Binomial Theorem

Binomial Theorem – As the power increases the expansion becomes lengthy and tedious to calculate. A binomial expression that has been raised to a very large power can be easily calculated with the help of Binomial Theorem.

Topics in Binomial Theorem

- Introduction to the Binomial Theorem
- Properties of Binomial Expansion
- Terms in the Binomial Expansion
- Binomial Theorem for any Index
- Applications of Binomial Theorem
- Multinomial Theorem
- Problems on Binomial Theorem

Introduction to the Binomial Theorem

The Binomial Theorem is the method of expanding an expression which has been raised to any finite power. A binomial Theorem is a powerful tool of expansion, which has application in Algebra (https://byjus.com/maths/algebra/), probability, etc.

Binomial Expression: A binomial expression is an algebraic expression which contains two dissimilar terms. Ex: a + b, $a^3 + b^3$, etc.

Binomial Theorem: Let n ∈ N,x,y,∈ R then

$^n\Sigma_{r=0}\ nC_r\ x^{n-r} \cdot y^r + nC_r\ x^{n-r} \cdot y^r + \ldots\ldots\ldots\ldots + nC_{n-1}\ x \cdot y^{n-1} + nC_n \cdot y^n$

i.e.$(x + y)^n = {}^n\Sigma_{r=0}\ nC_r\ x^{n-r} \cdot y^r$ where,

$$nC_r = \frac{n!}{(n-r)!r!}$$

Illustration 1: Expand $(x/3 + 2/y)^4$

Sol:

$$\left(\frac{x}{3} + \frac{2}{y}\right)^4 = 4c_0\left(\frac{x}{3}\right)^4 + 4c_1\left(\frac{x}{3}\right)^3\left(\frac{2}{y}\right) + 4c_2\left(\frac{x}{3}\right)^2\left(\frac{2}{y}\right)^2 + 4c_3\left(\frac{x}{3}\right)\left(\frac{2}{y}\right)^3 + 4c_4\left(\frac{2}{y}\right)^4$$

$$\Rightarrow \frac{x^4}{81} + \frac{8x^3}{27y} + \frac{8x^2}{3y^2} + \frac{32x}{3y^3} + \frac{16}{y^4}$$

Illustration 2: $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5$

Sol:

We have

$(x + y)^5 + (xy)^5 = 2[5C_0\ x^5 + 5C_2\ x^3 y^2 + 5C_4\ xy^4]$

$= 2(x^5 + 10\ x^3\ y^2 + 5xy^4)$

Now $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5 = 2[(\sqrt{2})^5 + 10(\sqrt{2})^3(1)^2 + 5(\sqrt{2})(1)^4]$

