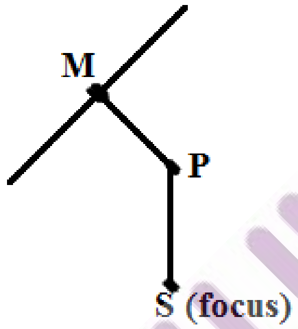
Lesson Name: Hyperbola
URL: https://byjus.com/jee/hyperbola/

Hyperbola

A hyperbola is defined as an open curve having two branches which are mirror images to each other. Hyperbolic curves are of special importance in astronomy & space studies. Here, we will be studying about hyperbola and characteristics of such curves.

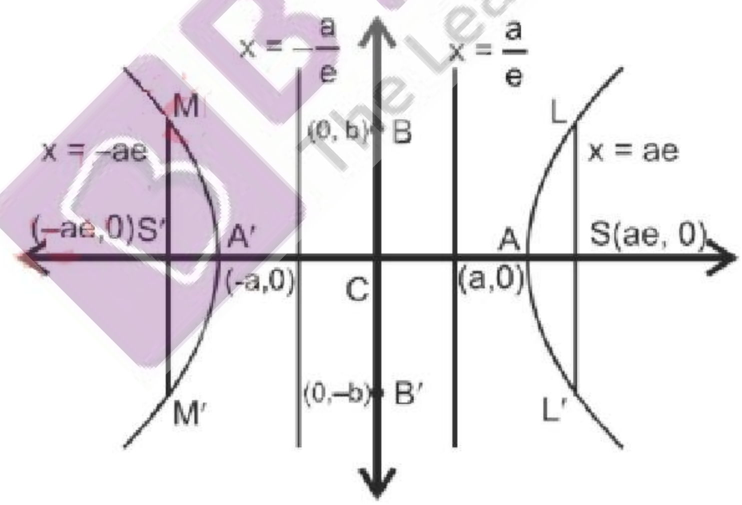
What is Hyperbola?

A hyperbola is a locus of points in such a way that the distance to each focus is a constant greater than one. In other words, the locus of a point moving in a plane in such a way that the ratio of its distance from a fixed point (focus) to that from a fixed line (directrix) is a constant greater than 1.



[Note: The point (focus) does not lie on the line (directrix)]

(PS/PM) = e > 1 eccentricity



Standard Equation of Hyperbola

The equation of the hyperbola is simplest when the centre of the hyperbola is at the origin and the foci are either on the x-axis or on the y-axis. The standard equation of a hyperbola is given as:

$[(x^2 / a^2) - (y^2 / b^2)] = 1$

where , $b^2 = a^2 (e^2 - 1)$

Important Terms and Formulas of Hyperbola

There are certain terms related to a hyperbola which needs to be thoroughly understood to be able to get confident with this concept. Some of the most important terms related to hyperbolae are:

- **Eccentricity (e):** $e^2 = 1 + (b^2 / a^2) = 1 + [(\text{conjugate axis})^2 / (\text{transverse axis})^2]$
- **Focii:** S = (ae, 0) & S' = (−ae, 0)
- **Directrix:** x=(a/e), x = (−a / e)
- **Transverse axis:**

The live segment A'A of length 2a in which the focii S' and S both lie is called the transverse axis of the hyperbola.

- **Conjugant axis:**

The line segment (https://byjus.com/maths/line-segment/) B'B of length 2b between the 2 points B' = (0, -b) & B = (0, b) is called the conjugate axis of the hyperbola.

- **Principal axes:**

The transverse axis & conjugate axis.

- **Vertices:**

A = (a, 0) & A' = (-a, 0)

- **Focal chord:**

A chord which passes through a focus is called a focal chord.

- **Double ordinate:**

Chord perpendicular to the transverse axis is called a double ordinate.

- **Latus Rectum:**

Focal chord $\perp^r$ to the transverse axis is called latus rectum.

Its length = $(2b^2 / a) = [(\text{conjugate})^2 / \text{transverse}] = 2a(e^2 - 1)$

The difference in focal distances is a constant

i.e. |PS−PS'| = 2a

Length of latus rectum = 2 e × (distance of focus from corresponding directrix)

End points of L.R : $(\pm ae, \pm b^2 / a)$

Centre:

The point which bisects every chord of the conic, drawn through it, is called the centre of the conic.

