Binomial Theorem

Binomial Theorem – As the power increases the expansion becomes lengthy and tedious to calculate. A binomial expression that has been raised to a very large power can be easily calculated with the help of Binomial Theorem.

Topics in Binomial Theorem

- Introduction to the Binomial Theorem
- Properties of Binomial Expansion
- Terms in the Binomial Expansion
- Binomial Theorem for any Index
- Applications of Binomial Theorem
- Multinomial Theorem
- Problems on Binomial Theorem

Introduction to the Binomial Theorem

The Binomial Theorem is the method of expanding an expression which has been raised to any finite power. A binomial Theorem is a powerful tool of expansion, which has application in Algebra (https://byjus.com/maths/algebra/), probability, etc.

Binomial Expression: A binomial expression is an algebraic expression which contains two dissimilar terms. Ex: $a + b$, $a^2 + b^3$, etc.

Binomial Theorem: Let $n \in \mathbb{N}, x, y \in \mathbb{R}$ then

$$\sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = (x + y)^n$$

i.e. $(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$ where,

$$\binom{n}{r} = \frac{n!}{(n-r)! r!}$$

Illustration 1: Expand $(x/3 + 2/y)^4$

Sol:

$$(\frac{x}{3} + \frac{2}{y})^4 = 4c_0 \left(\frac{x}{3}\right)^4 + 4c_1 \left(\frac{x}{3}\right)^3 \left(\frac{2}{y}\right) + 4c_2 \left(\frac{x}{3}\right)^2 \left(\frac{2}{y}\right)^2 + 4c_3 \left(\frac{x}{3}\right) \left(\frac{2}{y}\right)^3 + 4c_4 \left(\frac{2}{y}\right)^4$$

$$= \left(\frac{x^4}{81}\right) + 8 \left(\frac{x^3}{27y}\right) + 8 \left(\frac{x^2}{3y^2}\right) + 32 \left(\frac{x}{3y^3}\right) + 16 \left(\frac{1}{y^4}\right)$$

Illustration 2: $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5$

Sol:

We have

$$(x + y)^5 + (xy)^5 = 2[5C_0 x^5 + 5C_2 x^3 y^2 + 5C_4 xy^4]$$
= 2(x^5 + 10 x^3 y^2 + 5xy^4)

Now (√2 + 1)^5 + (√2 - 1)^5 = 2[(√2)^5 + 10(√2)^3(1)^2 + 5(√2)(1)^4]

=58√2

Properties of Binomial Expansion

There are a few main properties of binomial expansion. These properties are discussed below:

- The total number of terms in the expansion of (x+y)^n are (n+1)
- The sum of exponents of x and y is always n.
- nC_0, nC_1, nC_2, ... nC_n are called binomial coefficients and also represented by C_0, C_1, C_2, ... C_n
- The binomial coefficients which are equidistant from the beginning and from the ending are equal i.e. nC_0 = nC_n, nC_1 = nC_n - 1, nC_2 = nC_n - 2 .... etc.

To find binomial coefficients we can use Pascal Triangle also.

Some other useful expansions:

- (x + y)^n + (x−y)^n = 2[C_0 x^n + C_2 x^{n-1} y^2 + C_4 x^{n-4} y^4 + ...]
- (x + y)^n − (x−y)^n = 2[C_1 x^{n-1} y + C_3 x^{n-3} y^3 + C_5 x^{n-5} y^5 + ...]
- (1 + x)^n = Σ_{r=0}^{n} nC_r . x^r = [C_0 + C_1 x + C_2 x^2 + ... C_n x^n]
- (1+x)^n + (1−x)^n = 2[C_0 + C_2 x^2+C_4 x^4 + ...]
- (1+x)^n − (1−x)^n = 2[C_1 x + C_3 x^3 + C_5 x^5 + ...]
- The number of terms in the expansion of (x + a)^n + (x−a)^n is (n+2)/2 if "n" is even or (n+1)/2 if "n" is odd.
- The number of terms in the expansion of (x + a)^n − (x−a)^n is (n/2) if "n" is even or (n+1)/2 if "n" is odd.
Properties of Binomial Coefficients

