MAHARASHTRA BOARD CLASS 12
MATHS ANSWERS

Answers & Explanations

Section – I

Q1. (A) (i) Answer: d

Solution: Since $A - A^T = 0$, A is a symmetric matrix of order 2×2

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$[3x^2 + 10xy + 5y^2] = [x \quad y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$[3x^2 + 10xy + 5y^2] = [ax + by \quad bx + cy]\begin{bmatrix} x \\ y \end{bmatrix}$

$[3x^2 + 10xy + 5y^2] = [(ax + by)x + (bx + cy)y]$

$[3x^2 + 10xy + 5y^2] = [ax^2 + 2bxy + cy^2]$

Comparing the coefficients, $a = 3, 2b = 10, c = 5$

$\therefore A = \begin{bmatrix} 3 & 5 \\ 5 & 5 \end{bmatrix}$

(ii) Answer: c

Solution:

The straight line perpendicular to $2x + y = 3$ is $2y - x + C = 0$

Since it passes through (1, 1)

$2(1) - 1 + C = 0$

$C = -1$

Line is $2y - x - 1 = 0$

So y-intercept $= \frac{1}{2}$

(iii) Answer: c

Solution: Let $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, comparing the coefficients of $\hat{\imath},\ \hat{\jmath},\ \hat{k}$

$x = 1 + s - t$ ……(1)

$y = 2 - s$ ……(2)

$z = 3 - 2s + 2t$ ……(3)

Eliminating s and t from above equations, we get

Cartesian Equation of plane as $2x + z = 5$.

B. (i) (a) Here we have a biconditional statement,

If and only if both statements “Sandhya comes to the theatre” and “Poornima does not come to the theatre” are true, then the statement “Kalai comes to the theatre” is true.

Both statements r = “Sandhya comes to the theatre” and

Negation of p = “Poornima comes to the theatre” together forms one implicant

Here we use the logical connective “AND” and Negation

The statement s = “Kalai comes to the theatre” forms another implicant.

So the symbolic form is $s \leftrightarrow (r \wedge \sim p)$

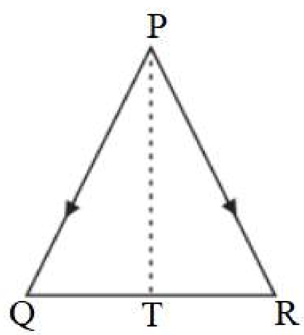
(b) Here there are two conditional statements

If Kalai does not come, Sandhya comes

The symbolic form of the statement is $\sim s \rightarrow r$
If Kalai comes, Lalitha does not come
The symbolic form of the statement is $s \rightarrow \sim q$
The logical connective AND of the above two statements gives the final symbolic form of the original statement is $(\sim s \rightarrow r) \wedge (s \rightarrow \sim q)$

(ii) Solution:
$\cos X = \frac{3}{5}$
$\cos Y = \frac{4}{5}$
$\sin X = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}$
$\sin Y = \frac{3}{5}$
As X and Y lies in fourth quadrant, sin X and sin Y has to be negative
$\sin X = -\frac{4}{5}$
$\sin Y = -\frac{3}{5}$
$2\sin X + 4\sin Y = -\frac{8}{5} - \frac{12}{5} = -\frac{20}{5} = -4$

(iii) Solution:



So the vectors of the three sides will be equal to 0
$PQ + QR + PR = 0$
$QR = PR - PQ$
$QT = \frac{PR - PQ}{2}$
Also,
$PQ + QT + TP = 0$
$PQ + \frac{PQ - PQ}{2} = PT$
$PT = \frac{3\hat{\imath} + 4\hat{k} + 5\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}{2}$
$PT = 4\hat{\imath} - \hat{\jmath} + 4\hat{k}$
$|PT| = \sqrt{4^2 + 1^2 + 4^2} = \sqrt{33}$
The length of the median PT is $\sqrt{33}$

(iv) Solution:
For co-planarity, the determinant of the 3 × 3 matrix formed by the coefficients of the three vectors should be zero.

$$\begin{vmatrix} -17 & -12 & 1 \\ 5 & \lambda & 1 \\ -6 & -11 & 1 \end{vmatrix} = 0$$

$5(-12 + 11) - \lambda(-17 + 6) + 1(187 - 72) = 0$
$-5 + 11\lambda + 115 = 0$
$11\lambda = -110$
$\Rightarrow \lambda = -10$

(v) We have to find a point on the first plane $x + y + 2z - 2 = 0$
Let $x = y = 0$
$2z = 2$
$z = 1$
(0, 0, 1) is on $x + y + 2z - 2 = 0$
Distance from point (0, 0, 1) to $x + y + 2z - 5 = 0$

$$d = \frac{|(1)(0) + (1)(0) + (2)(1) - 5|}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{2}}$$