C: (0, 0) is the centre of $[(x^2 / a^2) - (y^2 / b^2)] = 1$

Note:

You will notice that the results for ellipse (https://byjus.com/jee/ellipse/) are also applicable for a hyperbola. You need to replace b^2 by $(-b^2)$

Practice Problems on Hyperbola

Example 1:

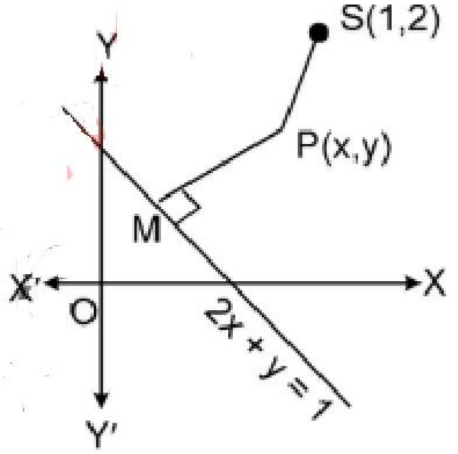
Find the equation of the hyperbola whose directrix is 2x + y = 1, focus (1, 2) and eccentricity √3.

Solution:

Let P(x, y) be any point on the hyperbola.

Draw PM perpendicular from P on the directrix, Then by definition SP=ePM.

$\Rightarrow (SP)^2 = e^2 (PM)^2$



$\Rightarrow (x - 1)^2 + (y - 2)^2 = 3\{(2x + y - 1) / \sqrt{(4+1)}\}^2$

$\Rightarrow 5 (x^2 + y^2 - 2x - 4y + 5)$

$= 3 (4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

Which is the required hyperbola.

Example 2:

Find the eccentricity of the hyperbola whose latus rectum is half of its transverse axis.

Solution:

Let the equation of hyperbola be $[(x^2 / a^2) - (y^2 / b^2)] = 1$

Then transverse axis = 2a and latus – rectum = $(2b^2 / a)$

According to question $(2b^2 / a) = (1/2) \times 2a$

$\Rightarrow 2b^2 = a^2$ (Since, $b^2 = a^2 (e^2 - 1)$)

$\Rightarrow 2a^2 (e^2 - 1) = a^2$

$\Rightarrow 2e^2 - 2 = 1$

$\Rightarrow e^2 = (3 / 2)$

$\therefore e = \sqrt{(3/2)}$

Hence the required eccentricity is $\sqrt{(3/2)}$

What is Conjugate Hyperbola?

2 hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & transverse axis of the other are called **conjugate hyperbola** of each other.

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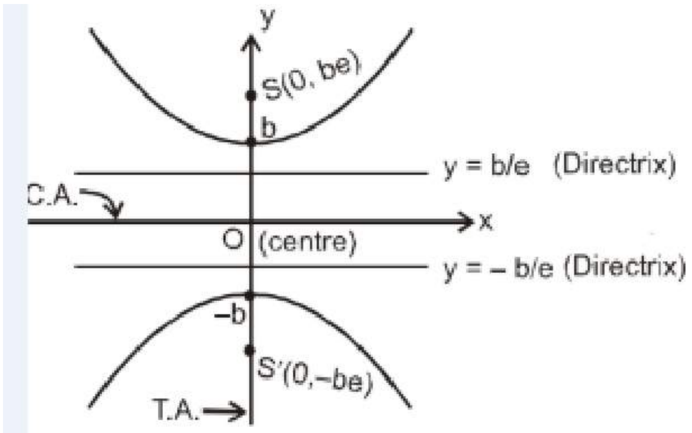
$(x^2 / a^2) - y^2 b^2 = 1$ & $(-x^2 / a^2) + (y^2 / b^2) = 1$ are conjugate hyperbolas of each other.

$(y^2 / b^2) - (x^2 / a^2) = 1$

$a^2 = b^2 (e^2 - 1)$

$$a^2 = b^2\left(e^2 - 1\right) \Rightarrow e = \sqrt{1 + \frac{a^2}{b^2}}$$

Vertices: Vertices: $(0,\pm b)$ L.R. = $(2a^2 / b)$



Some Important Conclusions on Conjugate Hyperbola

(a) If are eccentricities of the hyperbola & its conjugate, the

$(1 / e_1^2) + (1 / e_2^2) = 1$

(b) The foci of a hyperbola & its conjugate are concyclic & form the vertices of a square (https://byjus.com/maths/square/).

(c) 2 hyperbolas are similar if they have the same eccentricities.

(d) 2 similar hyperbolas are equal if they have the same latus rectum.

Example 3:

Find the lengths of transverse axis and conjugate axis, eccentricity, the co-ordinates of foci, vertices, length of the latus-rectum and equations of the directrices of the following hyperbola $16x^2 - 9y^2 = -144$.

