Chapter 16

PROBABILITY

❖ *Where a mathematical reasoning can be had, it is as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle in your hand. – JOHN ARBUTHNOT* ❖

16.1 Introduction

In earlier classes, we studied about the concept of probability as a measure of uncertainty of various phenomenon. We have obtained the probability of getting an even number in throwing a die as $\frac{3}{6}$ i.e., $\frac{1}{2}$. Here the total possible outcomes are 1,2,3,4,5 and 6 (six in number). The outcomes in favour of the event of 'getting an even number' are 2,4,6 (i.e., three in number). In general, to obtain the probability of an event, we find the ratio of the number of outcomes favourable to the event, to the total number of equally likely outcomes. This theory of probability is known as *classical theory of probability*.

Kolmogorov
(1903-1987)

In Class IX, we learnt to find the probability on the basis of observations and collected data. This is called *statistical approach of probability*.

Both the theories have some serious difficulties. For instance, these theories can not be applied to the activities/experiments which have infinite number of outcomes. In classical theory we assume all the outcomes to be equally likely. Recall that the outcomes are called equally likely when we have no reason to believe that one is more likely to occur than the other. In other words, we assume that all outcomes have equal chance (probability) to occur. Thus, to define probability, we used equally likely or equally probable outcomes. This is logically not a correct definition. Thus, another theory of probability was developed by A.N. Kolmogorov, a Russian mathematician, in 1933. He laid down some axioms to interpret probability, in his book 'Foundation of Probability' published in 1933. In this Chapter, we will study about this approach called *axiomatic approach of probability*. To understand this approach we must know about few basic terms viz. random experiment, sample space, events, etc. Let us learn about these all, in what follows next.

16.2 Random Experiments

In our day to day life, we perform many activities which have a fixed result no matter any number of times they are repeated. For example given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°.

We also perform many experimental activities, where the result may not be same, when they are repeated under identical conditions. For example, when a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained. Such experiments are called *random experiments*.

An experiment is called random experiment if it satisfies the following two conditions:

(i) It has more than one possible outcome.

(ii) It is not possible to predict the outcome in advance.

Check whether the experiment of tossing a die is random or not?

In this chapter, we shall refer the random experiment by experiment only unless stated otherwise.

16.2.1 ***Outcomes and sample space*** A possible result of a random experiment is called its *outcome*.

Consider the experiment of rolling a die. The outcomes of this experiment are 1, 2, 3, 4, 5, or 6, if we are interested in the number of dots on the upper face of the die.

The set of outcomes {1, 2, 3, 4, 5, 6} is called the *sample space of the experiment*.

Thus, the set of all possible outcomes of a random experiment is called the *sample space* associated with the experiment. Sample space is denoted by the symbol S.

Each element of the sample space is called a *sample point*. In other words, each outcome of the random experiment is also called *sample point*.

Let us now consider some examples.

Example 1 Two coins (a one rupee coin and a two rupee coin) are tossed once. Find a sample space.

Solution Clearly the coins are distinguishable in the sense that we can speak of the first coin and the second coin. Since either coin can turn up Head (H) or Tail(T), the possible outcomes may be

Heads on both coins = (H,H) = HH
Head on first coin and Tail on the other = (H,T) = HT
Tail on first coin and Head on the other = (T,H) = TH
Tail on both coins = (T,T) = TT
Thus, the sample space is S = {HH, HT, TH, TT}

Note The outcomes of this experiment are ordered pairs of H and T. For the sake of simplicity the commas are omitted from the ordered pairs.

Example 2 Find the sample space associated with the experiment of rolling a pair of dice (one is blue and the other red) once. Also, find the number of elements of this sample space.

Solution Suppose 1 appears on blue die and 2 on the red die. We denote this outcome by an ordered pair (1,2). Similarly, if '3' appears on blue die and '5' on red, the outcome is denoted by the ordered pair (3,5).

In general each outcome can be denoted by the ordered pair (x, y), where x is the number appeared on the blue die and y is the number appeared on the red die. Therefore, this sample space is given by

S = {(x, y): x is the number on the blue die and y is the number on the red die}.

The number of elements of this sample space is $6 \times 6 = 36$ and the sample space is given below:

{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6)
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6)
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

Example 3 In each of the following experiments specify appropriate sample space

(i) A boy has a 1 rupee coin, a 2 rupee coin and a 5 rupee coin in his pocket. He takes out two coins out of his pocket, one after the other.

(ii) A person is noting down the number of accidents along a busy highway during a year.

Solution (i) Let Q denote a 1 rupee coin, H denotes a 2 rupee coin and R denotes a 5 rupee coin. The first coin he takes out of his pocket may be any one of the three coins Q, H or R. Corresponding to Q, the second

draw may be H or R. So the result of two draws may be QH or QR. Similarly, corresponding to H, the second draw may be Q or R.
Therefore, the outcomes may be HQ or HR. Lastly, corresponding to R, the second draw may be H or Q. So, the outcomes may be RH or RQ.

Thus, the sample space is S={QH, QR, HQ, HR, RH, RQ}

(ii) The number of accidents along a busy highway during the year of observation can be either 0 (for no accident) or 1 or 2, or some other positive integer.
Thus, a sample space associated with this experiment is S= {0,1,2,...}

Example 4 A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail we throw a die. Describe the sample space of this experiment.

Solution Let us denote blue balls by B_1, B_2, B_3 and the white balls by W_1, W_2, W_3, W_4. Then a sample space of the experiment is

$$S = \{ HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6\}.$$

Here HB_i means head on the coin and ball B_i is drawn, HW_i means head on the coin and ball W_i is drawn. Similarly, Ti means tail on the coin and the number i on the die.

Example 5 Consider the experiment in which a coin is tossed repeatedly until a head comes up. Describe the sample space.

Solution In the experiment head may come up on the first toss, or the 2nd toss, or the 3rd toss and so on till head is obtained. Hence, the desired sample space is

S= {H, TH, TTH, TTTH, TTTTH,...}

EXERCISE 16.1

In each of the following Exercises 1 to 7, describe the sample space for the indicated experiment.

1. A coin is tossed three times.
2. A die is thrown two times.
3. A coin is tossed four times.
4. A coin is tossed and a die is thrown.
5. A coin is tossed and then a die is rolled only in case a head is shown on the coin.
6. 2 boys and 2 girls are in Room X, and 1 boy and 3 girls in Room Y. Specify the sample space for the experiment in which a room is selected and then a person.
7. One die of red colour, one of white colour and one of blue colour are placed in a bag. One die is selected at random and rolled, its colour and the number on its uppermost face is noted. Describe the sample space.
8. An experiment consists of recording boy–girl composition of families with 2 children.
   (i) What is the sample space if we are interested in knowing whether it is a boy or girl in the order of their births?
   (ii) What is the sample space if we are interested in the number of girls in the family?
9. A box contains 1 red and 3 identical white balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.
10. An experiment consists of tossing a coin and then throwing it second time if a head occurs. If a tail occurs on the first toss, then a die is rolled once. Find the sample space.
11. Suppose 3 bulbs are selected at random from a lot. Each bulb is tested and classified as defective (D) or non – defective(N). Write the sample space of this experiment.