Q2. (A) (i) Answer:
$3\pi < 4\beta < 4\pi$
$\frac{3\pi}{4} < \frac{4\beta}{4} < \frac{4\pi}{4}$
$\frac{3\pi}{4} < \beta < \pi$
$\sqrt{2cot\beta + \frac{1}{\sin^2 \beta}} = p - cot\beta$
$2cot\beta + cosec^2\beta = p^2 + \cot^2\beta - 2pcot\beta$
$2cot\beta + 1 + \cot^2\beta = p^2 + \cot^2\beta - 2pcot\beta$
$2(1 + p)cot\beta = (p^2 - 1)$
$cot\beta = \frac{p-1}{2}$
Since β lies in the second quadrant, so sine will be positive and cosine will be negative, so cot will also be negative
$\frac{3\pi}{4} < \beta < \pi$
$-\infty < cot\beta < -1$
$-\infty < \frac{p-1}{2} < -1$
$-\infty < p < -1$
Thus, the range of p is $-\infty < p < -1$.

(ii) Answer:
Let the equation of the line be $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} = k$
where (a_1, a_2, a_3) is a point on the line and $(b_1\ b_2, b_3)$ is the direction ratio of the line.

But given that this line passes through origin
$(a_1, a_2, a_3) = (0,0,0)$
Then the equation of the line
$\frac{x-0}{b_1} = \frac{y-0}{b_2} = \frac{z-0}{b_3} = k$ …… (1)
The equation of the other line is
$x - 2y - z + 3 = 0$ which can be rewritten as
$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z-0}{1} = j$ …… (2)
Let the point of intersection of the above two lines (1) and (2) be P (2j+3, j+3, j)
Direction ratio of the line segment from origin to point P is (2j+3, j+3, j)
Since line (i) also passes through the origin
Direction ratio $d_1 = (b_1, b_2, b_3) = (2\mathrm{j}+3, \mathrm{j}+3, \mathrm{j})$
Direction ratio $d_2 = (2,1,1)$
$d_1 d_2 = 2(2j+3) + 1(j+3) + 1.j = 6j + 9$
$|d_1| = \sqrt{(2j+3)^2 + (j+3)^2 + j^2} = \sqrt{6j^2 + 18j + 18}$
$|d_2| = \sqrt{4+1+1} = \sqrt{6}$
$\cos 60 = \left(\frac{d_1 d_2}{|d_1||d_2|}\right)$
$\frac{1}{2} = \frac{6j+9}{\sqrt{6j^2 + 18j + 18}\sqrt{6}}$
Solving for j, we get j=-1,-2

$d_1 = (1,2,-1)$ or $(-1,1,-2)$

(iii) Answer:
Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
We get
$a = 1, h = -2, b = 4, g = -\frac{3}{2}, f = 3, c = -10$

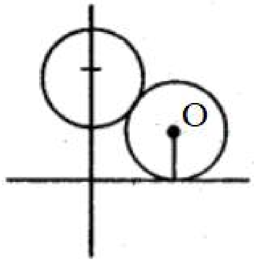
Consider
$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & -2 & -\frac{3}{2} \\ -2 & 4 & 3 \\ -\frac{3}{2} & 3 & -10 \end{vmatrix}$$
$= 1(-40-9) + 2\left(20 + \frac{9}{2}\right) - \frac{3}{2}(-6+6) = -49 + 49 + 0 = 0$
Hence the given equation represents a pair of lines.
The angle between the two lines is given
$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{4-4}}{1+4}\right| = 0$
Hence the two lines are parallel and they do no intersect.

(B) (i) Answer:
As the three points $A(p, 2-2p), B(-p+1, 2p), C(-4-p, 6-2p)$ are said to be collinear, we can form the 3×3 determinant where the first column is the x's for all the points, the second column is the y's for all the points, and the last column is all ones
Then equate that determinant to zero to get the value of p.

$$\begin{vmatrix} p & 2-2p & 1 \\ -p+1 & 2p & 1 \\ -4-p & 6-2p & 1 \end{vmatrix} = 0$$
$p((2p)(1)-(6-2p)(1)) - (-p+1)((2-2p)(1)-(6-2p)(1)) + (-4-p)((2-2p)(1)-(2p)(1)) = 0$
$p(2p-6+2p) - (-p+1)(2-2p-6+2p) + (-4-p)(2-2p-2p) = 0$
$p(4p-6) + (p-1)(-4) + (-4-p)(2-4p) = 0$
$4p^2 - 6p - 4p + 4 - 8 + 14p + 4p^2 = 0$
$8p^2 + 4p - 4 = 0$
$2p^2 + p - 1 = 0$
$(2p-1)(p+1) = 0$
$p = \frac{1}{2}, -1$
Substitute $p = \frac{1}{2}$ in each of the coordinates of A, B and C
We get $A\left(\frac{1}{2}, 1\right)$ and $B\left(\frac{1}{2}, 1\right)$
The coordinates of A and B are the same. So the points overlap.
Hence p cannot be $\frac{1}{2}$. So substitute p = -1 in given three points.
$A(p, 2-2p) = A\,(-1, 4)$
$B(-p+1, 2p) = B\,(2, -2)$
$C(-4-p, 6-2p) = C\,(-3, 8)$
Thus the required coordinates are $A\,(-1,4)$, $B\,(2,-2), C\,(-3,8)$

