Matrices

A rectangular array of m x n numbers (real or complex) in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as m x n matrix. Such an array is enclosed by [ ] or ( ) .

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Introduction to Matrices

An m x n matrix is usually written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

In brief, the above matrix is represented by $A = [a_{ij}]_{m \times n}$. The number $a_{ij}, a_{12}, \ldots$ etc., are known as the elements of the matrix A, where $a_{ij}$ belongs to the $i^{th}$ row and $j^{th}$ column and is called the $(i, j)^{th}$ element of the matrix $A = [a_{ij}]$.

Important Formulas for Matrices

If $A, B$ are square matrices of order $n$, and $I_n$ is a corresponding unit matrix, then

(a) $A(adj.A) = |A| I_n = (adj A) A$

(b) $|adj A| = |A|^{n-1}$ (Thus $A(adj A)$ is always a scalar matrix)

(c) $adj(adj.A) = |A|^{n-2} A$

(e) $|adj (adj. A)| = |A|^{(n-1)^2}$

(f) $adj(AB) = (adj B)(adj A)$

(g) $adj(A^m) = (adj A)^m$

(h) $adj(kA) = k^{n-1}(adj. A), k \in R$

(i) $adj(I_n) = I_n$

(j) $adj 0 = 0$

(k) $A$ is symmetric $\Rightarrow adj A$ is also symmetric

(l) $A$ is diagonal $\Rightarrow adj A$ is also diagonal

(m) $A$ is triangular $\Rightarrow adj A$ is also triangular
Matrices Introduction - Definition, Properties, Types and Examples

Types of Matrices

(i) **Symmetric Matrix**: A square matrix \( A = [a_{ij}] \) is called a symmetric matrix if \( a_{ij} = a_{ji} \), for all \( i, j \).

(ii) **Skew-Symmetric Matrix**: when \( a_{ij} = -a_{ji} \)

(iii) **Hermitian and skew – Hermitian Matrix**: \( A = A^\theta \) (Hermitian matrix)

(iv) **Orthogonal matrix**: if \( A^TA = I_n \)

(v) **Idempotent matrix**: \( A^2 = A \)

(vi) **Involuntary matrix**: \( A^2 = I \) or \( A^{-1} = A \)

(vii) **Nilpotent matrix**: if \( \exists p \in N \) such that \( A^p = 0 \)

Trace of matrix

(i) \( tr(\lambda A) = \lambda tr(A) \)

(ii) \( tr(A + B) = tr(A) + tr(B) \)

(iii) \( tr(AB) = tr(BA) \)

Transpose of matrix

(i) \( (A^T)^T = A \) 

(ii) \( (A + B)^T = A^T + B^T \) 

(iii) \( (AB)^T = B^T A^T \) 

(iv) \( (kA)^T = k(A)^T \) 

(v) \( (A_1 A_2 A_3 \ldots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \ldots A_2^T A_1^T \) 

(vi) \( I^T = I \) 

(vii) \( tr(A) = t(A^T) \) 

Properties of Matrix Multiplication

(i) \( AB \neq BA \) 

(ii) \( (AB)C = A(BC) \) 

(iii) \( A(B + C) = AB + AC \) 

Adjoint of a Matrix

(i) \( A(adj A) = (adj A)A = |A|I_n \) 

(ii) \( |adj A| = |A|^{n-1} \) 

(iii) \( (adj AB) = (adj B)(adj A) \) 

(iv) \( |adj A| = |A|^{n-2} \) 

Inverse of a Matrix

A matrix which has \( m \) rows and \( n \) columns is called a matrix of order \( m \times n \).

E.g. the order of \( \begin{bmatrix} 4 & -1 & 5 \\ 6 & 8 & -7 \end{bmatrix} \) matrix is \( 2 \times 3 \).
Note: (a) The matrix is just an arrangement of certain quantities.
(b) The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, then the matrix is called a real matrix.
(c) An m x n matrix has m.n elements.

Illustration 1: Construct a 3×4 matrix \( A = [a_{ij}] \), whose elements are given by \( a_{ij} = 2i + 3j \).

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix}; \quad \therefore a_{11} = 2 \times 1 + 3 \times 1 = 5; a_{12} = 2 \times 1 + 3 \times 2 = 8.
\]

Solution: In this problem, \( i \) and \( j \) are the number of rows and columns respectively. By substituting the respective values of rows and columns in \( a_{ij} = 2i + 3j \) we can construct the required matrix.

We have \( A = \).

Similarly, \( a_{13} = 11, a_{14} = 14, a_{21} = 7, a_{22} = 10, a_{23} = 13, a_{24} = 16, a_{31} = 9, a_{32} = 12, a_{33} = 15, a_{34} = 18 \)

\[
\therefore A = \begin{bmatrix} 5 & 8 & 11 & 14 \\ 7 & 10 & 13 & 16 \\ 9 & 12 & 18 & 18 \end{bmatrix}.
\]

Illustration 2: Construct a 3 x 4 matrix, whose elements are given by: \( a_{ij} = \frac{1}{2} |3i + j| \)

Solution:

Method for solving this problem is the same as in the above problem.