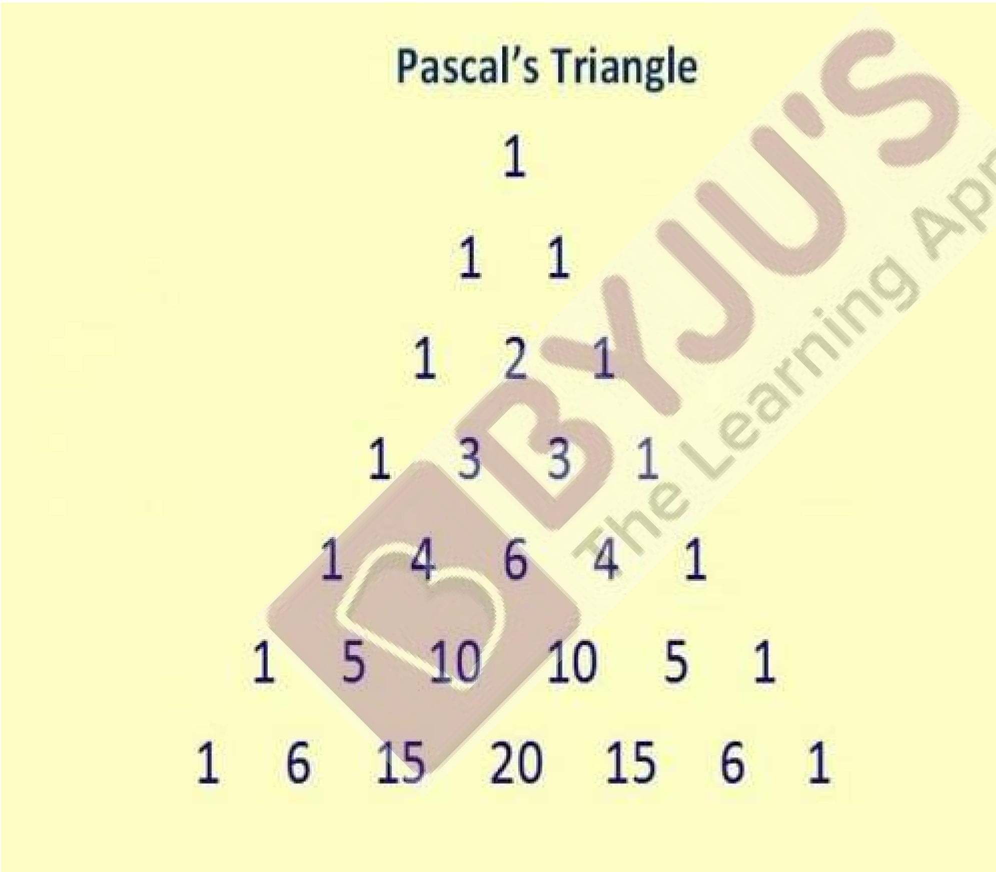
$=58\sqrt{2}$

Properties of Binomial Expansion

There are a few main properties of binomial expansion. These properties are discussed below:

- The total number of terms in the expansion of $(x+y)^n$ are $(n+1)$
- The sum of exponents of x and y is always n.
- nC_0, nC_1, nC_2, … nC_n are called binomial coefficients and also represented by C_0, C_1, C2, … C_n
- The binomial coefficients which are equidistant from the beginning and from the ending are equal i.e. $nC_0 = nC_n$, $nC_1 = nC_n - 1$, $nC_2 = nC_n - 2$ ….. etc.

To find binomial coefficients we can use **Pascal Triangle** also.



Some other useful expansions:

- $(x + y)^n + (x-y)^n = 2[C_0\ x^n + C_2\ x^{n-1}\ y^2 + C_4\ x^{n-4}\ y^4 + \ldots]$
- $(x + y)^n - (x-y)^n = 2[C_1\ x^{n-1}\ y + C_3\ x^{n-3}\ y^3 + C_5\ x^{n-5}\ y^5 + \ldots]$
- $(1 + x)^n = {}^n\Sigma_{r-0}\ nC_r .\ x^r = [C_0 + C_1\ x + C_2\ x^2 + \ldots\ C_n\ x_n]$
- $(1+x)^n + (1 - x)^n = 2[C_0 + C_2\ x^2 + C_4\ x^4 + \ldots]$
- $(1+x)^n - (1-x)^n = 2[C_1\ x + C_3\ x^3 + C_5\ x^5 + \ldots]$
- The number of terms in the expansion of $(x + a)^n + (x-a)^n$ is $(n+2)/2$ if "n" is even or $(n+1)/2$ if "n" is odd.
- The number of terms in the expansion of $(x + a)^n - (x-a)^n$ is $(n/2)$ if "n" is even or $(n+1)/2$ if "n" is odd.

Properties of Binomial Coefficients

Binomial coefficients refer to the integers which are coefficients in the binomial theorem. Some of the most important properties of binomial coefficients are:

- $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$
- $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$
- $C_0 - C_1 + C_2 - C_3 + \ldots +(-1)^n . nC_n = 0$
- $nC_1 + 2.nC_2 + 3.nC_3 + \ldots + n.nC_n = n.2^{n-1}$
- $C_1 - 2C_2 + 3C_3 - 4C_4 + \ldots +(-1)^{n-1} C_n = 0$ for $n > 1$
- $C_0^2 + C_1^2 + C_2^2 + \ldots C_n^2 = [(2n)!/ (n!)^2]$

Illustration: If $(1 + x)^{15} = a_0 + a_1x + ... + a_{15} x^{15}$ then, find the value of $\sum_{r=1}^{15} r.\frac{ar}{a_{r-1}}$