12. A coin is tossed. If the out come is a head, a die is thrown. If the die shows up an even number, the die is thrown again. What is the sample space for the experiment?

13. The numbers 1, 2, 3 and 4 are written separatly on four slips of paper. The slips are put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.

14. An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment.

15. A coin is tossed. If it shows a tail, we draw a ball from a box which contains 2 red and 3 black balls. If it shows head, we throw a die. Find the sample space for this experiment.

16. A die is thrown repeatedly untill a six comes up. What is the sample space for this experiment?

16.3 Event

We have studied about random experiment and sample space associated with an experiment. The sample space serves as an universal set for all questions concerned with the experiment.

Consider the experiment of tossing a coin two times. An associated sample space is

S = {HH, HT, TH, TT}.

Now suppose that we are interested in those outcomes which correspond to the occurrence of exactly one head. We find that HT and TH are the only elements of S corresponding to the occurrence of this happening (event). These two elements form the set E = { HT, TH}

We know that the set E is a subset of the sample space S . Similarly, we find the following correspondence between events and subsets of S.

Description of events	Corresponding subset of 'S'
Number of tails is exactly 2	A = {TT}
Number of tails is atleast one	B = {HT, TH, TT}
Number of heads is atmost one	C = {HT, TH, TT}
Second toss is not head	D = { HT, TT}
Number of tails is atmost two	S = {HH, HT, TH, TT}
Number of tails is more than two	ϕ

The above discussion suggests that a subset of sample space is associated with an event and an event is associated with a subset of sample space. In the light of this we define an event as follows.

Definition Any subset E of a sample space S is called *an event*.

16.3.1 ***Occurrence of an event*** Consider the experiment of throwing a die. Let E denotes the event " a number less than 4 appears". If actually '1' had appeared on the die then we say that event E has occurred. As a matter of fact if outcomes are 2 or 3, we say that event E has occurred

Thus, the event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that $\omega \in$ E. If the outcome ω is such that $\omega \notin$ E, we say that the event E has not occurred.

16.3.2 ***Types of events*** Events can be classified into various types on the basis of the elements they have.

1. Impossible and Sure Events The empty set ϕ and the sample space S describe events. In fact ϕ is called an *impossible event* and S, i.e., the whole sample space is called the *sure event*.

To understand these let us consider the experiment of rolling a die. The associated sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let E be the event " the number appears on the die is a multiple of 7". Can you write the subset associated with the event E?

Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensures the occurrence of the event E. Thus, we say that the empty set only correspond to the event E. In other words we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, the event E = ϕ is an impossible event.

Now let us take up another event F "the number turns up is odd or even". Clearly F = {1, 2, 3, 4, 5, 6,} = S, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, the event F = S is a sure event.

2. Simple Event If an event E has only one sample point of a sample space, it is called a *simple* (or *elementary*) *event*.

In a sample space containing *n* distinct elements, there are exactly *n* simple events.

For example in the experiment of tossing two coins, a sample space is

S={HH, HT, TH, TT}

There are four simple events corresponding to this sample space. These are

E_1= {HH}, E_2={HT}, E_3= { TH} and E_4={TT}.

3. Compound Event If an event has more than one sample point, it is called a *Compound event*.

For example, in the experiment of "tossing a coin thrice" the events

E: 'Exactly one head appeared'
F: 'Atleast one head appeared'
G: 'Atmost one head appeared' etc.

are all compound events. The subsets of S associated with these events are

E={HTT,THT,TTH}
F={HTT,THT, TTH, HHT, HTH, THH, HHH}
G= {TTT, THT, HTT, TTH}

Each of the above subsets contain more than one sample point, hence they are all compound events.

16.3.3 ***Algebra of events*** In the Chapter on Sets, we have studied about different ways of combining two or more sets, viz, union, intersection, difference, complement of a set etc. Like-wise we can combine two or more events by using the analogous set notations.

Let A, B, C be events associated with an experiment whose sample space is S.

1. Complementary Event For every event A, there corresponds another event A' called the complementary event to A. It is also called the *event 'not* A'.

For example, take the experiment 'of tossing three coins'. An associated sample space is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let A={HTH, HHT, THH} be the event 'only one tail appears'

Clearly for the outcome HTT, the event A has not occurred. But we may say that the event 'not A' has occurred. Thus, with every outcome which is not in A, we say that 'not A' occurs.

Thus the complementary event 'not A' to the event A is

A' = {HHH, HTT, THT, TTH, TTT}

or $A' = \{\omega : \omega \in S \text{ and } \omega \notin A\} = S - A$.

2. The Event 'A or B' Recall that union of two sets A and B denoted by $A \cup B$ contains all those elements which are either in A or in B or in both.

When the sets A and B are two events associated with a sample space, then '$A \cup B$' is the event 'either A or B or both'. This event '$A \cup B$' is also called 'A or B'.

Therefore Event 'A or B' = $A \cup B$

$= \{\omega : \omega \in A \text{ or } \omega \in B\}$

3. The Event 'A and B' We know that intersection of two sets $A \cap B$ is the set of those elements which are common to both A and B. i.e., which belong to both 'A and B'.

If A and B are two events, then the set $A \cap B$ denotes the event 'A and B'.

Thus, $A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$

For example, in the experiment of 'throwing a die twice' Let A be the event 'score on the first throw is six' and B is the event 'sum of two scores is atleast 11' then

$A = \{(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$, and $B = \{(5,6), (6,5), (6,6)\}$

so $A \cap B = \{(6,5), (6,6)\}$

Note that the set $A \cap B = \{(6,5), (6,6)\}$ may represent the event 'the score on the first throw is six and the sum of the scores is atleast 11'.

4. The Event 'A but not B' We know that A–B is the set of all those elements which are in A but not in B. Therefore, the set A–B may denote the event 'A but not B'.We know that

$A - B = A \cap B'$

Example 6 Consider the experiment of rolling a die. Let A be the event 'getting a prime number', B be the event 'getting an odd number'. Write the sets representing the events (i) Aor B (ii) A and B (iii) A but not B (iv) 'not A'.