Since \( a_{ij} = \frac{1}{2} |3i + j| \) we have
\[
a_{11} = \frac{1}{2} |3(1) + 1| = \frac{1}{2} |3 + 1| = \frac{1}{2} |4| = \frac{1}{2} \times 2 = 1
\]
\[
a_{12} = \frac{1}{2} |3(1) + 2| = \frac{1}{2} |3 + 2| = \frac{1}{2} |5| = \frac{1}{2} \times 5 = 2.5
\]
\[
a_{13} = \frac{1}{2} |3(1) + 3| = \frac{1}{2} |3 + 3| = \frac{1}{2} |6| = \frac{1}{2} \times 6 = 3
\]
\[
a_{14} = \frac{1}{2} |3(1) + 4| = \frac{1}{2} |3 + 4| = \frac{1}{2} |7| = \frac{1}{2} \times 7 = 3.5
\]
\[
a_{21} = \frac{1}{2} |3(2) + 1| = \frac{1}{2} |6 + 1| = \frac{1}{2} |7| = \frac{1}{2} \times 7 = 3.5
\]
\[
a_{22} = \frac{1}{2} |3(2) + 2| = \frac{1}{2} |6 + 2| = \frac{1}{2} |8| = \frac{1}{2} \times 8 = 4
\]
\[
a_{23} = \frac{1}{2} |3(2) + 3| = \frac{1}{2} |6 + 3| = \frac{1}{2} |9| = \frac{1}{2} \times 9 = 4.5
\]
\[
a_{24} = \frac{1}{2} |3(2) + 4| = \frac{1}{2} |6 + 4| = \frac{1}{2} |10| = \frac{1}{2} \times 10 = 5
\]
\[
a_{31} = \frac{1}{2} |3(3) + 1| = \frac{1}{2} |9 + 1| = \frac{1}{2} |10| = \frac{1}{2} \times 10 = 5
\]
\[
a_{32} = \frac{1}{2} |3(3) + 2| = \frac{1}{2} |9 + 2| = \frac{1}{2} |11| = \frac{1}{2} \times 11 = 5.5
\]
\[
a_{33} = \frac{1}{2} |3(3) + 3| = \frac{1}{2} |9 + 3| = \frac{1}{2} |12| = \frac{1}{2} \times 12 = 6
\]
\[
a_{34} = \frac{1}{2} |3(3) + 4| = \frac{1}{2} |9 + 4| = \frac{1}{2} |13| = \frac{1}{2} \times 13 = 6.5
\]

Hence, the required matrix is given by
\[
A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}
\]

Trace of a Matrix

Let \( A = [a_{ij}] \) and \( B = [b_{ij}] \) and \( \lambda \) be a scalar,

(i) \( \text{tr}(\lambda A) = \lambda \text{tr}(A) \) (ii) \( \text{tr}(A + B) = \text{tr}(A) + \text{tr}(B) \) (iii) \( \text{tr}(AB) = \text{tr}(BA) \)
Matrices Introduction - Definition, Properties, Types and Examples

Square Matrix

Transpose of Matrix

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called the transpose of matrix (https://byjus.com/maths/transpose-of-a-matrix/) A and is denoted by $A^T$ or $A'$. From the definition it is obvious that if the order of A is $m \times n$, then the order of $A^T$ becomes $n \times m$; E.g. transpose of matrix

\[
\begin{bmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3
\end{bmatrix}_{2\times3}
\]

is

\[
\begin{bmatrix}
  a_1 & b_1 \\
  a_2 & b_2 \\
  a_3 & b_3
\end{bmatrix}_{3\times2}
\]

Properties of Transpose of Matrix

(i) $(A^T)^T = A$  
(ii) $(A + B)^T = A^T + B^T$  
(iii) $(AB)^T = B^T A^T$  
(iv) $(kA)^T = k(A^T)$  
(v) $(A_1 A_2 A_3 \ldots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \ldots A_3^T A_2^T A_1^T$  
(vi) $I^T = I$  
(vii) $tr(A) = tr(A^T)$

Problems on Matrices

Illustration 3: If $A = \begin{bmatrix}
  1 & -2 & 3 \\
  -4 & 2 & 5
\end{bmatrix}$ and $B = \begin{bmatrix}
  1 & 3 \\
  -1 & 0 \\
  2 & 4
\end{bmatrix}$, then prove that $(AB)^T = B^T A^T$.