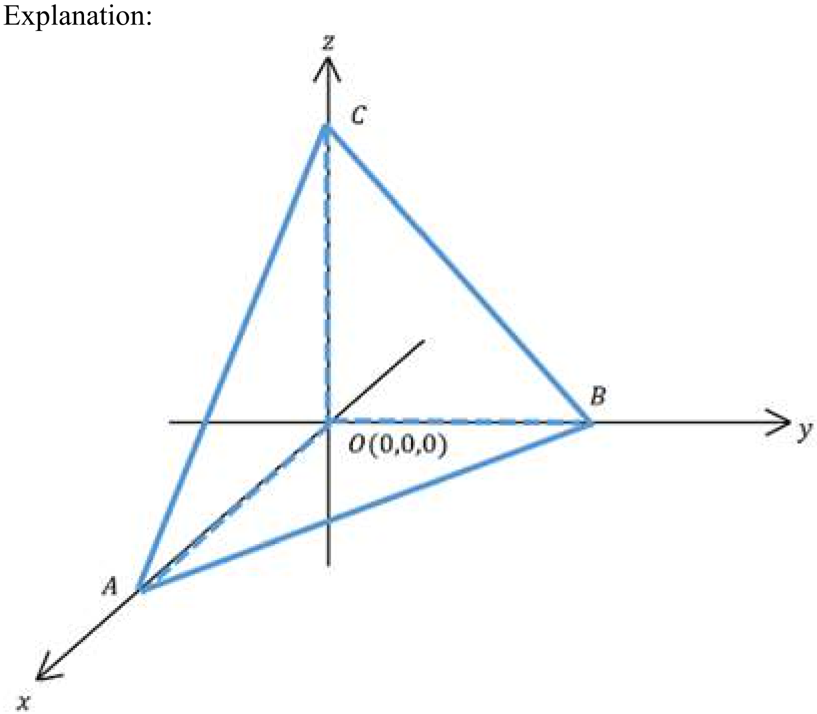
(ii) Answer:
$$\cot\left(\sum_{k=1}^{29} cot^{-1}(1+\sum_{m=1}^{k} 2m)\right) = \cot\left(\sum_{k=1}^{29} cot^{-1}\left(1+2\times\frac{k(k+1)}{2}\right)\right) = \cot\left(\sum_{k=1}^{29} cot^{-1}(1+k(k+1))\right)$$
$$= \cot\left(\sum_{k=1}^{29} tan^{-1}\frac{1}{1+k(k+1)}\right)$$
As we know that $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$
So,
$$\cot\left(\sum_{k=1}^{29} tan^{-1}\frac{1}{1+k(k+1)}\right) = \cot\left(\sum_{k=1}^{29} \tan^{-1}(k+1) - \tan^{-1}k\right)$$
On expansion we get
$$\cot(\tan^{-1}2 - \tan^{-1}1 + \tan^{-1}3 - \tan^{-1}2 + \tan^{-1}4 - \tan^{-1}3 + \cdots. + \tan^{-1}29 - \tan^{-1}28 + \tan^{-1}30 - \tan^{-1}29)$$
$$\cot(-\tan^{-1}1 + \tan^{-1}30) = \cot\left(\tan^{-1}\frac{30-1}{1+30\times 1}\right) = \frac{1}{\tan\left(tan^{-1}\frac{29}{31}\right)} = \frac{31}{29}$$

(iii) Answer:
Let us first draw the diagram as follows:



First let us determine the centre and radius of the circle whose equation is $x^2 + y^2 - 6y + 5 = 0$
Rewriting the equation as
$x^2 + y^2 - 6y + 9 - 4 = 0$
$x^2 + (y-3)^2 = 2^2$
Hence the centre is (0, 3) and radius is 2
Let the locus of the point O of the variable circle be (a, b)
Given that tangent to the variable circle is x-axis. So it touches the x-axis.
Hence equation of the circle is $(x-a)^2 + (y-b)^2 = b^2$
As the circles touch externally,
$\sqrt{a^2 + (3-b)^2} = 2 + b$
$a^2 + (b-3)^2 = b^2 + 4b + 4$
$a^2 = 10\left(b - \frac{1}{2}\right)$
So locus is $x^2 = 10\left(y - \frac{1}{2}\right)$ which is a parabola and it's vertex is $\left(0, \frac{1}{2}\right)$

