Adjoint and Inverse of a Matrix

The adjoint of a matrix (also called the adjugate of a matrix) is de ned as the transpose of the cofactor matrix of that particular matrix. For a matrix A, the adjoint is denoted as **adj (A)**. On the other hand, the inverse of a matrix A is that matrix which when multiplied by the matrix A give an identity matrix. The inverse of a Matrix A is denoted by $\mathbf{A^{-1}}$**.**

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Adjoint of a Matrix

Let the determinant of a square matrix A be $|A|$

$$IfA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad Then \quad |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The matrix formed by the cofactors of the elements in is $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

Where $A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}.\ a_{32}$

$$A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a\,3 \end{vmatrix} = -a_{21}.\ a_{33} + a_{23}.\ a_{31}; A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31};$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -a_{12}a_{33} + a_{13}.\ a_{32}; A_{22} = (-1)^{2+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}a_{33} - a_{13}.\ a_{31};$$

$$A_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -a_{11}a_{32} + a_{12}.\ a_{31}; A_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}a_{23} - a_{13}.\ a_{22};$$

$$A_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = -a_{11}a_{23} + a_{13}.\ a_{21}; A_{33} = (-1)^{3+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}.\ a_{21};$$

Then the transpose of the matrix of co-factors is called the adjoint of the matrix A and is written as

adj A. $adj\ A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

The **product of a matrix A and its adjoint** is equal to unit matrix multiplied by the determinant A.

Let A be a square matrix, then (Adjoint A). A = A. (Adjoint A) = | A |. I

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad and \;\; adj\ A \; = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

A. (adj. A) $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \times \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} & a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} & a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} \\ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} & a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} & a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} \\ a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} & a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} & a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \end{bmatrix}$$

= $\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I.$

Video Lesson on Finding the Inverse and Adjoint of a Matrix

Example Problems on How to Find the Adjoint of a Matrix

Example 1: If A= -A then x + y is equal to

(a) 2 (b) -1 (c) 0 (d) 12

Solution:

(c) A = -A; A is skew-symmetric matrix; diagonal elements of A are zeros

x = 0 , y = 0 ∴x + y = 0

Example 2: If A and B are two skew-symmetric matrices of order n, then,

(a) AB is a skew-symmetric matrix (b) AB is a symmetric matrix

(c) AB is a symmetric matrix if A and B commute (d)None of these

Solution:

(c) We are given A’ = -A and B’ = -B;

Now, (AB)’ = B’A’ = (-B) (-A) = BA = AB, if A and B commute.

Example 3: Let A and B be two matrices such that AB’ + BA’ = O. If A is skew symmetric ,then BA

(a) Symmetric (b) Skew symmetric (c) Invertible (d) None of these

Solution:

(c) we have, (BA)’ = A’B’ = -AB’ [A is skew symmetric]; = BA’ = B(-A) = -BA BA is skew symmetric.

Example 4: Let A $= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$,

then the co-factors of elements of A are given by –

Solution:

Co-factors of the elements of any matrix are obtain by eliminating all the elements of the same row and column and calculating the determinant of the remaining elements.

$$A_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = 3 \times 3 - 4 \times 4 = -7$$

$$A_{12} = -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1, A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1; A_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6, A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = -2, \quad A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1; \quad A_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1, \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

$$\therefore \quad Adj\ A = \begin{vmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{vmatrix}$$

Example 5: Which of the following statements are false –

(a) If | A | = 0, then | adj A | = 0;

(b) Adjoint of a diagonal matrix of order 3 × 3 is a diagonal matrix;

(c) Product of two upper triangular matrices is an upper triangular matrix;

(d) adj (AB) = adj (A) adj (B);

Solution:

(d) We have, adj (AB) = adj (B) adj (A) and not adj (AB) = adj (A) adj (B)

Inverse of a Matrix

If A and B are two square matrices of the same order, such that AB = BA = I (I = unit matrix)

Then B is called the inverse of A, i.e. B = A−1 and A is the inverse of B. Condition for a square matrix A to possess an inverse is that the matrix A is non-singular, i.e., | A | ≠ 0. If A is a square matrix and B is its inverse then AB = I. Taking determinant of both sides | AB | = | I | or | A | | B | = I. From this relation it is clear that | A | ≠ 0, i.e. the matrix A is non-singular.

How to find the inverse of a matrix by using the adjoint matrix?

We know that, $A.\,(Adj\;A) = |A|\,I \;\; or \;\; \frac{A.(Adj\;A)}{|A|} = I \;\; (Provided\,|A| \neq 0)$

And $A.\,A^{-1} = I;\, A^{-1} = \frac{1}{|A|}(Adj.\;A)$

Properties of Inverse and Adjoint of a Matrix

- **Property 1:** For a square matrix A of order n, A adj(A) = adj(A) A = |A|I, where I is the identitiy matrix of order n.
- **Property 2:** A square matrix A is invertible if and only if A is a non-singular matrix.

Problems on Finding the Inverse of a Matrix

Illustration 1: Let A $= \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$. What is inverse of A ?

Solution:

By using the formula A$^{-1} = \frac{adj\;A}{|A|}$ $we\ can\ obtain\ the\ value\ of\ A^{-1}$

We have $A_{11} = \begin{bmatrix} 4 & 5 \\ -6 & -7 \end{bmatrix} = 2 \;\; A_{12} = -\begin{bmatrix} 3 & 5 \\ 0 & -7 \end{bmatrix} = 21$

And similarly $A_{13} = -18, A_{31} = 4, A_{32} = -8, A_{33} = 4, A_{21} = +6, A_{22} = -7, A_{23} = 6$

adj A $= \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

Also $|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix} = \{4 \times (-7) - (-6) \times 5 - 3 \times (-6)\}$

=-28+30+18=20 $A^{-1} = \frac{adj\;A}{|A|} = \frac{1}{20}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

Illustration 2: If the product of a matrix A and $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ $is\ the\ matrix$ $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$,

then A^{-1} is given by:

$(a)\begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix} \quad (b)\begin{bmatrix} 0 & -1 \\ -2 & -4 \end{bmatrix} \quad (c)\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$

(d) None of these

Solution:

(a) We know if AB = C, then $B^{-1}A^{-1} = C^{-1} \Rightarrow A^{-1} = BC^{-1}$ by using this formula we will get value of A^{-1} in the above problem.