Solution:

The equation $16x^2 - 9y^2 = -144$ can be written as $(x^2 / 9) - (y^2 / 16) = -1$

This is of the form $(x^2 / a^2) - (y^2 / b^2) = -1$

$\therefore a^2 = 9, b^2 = 16$

$\Rightarrow a=3, b=4$

Length of transverse axis: The length of transverse axis = 2b = 8

Length of conjugate axis: The length of conjugate axis = 2a = 6

Eccentricity:

$$e = \sqrt{\left(1 + \frac{a^2}{b^2}\right)} = \sqrt{\left(1 + \frac{9}{16}\right)} = \frac{5}{4}$$

Foci: The co-ordinates of the foci are (0, ± be) i.e., (0, ± 5)

Vertices: The co-ordinates of the vertices are (0, ± b) i.e. (0, ± 4)

Lengths of latus-rectum: The length of latus-rectum = $(2a^2 / b) = [2(3)^2] / 4 = 9/2$

Equation of directrices: The equation of directrices are

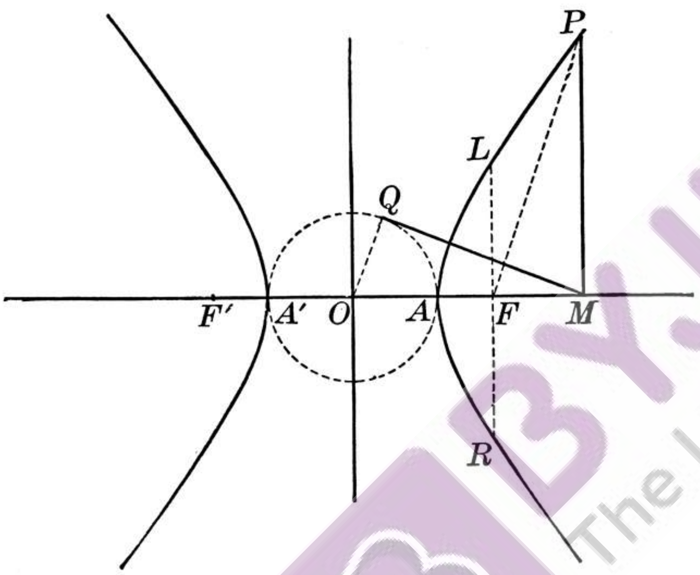
y = ± (b/e) ⇒y = ± [4 / (5/4)] ⇒ y ± (16 / 5)

Auxiliary Circles of the Hyperbola

A circle (https://byjus.com/maths/circles/) drawn with centre C & transverse axis as a diameter is called the **auxiliary circle of the hyperbola.** The auxilary circle of hyperbola equation is given as:

Equation of the auxiliary circle is $x^2 + y^2 = a^2$,

Note from the following figure that P & Q are called the "corresponding points" of the hyperbola & the auxiliary circle.



Parametric Representation:

x = a sec θ and y = b tan θ

i.e. If (a sec θ, b tan θ) is on the hyperbola, then Q : (a cos θ, a sin θ) lies on the auxiliary circle. The equation of chord joining 2 points P (α) and Q (f) is given by:

$$\frac{x}{a}\left(\frac{\alpha-\beta}{2}\right) - \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha+\beta}{2}\right)$$

The position of a point P

$S_1 = x_1^2/a^2 + y^2/b^2 = 1$ is positive, zero or negative accordingly as (x_1, y_1) lies inside, on or outside is positive, zero or negative accordingly as (x_1, y_1) lies inside, on or outside.

Example 4: Find the position of the point (5, -4) relative to the hyperbola $9x^2 - y^2 = 1$.

Solution:

Since $9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 > 0$, So, the point (5, -4) lies inside the hyperbola $9x^2 - y^2 = 1$.

Rectangular Hyperbola

The rectangular hyperbola is a hyperbola axes (or asymptotes) are perpendicular, or with its eccentricity is √2. Hyperbola with conjugate axis = transverse axis is a = b example of rectangular hyperbola.

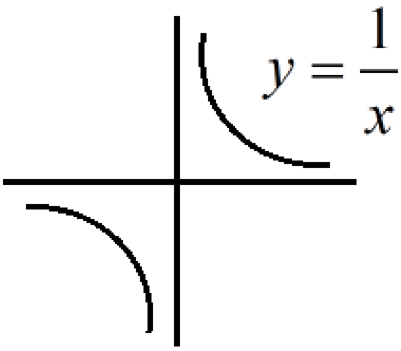
$x^2/a^2 - y^2/b^2$

$\Rightarrow x^2/a^2 - y^2/a^2 = 1$

Or, $x^2 - y^2 = a^2$

We know $b^2 = a^2 (e^2 - 1) a^2$

$= a^2 (e^2 - 1) e^2 = 2e = \sqrt{2}$.