Solution Here $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 3, 5\}$ and $B = \{1, 3, 5\}$

Obviously

(i) 'A or B' $= A \cup B = \{1, 2, 3, 5\}$

(ii) 'A and B' $= A \cap B = \{3,5\}$

(iii) 'A but not B' $= A - B = \{2\}$

(iv) 'not A' $= A' = \{1,4,6\}$

16.3.4 *Mutually exclusive events* In the experiment of rolling a die, a sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Consider events, A 'an odd number appears' and B 'an even number appears'

Clearly the event A excludes the event B and vice versa. In other words, there is no outcome which ensures the occurrence of events A and B simultaneously. Here

$A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$

Clearly $A \cap B = \phi$, i.e., A and B are disjoint sets.

In general, two events A and B are called *mutually exclusive* events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they can not occur simultaneously. In this case the sets A and B are disjoint.

Again in the experiment of rolling a die, consider the events A 'an odd number appears' and event B 'a number less than 4 appears'

Obviously $A = \{1, 3, 5\}$ and $B = \{1, 2, 3\}$

Now $3 \in A$ as well as $3 \in B$

Therefore, A and B are not mutually exclusive events.

Remark Simple events of a sample space are always mutually exclusive.

16.3.5 *Exhaustive events* Consider the experiment of throwing a die. We have S = {1, 2, 3, 4, 5, 6}. Let us define the following events

A: 'a number less than 4 appears',
B: 'a number greater than 2 but less than 5 appears'
and C: 'a number greater than 4 appears'.

Then A = {1, 2, 3}, B = {3,4} and C = {5, 6}. We observe that

$$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S.$$

Such events A, B and C are called exhaustive events. In general, if $E_1, E_2, ..., E_n$ are n events of a sample space S and if

$$E_1 \cup E_2 \cup E_3 \cup ... \cup E_n = \bigcup_{i=1}^{n} E_i = S$$

then $E_1, E_2, ..., E_n$ are called *exhaustive events*.In other words, events $E_1, E_2, ..., E_n$ are said to be exhaustive if atleast one of them necessarily occurs whenever the experiment is performed.

Further, if $E_i \cap E_j = \phi$ for $i \neq j$ i.e., events E_i and E_j are pairwise disjoint and $\bigcup_{i=1}^{n} E_i = S$, then events $E_1, E_2, ..., E_n$ are called *mutually exclusive and exhaustive events*.

We now consider some examples.

Example 7 Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events associated with this experiment

A: 'the sum is even'.
B: 'the sum is a multiple of 3'.
C: 'the sum is less than 4'.
D: 'the sum is greater than 11'.

Which pairs of these events are mutually exclusive?

Solution There are 36 elements in the sample space S = {(x, y): x, y = 1, 2, 3, 4, 5, 6}. Then

A = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)}

B = {(1, 2), (2, 1), (1, 5), (5, 1), (3, 3), (2, 4), (4, 2), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)}

C = {(1, 1), (2, 1), (1, 2)} and D = {(6, 6)}

We find that

$$A \cap B = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 6)\} \neq \phi$$

Therefore, A and B are not mutually exclusive events.

Similarly $A \cap C \neq \phi$, $A \cap D \neq \phi$, $B \cap C \neq \phi$ and $B \cap D \neq \phi$.

Thus, the pairs, (A, C), (A, D), (B, C), (B, D) are not mutually exclusive events.

Also $C \cap D = \phi$ and so C and D are mutually exclusive events.

Example 8 A coin is tossed three times, consider the following events.

A: 'No head appears', B: 'Exactly one head appears' and C: 'Atleast two heads appear'.

Do they form a set of mutually exclusive and exhaustive events?

Solution The sample space of the experiment is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

and A = {TTT}, B = {HTT, THT, TTH}, C = {HHT, HTH, THH, HHH}

Now

$A \cup B \cup C = \{TTT, HTT, THT, TTH, HHT, HTH, THH, HHH\} = S$

Therefore, A, B and C are exhaustive events.

Also, $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$

Therefore, the events are pair-wise disjoint, i.e., they are mutually exclusive.

Hence, A, B and C form a set of mutually exclusive and exhaustive events.

EXERCISE 16.2

1. A die is rolled. Let E be the event "die shows 4" and F be the event "die shows even number". Are E and F mutually exclusive?
2. A die is thrown. Describe the following events:
   (i) A: a number less than 7 (ii) B: a number greater than 7
   (iii) C: a multiple of 3 (iv) D: a number less than 4
   (v) E: an even number greater than 4 (vi) F: a number not less than 3
   Also find $A \cup B$, $A \cap B$, $B \cup C$, $E \cap F$, $D \cap E$, $A - C$, $D - E$, $E \cap F'$, F'
3. An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events:
   A: the sum is greater than 8, B: 2 occurs on either die
   C: the sum is at least 7 and a multiple of 3.
   Which pairs of these events are mutually exclusive?
4. Three coins are tossed once. Let A denote the event 'three heads show", B denote the event "two heads and one tail show", C denote the event" three tails show and D denote the event 'a head shows on the first coin". Which events are
   (i) mutually exclusive? (ii) simple? (iii) Compound?
5. Three coins are tossed. Describe
   (i) Two events which are mutually exclusive.
   (ii) Three events which are mutually exclusive and exhaustive.
   (iii) Two events, which are not mutually exclusive.
   (iv) Two events which are mutually exclusive but not exhaustive.
   (v) Three events which are mutually exclusive but not exhaustive.
6. Two dice are thrown. The events A, B and C are as follows:
   A: getting an even number on the first die.
   B: getting an odd number on the first die.
   C: getting the sum of the numbers on the dice ≤ 5.
   Describe the events
   (i) A' (ii) not B (iii) A or B (iv) A and B
   (v) A but not C (vi) B or C (vii) B and C (viii) $A \cap B' \cap C'$
7. Refer to question 6 above, state true or false: (give reason for your answer)
   (i) A and B are mutually exclusive
   (ii) A and B are mutually exclusive and exhaustive
   (iii) $A = B'$
   (iv) A and C are mutually exclusive
   (v) A and B' are mutually exclusive.
   (vi) A', B', C are mutually exclusive and exhaustive.

16.4 Axiomatic Approach to Probability

In earlier sections, we have considered random experiments, sample space and events associated with these experiments. In our day to day life we use many words about the chances of occurrence of events. Probability theory attempts to quantify these chances of occurrence or non occurrence of events.

In earlier classes, we have studied some methods of assigning probability to an event associated with an experiment having known the number of total outcomes.

Axiomatic approach is another way of describing probability of an event. In this approach some axioms or rules are depicted to assign probabilities.

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval [0,1] satisfying the following axioms

(i) For any event E, $P(E) \geq 0$ (ii) $P(S) = 1$

(iii) If E and F are mutually exclusive events, then $P(E \cup F) = P(E) + P(F)$.