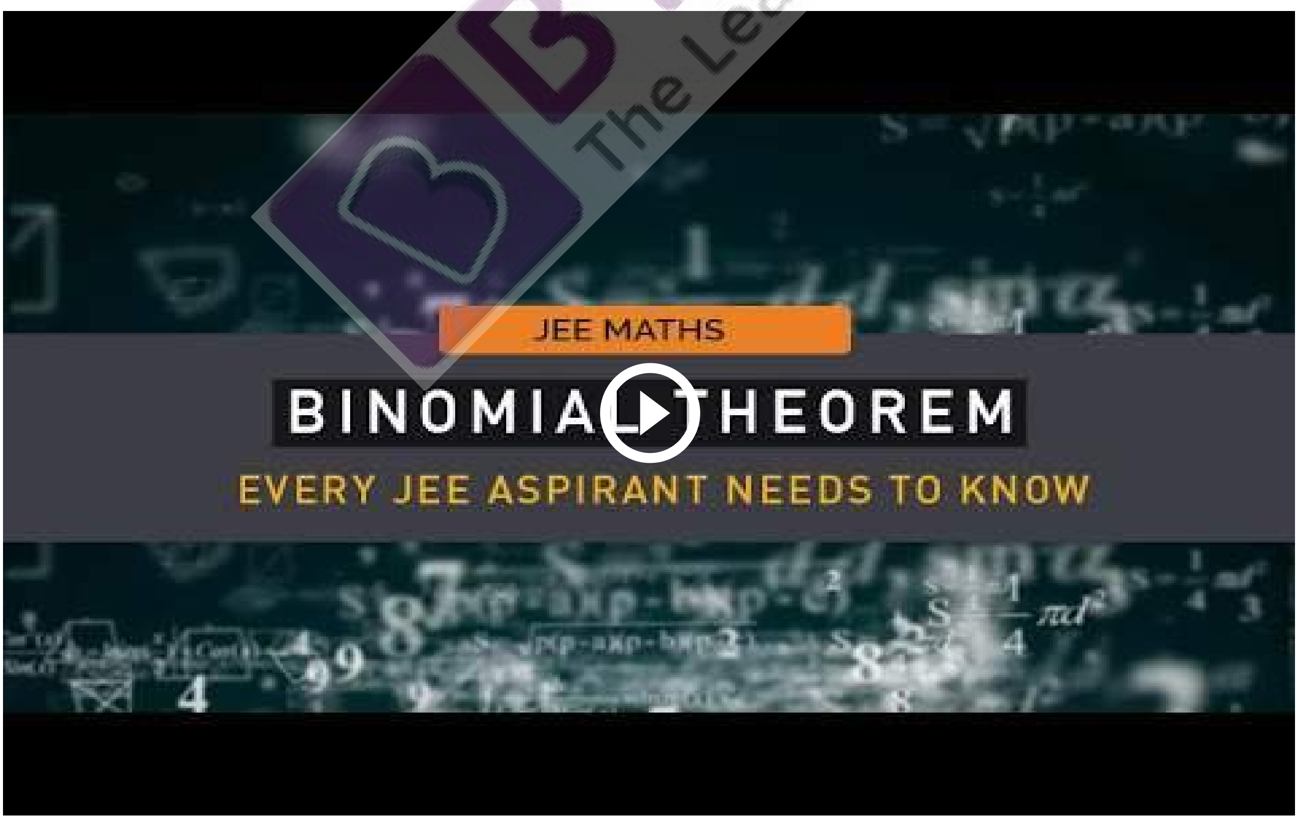
Sol:

$$\sum_{r=1}^{15} r.\frac{ar}{a_{r-1}} = 1.\frac{a_1}{a_0} + 2.\frac{a_2}{a_1} + 3.\frac{a_3}{a_2} + \ldots + 15.\frac{a_{15}}{a_{14}}$$

$= C_1/C_0 + 2\, C_2/C_1 + 3C_3/C_2 + ... + 15\, C_{15}/C_{14}$

$= 15 + 14 + 13 + ... + 1 = [15(15+1)]/2 = 120$

Binomial Theorem IIT JEE Video Lesson



Terms in the Binomial Expansion

In binomial expansion (https://byjus.com/maths/binomial-expansion/), it is often asked to find the middle term or th general term. Here, the different terms in the binomial expansion that are covered here include:

- General Term
- Middle Term
- Independent Term
- Determining a Particular Term
- Numerically greatest term
- Ratio of Consecutive Terms/Coefficients

General Term in binomial expansion:

We have $(x + y)^n = nC_0\, x^n + nC_1\, x^{n-1} . y + nC_2\, x^{n-2} . y^2 + \ldots + nC_n\, y^n$

General Term = $T_{r+1} = nC_r\, x^{n-r} . y^r$

- General Term in $(1 + x)^n$ is $nC_r\, x^r$
- In the binomial expansion of $(x + y)^n$ the r^{th} term from end is $(n – r + 2)^{th}$ term from the beginning.

Illustration: Find the number of terms in $(1 - 2x + x^2)^{50}$

Sol:

$(1 - 2x + x^2)^{50} = [(1 + x)^2]^{50} = (1 + x)^{100}$

The number of terms = (100 + 1) = 101

Illustration: Find the fourth term from the end in the expansion of $(2x – 1/x^2)^{10}$

Sol:

Required term $= T_{10 - 4 + 2} = T8 = 10C_7\, (2x)^3\, (-1/x^2)^7 = -960x^{-11}$

Middle Term(S) in the expansion of $(x+y)^{n.n}$

- If n is even then (n/2 + 1) Term is the middle Term.
- If n is odd then $[(n+1)/2]^{th}$ and $[(n+3)/2)^{th}$ terms are the middle terms.