Eccentricity of rectangular hyperbola

Also, xy = c

e.g. xy = 1, y = 1/x

Tangent of Rectangular hyperbola

The tangent of a rectangular hyperbola is a line that touches a point on the rectangular hyperbola's curve. The equation and slope form of a rectangular hyperbola's tangent is given as:

Equation of tangent

The y = mx + c write hyperbola $x^2/a^2 - y^2/b^2 = 1$ will be tangent if $c^2 = a^2/m^2 - b^2$.

Slope form of tangent

$y = mx \pm \sqrt{(a^2m^2 - b^2)}$

Secant

Secant will cut ellipse at 2 distinct points

$\Rightarrow c^2 > a^2 m^2 - b^2$

Neither Secant Nor Tangent

For line to be neither secant nor tangent, quadratic equation will give imaginary solution.

$\Rightarrow c^2 < a^2 m^2 - b^2$

Equation of tangent to hyperbola $x^2 /a^2 - y^2/b^2 = 1$ at point (x_1 ,y_1)

$= (xx_1)/a2 = (yy_1)/b_1 = 1$

Parametric form of tangent:

$(x \sec\theta)/a - (y \tan\theta)/b = 1$

Point of contact and examples on tangent

Compare:

$y = mx + c$

$(xx_1)/a^2 - (yy_1)/b^2 = 1 - mx + y = c$

$x_1 = (-a^2c)/m;$

$y_1 = -b^2/c$

$(x_1,y_1) = [(-a^2m)/c, -b^2/c]$

Solved Examples on Hyperbola

Example 5: Show that the line x cos α + y sin α = p touches the hyperbola $x^2/a^2 - y^2/b^2 = 1$. If $a^2 \cos^2 \alpha - b^2 \sin^2 \alpha = p^2$.

Solution:

Given line is $x \cos \alpha + y \sin\alpha = p$

$y = -x \cot \alpha + p \operatorname{cosec} \alpha$

Condition for y = mx + c to be tangent of hyperbola $x^2/a^2 - y^2/b^2 = 1$.

$c^2 = a^2/m^2 - b^2$

$p^2 \operatorname{cosec}2\alpha = \alpha^2 \cot2\alpha - b^2$

$p^2 = a^2 \cos^2\alpha - b^2 \sin^2 \alpha$

Director circle:

Locus of point of intersection of ⊥r tangents.

Director circle for $x^2/a^2 - y^2/b^2 = 1$ is

$x^2 + y^2 = a^2 - b^2$

Example 6: Find the equation of tangent to hyperbola x29−y2=1 whose slope is 5

Solution:

Slope of tangent m = 5, $a^2 = 9$, $b^2 = 1$

Equation of tangent in slope form is

$y = mx \pm \sqrt{(a^2m^2 - b^2)}$

$y = 5x \pm \sqrt{(9.5^2 - 1)}$

$y = 5x \pm 4\sqrt{14}$

[Note: For ellipse, director circle is $x^2 + y^2 = a^2 + b^2$, $x^2/a^2 + y^2/b^2 = 1$]

Normal:

Equation of normal of $x^2/a^2 - y^2/b^2 = 1$ at (x_1, y_1)

$a^2x/x_1 + b^2y/y_1 = a^2 + b^2$

Normal in parametric form:

$ax/\sec\theta + by/\tan\theta = a^2 + b^2$

Example 7: Find normal at the point (6, 3) to hyperbola $x^2/18 - y^2/9 = 1$

Solution:

Equation of Normal at point (x_1, y_1) is $a^2 = 18$, $b^2 = 9$

$a^2x/x_1 + b^2y/y_1 = a^2 + b^2$

Equation of Normal at point (6, 3) is

$18x/6 + 9y/3 = 18 + 9$

$x + y = 9$

Chord of contact:

$T = 0$

$xx1/a^2 - yy1/b^2 = 1$

Example 8: Find equation of chord of Contact of point (2, 3) to hyperbola $x^2/16 - y^2/9 = 1$

Solution:

Equation of chord of Contact is T = 0

i.e. $(xx_1)/a^2 - (yy_1)/b^2 - 1 = 0$

Or, $2x/16 - 3y/9 = 1$

Or, $x/8 - y/3 = 1$

Equation of chord when mid-point is given

$T = (xx_1)/a^2 - (yy_1)/b^2 - 1 = x_1^2/a^2 - y_1^2/b^2 - 1.$

Example 9: Find the equation of chord of hyperbola $x^2/9 - y^2/4 = 1$ whose midpoint is (5, 1).

Solution:

Equation of chord of the hyperbola whose mid points is (5, 1)

$T = (xx_1)/a^2 - (yy_1)/b^2 - 1 = x_1^2/a^2 - y_1^2/b^2 - 1$

$5x/9 - y/4 - 1 = 25/9 - ¼ - 1$

$5x/9 - y/4 = 91/36$