It follows from (iii) that $P(\phi) = 0$. To prove this, we take $F = \phi$ and note that E and ϕ are disjoint events. Therefore, from axiom (iii), we get

$P(E \cup \phi) = P(E) + P(\phi)$ or $P(E) = P(E) + P(\phi)$ i.e. $P(\phi) = 0$.

Let S be a sample space containing outcomes $\omega_1, \omega_2, ..., \omega_n$, i.e.,

$$S = \{\omega_1, \omega_2, ..., \omega_n\}$$

It follows from the axiomatic definition of probability that

(i) $0 \leq P(\omega_i) \leq 1$ for each $\omega_i \in S$

(ii) $P(\omega_1) + P(\omega_2) + ... + P(\omega_n) = 1$

(iii) For any event A, $P(A) = \sum P(\omega_i)$, $\omega_i \in A$.

Note It may be noted that the singleton $\{\omega_i\}$ is called elementary event and for notational convenience, we write $P(\omega_i)$ for $P(\{\omega_i\})$.

For example, in 'a coin tossing' experiment we can assign the number $\frac{1}{2}$ to each of the outcomes H and T.

i.e. $$P(H) = \frac{1}{2} \text{ and } P(T) = \frac{1}{2} \quad ... (1)$$

Clearly this assignment satisfies both the conditions i.e., each number is neither less than zero nor greater than 1 and

$$P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$$

Therefore, in this case we can say that probability of H = $\frac{1}{2}$, and probability of T = $\frac{1}{2}$

If we take $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$ ... (2)

Does this assignment satisfy the conditions of axiomatic approach?

Yes, in this case, probability of H = $\frac{1}{4}$ and probability of T = $\frac{3}{4}$.

We find that both the assignments (1) and (2) are valid for probability of H and T.

In fact, we can assign the numbers p and $(1 - p)$ to both the outcomes such that $0 \leq p \leq 1$ and $P(H) + P(T) = p + (1 - p) = 1$

This assignment, too, satisfies both conditions of the axiomatic approach of probability. Hence, we can say that there are many ways (rather infinite) to assign probabilities to outcomes of an experiment. We now consider some examples.

Example 9 Let a sample space be S = $\{\omega_1, \omega_2, ..., \omega_6\}$. Which of the following assignments of probabilities to each outcome are valid?

Outcomes	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6
(a)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
(b)	1	0	0	0	0	0
(c)	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{3}$
(d)	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{2}$
(e)	0.1	0.2	0.3	0.4	0.5	0.6

Solution (a) Condition (i): Each of the number p(ω_i) is positive and less than one.

Condition (ii): Sum of probabilities

$$= \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=1$$

Therefore, the assignment is valid

(b) Condition (i): Each of the number $p(\omega_i)$ is either 0 or 1.
Condition (ii) Sum of the probabilities = 1 + 0 + 0 + 0 + 0 + 0 = 1
Therefore, the assignment is valid

(c) Condition (i) Two of the probabilities $p(\omega_5)$ and $p(\omega_6)$ are negative, the assignment is not valid

(d) Since $p(\omega_6) = \frac{3}{2} > 1$, the assignment is not valid

(e) Since, sum of probabilities = 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 = 2.1, the assignment is not valid.

16.4.1 ***Probability of an event*** Let S be a sample space associated with the experiment 'examining three consecutive pens produced by a machine and classified as Good (non-defective) and bad (defective)'. We may get 0, 1, 2 or 3 defective pens as result of this examination.

A sample space associated with this experiment is

S = {BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG},

where B stands for a defective or bad pen and G for a non – defective or good pen.

Let the probabilities assigned to the outcomes be as follows

Sample point:	BBB	BBG	BGB	GBB	BGG	GBG	GGB	GGG
Probability:	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Let event A: there is exactly one defective pen and event B: there are atleast two defective pens.

Hence A = {BGG, GBG, GGB} and B = {BBG, BGB, GBB, BBB}

Now $\quad P(A) = \sum P(\omega_i), \forall \omega_i \in A$

$$= P(BGG) + P(GBG) + P(GGB) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

and $\quad P(B) = \sum P(\omega_i), \forall \omega_i \in B$

$$= P(BBG) + P(BGB) + P(GBB) + P(BBB) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$

Let us consider another experiment of 'tossing a coin "twice"

The sample space of this experiment is S = {HH, HT, TH, TT}

Let the following probabilities be assigned to the outcomes

$$P(HH) = \frac{1}{4},\ P(HT) = \frac{1}{7},\ P(TH) = \frac{2}{7},\ P(TT) = \frac{9}{28}$$

Clearly this assignment satisfies the conditions of axiomatic approach. Now, let us find the probability of the event E: 'Both the tosses yield the same result'.

Here $\quad E = \{HH, TT\}$

Now $\quad P(E) = \Sigma\, P(w_i)$, for all $w_i \in E$

$$= P(HH) + P(TT) = \frac{1}{4}+\frac{9}{28}=\frac{4}{7}$$

For the event F: 'exactly two heads', we have F = {HH}

and $\quad P(F) = P(HH) = \dfrac{1}{4}$

16.4.2 *Probabilities of equally likely outcomes*

Let a sample space of an experiment be

$$S = \{\omega_1, \omega_2, ..., \omega_n\}.$$

Let all the outcomes are equally likely to occur, i.e., the chance of occurrence of each simple event must be same.

i.e. $\quad P(\omega_i) = p$, for all $\omega_i \in S$ where $0 \le p \le 1$

Since $\quad \displaystyle\sum_{i=1}^{n} P(\omega_i)=1$ i.e., $p + p + ... + p$ (n times) = 1

or $\quad np = 1$ i.e., $p = \dfrac{1}{n}$

Let S be a sample space and E be an event, such that $n(S) = n$ and $n(E) = m$. If each out come is equally likely, then it follows that

$$P(E) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to E}}{\text{Total possible outcomes}}$$

16.4.3 *Probability of the event 'A or B'*

Let us now find the probability of event 'A or B', i.e., P (A ∪ B)

Let $\quad$ A = {HHT, HTH, THH} and B = {HTH, THH, HHH} be two events associated with 'tossing of a coin thrice'

Clearly $A \cup B = \{HHT, HTH, THH, HHH\}$

Now $P(A \cup B) = P(HHT) + P(HTH) + P(THH) + P(HHH)$

If all the outcomes are equally likely, then

$$P(A \cup B) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Also $P(A) = P(HHT) + P(HTH) + P(THH) = \frac{3}{8}$

and $P(B) = P(HTH) + P(THH) + P(HHH) = \frac{3}{8}$

Therefore $P(A) + P(B) = \frac{3}{8} + \frac{3}{8} = \frac{6}{8}$

It is clear that $P(A \cup B) \neq P(A) + P(B)$

The points HTH and THH are common to both A and B . In the computation of P(A) + P(B) the probabilities of points HTH and THH, i.e., the elements of $A \cap B$ are included twice. Thus to get the probability $P(A \cup B)$ we have to subtract the probabilities of the sample points in $A \cap B$ from P(A) + P(B)

i.e. $$P(A \cup B) = P(A) + P(B) - \sum P(\omega_i),\ \forall \omega_i \in A \cap B$$
$$= P(A) + P(B) - P(A \cap B)$$