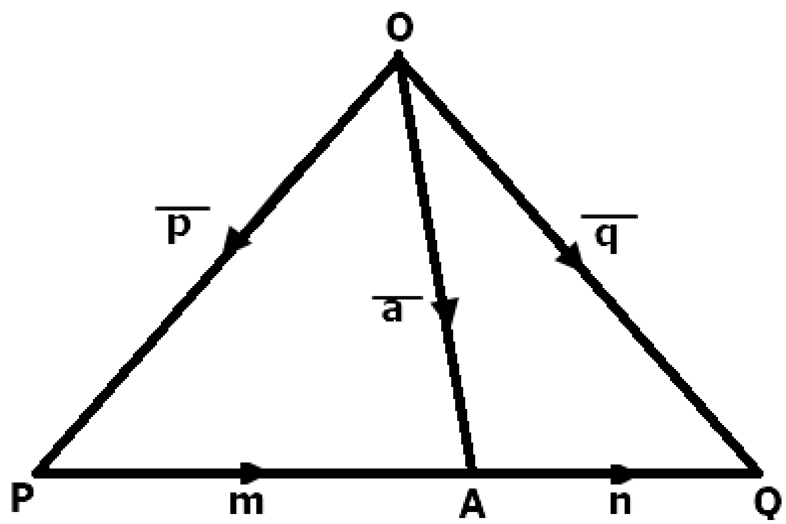
Q3. (A)
(i) Answer:
Explanation:



The four vertices of the tetrahedron would be obtained by plotting the points where the given plane 2x+4y+6z=8 intersects the axes
When it intersects x-axis where y = 0 and z = 0,

So, 2x = 8 which implies x = 4. This is the point (4, 0, 0).
Similarly it goes through (0, 2, 0) and (0, 0, $\frac{4}{3}$).
Hence four vertices of the tetrahedron are (0, 0, 0), (4, 0, 0), (0, 2, 0) and (0, 0, $\frac{4}{3}$).
Let $a = 0i + 0j + 0k$, $b = 4i + 0j + 0k$, $c = 0i + 2j + 0k$, $d = 0i + 0j + \frac{4}{3}k$
Now OA = $b - a = 4i$
OB=$c - a = 2j$
OC=$d - a = \frac{4}{3}k$
Volume=$\frac{1}{6}|OA(OB \times OC)|$=$\frac{1}{6}\begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{4}{3} \end{vmatrix} = \frac{1}{6}\left[4\left(2 \times \frac{4}{3}\right)\right] = \frac{16}{9}$
Volume of tetrahedron so formed is $\frac{16}{9}$.

(ii) Answer:
a) The line segment PQ with point A dividing it in the ration m : n is shown in the figure



$\overline{OP}, \overline{OQ}$ and $\overline{OA}$ are the position vectors of the points P, Q and A respectively.
Given that
$\frac{PA}{AQ} = \frac{m}{n}$
$nPA = mAQ$
PA and AQ are in the same direction.
So $n(\overline{PA}) = m(\overline{AQ})$

$n(\bar{a} - \bar{p}) = m(\bar{q} - \bar{a})$
$n\bar{a} - n\bar{p} = m\bar{q} - m\bar{a}$
On solving the above equation we get
$\bar{a} = \frac{m\bar{q} + n\bar{p}}{m + n}$
This is the section formula for internal division
Hence proved.

b) The position vector of point P(1,-2,1) is given as $\hat{\imath} - 2\hat{\jmath} + \hat{k}$
The position vector of point Q(1,4,-2) is given as $\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$
Given $\frac{m}{n} = \frac{2}{1}$
So from the proved result in (a), we get

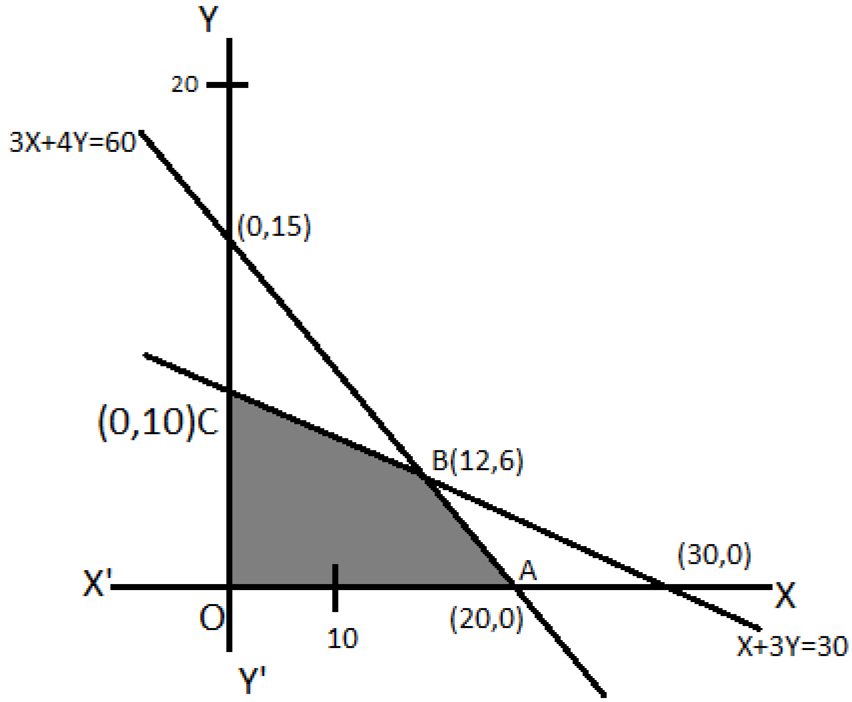
$$\bar{a} = \frac{m\bar{q} + n\bar{p}}{m+n}$$