Binomial coefficients refer to the integers which are coefficients in the binomial theorem. Some of the most important properties of binomial coefficients are:

- \( C_0 + C_1 + C_2 + \ldots + C_n = 2^n \)
- \( C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1} \)
- \( C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^n \cdot nC_n = 0 \)
- \( nC_1 + 2nC_2 + 3nC_3 + \ldots + nC_n = n \cdot 2^{n-1} \)
- \( C_1 - 2C_2 + 3C_3 - 4C_4 + \ldots + (-1)^n \cdot C_n = 0 \) for \( n > 1 \)
- \( C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \left(\frac{2n!}{n!}\right) \)

Illustration: If \((1 + x)^{15} = a_0 + a_1x + \ldots + a_{15}x^{15}\) then, find the value of \( \sum_{r=1}^{15} r \cdot \frac{a_r}{a_{r-1}} \)

Sol:

\[
\sum_{r=1}^{15} r \cdot \frac{a_r}{a_{r-1}} = 1 \cdot \frac{a_1}{a_0} + 2 \cdot \frac{a_2}{a_1} + 3 \cdot \frac{a_3}{a_2} + \ldots + 15 \cdot \frac{a_{15}}{a_{14}}
\]

\[
= \frac{C_1}{C_0} + 2 \frac{C_2}{C_1} + 3 \frac{C_3}{C_2} + \ldots + 15 \frac{C_{15}}{C_{14}}
\]

\[
= 15 + 14 + 13 + \ldots + 1 = \frac{15(15+1)}{2} = 120
\]

Binomial Theorem IIT JEE Video Lesson

Terms in the Binomial Expansion

In binomial expansion (https://byjus.com/maths/binomial-expansion/), it is often asked to find the middle term or th general term. Here, the different terms in the binomial expansion that are covered here include:
- General Term
- Middle Term
- Independent Term
- Determining a Particular Term
- Numerically greatest term
- Ratio of Consecutive Terms/Coefficients

**General Term in binomial expansion:**

We have \((x + y)^n = nC_0 \ x^n + nC_1 \ x^{n-1} \cdot y + nC_2 \ x^{n-2} \cdot y^2 + \ldots + nC_n \ y^n\)

General Term = \(T_{r+1} = nC_r \ x^{n-r} \cdot y^r\)

- General Term in \((1 + x)^n\) is \(nC_r \ x^r\)
- In the binomial expansion of \((x + y)^n\) the \(r^{th}\) term from end is \((n - r + 2)^{th}\) term from the beginning.

**Illustration:** Find the number of terms in \((1 − 2x + x^2)^{50}\)

**Sol:**

\[(1 − 2x + x^2)^{50} = [(1 + x)^2]^{50} = (1 + x)^{100}\]

The number of terms = \((100 + 1) = 101\)

**Illustration:** Find the fourth term from the end in the expansion of \((2x − 1/x)^{10}\)

**Sol:**

Required term = \(T_{10 − 4 + 2} = T_8 = 10C_7 \ (2x)^3 (-1/x)^7 = -960x^{11}\)

**Middle Term(S) in the expansion of \((x+y)^n\):**

- If \(n\) is even then \((n/2 + 1)\) Term is the middle Term.
- If \(n\) is odd then \([(n+1)/2]\) and \([(n+3)/2]\) terms are the middle terms.

**Illustration:** Find the middle term of \((1 −3x + 3x^2 − x^3)^{2n}\)

**Sol:**

\[(1 − 3x + 3x^2 − x^3)^{2n} = [(1 − x)^3]^{2n} = (1 − x)^{6n}\]

Middle Term = \([(6n/2) + 1]\) term = \(6nC_{3n} (-x)^{3n}\)

**Determining a Particular Term:**

- In the expansion of \((ax^p + b/x^q)^n\) the coefficient of \(x^m\) is the coefficient of \(T_{r+1}\) where \(r = [(np−m)/(p+q)]\)
- In the expansion of \((x + a)^n\), \(T_{r+1}/Tr = (n − r + 1)/r \cdot a/x\)

**Independent Term**

The term Independent of in the expansion of \([ax^p + (b/x^q)]^n\) is

\(T_{r+1} = nC_r \ a^{n−r} b^r\), where \(r = (np/p+q)\) (integer)