Thus we observe that, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

In general, if A and B are any two events associated with a random experiment, then by the definition of probability of an event, we have

$$P(A \cup B) = \sum p(\omega_i),\ \forall \omega_i \in A \cup B.$$

Since $A \cup B = (A–B) \cup (A \cap B) \cup (B–A)$,

we have

$P(A \cup B) = [\sum P(\omega_i) \forall \omega_i \in (A–B)] + [\sum P(\omega_i) \forall \omega_i \in A \cap B] + [\sum P(\omega_i) \forall \omega_i \in B – A]$

(because A–B, $A \cap B$ and B – A are mutually exclusive) ... (1)

Also $P(A) + P(B) = [\sum p(\omega_i)\ \forall \omega_i \in A] + [\sum p(\omega_i)\ \forall \omega_i \in B]$

$= [\sum P(\omega_i) \forall \omega_i \in (A–B) \cup (A \cap B)] + [\sum P(\omega_i) \forall \omega_i \in (B – A) \cup (A \cap B)]$

$= [\sum P(\omega_i) \forall \omega_i \in (A – B)] + [\sum P(\omega_i) \forall \omega_i \in (A \cap B)] + [\sum P(\omega_i) \forall \omega_i \in (B–A)] +$

$[\sum P(\omega_i) \forall \omega_i \in (A \cap B)]$

$= P(A \cup B) + [\sum P(\omega_i) \forall \omega_i \in A \cap B]$ [using (1)]

$= P(A \cup B) + P(A \cap B)$.

Hence $P(A \cup B) = P(A) + P(B) – P(A \cap B)$.

Alternatively, it can also be proved as follows:

$A \cup B = A \cup (B – A)$, where A and B – A are mutually exclusive,

and $B = (A \cap B) \cup (B – A)$, where $A \cap B$ and B – A are mutually exclusive.

Using Axiom (iii) of probability, we get

$P(A \cup B) = P(A) + P(B – A)$ ... (2)

and $P(B) = P(A \cap B) + P(B – A)$ ... (3)

Subtracting (3) from (2) gives

$$P(A \cup B) - P(B) = P(A) - P(A \cap B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The above result can further be verified by observing the Venn Diagram (Fig 16.1)

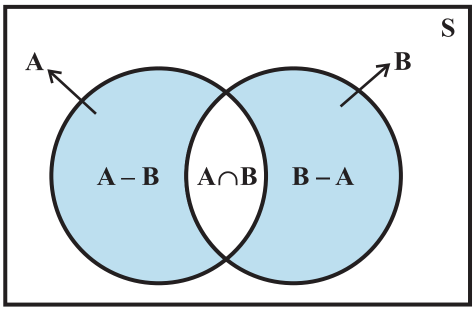

Fig 16.1

If A and B are disjoint sets, i.e., they are mutually exclusive events, then $A \cap B = \phi$

Therefore $\quad P(A \cap B) = P(\phi) = 0$

Thus, for mutually exclusive events A and B, we have

$$P(A \cup B) = P(A) + P(B),$$

which is Axiom (iii) of probability.

16.4.4 ***Probability of event 'not A'*** Consider the event A = {2, 4, 6, 8} associated with the experiment of drawing a card from a deck of ten cards numbered from 1 to 10. Clearly the sample space is S = {1, 2, 3, ...,10}

If all the outcomes 1, 2, ...,10 are considered to be equally likely, then the probability of each outcome is $\frac{1}{10}$

Now $\quad P(A) = P(2) + P(4) + P(6) + P(8)$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{4}{10} = \frac{2}{5}$$

Also event 'not A' = $A' = \{1, 3, 5, 7, 9, 10\}$

Now $\quad P(A') = P(1) + P(3) + P(5) + P(7) + P(9) + P(10)$

$$= \frac{6}{10} = \frac{3}{5}$$

Thus, $\quad P(A') = \frac{3}{5} = 1 - \frac{2}{5} = 1 - P(A)$

Also, we know that A' and A are mutually exclusive and exhaustive events i.e., $A \cap A' = \phi$ and $A \cup A' = S$

or $\quad P(A \cup A') = P(S)$

Now $\quad P(A) + P(A') = 1$, by using axioms (ii) and (iii).

or $\quad P(A') = P(\text{not } A) = 1 - P(A)$

We now consider some examples and exercises having equally likely outcomes unless stated otherwise.

Example 10 One card is drawn from a well shuffled deck of 52 cards. If each outcome is equally likely, calculate the probability that the card will be

(i) a diamond (ii) not an ace
(iii) a black card (i.e., a club or, a spade) (iv) not a diamond
(v) not a black card.

Solution When a card is drawn from a well shuffled deck of 52 cards, the number of possible outcomes is 52.

(i) Let A be the event 'the card drawn is a diamond'
Clearly the number of elements in set A is 13.

Therefore, $P(A) = \frac{13}{52} = \frac{1}{4}$

i.e. Probability of a diamond card $= \frac{1}{4}$

(ii) We assume that the event 'Card drawn is an ace' is B
Therefore 'Card drawn is not an ace' should be B′.

We know that $P(B') = 1 - P(B) = 1 - \frac{4}{52} = 1 - \frac{1}{13} = \frac{12}{13}$

(iii) Let C denote the event 'card drawn is black card'
Therefore, number of elements in the set C = 26

i.e. $P(C) = \frac{26}{52} = \frac{1}{2}$

Thus, Probability of a black card $= \frac{1}{2}$.

(iv) We assumed in (i) above that A is the event 'card drawn is a diamond',
so the event 'card drawn is not a diamond' may be denoted as A' or 'not A'

Now $P(\text{not } A) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$

(v) The event 'card drawn is not a black card' may be denoted as C′ or 'not C'.

We know that $P(\text{not } C) = 1 - P(C) = 1 - \frac{1}{2} = \frac{1}{2}$

Therefore, Probability of not a black card $= \frac{1}{2}$

Example 11 A bag contains 9 discs of which 4 are red, 3 are blue and 2 are yellow. The discs are similar in shape and size. A disc is drawn at random from the bag. Calculate the probability that it will be (i) red, (ii) yellow, (iii) blue, (iv) not blue, (v) either red or blue.

