# Matrices

A rectangular array of m x n numbers (real or complex) in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n, written as m x n matrix. Such an array is enclosed by [ ] or ( ) |.

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## Introduction to Matrices

An m x n matrix is usually written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

In brief, the above matrix is represented by A = $[a_{ij}]_{m \times n}$. The number $a_{11}$, $a_{12}$, ….. etc., are known as the elements of the matrix A, where $a_{ij}$ belongs to the $i^{th}$ row and $j^{th}$ column and is called the $(i, j)^{th}$ element of the matrix A = $[a_{ij}]$.

## Important Formulas for Matrices

If A, B are square matrices of order n, and $I_n$ is a corresponding unit matrix (https://byjus.com/maths/identity-matrix/), then

**(a)** A(adj.A) = | A | $I_n$ = (adj A) A

**(b)** | adj A | = | A |$n^{-1}$ (Thus A (adj A) is always a scalar matrix)

**(c)** adj (adj.A) = | A |$^{n-2}$ A

**(e)** $|adj\,(adj.\,A)| = |A|^{(n-1)^2}$

**(f)** adj (AB) = (adj B) (adj A)

**(g)** adj ($A^m$) = (adj A)$^m$,

**(h)** $adj(kA) = k^{n-1}(adj.\,A), k \in R$

**(i)** adj$(I_n) = I_n$

**(j)** adj 0 = 0

**(k)** A is symmetric ⇒adj A is also symmetric

**(l)** A is diagonal ⇒adj A is also diagonal

**(m)** A is triangular ⇒adj A is also triangular

**(n)** A is singular ⇒| adj A | = 0

## Types of Matrices

**(i) Symmetric Matrix:** A square matrix A $= [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$, for all i, j.

**(ii) Skew-Symmetric Matrix:** when $a_{ij} = -a_{ji}$

**(iii) Hermitian and skew – Hermitian Matrix:** $A = A^{\theta}$ (Hermitian matrix)

$A^{\theta} = -A$ (skew-Hermitian matrix)

**(iv) Orthogonal matrix:** if $AA^T = I_n = A^T A$

**(v) Idempotent matrix:** if $A^2 = A$

**(vi) Involuntary matrix:** if $A^2 = I \ or \ A^{-1} = A$

**(vii) Nilpotent matrix:** if $\exists \ p \in N$ such that $A^P = 0$

### Trace of matrix

**(i)** $tr\ (\lambda A) = \lambda tr\ (A)$

**(ii)** $tr(A + B) = tr(A) + tr(B)$

**(iii)** $tr(AB) = tr(BA)$

### Transpose of matrix

$(i)(A^T)^T = A \quad (ii)(A \pm B)^T = A^T \pm B^T \quad (iii)(AB)^T = B^T A^T$
$(iv)(kA)^T = k(A)^T \quad (v)(A_1 A_2 A_3 \ldots\ldots A_{n-1} A_n)^T = A_n^T A_{n-1}^t \ldots\ldots A_3^T A_2^T A_1^T$

$(vi)I^T = I \quad (vii)tr(A) = t(A^T)$

### Properties of Matrix Multiplication

$(i)AB \neq BA \quad (ii)(AB)C = A(BC) \quad (iii)A.(B + C) = A.B + A.C$

### Adjoint of a Matrix

$(i)A(adj\,A) = (adj\,A)A = |A|I_n \quad (ii)|adj\,A| = |A|^{n-1}$

$(iii)(adj\ AB) = (adj\ B)(adj\ A) \quad (iv)adj\,(adj\,A) = |A|^{n-2}$

## Inverse of a Matrix

**A$^{-1}$** exists if A is non singular i.e. $|A| \neq 0$

$(i)A^{-1} = \frac{1}{|A|}(Adj.\,A) \quad (ii)A^{-1}A = I_n = AA^{-1} \quad (iii)(A^T)^{-1} = (A^{-1})^T \quad (iv)(A^{-1})^{-1} = A$

$(v)|A^{-1}| = |A|^{-1} = \frac{1}{|A|}$

### Order of a Matrix

A matrix which has m rows and n columns is called a matrix of order (https://byjus.com/maths/determine-the-order-of-matrix/) m x n

E.g. the order of $\begin{bmatrix} 4 & -1 & 5 \\ 6 & 8 & -7 \end{bmatrix}$ matrix is 2 x 3.