**Illustration:** Find the independent term of \(x\) in \((x+1/x)^6\)

**Sol:**

\(r = (6(1)+1) = 3\)
The independent term is \(6C_3 = 20\)

**Illustration:** Find the independent term in the expansion of:

\[
\left( \frac{x + 1}{x^{2/3} - x^{1/3} + 1} - \frac{x - 1}{x - \sqrt{x}} \right)^{10}
\]

**Sol:**

\[
\left( \frac{x + 1}{x^{2/3} - x^{1/3} + 1} - \frac{x - 1}{x - \sqrt{x}} \right)^{10} = \left[ \left(x^{1/3} + 1\right) - \frac{\sqrt{x} + 1}{\sqrt{x}} \right]^{10}
\]

\[
(x^{1/3} + 1 - 1 - 1/\sqrt{x})^{10} = (x^{1/3} - 1/\sqrt{x})^{10}
\]

\[
r = \left[\frac{10(1/3)}{1/3 + 1/2}\right] = 4
\]

\[
\therefore T_5 = {10C_4} = 210
\]

**Numerically greatest term in the expansion of \((1+x)^n\):**

- If \([n + 1]|x|/|x|+1\) is a positive integer then \(P^{th}\) term and \((P+1)^{th}\) terms are numerically greatest terms in the expansion of \((1+x)^n\).
- If \([n + 1]|x|/|x|+1\) = \(P + F\), where \(P\) is a positive integer and \(0 < F < 1\) then \((P+1)^{th}\) term is numerically greatest term in the expansion of \((1+x)^n\).

**Illustration:** Find the numerically greatest term in \((1-3x)^{10}\) when \(x = (1/2)\)

**Sol:**

\[
[n + 1]|a|/|a|+1 = (11 \times 3/2)/(3/2+1) = 33/5 = 6.6
\]

Therefore, \(T_7\) is the numerically greatest term.

\[T_{6+1} = 10C_6 \cdot (-3x)^6 = 10C_6 \cdot (3/2)^6\]

**Ratio of Consecutive Terms/Coefficients:**

Coefficients of \(x^r\) and \(x^{r+1}\) are \(nC_{r-1}\) and \(nC_r\) respectively.

\[
(nC_r / nC_{r-1}) = (n-r+1)/r
\]

**Illustration:** If the coefficients of three consecutive terms in the expansion of \((1+x)^n\) are in the ratio 1:7:42 then find the value of \(n\).

**Sol:**

Let \((r+1)^{th}\), \((r+2)^{th}\) and \((r+3)^{th}\) be the three consecutive terms.

Then \(nC_r : nC_{r+1} : nC_{r+2} = 1:7:42\)

Now \((nC_r / nC_{r-1}) = 1/7\)

\[
(nC_r / nC_{r-1}) = 1/7 \Rightarrow [(r+1)/(n-r)] = 1/7 \Rightarrow n-8r = 7 \rightarrow (1)
\]

And,
\[
\left( \frac{nC_r}{nC_{r-1}} \right) = \left( \frac{7}{42} \right) \Rightarrow \frac{(r+2)/(n-r-1)}{1/6} = \frac{n-7r}{13} \Rightarrow (2)
\]

From (1) & (2), \( n = 55 \)

**Applications of Binomial Theorem**

Binomial theorem has a wide range of application in mathematics like finding the remainder, finding digits of a number, etc. The most common binomial theorem applications are:

**Finding Remainder using Binomial Theorem**

**Illustration:** Find the remainder when \( 7^{103} \) is divided by 25

**Sol:**

\[
\left( \frac{7^{103}}{25} \right) = \left[ \frac{7(49)^{51}}{25} \right] = \left[ \frac{7(50-1)^{51}}{25} \right] = \left[ \frac{7(25K-1)}{25} \right] = \left[ \frac{(175K-25+25-7)}{25} \right] = \left[ \frac{(25(7K-1)+18)}{25} \right]
\]

\( \therefore \) The remainder = 18.

**Illustration:** If the fractional part of the number \( \left( \frac{2^{403}}{15} \right) \) is \( \frac{K}{15} \), then find \( K \).