Solution There are 9 discs in all so the total number of possible outcomes is 9.
Let the events A, B, C be defined as

A: 'the disc drawn is red'

B: 'the disc drawn is yellow'

C: 'the disc drawn is blue'.

(i) The number of red discs = 4, i.e., n (A) = 4

Hence $\qquad P(A) = \frac{4}{9}$

(ii) The number of yellow discs = 2, i.e., n (B) = 2

Therefore, $\qquad P(B) = \frac{2}{9}$

(iii) The number of blue discs = 3, i.e., n(C) = 3

Therefore, $\qquad P(C) = \frac{3}{9} = \frac{1}{3}$

(iv) Clearly the event 'not blue' is 'not C'. We know that P(not C) = 1 – P(C)

Therefore $\qquad P(\text{not } C) = 1 - \frac{1}{3} = \frac{2}{3}$

(v) The event 'either red or blue' may be described by the set 'A or C'

Since, A and C are mutually exclusive events, we have

$$P(A \text{ or } C) = P(A \cup C) = P(A) + P(C) = \frac{4}{9} + \frac{1}{3} = \frac{7}{9}$$

Example 12 Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination.

(b) Atleast one of them will not qualify the examination and

(c) Only one of them will qualify the examination.

Solution Let E and F denote the events that Anil and Ashima will qualify the examination, respectively. Given that

$P(E) = 0.05$, $P(F) = 0.10$ and $P(E \cap F) = 0.02$.

Then

(a) The event 'both Anil and Ashima will not qualify the examination' may be expressed as $E' \cap F'$.

Since, E' is 'not E', i.e., Anil will not qualify the examination and F' is 'not F', i.e., Ashima will not qualify the examination.

Also $\qquad E' \cap F' = (E \cup F)'$ (by Demorgan's Law)

Now $\qquad P(E \cup F) = P(E) + P(F) - P(E \cap F)$

or $\qquad P(E \cup F) = 0.05 + 0.10 - 0.02 = 0.13$

Therefore $P(E' \cap F') = P(E \cup F)' = 1 - P(E \cup F) = 1 - 0.13 = 0.87$

(b) P (atleast one of them will not qualify)

= 1 – P(both of them will qualify)

= 1 – 0.02 = 0.98

(c) The event only one of them will qualify the examination is same as the event either (Anil will qualify, and Ashima will not qualify) or (Anil will not qualify and Ashima will qualify) i.e., $E \cap F'$ or $E' \cap F$, where $E \cap F'$ and $E' \cap F$ are mutually exclusive.

Therefore, P(only one of them will qualify) = P(E ∩ F′ or E′ ∩ F)

= P(E ∩ F′) + P(E′ ∩ F) = P (E) – P(E ∩ F) + P(F) – P (E ∩ F)

= 0.05 – 0.02 + 0.10 – 0.02 = 0.11

Example 13 A committee of two persons is selected from two men and two women. What is the probability that the committee will have (a) no man? (b) one man? (c) two men?

Solution The total number of persons = 2 + 2 = 4. Out of these four person, two can be selected in ${}^4C_2$ ways.

(a) No men in the committee of two means there will be two women in the committee. Out of two women, two can be selected in ${}^2C_2 = 1$ way.

Therefore $$\text{P(no man)} = \frac{{}^2C_2}{{}^4C_2} = \frac{1\times 2\times 1}{4\times 3} = \frac{1}{6}$$

(b) One man in the committee means that there is one woman. One man out of 2 can be selected in ${}^2C_1$ ways and one woman out of 2 can be selected in ${}^2C_1$ ways. Together they can be selected in ${}^2C_1 \times {}^2C_1$ ways.

Therefore $$\text{P(One man)} = \frac{{}^2C_1 \times {}^2C_1}{{}^4C_2} = \frac{2\times 2}{2\times 3} = \frac{2}{3}$$

(c) Two men can be selected in ${}^2C_2$ way.

Hence $$\text{P(Two men)} = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{{}^4C_2} = \frac{1}{6}$$

EXERCISE 16.3

1. Which of the following can not be valid assignment of probabilities for outcomes of sample Space S = $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7\}$

Assignment	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7
(a)	0.1	0.01	0.05	0.03	0.01	0.2	0.6
(b)	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
(c)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
(d)	– 0.1	0.2	0.3	0.4	– 0.2	0.1	0.3
(e)	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{15}{14}$

2. A coin is tossed twice, what is the probability that atleast one tail occurs?
3. A die is thrown, find the probability of following events:
   (i) A prime number will appear,

(ii) A number greater than or equal to 3 will appear,
(iii) A number less than or equal to one will appear,
(iv) A number more than 6 will appear,
(v) A number less than 6 will appear.

4. A card is selected from a pack of 52 cards.
(a) How many points are there in the sample space?
(b) Calculate the probability that the card is an ace of spades.
(c) Calculate the probability that the card is (i) an ace (ii) black card.

5. A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. find the probability that the sum of numbers that turn up is (i) 3 (ii) 12

6. There are four men and six women on the city council. If one council member is selected for a committee at random, how likely is it that it is a woman?

7. A fair coin is tossed four times, and a person win Re 1 for each head and lose Rs 1.50 for each tail that turns up.
From the sample space calculate how many different amounts of money you can have after four tosses and the probability of having each of these amounts.

8. Three coins are tossed once. Find the probability of getting
(i) 3 heads (ii) 2 heads (iii) atleast 2 heads
(iv) atmost 2 heads (v) no head (vi) 3 tails
(vii) exactly two tails (viii) no tail (ix) atmost two tails

9. If $\frac{2}{11}$ is the probability of an event, what is the probability of the event 'not A'.

10. A letter is chosen at random from the word 'ASSASSINATION'. Find the probability that letter is (i) a vowel (ii) a consonant

11. In a lottery, a person choses six different natural numbers at random from 1 to 20, and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize. What is the probability of winning the prize in the game? [**Hint** order of the numbers is not important.]

12. Check whether the following probabilities P(A) and P(B) are consistently defined
(i) $P(A) = 0.5$, $P(B) = 0.7$, $P(A \cap B) = 0.6$
(ii) $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cup B) = 0.8$

13. Fill in the blanks in following table:

	P(A)	**P(B)**	**P(A ∩ B)**	**P(A ∪ B)**
(i)	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{15}$	. . .
(ii)	0.35	. . .	0.25	0.6
(iii)	0.5	0.35	. . .	0.7

14. Given $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find P(A or B), if A and B are mutually exclusive events.

15. If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E \text{ and } F) = \frac{1}{8}$, find
(i) P(E or F), (ii) P(not E and not F).

