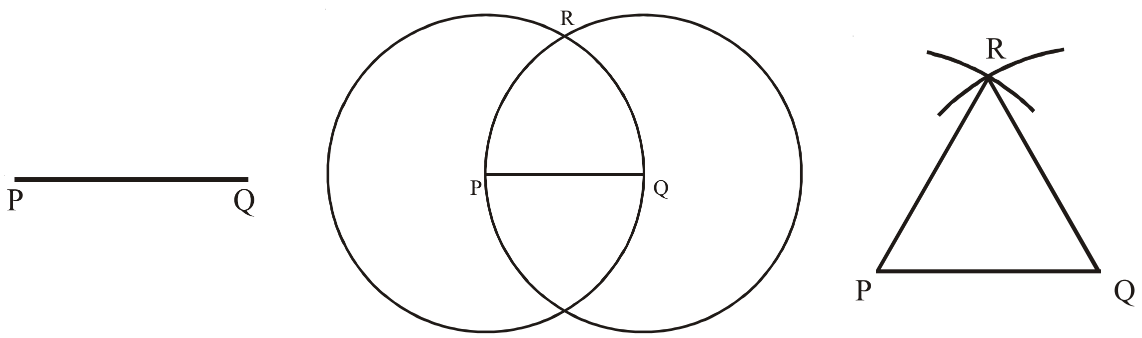
Substituting this value of AC in the given equation AC - AB = BC

$\cancel{A}B + BC - \cancel{A}B = BC$

Note that in this solution, it has been assumed that there is a unique line passing through two points.

Propostion -1. Prove that an equilateral triangle can be constructed on any given line segment.

Solution : It is given that; a line segment of any length say PQ



From Euclid's 3^{rd} postulate, we can draw a circle with any centre and any radius. So, we can draw a circle with centre P and radius PQ. Draw another circle with centre Q and radius QP. The two circles meet at R. Join 'R' to P and Q to form Δ PQR.

Now we require to prove the triangle thus formed is equilateral i.e., PQ = QR = RP.

PQ = PR (radii of the circle with centre P). Similarly, PQ = QR (radii of the circle with centre Q)

From Euclid's axiom, two things which are equal to same thing are equal to each another, we have PQ = QR = RP, so ΔPQR is an equilateral triangle. Note that here Euclid has assumed, without mentioning anywhere, that the two circles drawn with centre P and Q will meet each other at a point.

Let us now prove a theorem.

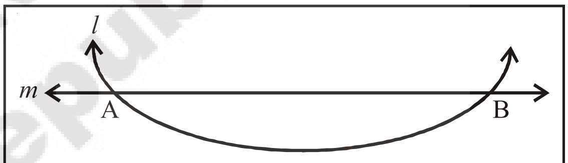
Example-3. Two distinct lines cannot have more than one point in common.

Given : Two lines l and m.

Required to Prove (RTP): They have only one point in common.

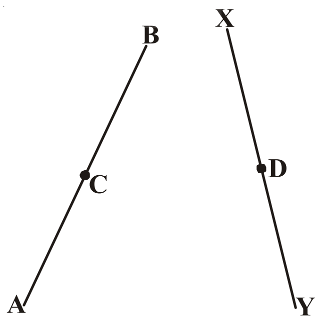
Proof: Let us assume that two lines intersect in two distinct points say A and B.

Now we have two lines passing through A and B. This assumption contradicts with the Euclid's axiom that only one line can pass through two distinct points. This contradiction arose due to our assumption that two lines can pass through two distinct points. So we can conclude that two distinct lines cannot have more than one point in common.



Note that Euclid state this for straight lines only, not for the curved lines. Where ever line is written always assume that we are talking about straight line.

Example-4. In the adjacent figure, we have AC = XD, C and D are mid points of AB and XY respectively. Show that AB = XY.



Solution : Given AB = 2 AC (C is mid point of AB)

XY = 2 XD (D is mid point of XY)

and AC = XD (given)

therefore, AB = XY

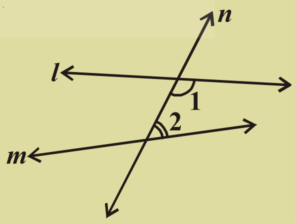
Since things which are double of the same things are equal to one another.

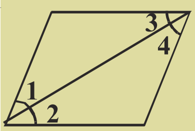
Exercise - 3.1

1. Answer the following:
   - i. How many dimensions a solid has?
   - ii. How many books are there in Euclid's Elements ?
   - iii. Write the number of faces of a cube and cuboid.
   - iv. What is sum of interior angles of a triangle ?
   - v. Write three un-defined terms of geometry.
2. State whether the following statements are true or false? Also give reasons for your answers.
   - a) Only one line can pass through a given point.
   - b) All right angles are equal.
   - c) Circles with same radii are equal.
   - d) A finite line can be extended on its both sides endlessly to get a straight line.