**Note:** (a) The matrix is just an arrangement of certain quantities.

(b) The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, then the matrix is called a real matrix.

(c) An m x n matrix has m.n elements.

**Illustration 1:** Construct a 3×4 matrix A = $[a_{ij}]$, whose elements are given by $a_{ij}$ = 2i + 3j.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}; \quad \therefore a_{11} = 2 \times 1 + 3 \times 1 = 5; a_{12} = 2 \times 1 + 3 \times 2 = 8.$$

**Solution:** In this problem, I and j are the number of rows and columns respectively. By substituting the respective values of rows and columns in $a_{ij}$ = 2i + 3j we can construct the required matrix.

We have A =.

Similarly, $a_{13}$ = 11, $a_{14}$=14, $a_{21}$ = 7, $a_{22}$=10, $a_{23}$=13, $a_{24}$=16,$a_{31}$=9, $a_{32}$=12, $a_{33}$=15, $a_{34}$=18

$$\therefore A = \begin{bmatrix} 5 & 8 & 11 & 14 \\ 7 & 10 & 13 & 16 \\ 9 & 12 & 18 & 18 \end{bmatrix}.$$

**Illustration 2:** Construct a 3 x 4 matrix, whose elements are given by: $a_{ij} = \frac{1}{2}|3i + j|$

**Solution:**

Method for solving this problem is the same as in the above problem.

Since $a_{ij} = \frac{1}{2}|-3i+j| we\ have\ a_{11} = \frac{1}{2}|-3(1)+1| = \frac{1}{2}|-3+1| = \frac{1}{2}|-2| = \frac{2}{2} = 1$

$$a_{12} = \frac{1}{2}|-3(1)+2| = \frac{1}{2}|-3+2| = \frac{1}{2}|-1| = \frac{1}{2}$$

$$a_{13} = \frac{1}{2}|-3(1)+3| = \frac{1}{2}|-3+3| = \frac{1}{2}(0) = 0$$

$$a_{14} = \frac{1}{2}|-3(1)+4| = \frac{1}{2}|-3+4| = \frac{1}{2}; \qquad a_{21}\frac{1}{2}|-3(2)+1| = \frac{1}{2}|-6+1| = \frac{5}{2}$$

$$a_{22} = \frac{1}{2}|-3(2)+2| = \frac{1}{2}|-6+2| = \frac{4}{2} = 2; \qquad a_{23}\frac{1}{2}|-3(2)+3| = \frac{1}{2}|-6+3| = \frac{3}{2}$$

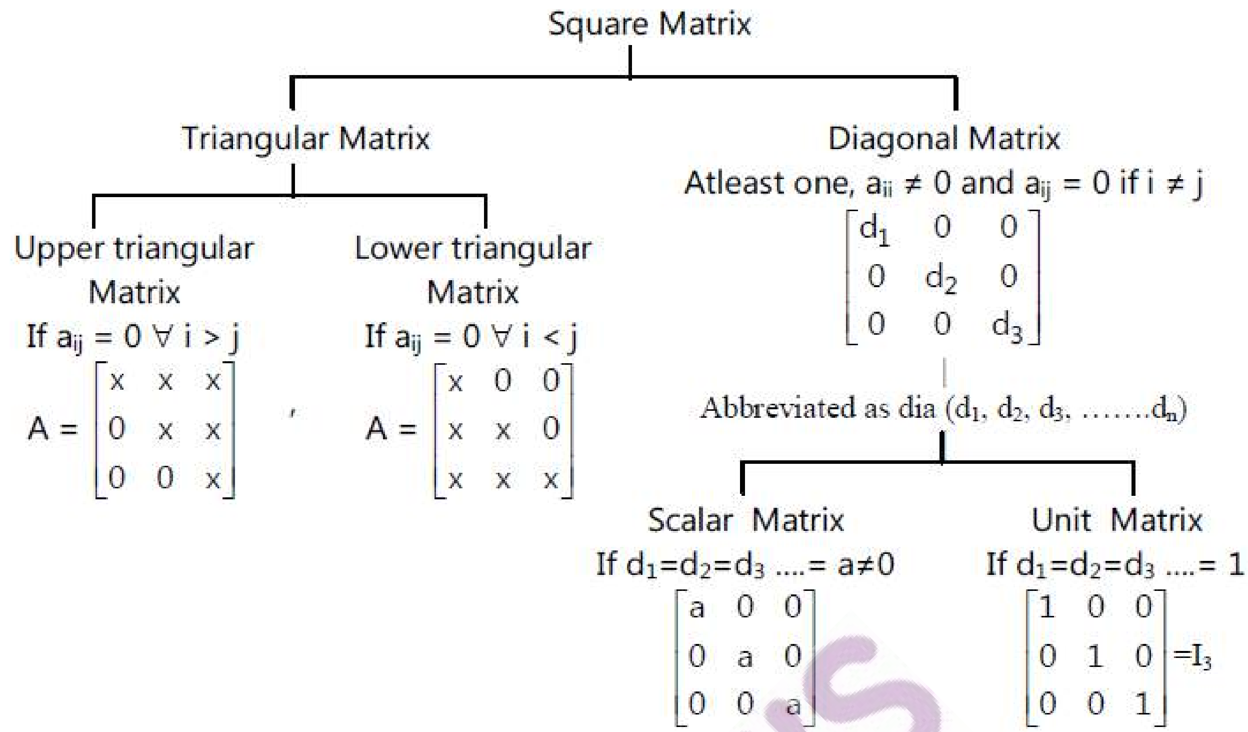
$$a_{24} = \frac{1}{2}|-3(2)+4| = \frac{1}{2}|-6+4| = \frac{2}{2} = 1; \quad Similarly a_{31} = 4, a_{32} = \frac{7}{2}, a_{33} = 3, a_{34} = \frac{5}{2}$$