16. Events E and F are such that P(not E or not F) = 0.25, State whether E and F are mutually exclusive.

17. A and B are events such that P(A) = 0.42, P(B) = 0.48 and P(A and B) = 0.16. Determine (i) P(not A), (ii) P(not B) and (iii) P(A or B)

18. In Class XI of a school 40% of the students study Mathematics and 30% study Biology. 10% of the class study both Mathematics and Biology. If a student is selected at random from the class, find the probability that he will be studying Mathematics or Biology.

19. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

20. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, what is the probability of passing the Hindi examination?

21. In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. If one of these students is selected at random, find the probability that

(i) The student opted for NCC or NSS.
(ii) The student has opted neither NCC nor NSS.
(iii) The student has opted NSS but not NCC.

Miscellaneous Examples

Example 14 On her vacations Veena visits four cities (A, B, C and D) in a random order. What is the probability that she visits

(i) A before B?
(ii) A before B and B before C?
(iii) A first and B last?
(iv) A either first or second?
(v) A just before B?

Solution The number of arrangements (orders) in which Veena can visit four cities A, B, C, or D is 4! i.e., 24.Therefore, n (S) = 24.

Since the number of elements in the sample space of the experiment is 24 all of these outcomes are considered to be equally likely. A sample space for the experiment is

S = {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB
BACD, BADC, BDAC, BDCA, BCAD, BCDA
CABD, CADB, CBDA, CBAD, CDAB, CDBA
DABC, DACB, DBCA, DBAC, DCAB, DCBA}

(i) Let the event 'she visits A before B' be denoted by E

Therefore, E = {ABCD, CABD, DABC, ABDC, CADB, DACB
ACBD, ACDB, ADBC, CDAB, DCAB, ADCB}

Thus $$\mathrm{P(E)} = \frac{n(\mathrm{E})}{n(\mathrm{S})} = \frac{12}{24} = \frac{1}{2}$$

(ii) Let the event 'Veena visits A before B and B before C' be denoted by F.
Here F = {ABCD, DABC, ABDC, ADBC}

Therefore, $$\mathrm{P(F)} = \frac{n(\mathrm{F})}{n(\mathrm{S})} = \frac{4}{24} = \frac{1}{6}$$

Students are advised to find the probability in case of (iii), (iv) and (v).

Example 15 Find the probability that when a hand of 7 cards is drawn from a well shuffled deck of 52 cards, it contains (i) all Kings (ii) 3 Kings (iii) atleast 3 Kings.

Solution Total number of possible hands = ${}^{52}C_7$

(i) Number of hands with 4 Kings = ${}^{4}C_4 \times {}^{48}C_3$ (other 3 cards must be chosen from the rest 48 cards)

Hence P (a hand will have 4 Kings) = $\dfrac{{}^{4}C_4 \times {}^{48}C_3}{{}^{52}C_7} = \dfrac{1}{7735}$

(ii) Number of hands with 3 Kings and 4 non-King cards = ${}^{4}C_3 \times {}^{48}C_4$

Therefore P (3 Kings) = $\dfrac{{}^{4}C_3 \times {}^{48}C_4}{{}^{52}C_7} = \dfrac{9}{1547}$

(iii) P(atleast 3 King) = P(3 Kings or 4 Kings)
= P(3 Kings) + P(4 Kings)

$$= \frac{9}{1547} + \frac{1}{7735} = \frac{46}{7735}$$

Example 16 If A, B, C are three events associated with a random experiment, prove that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C)$$
$$- P(B \cap C) + P(A \cap B \cap C)$$

Solution Consider E = B ∪ C so that

$$P(A \cup B \cup C) = P(A \cup E)$$
$$= P(A) + P(E) - P(A \cap E) \quad \text{... (1)}$$

Now

$$P(E) = P(B \cup C)$$
$$= P(B) + P(C) - P(B \cap C) \quad \text{... (2)}$$

Also $A \cap E = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [using distribution property of intersection of sets over the union]. Thus

$$P(A \cap E) = P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$
$$= P(A \cap B) + P(A \cap C) - P[A \cap B \cap C] \quad \text{... (3)}$$

Using (2) and (3) in (1), we get

$$P[A \cup B \cup C] = P(A) + P(B) + P(C) - P(B \cap C)$$
$$- P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Example 17 In a relay race there are five teams A, B, C, D and E.

(a) What is the probability that A, B and C finish first, second and third, respectively.

(b) What is the probability that A, B and C are first three to finish (in any order)
(Assume that all finishing orders are equally likely)

Solution If we consider the sample space consisting of all finishing orders in the first three places, we will have ${}^5P_3$, i.e., $\frac{5!}{(5-3)!} = 5 \times 4 \times 3 = 60$ sample points, each with a probability of $\frac{1}{60}$.

(a) A, B and C finish first, second and third, respectively. There is only one finishing order for this, i.e., ABC.

Thus P(A, B and C finish first, second and third respectively) = $\frac{1}{60}$.

(b) A, B and C are the first three finishers. There will be 3! arrangements for A, B and C. Therefore, the sample points corresponding to this event will be 3! in number.

So P (A, B and C are first three to finish) $= \frac{3!}{60} = \frac{6}{60} = \frac{1}{10}$

Miscellaneous Exercise on Chapter 16

1. A box contains 10 red marbles, 20 blue marbles and 30 green marbles. 5 marbles are drawn from the box, what is the probability that
(i) all will be blue? (ii) atleast one will be green?
2. 4 cards are drawn from a well – shuffled deck of 52 cards. What is the probability of obtaining 3 diamonds and one spade?
3. A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once, determine
(i) P(2) (ii) P(1 or 3) (iii) P(not 3)
4. In a certain lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (a) one ticket (b) two tickets (c) 10 tickets.
5. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among the 100 students, what is the probability that
(a) you both enter the same section?
(b) you both enter the different sections?
6. Three letters are dictated to three persons and an envelope is addressed to each of them, the letters are inserted into the envelopes at random so that each envelope contains exactly one letter. Find the probability that at least one letter is in its proper envelope.
7. A and B are two events such that P(A) = 0.54, P(B) = 0.69 and $P(A \cap B) = 0.35$.
Find (i) $P(A \cup B)$ (ii) $P(A' \cap B')$ (iii) $P(A \cap B')$ (iv) $P(B \cap A')$
8. From the employees of a company, 5 persons are selected to represent them in the managing committee of the company. Particulars of five persons are as follows:

S. No.	Name	Sex	Age in years
1.	Harish	M	30
2.	Rohan	M	33
3.	Sheetal	F	46
4.	Alis	F	28
5.	Salim	M	41

A person is selected at random from this group to act as a spokesperson. What is the probability that the spokesperson will be either male or over 35 years?

9. If 4-digit numbers greater than 5,000 are randomly formed from the digits 0, 1, 3, 5, and 7, what is the probability of forming a number divisible by 5 when, (i) the digits are repeated? (ii) the repetition of digits is not allowed?