**Sol:**

\[
\left( \frac{2^{403}}{15} \right) = \left[ \frac{2^3 (2^4)^{100}}{15} \right] = \frac{8}{15} (15+1)^{100} = \frac{8}{15} (15\lambda + 1) = 8\lambda + 8/15
\]

\( \therefore \) 8\( \lambda \) is an integer, fractional part = 8/15

So, \( K = 8 \).

**Finding Digits of a Number**

**Illustration:** Find the last two digits of the number \( (13)^{10} \)

**Sol:**

\[
(13)^{10} = (169)^5 = (170 - 1)^5
\]

\[
= 5C_0 (170)^5 - 5C_1 (170)^4 + 5C_2 (170)^3 - 5C_3 (170)^2 + 5C_4 (170) - 5C_5
\]

\[
= 5C_0 (170)^5 - 5C_1 (170)^4 + 5C_2 (170)^3 - 5C_3 (170)^2 + 5(170) - 1
\]

A multiple of 100 + 5(170) – 1 = 100\( K \) + 849

\( \therefore \) The last two digits are 49.

**Relation Between two Numbers**

**Illustration:** Find the larger of \( 99^{50} + 100^{50} \) and \( 101^{50} \)

**Sol:**

\[
101^{50} = (100 + 1)^{50} = 100^{50} + 50 \cdot 100^{49} + 25 \cdot 49 \cdot 100^{48} + ... \Rightarrow 99^{50} = (100 - 1)^{50} = 100^{50} - 50 \cdot 100^{49} + 25 \cdot 49 \cdot 100^{48} - ... \Rightarrow 101^{50} - 99^{50} = 2[50 \cdot 100^{49} + 25(49)(16) 100^{47} + ...] = 100^{50} + 50 \cdot 49 \cdot 16 \cdot 100^{47} + ... >100^{50} \]

\( \therefore 101^{50} > 99^{50} \)
\[ 101^{50} > 100^{50} + 99^{50} \]

### Divisibility Test

**Illustration:** Show that \(11^9 + 9^{11}\) is divisible by 10.

**Sol:**

\[ 11^9 + 9^{11} = (10 + 1)^9 + (10 - 1)^{11} \]

\[ = (9C_0 \cdot 10^9 + 9C_1 \cdot 10^8 + \ldots + 9C_9) + (11C_0 \cdot 10^{11} - 11C_1 \cdot 10^{10} + \ldots - 11C_{11}) \]

\[ = 9C_0 \cdot 10^9 + 9C_1 \cdot 10^8 + \ldots + 9C_9 + 10 + 1 + 10^{10} - 11C_1 \cdot 10^{10} + \ldots + 11C_{10} \cdot 10^{10} - 1 \]

\[ = 10[9C_0 \cdot 10^8 + 9C_1 \cdot 10^7 + \ldots + 9C_9 + 11C_0 \cdot 10^9 - 11C_1 \cdot 10^9 + \ldots + 11C_{10}] \]

\[ = 10K, \text{ which is divisible by 10.} \]

**Formulae:**

- The number of terms in the expansion of \((x_1 + x_2 + \ldots x_r)^n\) is \((n + r - 1)C_{r-1}\)
- Sum of the coefficients of \((ax + by)^n\) is \((a + b)^n\)

If \(f(x) = (a_0 + a_1x + a_2x^2 + \ldots + a_mx^m)^n\) then

- (a) Sum of coefficients = \(f(1)\)
- (b) Sum of coefficients of even powers of \(x\) is: \([f(1) + f(-1)] / 2\)
- (c) Sum of coefficients of odd powers of \(x\) is \([f(1) - f(-1)] / 2\)

### Binomial Theorem for any Index

Let \(n\) be a rational number and \(x\) be a real number such that \(|x| < 1\) Then

\[
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r + \ldots + \infty
\]

**Proof:**

Let \(f(x) = (1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_rx^r + \ldots (1)\)

\(f(0) = (1 + 0)^n = 1\)

Differentiating \((1)\) w.r.t. \(x\) on both sides, we get

\(n(1 + x)^{n-1} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + ra_rx^{r-1} + \ldots (2)\)