Illustration: Find the middle term of $(1 - 3x + 3x^2 – x^3)^{2n}$

Sol:

$(1 - 3x + 3x^2 – x^3)^{2n} = [(1 - x)^3]^{2n} = (1 - x)^{6n}$

Middle Term = [(6n/2) + 1] term = $6nC_{3n}\, (-x)^{3n}$

Determining a Particular Term:

- In the expansion of $(ax^p + b/x^q)^n$ the coefficient of x^m is the coefficient of T_{r+1} where $r = [(np-m)/(p+q)]$
- In the expansion of $(x + a)^n$, $T_{r+1}/Tr = (n – r + 1)/r . a/x$

Independent Term

The term Independent of in the expansion of $[ax^p + (b/x^q)]^n$ is

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$, where r = (np/p+q) (integer)

Illustration: Find the independent term of x in $(x+1/x)^6$

Sol:

r = [6(1)/1+1] = 3

The independent term is $6C_3 = 20$

Illustration: Find the independent term in the expansion of:

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-\sqrt{x}}\right)^{10}$$

Sol:

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-\sqrt{x}}\right)^{10} = \left[\left(x^{1/3}+1\right) - \frac{\sqrt{x}+1}{\sqrt{x}}\right]^{10}$$

$(x^{1/3} + 1 - 1 - 1/\sqrt{x})^{10} = (x^{1/3} - 1/\sqrt{x})^{10}$

$r = [10(1/3)]/[1/3+1/2] = 4$

∴ $T_5 = {}^{10}C_4 = 210$

Numerically greatest term in the expansion of $(1+x)^n$:

- If $[(n+1)|x|]/[|x|+1] = P$, is a positive integer then P^{th} term and $(P+1)^{th}$ terms are numerically greatest terms in the expansion of $(1+x)^n$
- If$[(n+1)|x|]/[|x|+1] = P + F$, where P is a positive integer and $0 < F < 1$ then $(P+1)^{th}$ term is numerically greatest term in the expansion of $(1+x)^n$.

Illustration: Find the numerically greatest term in $(1-3x)^{10}$ when $x = (1/2)$

Sol:

$[(n + 1)|\alpha|] / [|\alpha| + 1] = (11 \times 3/2)/(3/2+1) = 33/5 = 6.6$

Therefore, T_7 is the numerically greatest term.

$T_{6+1} = 10C_6 . (-3x)^6 = 10C_6 . (3/2)^6$

Ratio of Consecutive Terms/Coefficients:

Coefficients of x^r and x^{r+1} are nC_{r-1} and nC_r respectively.

$(nC_r / nC_{r-1}) = (n - r + 1) / r$

Illustration: If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1:7:42 then find the value of n.

Sol:

Let $(r + 1)^{th}$, $(r + 2)^{th}$ and $(r + 3)^{th}$ be the three consecutive terms.

Then $nC_r : nC_{r+1} : nC_{r+2} = 1:7:42$

Now $(nC_r / nC_{r-1}) = (1/7)$

$(nC_r / nC_{r-1}) = (1/7) \Rightarrow [(r + 1)/(n - r)] = (1/7) \Rightarrow n-8r = 7 \rightarrow (1)$

And,

$(nC_r / nC_{r-1}) = (7/42) \Rightarrow [(r+2)/(n - r - 1)] = (1/6) \Rightarrow n-7r = 13 \rightarrow (2)$

From (1) & (2), n = 55

Applications of Binomial Theorem

Binomial theorem has a wide range of application in mathematics like finding the remainder, finding digits of a number, etc. The most common binomial theorem applications are:

Finding Remainder using Binomial Theorem

Illustration: Find the remainder when 7^{103} is divided by 25

Sol:

$(7^{103} / 25) = [7(49)^{51} / 25)] = [7(50 - 1)^{51} / 25]$

$= [7(25K - 1) / 25] = [(175K - 25 + 25-7) / 25]$

$= [(25(7K - 1) + 18) / 25]$

∴ The remainder = 18.

Illustration: If the fractional part of the number $(2^{403} / 15)$ is (K/15), then find K.

Sol:

$(2^{403} / 15) = [2^3 (2^4)^{100} / 15]$

$= 8/15 (15 + 1)^{100} = 8/ 15 (15\lambda + 1) = 8\lambda + 8/15$

∵ 8λ is an integer, fractional part = 8/15

So, K = 8.

Finding Digits of a Number

Illustration: Find the last two digits of the number $(13)^{10}$

Sol:

$(13)^{10} = (169)^5 = (170 - 1)^5$

$= 5C_0 (170)^5 - 5C_1 (170)^4 + 5C_2 (170)^3 - 5C_3 (170)^2 + 5C_4 (170) - 5C_5$

$= 5C_0 (170)^5 - 5C_1 (170)^4 + 5C_2 (170)^3 - 5C_3 (170)^2 + 5(170) - 1$

A multiple of 100 + 5(170) – 1 = 100K + 849

∴ The last two digits are 49.