Solution:

By obtaining the transpose of AB i.e. $(AB)^T$ and multiplying $B^T$ and $A^T$ we can easily get the result.
Here, \( AB = \)

\[
A = \begin{bmatrix}
1 & -2 & 3 \\
-4 & 2 & 5
\end{bmatrix}
\text{ and } B = \begin{bmatrix}
1 & 3 \\
-1 & 0 \\
2 & 4
\end{bmatrix} = \begin{bmatrix}
1(1) - 2(-1) + 3(2) & 1(3) - 2(0) + 3(4) \\
-4(1) + 2(-1) + 5(2) & -4(3) + 2(0) + (4)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
9 & 15 \\
4 & 8
\end{bmatrix}
\]

\[
\therefore (AB)^T = \begin{bmatrix}
9 & 4 \\
15 & 8
\end{bmatrix}; B^T A^T = \begin{bmatrix}
1 & -1 & 2 \\
3 & 0 & 4
\end{bmatrix} \begin{bmatrix}
1 & -4 \\
2 & 2 \\
3 & 5
\end{bmatrix} = \begin{bmatrix}
9 & 4 \\
15 & 8
\end{bmatrix} = (AB)^T
\]

Illustration 4: If \( A = \begin{bmatrix}
5 & -1 & 3 \\
0 & 1 & 2
\end{bmatrix} \text{ and } B = \begin{bmatrix}
0 & 2 \\
1 & -1 \\
3 & 4
\end{bmatrix} \) then what is \( AB \) equal to?

Solution:

In this problem, we use the properties of the transpose of a matrix to get the required result.

We have \((B')'A' = BA' = \begin{bmatrix}
0 & 2 \\
1 & -1 \\
3 & 4
\end{bmatrix} \begin{bmatrix}
5 & 0 \\
-1 & 1 \\
3 & 2
\end{bmatrix} = \begin{bmatrix}
7 & 8 \\
18 & 7
\end{bmatrix}.
\]

Illustration 5: If the matrix \( A = \begin{bmatrix}
3 - x & 2 & 2 \\
2 & 4 - x & 1 \\
-2 & -4 & -1 - x
\end{bmatrix} \) is a singular matrix then find \( x \). Verify whether \( AA^T = I \) for that value of \( x \).

Solution:

Using the condition of a singular matrix, i.e. \( |A| = 0 \), we get the value of \( x \) and then substituting the value of \( x \) in matrix \( A \) and multiplying it to its transpose we will obtain the required result.

Here, \( A \) is a singular matrix if \( |A| = 0 \), i.e.,

\[
\begin{vmatrix}
3 - x & 2 & 2 \\
2 & 4 - x & 1 \\
-2 & -4 & -1 - x
\end{vmatrix}
\]

\[
= 0 \quad \Rightarrow \quad u \sin g R_3 \rightarrow R_3 + R_2 \quad \text{or} \quad R_2 \rightarrow R_2 - x
\]

\[
\begin{vmatrix}
3 - x & 0 & 2 \\
2 & 3 - x & 1 \\
0 & 0 & -x
\end{vmatrix}
\]

\[
= 0, \quad u \sin g C_2
\]

\[
\rightarrow C_2 - C_3
\]

\[
\text{or} \quad (3x)^2 = 0, \quad x = 0, 3.
\]

When \( x = 0 \), \( A = \begin{bmatrix}
3 & 2 & 2 \\
2 & 4 & 1 \\
-2 & -4 & -1
\end{bmatrix}
\)

\[
\therefore AA^T = \begin{bmatrix}
3 & 2 & 2 \\
2 & 4 & 1 \\
-2 & -4 & -1
\end{bmatrix} \begin{bmatrix}
3 & 2 & -2 \\
2 & 4 & -4 \\
-2 & 1 & -1
\end{bmatrix}
\]
= \begin{bmatrix} 17 & 16 & -16 \\ 16 & 21 & -21 \\ -16 & -21 & 21 \end{bmatrix} \neq I

When x = 3, A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix} ;

\therefore AA^T = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & -4 \\ -2 & -4 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 4 & -16 \\ 4 & 6 & -12 \\ -16 & -12 & 36 \end{bmatrix} \neq I

Note: simple way to solve is that if A is a singular matrix then |A| = 0 and |A^T| = 0. But |I| is 1.

Hence, AA^T \neq I if |A| = 0.

Illustration 6: If the matrix A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}

where a, b, c, are positive real numbers such that abc = 1 and ATA = I then find the value of a^3 + b^3 + c^3.

Solution:

Here, A= \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} . So, A^T = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} ,

Interchanging rows and columns.

⇒ A^T A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = A^2

⇒ |A^T A| = |A^2|; But A^T A = I (given).

∴ |I| = |A|^2 ⇒ 1 = |A|^2

Now, |A| = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a + b + c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}, R_1 \rightarrow R_1 + R_2 + R_3

= (a + b + c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-a & b-c \\ c & a-b & b-c \end{vmatrix}, C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1

= (a + b + c) ((c b) (b c) - (a b) (a c)) = (a + b + c) (b^2 c^2 + 2 b c a + a c + a b c)

= (a + b + c) (a^2 + b^2 + c^2) b c a = (a^3 + b^3 + c^3) a b c

= (a^3 + b^3 + c^3) (a b c = 1) |A|^2 = 1 (a^3 + b^3 + c^3)^2 = 1 ....(i)

As a, b, c are positive, \(\frac{a^3 + b^3 + c^3}{3} > \sqrt[3]{a^3 b^3 c^3}\)

Since, abc = 1 \(\Rightarrow (a^3)^{(3)} + (b^3)^{(3)} + (c^3)^{(3)} > 3\) \(\Rightarrow a^3 + b^3 + c^3 - 3 = 1\)
\[ \therefore a^3 + b^3 + c^3 = 4 \]