Put \(x = 0\), we get \(n = a_1\)

Differentiating \((2)\) w.r.t. \(x\) on both sides, we get

\(n(n-1)(1 + x)^{n-2} = 2a_2 + 6a_3x + 12a_4x^2 + \ldots + r(r-1)a_rx^{r-2} + \ldots (3)\)
Put \( x = 0 \), we get \( a_2 = \frac{[n(n-1)]}{2!} \)

Differentiating (3), w.r.t. \( x \) on both sides, we get

\[
n(n-1)(n-2)(1+x)^{n-3} = 6a_3 + 24a_4 x + \ldots + r(r-1)(r-2) a_r x_{r-3} + \ldots
\]

Put \( x = 0 \), we get \( a_3 = \frac{[n(n-1)(n-2)]}{3!} \)

Similarly, we get \( a_4 = \frac{[n(n-1)(n-2)(n-3)]}{4!} \) and so on

\[
\therefore a_r = \frac{[n(n-1)(n-2)\ldots(n-r+1)]}{r!}
\]

Putting the values of \( a_0, a_1, a_2, a_3, \ldots, a_r \) obtained in (1), we get

\[
(1 + x)^n = 1 + nx + \frac{[n(n-1)]}{2!} x^2 + \frac{[n(n-1)(n-2)]}{2!} x^3 + \ldots + \frac{[n(n-1)(n-2)\ldots(n-r+1)]}{r!} x^r + \ldots
\]

**Binomial Theorem for Rational Index**

The number of rational terms in the expression of \((a^{1/l} + b^{1/k})^n\) is \( \frac{n}{\text{LCM of } \{l,k\}} \) when none of and is a factor of and when at least one of and is a factor of is \( \frac{n}{\text{LCM of } \{l,k\}} + 1 \) where \([\cdot]\) is the greatest integer function.

**Illustration:** Find the number of irrational terms in \((\sqrt{5} + \sqrt{2})^{100}\).

**Sol:**

\( T_{r+1} = 100C_r (\sqrt{5})^{100 - r} . (\sqrt{2})^r = 100C_r . 5^{[(100 - r)/8]} . 2^{r/6} \)

\( \therefore r = 12,36,60,84 \)

The number of rational terms = 4

Number of irrational terms = 101 – 4 = 97

**Binomial Theorem for Negative Index**

1. If rational number and \(-1 < x < 1\) then,

\[
\begin{align*}
(1 - x)^{-1} &= 1 + x + x^2 + x^3 + \ldots + x^r + \ldots \infty \\
(1 + x)^{-1} &= 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots \infty \\
(1 - x)^{-2} &= 1 + 2x + 3x^2 - 4x^3 + \ldots + (r + 1)x^r + \ldots \infty \\
(1 + x)^{-2} &= 1 - 2x + 3x^2 - 4x^3 + \ldots + (-1)^r (r + 1)x^r + \ldots \infty
\end{align*}
\]

2. Number of terms in \((1 + x)^n\) is

- ‘\(n+1\)’ when positive integer.
- Infinite when is not a positive integer & \(|x| < 1\)

https://byjus.com/jee/binomial-theorem/
3. First negative term in \((1 + x)^{\frac{p}{q}}\) when \(0 < x < 1\), \(p, q\) are positive integers & 'p' is not a multiple of 'q' is \(T_{\lfloor \frac{p}{q} \rfloor + 3}\)

**Multinomial Theorem**

Using binomial theorem, we have

\[(x + a)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} a^r, n \in \mathbb{N}\]

\[= \sum_{r=0}^{n} \left[ \frac{n!}{(n-r)!r!} \right] x^{n-r} a^r, \text{ where } s = n - r.\]

This result can be generalized in the following form:

\[(x_1 + x_2 + \ldots + x_k)^n = \sum \frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}\]

The general term in the above expansion is

\[
\left[ \frac{n!}{r_1!r_2!\ldots r_k!} \right] x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}
\]

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation.

\[r_1 + r_2 + \ldots + r_k = n, \text{ because each solution of this equation gives a term in the above expansion. The number of such solutions is } \binom{n+k-1}{k-1}.\]

**PARTICULAR CASES**

**Case-1:**

\[
(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t
\]

The above expansion has \(n+3-1C_3-1 = n+2C_2\) terms.