Relation Between two Numbers

Illustration: Find the larger of $99^{50} + 100^{50}$ and 101^{50}

Sol:

$101^{50} = (100 + 1)^{50} = 100^{50} + 50 . 100^{49} + 25 . 49 . 100^{48} + \dots$

$\Rightarrow 99^{50} = (100 - 1)^{50} = 100^{50} - 50 . 100^{49} + 25 . 49 . 100^{48} - \dots.$

$\Rightarrow 101^{50} - 99^{50} = 2[50 . 100^{49} + 25(49) (16) 100^{47} + \dots]$

$= 100^{50} + 50 . 49 . 16 . 100^{47} + \dots > 100^{50}$

$\therefore 101^{50} - 99^{50} > 100^{50}$

$\Rightarrow 101^{50} > 100^{50} + 99^{50}$

Divisibility Test

Illustration: Show that $11^9 + 9^{11}$ is divisible by 10.

Sol:

$11^9 + 9^{11} = (10 + 1)^9 + (10 - 1)^{11}$

$= (9C_0 . 10^9 + 9C_1 . 10^8 + \dots 9C_9) + (11C_0 . 10^{11} - 11C_1 . 10^{10} + \dots -11C_{11})$

$= 9C_0 . 10^9 + 9C_1 . 10^8 + \dots + 9C_8 . 10 + 1 + 10^{11} - 11C_1 . 10^{10} + \dots + 11C_{10} . 10-1$

$= 10[9C_0 . 10^8 + 9C_1 . 10^7 + \dots + 9C_8 + 11C_0 . 10^{10} - 11C_1 . 10^9 + \dots + 11C_{10}]$

= 10K, which is divisible by 10.

Formulae:

- The number of terms in the expansion of $(x_1 + x_2 + \dots x_r)^n$ is $(n + r - 1)C_{r-1}$
- Sum of the coefficients of $(ax + by)^n$ is $(a + b)^n$

If $f(x) = (a_0 + a_1x + a_2x^2 + \dots. + a_mx^m)^n$ then

- (a) Sum of coefficients = f(1)
- (b) Sum of coefficients of even powers of x is: $[f(1) + f(-1)] / 2$
- (c) Sum of coefficients of odd powers of x is $[f(1) - f(-1)] / 2$

Binomial Theorem for any Index

Let n be a rational number (https://byjus.com/maths/rational-numbers/) and x be a real number such that | x | < 1 Then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \\ + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots\infty$$

Proof:

Let $f(x) = (1 + x)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_r x^r + \dots$ (1)

$f(0) = (1 + 0)^n = 1$

Differentiating (1) w.r.t. x on both sides, we get

$n(1 + x)^{n-1}$

$= a_1 + 2a_2 x + 3a_3 x^3 + 4a_4 x^3 + \dots + ra_r x^{r-1} + \dots$ (2)

Put x = 0, we get $n = a_1$

Differentiating (2) w.r.t. x on both sides, we get

$n(n - 1)(1 + x)^{n-2}$

$= 2a_2 + 6a_3 x + 12a_4 x^2 + \dots + r(r-1) a_r x^{r-2} + \dots$ (3)

Put x = 0, we get $a_2 = [n(n-1)] / 2!$

Differentiating (3), w.r.t. x on both sides, we get

$n(n - 1)(n - 2)(1 + x)^{n-3} = 6a_3 + 24a_4 x + \dots + r(r - 1)(r - 2) a_r x_{r-3} + \dots$

Put x = 0, we get $a_3 = [n(n-1)(n-2)] / 3!$

Similarly, we get $a_4 = [n(n-1)(n-2)(n-3)] / r!$ and so on

$\therefore a_r = [n(n-1)(n-2)\dots(n-r+1)] / r!$

Putting the values of $a_0, a_1, a_2, a_3, \dots, a_r$ obtained in (1), we get

$(1 + x)^n = 1 + nx + [\{n(n-1)\} / 2!] x^2 + [\{n(n - 1)(n - 2)\} / 2!] x^3 + \dots + [\{n(n - 1)(n - 2) \dots (n - r + 1)\}/ r!] x^r + \dots$

Binomial theorem for Rational Index

The number of rational terms in the expression of $(a^{1/l} + b^{1/k})^n$ is [n / LCM of {l,k}] when none of and is a factor of and when at least one of and is a factor of is [n / LCM of {l,k}] + 1 where [.] is the greatest integer function.

Illustration: Find the number of irrational terms in $({}^8\sqrt{5} + {}^6\sqrt{2})^{100}$.

Sol:

$T_{r+1} = 100C_r ({}^8\sqrt{5})^{100-r} . ({}^6\sqrt{2})^r = 100C_r . 5[(100 - r)/8] .2^{r/6}$.