$$\bar{a} = \frac{2(\hat{\imath} + 4\hat{\jmath} - 2\hat{k}) + 1(\hat{\imath} - 2\hat{\jmath} + \hat{k})}{2+1}$$

$$\bar{a} = \frac{3\hat{\imath} + 6\hat{\jmath} - 3\hat{k}}{3} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$$

The position vector of A is $\bar{a} = \hat{\imath} + 2\hat{\jmath} - \hat{k}$

(iii) Solution:
Let us consider x is the number of pieces of model A and y is the number of pieces of model B.
Then, the total profit becomes,
Total profit $(P) = 4000x + 6000y$ (in Rs)
Now we have the following mathematical model for the given problem.
Maximize $P = 4000x + 6000y \quad \rightarrow (1)$
And the subject to the constraints:
$6x + 8y \leq 120$ (Fabricating constant)
i.e. $3x + 4y \leq 60 \quad \rightarrow \quad (2)$
$2x + 6y \leq 60$ (Finishing constraint)
$x + 3y \leq 30 \quad \rightarrow \quad (3)$
$x \geq 0, y \geq 0$ (Non-negative constraint) $\longrightarrow \quad (4)$
The shaded region OABC resolved by the linear inequalities (2) to (4) is shown in figure.



Let's calculate the function Z

Corner Point	$P = 4000x + 6000y$
$0(0,0)$	0
$A(20,0)$	80000
$B(12,6)$	84000 ⟶(MAXIMUM)
$C(0,10)$	60000

We find that the maximum value of P is 84000 at B (12, 6). The company should be manufactured 12 pieces of Model A and 6 pieces of Model B to make a maximum profit, which will be Rs.8400.

(B) (i) Answer:
Given that we assume the center of the ellipse at (0, 0) with the major axis along the x axis.
The semi major axis is $a = \frac{7.4 + 4.4}{2} = 5.9$
This will be the value of 'a' in the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The minor axis b is given by $b^2 = a^2 - c^2$, where c is the distance of either focus from the middle, which in this case is 5.9 - 4.4 = 1.5.
That means $b^2 = 32.56$; b = 5.71
The equation of the orbit of the planet X:

$$\left(\frac{x}{5.9}\right)^2 + \left(\frac{y}{5.71}\right)^2 = 1$$

(iii) Answer:
Given P(3,0,2), Q(4,3,0) and R(8,1,-1)
Given $\overrightarrow{OP} = \bar{p} = 3\hat{\imath} + 2\hat{k},$
$\overrightarrow{OQ} = \bar{q} = 4\hat{\imath} + 3\hat{\jmath},$
$\overrightarrow{OR} = \bar{r} = 8\hat{\imath} + \hat{\jmath} - \hat{k},$
$\overrightarrow{PQ} = \bar{q} - \bar{p} = 4\hat{\imath} + 3\hat{\jmath} - (3\hat{\imath} + 2\hat{k}) = \hat{\imath} + 3\hat{\jmath} - 2\hat{k}$
$\overrightarrow{QR} = \bar{r} - \bar{q} = 8\hat{\imath} + \hat{\jmath} - \hat{k} - (4\hat{\imath} + 3\hat{\jmath}) = 4\hat{\imath} - 2\hat{\jmath} - \hat{k}$
$\overrightarrow{RP} = \bar{p} - \bar{r} = 3\hat{\imath} + 2\hat{k} - (8\hat{\imath} + \hat{\jmath} - \hat{k}) = -5\hat{\imath} - \hat{\jmath} + 3\hat{k}$
$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \hat{\imath} + 3\hat{\jmath} - 2\hat{k} + 4\hat{\imath} - 2\hat{\jmath} - \hat{k} + (-5\hat{\imath} - \hat{\jmath} + 3\hat{k}) = 0$
Hence, Triangle law is satisfied.
Thus $\bar{p}, \bar{q}$ *and* $\bar{r}$ form a triangle.
$|\overrightarrow{PQ}| = \sqrt{1^2 + 3^2 + (-2)^2} = \sqrt{14}$
$|\overrightarrow{QR}| = \sqrt{4^2 + (-2)^2 + (-1)^2} = \sqrt{21}$
$|\overrightarrow{RP}| = \sqrt{(-5)^2 + (-1)^2 + 3^2} = \sqrt{35}$
$|\overrightarrow{PQ}|^2 + |\overrightarrow{QR}|^2 = 21 + 14 = 35 = |\overrightarrow{RP}|^2$
Hence, Pythagoras theorem satisfied.
Thus $\bar{p}, \bar{q}$ *and* $\bar{r}$ form a right triangle.