**Case-2:**

\[
(x + y + z + u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s
\]

There are \(n+4-1C_4-1 = n+3C_3\) terms in the above expansion.

**REMARK:** The greatest coefficient in the expansion of \((x_1 + x_2 + \ldots + x_m)^n\) is \[\frac{n!}{(q!)^{m-\lfloor (q+1)! \rfloor}},\] where \(q\) and \(r\) are the quotient and remainder respectively when \(n\) is divided by \(m\).

**Multinomial Expansions**

Consider the expansion of \((x + y + z)^{10}\). In the expansion, each term has different powers of \(x, y,\) and \(z\) and the sum of these powers is always 10.
One of the terms is $\lambda x^2y^3z^5$. Now, the coefficient of this term is equal to the number of ways 2x's, 3y's, and 5z's are arranged, i.e., $10! \cdot (2! \cdot 3! \cdot 5!)$. Thus,

$$(x+y+z)^{10} = \sum (10!) / (P1! \cdot P2! \cdot P3!) \cdot x^{P1} \cdot y^{P2} \cdot z^{P3}$$

Where $P1 + P2 + P3 = 10$ and $0 \leq P1, P2, P3 \leq 10$

In general,

$$(x_1 + x_2 + \ldots + x_r)^n = \sum (n!) / (P1! \cdot P2! \ldots \cdot Pr!) \cdot x_1^{P1} \cdot x_2^{P2} \ldots \cdot x_r^{Pr}$$

Where $P1 + P2 + \ldots + Pr = n$ and $0 \leq P1, P2, \ldots, Pr \leq n$

**Number of Terms in the Expansion of $(x_1 + x_2 + \ldots + x^n)^n$**

From the general term of the above expansion, we can conclude that the number of terms is equal to the number of ways different powers can be distributed to $x_1, x_2, x_3, \ldots, x_n$ such that the sum of powers is always "$n$".

Number of non-negative integral solutions of $x_1 + x_2 + \ldots + x_r = n$ is $n + r - 1 \choose r - 1$.

For example, number of terms in the expansion of $(x + y + z)^3$ is $3 + 3 + 1 = 5 \choose 2 = 10$

As in the expansion, we have terms such as $x^0y^0z^0, x^0y^1z^2, x^0y^2z^1, x^1y^0z^0, x^1y^1z^1, x^1y^2z^0, x^2y^0z^0, x^1y^1z^0, x^2y^0z^0$.

Number of terms in $(x + y + z)^n$ is $n + 3 - 1 \choose 3 - 1 = n + 2 \choose 2$.

Number of terms in $(x + y + z + w)^n$ is $n + 4 - 1 \choose 4 - 1 = n + 3 \choose 3$ and so on.

**Problems on Binomial Theorem**

**Q.1:** If the third term in the binomial expansion of $\left(1 + x^{\frac{1}{2}} \cdot \log_2 s\right)^5$ equals 2560, find $x$.

**Sol:**

$$T_3 = 5C_2 \cdot \left(x^{\frac{\log_2 s}{2}}\right)^2 = 2560 \Rightarrow 10x^{2\log_2 s} = 2560 \Rightarrow x^{2\log_2 s} = 256$$

$$\Rightarrow (\log_2 x)^2 = 4$$

$$\Rightarrow \log_2 x = 2 \text{ or } -2$$

$$\Rightarrow x = 4 \text{ or } 1/4.$$  

**Q.2:** Find the positive value of $\lambda$ for which the coefficient of $x^2$ in the expression $x^2[\sqrt{x} + (\lambda/x^2)]^{10}$ is 720.

**Sol:**

$$\Rightarrow x^2 \left[10C_1 \cdot \left(\sqrt{x}\right)^{10-r} \cdot (\lambda/x^2)^r\right] = x^2 \left[10C_1 \cdot \lambda^r \cdot x^{(10-r)/2} \cdot x^{-2r}\right]$$

$$= x^2 \left[10C_1 \cdot \lambda^r \cdot x^{(10-r)/2}\right]$$

Therefore, $r = 2$

Hence, $10C_2 \cdot \lambda^2 = 720$

$$\Rightarrow \lambda^2 = 16$$

$$\Rightarrow \lambda = \pm 4.$$
Q.3: The sum of the real values of x for which the middle term in the binomial expansion of \((x^3/3 + 3/x)^8\) equals 5670 is?