Hence, the required matrix is given by $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$

## Trace of a Matrix

Let A = $[a_{ij}]_{nxn}$ and B = $[b_{ij}]_{nxn}$ and λ be a scalar,

(i) tr(λA) = λ tr(A) (ii) tr(A + B) = tr(A) + tr(B) (iii) tr(AB) = tr(BA)




## Transpose of Matrix

The matrix obtained from a given matrix A by changing its rows into columns or columns into rows is called the transpose of matrix (https://byjus.com/maths/transpose-of-a-matrix/) A and is denoted by $A^T$ or A'. From the definition it is obvious that if the order of A is m x n, then the order of $A^T$ becomes n x m; E.g. transpose of matrix

$$\begin{bmatrix} a1 & a2 & a3 \\ b1 & b2 & b3 \end{bmatrix}_{2\times 3} is \begin{bmatrix} a1 & b1 \\ a2 & b2 \\ a3 & b3 \end{bmatrix}_{3\times 2}.$$

### Properties of Transpose of Matrix

(i) $(A^T)^T = A$ (ii) $(A + B)^T = A^T + B^T$ (iii) $(AB)^T = B^TA^T$ (iv) $(kA)^T = k(A)^T$

(v) $(A_1A_2A_3 \ldots\ldots A_{n-1}A_n)^T = A_n^T A_{n-1}^T \ldots\ldots A_3^T A_2^T A_1^T$ (vi) $I^T = I$ (vii) $tr(A) = tr(A^T)$

## Problems on Matrices

**Illustration 3:** If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} and B = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}$. then prove that $(AB)^T = B^TA^T$.

**Solution:**

By obtaining the transpose of AB i.e. $(AB)^T$ and multiplying $B^T$ and $A^T$ we can easily get the result.

Here, AB =

$$A=\begin{bmatrix}1 & -2 & 3\\ -4 & 2 & 5\end{bmatrix} andB=\begin{bmatrix}1 & 3\\ -1 & 0\\ 2 & 4\end{bmatrix}=\begin{bmatrix}1(1)-2(-1)+3(2) & 1(3)-2(0)+3(4)\\ -4(1)+2(-1)+5(2) & -4(3)+2(0)+(4)\end{bmatrix}.$$

$$=\begin{bmatrix}9 & 15\\ 4 & 8\end{bmatrix}$$

$$\therefore (AB)^T=\begin{bmatrix}9 & 4\\ 15 & 8\end{bmatrix}; B^TA^T==\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & 4\end{bmatrix}\begin{bmatrix}1 & -4\\ -2 & 2\\ 3 & 5\end{bmatrix}$$

$$=\begin{bmatrix}1(1)-1(-2)+2(3) & 1(-4)-1(2)+2(5)\\ 3(1)+0(-2)+4(3) & 3(-4)+0(2)+4(5)\end{bmatrix}=\begin{bmatrix}9 & 4\\ 15 & 8\end{bmatrix}=(AB)^T$$

**Illustration 4:** If A =$A=\begin{bmatrix}5 & -1 & 3\\ 0 & 1 & 2\end{bmatrix} andB=\begin{bmatrix}0 & 2 & 3\\ 1 & -1 & 4\end{bmatrix}$. then what is is equal to?