(iii) Solution: Suppose D divides BC in the ratio k:1, then $D = \left(\frac{2k}{k+1}, \frac{-3k-11}{k+1}, \frac{k+4}{k+1}\right)$
Direction ratios of AD are $\left(\frac{k-1}{k+1}, \frac{-11k-19}{k+1}, \frac{-3k}{k+1}\right)$
Direction ratios of BC are $(2, 8, -3)$
AD is perpendicular to BC,
$$2\left(\frac{k-1}{k+1}\right) + 8\left(\frac{-11k-19}{k+1}\right) - 3\left(\frac{-3k}{k+1}\right) = 0$$
$77k + 154 = 0, \quad k = -2$
$\therefore D = (4, 5, -2)$

SECTION – II
Q4. (A) (i) Answer: (a)
Rewriting the given equation $v = u\frac{dv}{du} + \frac{2}{\frac{dv}{du}}$

We get
$$v\frac{dv}{du} = u\left(\frac{dv}{du}\right)^2 + 2$$
Order = 1, degree = 2

(ii) Answer:
Since $f(-\theta) = f(\theta)$
Therefore, $f(\theta)$ is an even function
Now $I_a = \int_0^{\pi} f(\cos\theta)d\theta$
$$I_a = \int_0^{\pi} f(\cos(\pi - \theta)d\theta = \int_0^{\pi} f(-\cos\theta)d\theta = \int_0^{\pi} f(\cos\theta)d\theta$$
$$I_a = 2\int_0^{\frac{\pi}{2}} f(\cos\theta)d\theta = 2\int_0^{\frac{\pi}{2}} f(\cos(\frac{\pi}{2} - \theta)d\theta$$

$= 2\int_0^{\frac{\pi}{2}} f(\sin\theta)d\theta = 2I_b$

$\frac{I_a}{I_b} = 2$

Option (b) is correct.

(iii) Answer: (a)

Given that Mean = 20

$\lambda = \frac{1}{20} = 0.05$

We know that $P(X < x) = 1 - e^{-\lambda x}$

$P(15 < X < 25) = P(X < 25) - P(X < 15)$

$= \left[1 - e^{-0.05(25)}\right] - \left[1 - e^{-0.05(15)}\right]$

$= 0.7135 - 0.5276 = 0.1859$

So, Option (a) is correct

(B) (i) Answer:

$v = g(t^2 + 2)$

$v' = g'(t^2 + 2) \times 2t$

$v' = g'(1^2 + 2) \times 2 \times 1$

$= 2g'(3) = 2 \times 5 = 10$

(ii) Answer:

All that we're really being asked to do here is to maximize the profit subject to the constraint that x must be in the range $0 \le x \le 125$

First, we'll need the derivative and the critical point(s) that fall in the range $0 \le x \le 125$

$P'(x) = -8x + 800$

$-8x + 800 = 0$

$x = 100$

Since the profit function is continuous and we have an interval with finite bounds we can find the maximum value by simply plugging in the only critical point that we have (which nicely enough in the range of acceptable answers) and the end points of the range.

$P(0) = -20000$

$P(125) = 17500$

$P(100) = 20000$

So, it looks like they will generate the most profit if they only rent out 100 of the apartments instead of all 125 of them.

(iii) Answer: $\int \sqrt{(1 + cos\alpha)}d\alpha$

$= \int \sqrt{\left(2\cos^2\frac{\alpha}{2}\right)}d\alpha$

$= \sqrt{2}\int \left(\cos\frac{\alpha}{2}\right)d\alpha$

$= \frac{\sqrt{2}\sin\frac{\alpha}{2}}{\frac{1}{2}} + C$

$= 2\sqrt{2}\sin\frac{\alpha}{2} + C$

(iv) Answer:
Let X be the random variable that denotes "length of the rod in the unit of centimetres" where Y~u[7,10]
Let Y be the random variable that "Number of rods shorter than 0.076 m"
0.076 m = 7.6 cm
Where Y~B(6,0.2)
Probability of winning a clock = P(Y>4)
$1 - P(Y \le 4) = 1 - 0.9984 = 0.0016$
(v) Answer:
$\sqrt{p} = qe^{\beta \cot\theta}$
Taking logarithms on both sides
$\frac{1}{2}\log p = \log q + \beta \cot\theta$
Differentiating we get
$\frac{1}{2p}\frac{dp}{d\beta} = \cot\theta$
$\frac{dp}{d\beta} = 2p\cot\theta$ .....(1)
$\frac{d^2p}{d\beta^2} = \frac{2dp}{d\beta}\cot\theta$ ......(2)
Using (1) and (2)
$\frac{d^2p}{d\beta^2} = \frac{2dp}{d\beta}\cot\theta = 2 \times 2p\cot\theta \times \cot\theta = 4p\cot^2\theta$
So $\frac{d^2p}{d\beta^2} - 4pcot^2\theta = 0$
Thus the value of $\frac{d^2p}{d\beta^2} - 4pcot^2\theta$ is 0