Sol:
\[ T_5 = ^8C_4 \times \left(\frac{x^{12}}{81}\right) \times \frac{1}{x^4} = 5670 \]
\[ \Rightarrow 70x^8 = 5670 \]
\[ \Rightarrow x = \pm \sqrt{3}. \]

Q.4: Let \((x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}\) for all \(x \in \mathbb{R}\), then \(a_2/a_0\) is equal to?

Sol:
\[ \Rightarrow (x + 10)^{50} + (x - 10)^{50}. \]
\[ a_2 = 2 \times ^{50}C_2 \times 10^{48} \]
\[ a_0 = 2 \times 10^{50} \]
\[ \Rightarrow \frac{a_2}{a_0} = \frac{^{50}C_2}{10^{2}} = 12.25. \]

Q.5: Find the coefficient of \(x^9\) in the expansion of \((1 + x)(1 + x^2)(1 + x^3)\ldots(1 + x^{100})\).

Sol:
\(x^9\) can be formed in 8 ways.
\[ \text{i.e., } x^9, x^{1+8}, x^{2+7}, x^{3+6}, x^{4+5}, x^{1+3+4}, x^{2+3+4} \]
\[ \therefore \text{The coefficient of } x^9 = 1 + 1 + 1 + \ldots + 8 \text{ times} = 8. \]

Q.6: The coefficients of three consecutive terms of \((1 + x)^{n+5}\) are in the ratio 5:10:14, find \(n\).

Sol:
Let \(T_{r-1}, T_r, T_{r+1}\) are three consecutive terms of \((1 + x)^{n+5}\)
\[ \Rightarrow T_{r-1} = (n+5) \times ^{r-2}C_r \times x^{r-2} \]
\[ \Rightarrow T_r = (n+5) \times ^{r-1}C_r \times x^{r-1} \]
\[ \Rightarrow T_{r+1} = (n+5) \times ^rC_r \times x^r \]

Given
\[ (n+5) \times ^{r-2}C_r : (n+5) \times ^{r-1}C_r : (n+5) \times ^rC_r = 5 : 10 : 14 \]
Therefore, \[\frac{(n+5) \times ^{r-2}C_r}{5} = \frac{(n+5) \times ^{r-1}C_r}{10} = \frac{(n+5) \times ^rC_r}{14}\]
Comparing first two results we have \(n - 3r = -9 \ldots \ldots (1)\)
Comparing last two results we have \(5n - 12r = -30 \ldots \ldots (2)\)
From equation (1) and (2) \(n = 6\).

Q.7: The digit in the units place of the number \(183! + 3^{183}\).

Sol:
\[ \Rightarrow 3^{183} = (3^4)^{45} \times 3^3 \]
\[ \Rightarrow \text{unit digit} = 7; \text{and 183! ends with 0} \]
\[ \therefore \text{Units digit of } 183! + 3^{183} \text{ is 7.} \]

Q.8: Find the total number of terms in the expansion of \((x + a)^{100} + (x - a)^{100}\).
Q.9: Find the coefficient of $t^4$ in the expansion of $[(1-t^6)/(1 - t)]$.

Sol:

$\Rightarrow [(1-t^6)/(1 - t)] = (1 - t^{18} - 3t^6 + 3t^{12}) (1 - t)^{-3}$

Coefficient of $t$ in $(1 - t)^{-3} = 3 + 4 - 1$

$C_4 = 6C_2 = 15$

The Coefficient of $x^r$ in $(1 - x)^n = (r + n - 1) C_r$

Q.10: Find the ratio of the 5th term from the beginning to the 5th term from the end in the binomial expansion of $[2^{1/3} + 1/(2.3^{1/3})]^{10}$.