**Solution:**

In this problem, we use the properties of the transpose of a matrix to get the required result.

We have =$(B')'A'=BA'=\begin{bmatrix}0 & 2 & 3\\ 1 & -1 & 4\end{bmatrix}\begin{bmatrix}5 & 0\\ -1 & 1\\ 3 & 2\end{bmatrix}=\begin{bmatrix}7 & 8\\ 18 & 7\end{bmatrix}$.

**Illustration 5:** If the matrix $A=\begin{bmatrix}3-x & 2 & 2\\ 2 & 4-x & 1\\ -2 & -4 & -1-x\end{bmatrix}$

is a singular matrix then find x. Verify whether $AA^T = I$ for that value of x.

**Solution:**

Using the condition of a singular matrix, i.e. |A| = 0, we get the value of x and then substituting the value of x in matrix A and multiplying it to its transpose we will obtain the required result.

Here, A is a singular matrix if |A| = 0, i.e., $\begin{vmatrix}3-x & 2 & 2\\ 2 & 4-x & 1\\ -2 & -4 & -1-x\end{vmatrix}=0$

$$or\begin{vmatrix}3-x & 2 & 2\\ 2 & 4-x & 1\\ 0 & -x & -x\end{vmatrix}=0, u\sin g\,R_3\to R_3+R_2\,or\begin{vmatrix}3-x & 0 & 2\\ 2 & 3-x & 1\\ 0 & 0 & -x\end{vmatrix}=0, u\sin g\,C_2$$

$\to C_2-C_3$

orx$(3x)^2$ = 0, x = 0, 3.

When x = 0, A = $\begin{bmatrix}3 & 2 & 2\\ 2 & 4 & 1\\ -2 & -4 & -1\end{bmatrix}$

$$\therefore AA^T=\begin{bmatrix}3 & 2 & 2\\ 2 & 4 & 1\\ -2 & -4 & -1\end{bmatrix}\begin{bmatrix}3 & 2 & -2\\ 2 & 4 & -4\\ 2 & 1 & -1\end{bmatrix}$$

= $\begin{bmatrix} 17 & 16 & -16 \\ 16 & 21 & -21 \\ -16 & -21 & 21 \end{bmatrix} \neq I$

When x = 3, A = $\begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix}$;

$\therefore AA^T = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 1 & 1 \\ -2 & -4 & -4 \end{bmatrix} \begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & -4 \\ 2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 4 & -16 \\ 4 & 6 & -12 \\ -16 & -12 & 36 \end{bmatrix} \neq I$

**Note:** simple way to solve is that if A is a singular matrix then |A| = 0 and $|A^T|$ = 0. But |I| is 1.

Hence, $AA^T \neq$ I if |A| = 0.

**Illustration 6:** If the matrix A = $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

where a, b, c, are positive real numbers such that abc = 1 and ATA = I then find the value of $a^3 + b^3 + c^3$.

**Solution:**

Here, A= $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} . So, A^T = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$,

Interchanging rows and columns.

$\Rightarrow A^T A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}^2 = A^2$

$\Rightarrow |A^T A| = |A^2|; But A^T A = I(given).$

$\therefore |I| = |A|^2 \Rightarrow 1 = |A|^2$

Now, |A| = $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}, R_1 \to R_1 + R_2 + R_3$

= $(a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}, \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$

= (a + b + c) {(c b) (b c) − (a b) (a c)} = (a + b +c) ($b^2c^2$ + 2bc$a^2$ + ac + abbc)

= (a + b + c) ($a^2 + b^2 + c^2$bccaab) =($a^3 + b^3 + c^3$3 abc)

= ($a^3 + b^3 + c^3$3) (abc = 1) $|A|^2$ = 1($a^3 + b^3 + c^3$3)$^2$ = 1 .....(i)

As a, b, c are positive, $\frac{a^3+b^3+c^3}{3} > \sqrt{a^3b^3c^3}$

Since, abc=1 ∴ {{a}^{3}}+{{b}^{3}}+{{c}^{3}}>3\) $(i) \Rightarrow a^3 + b^3 + c^3 - 3 = 1$

$\therefore\ a^3 + b^3 + c^3 = 4$