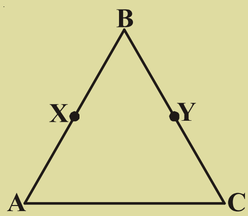
   - e) From the figure, AB > AC
3. In the figure given below, show that length AH > AB + BC + CD.

4. If a point Q lies between two points P and R such that PQ = QR, prove that PQ = $\frac{1}{2}$ PR.
5. Draw an equilateral triangle whose sides are 5.2 cm. each.
6. What is a conjecture ? Give an example for it.
7. Mark two points P and Q. Draw a line through P and Q. Now how many lines are parallel to PQ, can you draw?
8. In the adjacent figure, a line *n* falls on lines *l* and *m* such that the sum of the interior angles 1 and 2 is less than 180°, then what can you say about lines *l* and *m*.



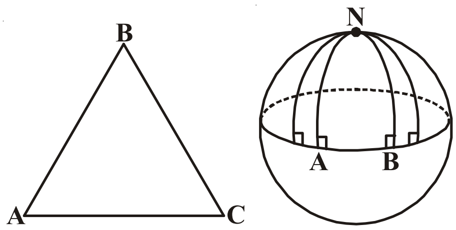


9. In the adjacent figure, if $\angle 1 = \angle 3$, $\angle 2 = \angle 4$ and $\angle 3 = \angle 4$, write the relation between $\angle 1$ and $\angle 2$ using an Euclid's postulate.
10. In the adjacent figure, we have BX = $\frac{1}{2}$ AB, BY = $\frac{1}{2}$ BC and AB = BC. Show that BX = BY



NON-EUCLIDIAN GEOMETRY

The failure of attempts to prove the V postulate, gave new thoughts to Carl Fedrick Gauss, Lobachevsky and Bolyai. They thought V postulate is true or some contrary postulate can be subtituted for it. If subtituted with other, we obtain, Geometry different from Euclid's Geometry, hence called non-Euclidian Geometry.



(i) (ii)

If plane is not flat what happens to our theorems?

Let us observe.

Take a ball and try to draw a triangle on it? What difference do you find between triangle on plane and on a ball. You observe that lines of a triangle on paper are straight but not on ball.

See in figure (ii), the lines AN and BN (which are parts of great circles of a sphere) are perpendicular to the same line AB. But they are meeting at N, even though the sum of the angles on the same side of line AB is not less than two right angles (in fact, it is $90° + 90° = 180°$). Also, note that the sum of the angles of the triangle NAB on sphere is greater than 180°, as $\angle A + \angle B = 180°$.

We call the plane on a sphere as a spherical plane. Can any parallel lines exist on a sphere? Similarly by taking different planes and related axioms new geometries arise.

WHAT WE HAVE DISCUSSED

- The three building blocks of geometry are Points, Lines and Planes, which are undefined terms.
- Ancient mathematicians including Euclid tried to define these undefined terms.
- Euclid developed a system of thought in his "The Elements" that serves as the foundation for development of all subsequent mathematics.
- Some of Euclid's axioms are
  - Things which are equal to the same things are equal to one another
  - If equals are added to equals, the wholes are equal

- If equals are subtracted from equals, the remainders are equal.
- Things which coincide with one another are equal to one another.
- The whole is greater than the part
- Things which are double of the same things are equal to one another
- Things which are halves of the same things are equal to one another

- Euclid's postulates are

Postulate - 1: To draw a straight line from any point to any point

Postulate - 2: A terminated line can be produced infinitely

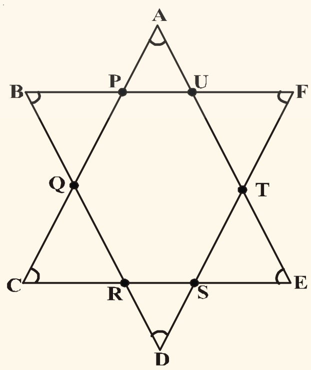
Postulate - 3: To describe a circle with any centre and radius

Postulate - 4: That all right angles equal to one another

Postulate - 5: If a straight line falling on two straight lines makes the interior angles on the same side of it taken together is less than two right angles, then the two straight lines, if produced infinitely, meet on that side on which the sum of the angles is less than two right angles.

Brain teaser

1. What is the measure of $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$ in the figure given below. Give reason to your answer.



2. If the diagonal of a square is 'a' units, what is the diagonal of the square, whose area is doubel that of the first square?