Sol:

$$\frac{T_5}{T_5} = \frac{10C_4 \left(\frac{2^{1/3}}{3^{1/3}}\right)^{10-4} \left(\frac{1}{2^{1/3}}\right)^4}{10C_4 \left(\frac{1}{2^{1/3}}\right)^{10-4} \left(\frac{2^{1/3}}{3^{1/3}}\right)^4} = 4 \left(\frac{36}{13}\right)^{1/3}$$

Q.11: Find the coefficient of $a^2b^2c^4d$ in the expansion of $(a-b-c+d)^{10}$.

Sol:

Expand $(a - b - c + d)^{10}$ using multinomial theorem and by using coefficient property we can obtain the required result.

Using multinomial theorem, we have

$$(a-b-c+d)^{10} = \sum_{r_1+r_2+r_3+r_4=10}^{(10)!} \binom{10}{r_1 r_2 r_3 r_4} (a)^{r_1} (-b)^{r_2} (-c)^{r_3} (d)^{r_4}$$

We want to get coefficient of $a^2b^2c^4d$ this implies that $r_1 = 3, r_2 = 2, r_3 = 4, r_4 = 1$.

$\therefore$ The coefficient of $a^2b^2c^4d$ is $[(10)!/(3!2!4!)] (-1)^2 (-1)^4 = 12600$.

Q.12: Find the coefficient of in the expansion of $(1 + x + x^2 + x^3)^{11}$.

Sol:

By expanding given equation using expansion formula we can get the coefficient $x^4$

i.e. $1 + x + x^2 = x^3 = (1 + x) + x^2 (1 + x) = (1 + x) (1 + x^2$)

$\Rightarrow (1 + x + x^2 + x^3) x^{11} = (1+x)^{11} (1+x^2)^{11}$

$= 1 + 11C_1 x^2 + 11C_2 x^2 + 11C_3 x^3 + 11C_4 x^4 \ldots . . . . . .$

$= 1 + 11C_1 x^2 + 11C_2 x^4 + \ldots . . . . . .$

To find term in from the product of two brackets on the right-hand-side, consider the following products terms as

$= 1 \times 11C_2 x^4 + 11C_2 x^2 \times 11C_1 x^2 + 11C_4 x^4$
\[ C_2 + 11C_2 \times 11C_1 + 11C_4 \] \ x^4

\Rightarrow [55 + 605 + 330] \ x^4 = 990x^4

\therefore \text{The coefficient of } x^4 \text{ is } 990.

**Q.13:** Find the number of terms free from the radical sign in the expansion of \((\sqrt{5} + 4\sqrt{n})^{100}\).

**Sol:**

\[ T_{r+1} = ^{100}C_r \cdot 5^{(100 - r)/2} \ n^{r/4} \]

Where \( r = 0, 1, 2, \ldots, 100 \)

\( r \) must be 0, 4, 8, … 100

Number of rational terms = 26

**Q.14:** Find the degree of the polynomial \([x + \{\sqrt{3(3-1)}\}]^5 + [x + \{\sqrt{3(3-1)}\}]^5\).

**Sol:**

\[ [x + \{\sqrt{3(3-1)}\}]^5. \]

\[ = 2 [^5C_0 \ x^5 + ^5C_2 \ x^5 (x^3 - 1) + ^5C_4 \ x \cdot (x^3 - 1)^2] \]

Therefore, the highest power = 7.

**Q.15:** Find the last three digits of 27^{26}.

**Sol:**

By reducing 27^{26} into the form \((730 – 1)^n\) and using simple binomial expansion we will get required digits.

We have \(27^2 = 729\)

Now \(27^{26} = (729)^{13} = (730 – 1)^{13}\)

\[ = ^{13}C_0 (730)^{13} – ^{13}C_1 (730)^{12} + ^{13}C_2 (730)^{11} - \ldots - ^{13}C_{10} (730)^3 + ^{13}C_{11} (730)^2 - ^{13}C_{12} (730) + 1 \]

\[ = 1000m + [(13 \times 12)/2] \times (14)^2 - (13) \times (730) + 1 \]

Where \(m\) is a positive integer

\[ = 1000m + 15288 – 9490 = 1000m + 5799 \]

Thus, the last three digits of 17^{256} are 799.