$\therefore$ r = 12,36,60,84

The number of rational terms = 4

Number of irrational terms = 101 – 4 = 97

Binomial Theorem for Negative Index

1. If rational number and -1 < x <1 then,

- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots + x^r + \dots \infty$
- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots (-1)^r x^r + \dots \infty$
- $(1 - x)^{-2} = 1 + 2x + 3x^2 - 4x^3 + \dots + (r + 1)x^r + \dots \infty$
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots + (-1)^r (r + 1)x^r + \dots \infty$

- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-2)\dots\dots(n-4+1)}{r!} \dots\dots\infty$
- $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \dots + (-1)^r\frac{n(n-1)(n-2)\dots\dots(n-r+1)}{r!}x^r + \dots\dots\infty$
- $(1-x)^n = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots + \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r + \dots\infty$
- $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 \dots \frac{n(n+1)(n+2)}{3!}x^3 + \dots + (-1)^r\frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r + \dots\infty$
- $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + \frac{(r+1)(r+2)}{r!} + \dots\infty$
- $(1+x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \dots + (-1)^r\frac{(r+1)(r+2)}{r!} + \dots\infty$

2. Number of terms in $(1 + x)^n$ is

- ‘n+1 when positive integer.
- Infinite when is not a positive integer & | x | < 1

3. First negative term in $(1 + x)^{p/q}$ when $0 < x < 1$, p, q are positive integers & 'p' is not a multiple of 'q' is $T_{[p/q]} + 3$

Multinomial Theorem

Using binomial theorem, we have

$(x + a)^n$

$= {}^n\sum_{r = 0} nC_r \, x^{n - r} a^r$, $n \in N$

$= {}^n\sum_{r = 0} [n! / (n - r)!r!] \, x^{n - r} a^r$

$= {}^n\sum_{r + s = n} [n! / r!s!] \, x^s a^r$, where $s = n - r$.

This result can be generalized in the following form:

$(x_1 + x_2 + \ldots + x_k)^n$

$= \sum_{r1 + r2 + \ldots + rk = n} [n! / r1!r2!\ldots rk!] \, x_1^{r1} x_2^{r2} \ldots x_k^{rk}$

The general term in the above expansion is

$[(n!) / (r_1! \, r_2! \, r_3! \ldots r_k!)] \, x_1^{r1} x_2^{r2} x_3^{r3} \ldots x_k^{rk}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation.

$r_1 + r_2 + \ldots + r_k = n$, because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n + k - 1}C_k - 1$.

PARTICULAR CASES

Case-1:

$$(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms.

Case-2:

$$(x + y + z + u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$$

There are ${}^{n + 4 - 1}C_{4 - 1} = {}^{n + 3}C_3$ terms in the above expansion.

REMARK: The greatest coefficient in the expansion of $(x_1 + x_2 + \ldots + x_m)^n$ is $[(n!) / (q!)^{m - r}\{(q+1)!\}^r]$, where q and r are the quotient and remainder respectively when n is divided by m.

Multinomial Expansions

Consider the expansion of $(x + y + z)^{10}$. In the expansion, each term has different powers of x,y, and z and the sum of these powers is always 10.

One of the terms is $\lambda x^2y^3z^5$. Now, the coefficient of this term is equal to the number of ways 2x's, 3y's, and 5z's are arranged, i.e., 10! (2! 3! 5!). Thus,

$(x+y+z)^{10} = \sum(10!) / (P1!\ P2!\ P3!)\ x^{P1}\ y^{P2}\ z^{P3}$

Where P1 + P2 + P3 = 10 and 0 ≤ P1, P2, P3 ≥ 10

In general,

$(x_1 + x_2 + \ldots x_r)^n = \sum (n!) / (P1!\ P2!\ \ldots\ Pr!)\ x^{P1}\ x^{P2} \ldots x^{Pr}$

Where P1 + P2 + P3 + … + Pr = n and 0 ≤ P1, P2, … Pr ≥ n

Number of Terms in the Expansion of $(x_1 + x2 + \ldots + xr)^n$

From the general term of the above expansion, we can conclude that the number of terms is equal to the number of ways different powers can be distributed to x_1, x_2, x_3 …., x_n such that the sum of powers is always "n".

Number of non-negative integral solutions of $x_1 + x_2 + \ldots + x_r = n$ is ${}^{n+r-1}C_{r-1}$.

For example, number of terms in the expansion of $(x + y + z)^3$ is ${}^{3+3-1}C_{3-1} = {}^5C_2 = 10$

As in the expansion, we have terms such as

As $x^0 y^0 z^0$, $x^0 y^1 z^2$, $x^0 y^2 z^1$, $x^0 y^3 z^0$, $x^1 y^0 z^2$, $x^1 y^1 z^1$, $x^1 y^2 z^0$, $x^2 y^0 z^1$, $x^2 y^1 z^0$, $x^3 y^0 z^0$.