Q5. (A) (i) Answer:
We note that the functions are continuous on their domains, so we check that the left- and right-hand limits agree at the boundary t-values. At t = 0,
$\lim_{t\to0^-} f(t) = \lim_{t\to0^-} me^t + n + 1 = m + n + 1$
$\lim_{t\to0^+} f(t) = \lim_{t\to0^+} mt^2 + n(t+3) = 3n$
So $m + n + 1 = 3n$ and $m = 2n - 1$ ..... (1)
Next at $t = 1$
$\lim_{t\to1^-} f(t) = \lim_{t\to1^-} mt^2 + n(t+3) = m + 4n$
$\lim_{t\to1^+} f(t) = \lim_{t\to1^+} mcos(\pi t) + 7nt = -m + 7n$
So $m + 4n = -m + 7n$ and $2m = 3n$ ..... (2)
Solving (1) and (2), we get $n = 2, m = 3$

(ii) Answer:
$t = e^{\theta + e^{\theta + e^{\theta + \cdots}}} = e^{t+\theta}$
Hence $t = e^{t+\theta}$
Differentiating both sides
$1 = e^{t+\theta}[t' + 1]$
$e^{-t-\theta} - 1 = t'$

$\ln\theta = t + \theta$
$e^{-\ln\theta} - 1 = t'$
$t = \ln\theta - \theta$
$\theta^{-1} - 1 = t'$
$\frac{1}{\theta} - 1 = t'$
Hence $t' = \frac{dt}{d\theta} = \frac{1-\theta}{\theta}$

(iii) Answer:

$$\int_0^1 \left(\frac{dt}{1+t+t^2}\right) = \int_0^1 \left(\frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2x+1}{\sqrt{3}}\right]_0^1$$

$$\frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right] = \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} \times \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

(B) (i) **Solution:**
Differentiate $x^2 + 2xy - 3y^2 = 0$ w.r.t. x

$$2x + 2\left[y + x\frac{dy}{dx}\right] - 6y\frac{dy}{dx} = 0$$

$$(3y - x)\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} = \frac{x+y}{3y-x}$$

Slope of tangent at (1, 1) is $\left(\frac{dy}{dx}\right)_{(1,1)} = 1$

Slope of normal at (1,1) is -1
Equation normal is $y = -x + 2$
Solving normal and curve we get,
$x^2 + 2x(2-x) - 3(2-x)^2 = 0$
$x^2 - 4x + 3 = 0$
$x = 1, x = 3$
Substitute x=3 in normal, we get $y = -1$
$\therefore (3, -1)$ is the other point which is in 4th quadrant where the normal cuts the curve again.

(ii) **Solution:** $\int \frac{x^2-1}{x^4+1}\, dx = \int \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}\right)}\, dx = \int \frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2 - 2}\, dx$

Let $x + \frac{1}{x} = t$
$\left(1 - \frac{1}{x^2}\right) dx = dt$

$= \int \frac{dt}{t^2-\sqrt{2}^2} = \frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c$

$= \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + c = \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + c.$

(iii) **Solution:** Let $k = \lim\limits_{n\to\infty}\left[\frac{(n+1)(n+2)......(4n)}{n^{3n}}\right]^{1/n}$

$\log k = \lim\limits_{n\to\infty}\frac{1}{n}\log\left[\frac{(n+1)(n+2)\ldots\ldots(4n)}{n^{3n}}\right]$

$\log k = \lim\limits_{n\to\infty}\frac{1}{n}\log\left\{\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\ldots\ldots\ldots\left(\frac{n+3n}{n}\right)\right\}$

$\log k = \lim\limits_{n\to\infty}\frac{1}{n}\left\{\log\left(\frac{n+1}{n}\right)+\log\left(\frac{n+2}{n}\right)+\cdots\ldots\ldots+\log\left(\frac{n+3n}{n}\right)\right\}$

$\log k = \lim\limits_{n\to\infty}\frac{1}{n}\left\{\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\cdots\ldots\ldots+\log\left(1+\frac{3n}{n}\right)\right\}$

$\log k = \lim\limits_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$

$\log k = \int_0^3 \log(1+x)\,dx$

$\log k = \log(1+x)\,.\int_0^3 dx - \int_0^3\left[\frac{d}{dx}(\log(1+x)).dx\right]$

$\log k = (3\log 4 - 0) - \int_0^3\left(1-\frac{1}{1+x}\right)dx$

$\log k = \log 4^3 - (3-\log 4) = \log 256 - 3\log e = \log\left(\frac{256}{e^3}\right)$