Here,

$$A\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$$

Illustration 3: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix}$ $and\ B = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$. $Prove\ that\ (AB)^{-1} = B^{-1}A^{-1}.$

Solution:

By obtaining | AB | and adj AB we can obtain $(AB)^{-1}$ by using the formula $(AB)^{-1} = \frac{adj\ AB}{|AB|}$. Similarly we can also obtain the values of B^{-1} and A^{-1} Then by multiplying B^{-1} and A^{-1} we can prove the given problem.

Here,

$$AB = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2+1 & 4+3-1 & 10+1-1 \\ 0+2+0 & 0+3+0 & 0+1+0 \\ 1+6+1 & 2+9-1 & 5+3-1 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 10 \\ 2 & 3 & 1 \\ 8 & 10 & 7 \end{bmatrix}$$

Now,

$$|AB| = \begin{vmatrix} 5 & 6 & 10 \\ 2 & 3 & 1 \\ 8 & 10 & 7 \end{vmatrix} = 5(21-10) - 6(14-8) + 10(20-24) = 55 - 36 - 40 = -21.$$

The matrix of cofactors of | AB | is =

$$\begin{bmatrix} 3(7) - 1(10) & -\{2(7) - 8(1)\} & a2(10) - 3(8) \\ -\{6(7) - 10(10)\} & 5(7) - 8(10) & -\{5(10) - 6(8)\} \\ 6(1) - 10(3) & -\{5(1) - 2(10)\} & 5(3) - 6(2) \end{bmatrix} = \begin{bmatrix} 11 & -6 & -4 \\ 58 & -45 & -2 \\ -24 & 15 & 3 \end{bmatrix}$$

adj AB = $\begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix}$ $So,\ (AB)^{-1} = \frac{adj\ AB}{|AB|} = \frac{-1}{21} \begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix}$

Next, $|B| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1(3-1) - 2(2+1) + 5(2+3) = 21$

$$\therefore B^{-1} \frac{adj\ B}{|B|} = \frac{1}{21} \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}; \quad |A| = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix} = 1(-2+1) = -1$$

$$\therefore \ A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{-1} \begin{bmatrix} -1 & -2 & 1 \\ 0 & -1 & 0 \\ -1 & -5 & 2 \end{bmatrix}$$

$$\therefore B^{-1}A^{-1} = -\frac{1}{21} \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 1 \\ 0 & -1 & 0 \\ -1 & -5 & 2 \end{bmatrix}$$

$$= -\frac{1}{21} \begin{bmatrix} 11 & 58 & -24 \\ -6 & -45 & 15 \\ -4 & -2 & 3 \end{bmatrix} \quad Thus, \quad (AB)^{-1} = B^{-1}A^{-1}$$

Illustration 4: If A $= \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ $satisfies\ A' = A^{-1},$

then

$(a) x = \pm 1/\sqrt{6}, y = \pm 1/\sqrt{6}, z = \pm 1/\sqrt{3}$ $(b) x = \pm 1/\sqrt{2}, y = \pm 1/\sqrt{6}, z = \pm 1/\sqrt{3}$

$(c) x = \pm 1/\sqrt{6}, y = \pm 1/\sqrt{2}, z = \pm 1/\sqrt{3}$ $(d) x = \pm 1/\sqrt{2}, y = \pm 1/3, z = \pm 1/\sqrt{2}$

Solution:

(b) Given that $A' = A^{-1}$ and we know that $AA^{-1} = I$ and therefore $AA' = I.$ Using the multiplication method we can obtain values of x, y and z.

$$A' = A^{-1} \Leftrightarrow AA' = 1$$

Now,

$$AA' = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 4y^2 + z^2 & 2y^2 - z^2 & -2y^2 + z \\ 2y^2 - z^2 & x^2 + y^2 + z^2 & x^2 - y^2 - z^2 \\ -2y^2 + z^2 & x^2 - y^2 - z^2 & x^2 + y^2 + z^2 \end{bmatrix}$$

Thus, $AA' = I \quad \Rightarrow 4y^2 + z^2 = 1, 2y^2 - z^2 = 0, \quad x^2 + y^2 + z^2 = 1, x^2 - y^2 - z^2 = 0$

$$x = \pm 1/\sqrt{2}, y = \pm 1/\sqrt{6}, z = \pm 1/\sqrt{3}$$

Illustration 5: If A $= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{bmatrix} \ and \ A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & y \\ 5/2 & -3/2 & 1/2 \end{bmatrix},$

then

$(a) x = 1, y = -1 \quad (b) x = -1, y = 1 \quad (c) x = 2, y = -1/2 \quad (d) x = 1/2, y = \frac{1}{2}$

Solution:

(a) We know $AA^{-1} = I$, hence by solving it we can obtain the values of x and y.

We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & y \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & y + 1 \\ 0 & 1 & 2(y + 1) \\ 4(1 - x) & 3(x - 1) & 2 + xy \end{bmatrix}$$

$$\Rightarrow \quad 1 - x = 0, x - 1 = 0; y + 1 = 0, y + 1 = 0, 2 + xy = 1$$

∴ x = 1, y = -1