Number of terms in $(x + y + z)^n$ is ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$.

Number of terms in $(x + y + z + w)^n$ is ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ and so on.

Problems on Binomial Theorem

Q.1: If the third term in the binomial expansion of $\left(1 + x^{\log_2^x}\right)^5$ **equals 2560, find x.**

Sol:

$$T_3 = 5C_2.\left(x^{\log_2^x}\right)^2 = 2560 \Rightarrow 10.\ x^{2\log_2^x} = 2560 \Rightarrow x^{2\log_2^x} = 256$$

$\Rightarrow (\log_2^x)^2 = 4$

$\Rightarrow \log_2^x = 2$ or -2

$\Rightarrow x = 4$ or $1/4$.

Q.2: Find the positive value of λ for which the coefficient of x^2 in the expression $x^2[\sqrt{x} + (\lambda/x^2)]^{10}$ is 720.

Sol:

$\Rightarrow x^2\ [{}^{10}C_r\ .\ (\sqrt{x})^{10-r}\ .\ (\lambda/x^2)^r] = x^2\ [{}^{10}C_r\ .\ \lambda^r\ .\ x^{(10-r)/2}\ .\ x^{-2r}]$

$= x^2\ [{}^{10}C_r\ .\ \lambda^r\ .\ x^{(10-5r)/2}]$

Therefore, r = 2

Hence, ${}^{10}C_2\ .\ \lambda^2 = 720$

$\Rightarrow \lambda^2 = 16$

$\Rightarrow \lambda = \pm 4$.

Q.3: The sum of the real values of x for which the middle term in the binomial expansion of $(x^3/3 + 3/x)^8$ equals 5670 is?

Sol:

$T_5 = {}^8C_4 \times (x^{12}/81) \times (81/x^4) = 5670$

$\Rightarrow 70\,x^8 = 5670$

$\Rightarrow x = \pm \sqrt{3}$.

Q.4: Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2\,x_2 + \ldots\ldots + a_{50}\,x^{50}$ for all $x \in R$, then a_2/a_0 is equal to?

Sol:

$\Rightarrow (x + 10)^{50} + (x - 10)^{50}$:

$a_2 = 2 \times {}^{50}C_2 \times 10^{48}$

$a_0 = 2 \times 10^{50}$

$\Rightarrow a_2/a_0 = {}^{50}C_2/10^2 = 12.25$.

Q.5: Find the coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)\ldots\ldots(1 + x^{100})$.

Sol:

x^9 can be formed in 8 ways.

i.e., $x^9\ x^{1+8}\ x^{2+7}\ x^{3+6}\ x^{4+5},\ x^{1+3+5},\ x^{2+3+4}$

∴ The coefficient of $x^9 = 1 + 1 + 1 + \ldots. + 8$ times $= 8$.

Q.6: The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5:10:14, find n.

Sol:

Let T_{r-1}, T_r, T_{r+1} are three consecutive terms of $(1 + x)^{n+5}$

$\Rightarrow\ T_{r-1} = (n+5)\,C_{r-2}\,.\,x^{r-2}$

$\Rightarrow\ T_r = (n+5)\,C_{r-1}\,.\,x^{r-1}$

$\Rightarrow\ T_{r+1} = (n+5)\,C_r\,.\,x^r$

Given

$(n+5)\,C_{r-2} : (n+5)\,C_{r-1} : (n+5)\,C_r = 5 : 10 : 14$

Therefore, $[(n+5)\,C_{r-2}]/5 = [(n+5)\,C_{r-1}]/10 = (n+5)\,C_r/14$

Comparing first two results we have $n - 3r = -9$ . . . . . . (1)

Comparing last two results we have $5n - 12r = -30$ . . . . . . (2)

From equation (1) and (2) n = 6.

Q.7: The digit in the units place of the number $183! + 3^{183}$.

Sol:

$\Rightarrow 3^{183} = (3^4)^{45}.3^3$

⇒ unit digit = 7 and 183! ends with 0

∴ Units digit of $183! + 3^{183}$ is 7.

Q.8: Find the total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$.

Sol:

⇒ $\mathbf{(x + a)^{100} + (x - a)^{100}} = 2[{}^{100}C_0\, x^{100}.\ {}^{100}C_2\ x^{98}\ .\ a^2 + \ldots\ldots + {}^{100}C_{100}\, a^{100}]$

∴ Total Terms = 51.

Q.9: Find the coefficient of t^4 in the expansion of $[(1\text{-}t^6)/(1 - t)]$.