$\therefore k = \frac{256}{e^3}.$

Q6. (A) (i) Solution: Given, $\int_1^b f(x)\,dx = \log\left(b+\sqrt{b^2+1}\right)$

Differentiate both sides w.r.t. b

$(1)\times f(b) - 0\times f(1) = \frac{1}{b+\sqrt{b^2+1}}\left(1+\frac{1}{2\sqrt{b^2+1}}\times 2b\right)$

$f(b) = \frac{1}{b+\sqrt{b^2+1}}\left(\frac{\sqrt{b^2+1}+b}{\sqrt{b^2+1}}\right) = \frac{1}{\sqrt{b^2+1}}$

$\therefore\ f(x) = \frac{1}{\sqrt{b^2+1}}$

(ii) Answer:

As the question is asked in minutes, conversion of 600 seconds to minutes is done. Hence we get 10 minutes.

$\lambda = \frac{1}{10} = 0.1$

So the probability that Maria will wait for more than x minutes is given by $e^{-\lambda x}$

$e^{-(0.1)x} = 0.10$

$x = -10\times\ ln0.1$

$x = 23.03$

(iii) Solution: W.K.T. $p+q=1$

Given $P(X=2)=9\,P(X=4)$

$6_{C_2}p^2q^4=9\times 6_{C_4}p^4q^2$

$q^2=9\times p^2$

$q=3p$

$1-p=3p$

$p=\frac{1}{4}\ and\ q=\frac{3}{4}$

Hence Variance= $npq=6\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)=\frac{9}{8}$

(B) (i) Solution: It is a linear differential equation in 'y'

Comparing with $\frac{dy}{dx}+P(x).y=Q(x)$, where $P(x)=\frac{2}{x}, Q(x)=\sin x$

I.F. $=e^{\int P(x)dx}=e^{2\int\frac{1}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$

Solution is $y\,(I.F)=\int Q(x).(I.F)dx$

$y\,x^2=\int \sin x\,.x^2\,dx$

$y\,x^2=x^2\int \sin x\ dx-\int\frac{d}{dx}x^2.\int\sin x\ dx.dx$

$x^2y=-x^2\cos x+2\int x\cos x\ dx,$

$x^2y=-x^2\cos x+2\left[x\int\cos x\ dx-\int\frac{d}{dx}(x).\int\cos x\,dx\,.\,dx\right]$

$x^2y=-x^2\cos x+2x\sin x+2\cos x+c$

(ii) Answer:

Let $A=a_i, B=b_i$

	a_i	b_i	a_i^2	b_i^2	a_ib_i
	9	6	81	36	54
	8	9	64	81	72
	7	8	49	64	56
	6	7	36	49	42
	11	10	121	100	110
	10	11	100	121	110
	5	5	25	25	25
Total	56	56	476	476	469

From the table, we get the following values

$n=7,$

$$\sum_{i=1}^{7}a_i=56$$

$$\sum_{i=1}^{7}b_i=56$$

$$\sum_{i=1}^{7}a_i^2=476$$

$$\sum_{i=1}^{7} b_i^2 = 476$$
$$\sum_{i=1}^{7} a_i b_i = 469$$
$$\bar{a} = \frac{1}{n}\sum_{i=1}^{7} a_i = \frac{56}{7} = 8$$
$$\bar{b} = \frac{1}{n}\sum_{i=1}^{7} b_i = \frac{56}{7} = 8$$
$$Corr(X,Y) = \frac{\frac{1}{n}\sum_{i=1}^{7} a_i b_i - \bar{a}\bar{b}}{\sqrt{\frac{1}{n}\sum_{i=1}^{7} a_i^2 - \bar{a}^2}\ \sqrt{\frac{1}{n}\sum_{i=1}^{7} b_i^2 - \bar{b}^2}}$$
$$= \frac{\frac{469}{7} - 8\times 8}{\sqrt{\frac{476}{7} - (8)^2}\ \sqrt{\frac{476}{7} - (8)^2}} = \frac{3}{2\times 2} = 0.75$$

(iii) Solution: Let A_1 be the event of selecting bag-1 and A_2 be the event of selecting bag-2
$P(A_1)$= probability of getting 1 or 3 when a die is rolled $= \frac{2}{6} = \frac{1}{3}$
$P(A_2) = 1 - \frac{1}{3} = \frac{2}{3}$
Now A_1 and A_2 are mutually exclusive and exhaustive events.
Let E be the event of drawing a black ball from selected bag
$P\left(\frac{E}{A_1}\right)$ = Probability of drawing black ball from bag-1=$\frac{3}{7}$
$P\left(\frac{E}{A_2}\right)$ = Probability of drawing black ball from bag-2=$\frac{4}{7}$
$P(E) = P(A_1)P\left(\frac{E}{A_1}\right) + P(A_2)P\left(\frac{E}{A_2}\right) = \frac{1}{3}\times\frac{3}{7} + \frac{2}{3}\times\frac{4}{7} = \frac{11}{21}$.