Sol:

⇒ $\mathbf{[(1\text{-}t^6)/(1 - t)]} = (1 - t^{18} - 3t^6 + 3t^{12})\ (1 - t)^{-3}$

Coefficient of t in $(1 - t)^{-3} = 3 + 4 - 1$

$C_4 = {}^6C_2 = 15$

The Coefficient of x^r in $\mathbf{(1 - x)^{-n}} = (r + n - 1)\ C_r$

Q.10: Find the ratio of the 5^{th} term from the beginning to the 5^{th} term from the end in the binomial expansion of $[2^{1/3} + 1/\{2.(3)^{1/3}\}]^{10}$**.**

Sol:

$$\frac{T_5}{T_5^1} = \frac{10C_4\,(2^{1/3})^{10-4}\left[\frac{1}{2(3)^{1/3}}\right]^4}{10C_4\left(\frac{1}{2(3^{1/3})}\right)^{10-4}.(2^{1/3})^4} = 4.(36)^{1/3}$$

Q.11: Find the coefficient of $a^3b^2c^4d$ **in the expansion of $(a\text{-}b\text{-}c\text{+}d)^{10}$**.

Sol:

Expand $\mathbf{(a - b - c + d)^{10}}$ using multinomial theorem and by using coefficient property we can obtain the required result.

Using multinomial theorem, we have

$$(a-b-c+d)^{10} = \sum_{r_1+r_2+r_3+r_4=10} \frac{(10)!}{r_1!r_2!r_3!r_4!}(a)^{r_1}(-b)^{r2}(-c)^{r3}(d)^{r_4}$$

We want to get coefficient of $a^3b^2c^4d$ this implies that $r_1 = 3$, $r_2 = 2$, $r_3 = 4$, $r_4 = 1$,

∴ The coefficient of $a^3b^2c^4d$ is $[(10)!/(3!.2!.4)]\ (-1)^2\ (-1)^{-4} = 12600$.

Q.12: Find the coefficient of in the expansion of $(1 + x + x^2 +x^3)^{11}$.

Sol:

By expanding given equation using expansion formula we can get the coefficient x^4

i.e. $1 + x + x^2 = x^3 = (1 + x) + x^2\,(1 + x) = (1 + x)\ (1 + x^2)$

⇒ $(1 + x + x^2 + x^3)\ x^{11} = (1+x)^{11}\ (1+x^2)^{11}$

$= 1 + {}^{11}C_1\, x^2 + {}^{11}C_2\, x^2 + {}^{11}C_3\, x^3 + {}^{11}C_4\, x^4 \ldots\ldots.$

$= 1 + {}^{11}C_1\, x^2 + {}^{11}C_2\, x^4 + \ldots\ldots$

To find term in from the product of two brackets on the right-hand-side, consider the following products terms as

$= 1 \times {}^{11}C_2\, x^4 + {}^{11}C_2\, x^2 \times {}^{11}C_1\, x^2 + {}^{11}C_4\, x^4$

= $C_2 + {}^{11}C_2 \times {}^{11}C_1 + {}^{11}C_4]\ x^4$

⇒ $[55 + 605 + 330]\ x^4 = 990x^4$

∴ The coefficient of x^4 is 990.

Q.13: Find the number of terms free from the radical sign in the expansion of $(\sqrt{5} + {}^4\sqrt{n})^{100}$.

Sol:

$T_{r+1} = {}^{100}C_r \, . \, 5^{(100 - r)/2}\, n^{r/4}$

Where r = 0, 1, 2, ... , 100

r must be 0, 4, 8, … 100

Number of rational terms = 26

Q.14: Find the degree of the polynomial $[x + \{\sqrt{(3^{(3-1)})}\}^{1/2}]^5 + [x + \{\sqrt{(3^{(3-1)})}\}^{1/2}]^5$.

Sol:

$[x + \{ \sqrt{(3^{(3-1)})} \}^{1/2}]^5$:

$= 2\ [{}^5C_0\ x^5 + {}^5C_2\ x^5\ (x^3 - 1) + {}^5C_4 \, . \, x \, . \, (x^3 - 1)^2]$

Therefore, the highest power = 7.

Q.15: Find the last three digits of 27^{26}.

Sol:

By reducing 27^{26} into the form $(730 - 1)^n$ and using simple binomial expansion we will get required digits.

We have **27^2 =729**

Now $27^{26} = (729)^{13} = (730 - 1)^{13}$

$= {}^{13}C_0\,(730)^{13} - {}^{13}C_1\,(730)^{12} + {}^{13}C_2\,(730)^{11} - \ldots\ldots - {}^{13}C_{10}\,(730)^3 + {}^{13}C_{11}(730)^2 - {}^{13}C_{12}\,(730) + 1$

$= 1000m + [(13 \times 12)]/2] \times (14)^2 - (13) \times (730) + 1$

Where ‘m’ is a positive integer

= 1000m + 15288 − 9490 = 1000m + 5799

Thus, the last three digits of 17^{256} are 799.