10. The number lock of a suitcase has 4 wheels, each labelled with ten digits i.e., from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase?

Summary

In this Chapter, we studied about the axiomatic approach of probability. The main features of this Chapter are as follows:

- ***Sample space***: The set of all possible outcomes
- ***Sample points***: Elements of sample space
- ***Event***: A subset of the sample space
- ***Impossible event*** : The empty set
- ***Sure event***: The whole sample space
- ***Complementary event or 'not event'*** : The set A′ or S – A
- ***Event* A *or* B**: The set $A \cup B$
- ***Event* A *and* B**: The set $A \cap B$
- ***Event* A *and* not B**: The set A – B
- ***Mutually exclusive event***: A and B are mutually exclusive if $A \cap B = \phi$
- ***Exhaustive and mutually exclusive events***: Events $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive if $E_1 \cup E_2 \cup ... \cup E_n = S$ and $E_i \cap E_j = \phi \ \forall\, i \neq j$
- ***Probability***: Number P (ω_i) associated with sample point ω_i such that

  (i) $0 \le P(\omega_i) \le 1$ (ii) $\sum P(\omega_i)$ for all $\omega_i \in S = 1$

  (iii) $P(A) = \sum P(\omega_i)$ for all $\omega_i \in A$. The number P (ω_i) is called *probability of the outcome* ω_i.
- ***Equally likely outcomes***: All outcomes with equal probability

- ***Probability of an event***: For a finite sample space with equally likely outcomes Probability of an event $P(A) = \frac{n(A)}{n(S)}$, where $n(A)$ = number of elements in the set A, $n(S)$ = number of elements in the set S.
- If A and B are any two events, then

  P(A or B) = P(A) + P(B) – P(A and B)

  equivalently, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If A and B are mutually exclusive, then P(A or B) = P(A) + P(B)
- If A is any event, then

  P(not A) = 1 – P(A)

Historical Note

Probability theory like many other branches of mathematics, evolved out of practical consideration. It had its origin in the 16th century when an Italian physician and mathematician Jerome Cardan (1501–1576) wrote the first book on the subject "Book on Games of Chance" (Biber de Ludo Aleae). It was published in 1663 after his death.

In 1654, a gambler Chevalier de Metre approached the well known French Philosopher and Mathematician Blaise Pascal (1623–1662) for certain dice problem. Pascal became interested in these problems and discussed with famous French Mathematician Pierre de Fermat (1601–1665). Both Pascal and Fermat solved the problem independently. Besides, Pascal and Fermat, outstanding contributions to probability theory were also made by Christian Huygenes (1629–1665), a Dutchman, J. Bernoulli (1654–1705), De Moivre (1667–1754), a Frenchman Pierre Laplace (1749–1827), the Russian P.L Chebyshev (1821–1897), A. A Markov (1856–1922) and A. N Kolmogorove (1903–1987). Kolmogorov is credited with the axiomatic theory of probability. His book 'Foundations of Probability' published in 1933, introduces probability as a set function and is considered a classic.

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Appendix 1

INFINITE SERIES

A.1.1 Introduction

As discussed in the Chapter 9 on Sequences and Series, a sequence $a_1, a_2, ..., a_n, ...$ having infinite number of terms is called *infinite sequence* and its indicated sum, i.e., $a_1 + a_2 + a_3 + ... + a_n + ...$ is called an *infinte series* associated with infinite sequence. This series can also be expressed in abbreviated form using the sigma notation, i.e.,

$$a_1 + a_2 + a_3 + \ . \ . \ . \ + a_n + \ . \ . \ . = \sum_{k=1}^{\infty} a_k$$

In this Chapter, we shall study about some special types of series which may be required in different problem situations.

A.1.2 Binomial Theorem for any Index

In Chapter 8, we discussed the Binomial Theorem in which the index was a positive integer. In this Section, we state a more general form of the theorem in which the index is not necessarily a whole number. It gives us a particular type of infinite series, called *Binomial Series*. We illustrate few applications, by examples.

We know the formula

$$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + \ . \ . \ . \ + {}^nC_n\, x^n$$

Here, n is non-negative integer. Observe that if we replace index n by negative integer or a fraction, then the combinations ${}^nC_r$ do not make any sense.

We now state (without proof), the Binomial Theorem, giving an infinite series in which the index is negative or a fraction and not a whole number.

Theorem The formula

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + ...$$

holds whenever $|x| < 1$.

Remark 1. Note carefully the condition $|x| < 1$, i.e., $-1 < x < 1$ is necessary when m is negative integer or a fraction. For example, if we take $x = -2$ and $m = -2$, we obtain

$$(1-2)^{-2}=1+(-2)(-2)+\frac{(-2)(-3)}{1.2}(-2)^2+...$$

or $\qquad 1= 1 + 4 + 12 + . . .$

This is not possible

2. Note that there are infinite number of terms in the expansion of $(1+x)^m$, when m is a negative integer or a fraction

Consider
$$(a+b)^m = \left[a\left(1+\frac{b}{a}\right)\right]^m = a^m\left(1+\frac{b}{a}\right)^m$$

$$= a^m\left[1+m\frac{b}{a}+\frac{m(m-1)}{1.2}\left(\frac{b}{a}\right)^2+...\right]$$

$$= a^m + ma^{m-1}b+\frac{m(m-1)}{1.2}a^{m-2}b^2+...$$

This expansion is valid when $\left|\frac{b}{a}\right|<1$ or equivalently when $| b | < | a |$.

The general term in the expansion of $(a + b)^m$ is

$$\frac{m(m-1)(m-2)...(m-r+1)a^{m-r}b^r}{1.2.3...r}$$

We give below certain particular cases of Binomial Theorem, when we assume $|x|<1$, these are left to students as exercises:

1. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + . . .$
2. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + . . .$
3. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + . . .$
4. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + . . .$

Example 1 Expand $\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}$, when $| x | < 2$.

Solution We have

$$\left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = 1+\frac{\left(-\frac{1}{2}\right)}{1}\left(\frac{-x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2}\left(-\frac{x}{2}\right)^2+...$$

$$= 1+\frac{x}{4}+\frac{3x^2}{32}+...$$

A.1.3 Infinite Geometric Series

From Chapter 9, Section 9.5, a sequence $a_1, a_2, a_3, ..., a_n$ is called G.P., if $\frac{a_{k+1}}{a_k} = r$ (constant) for $k = 1, 2, 3, ..., n-1$. Particularly, if we take $a_1 = a$, then the resulting sequence $a, ar, ar^2, ..., ar^{n-1}$ is taken as the standard form of G.P., where a is first term and r, the common ratio of G.P.

Earlier, we have discussed the formula to find the sum of finite series $a + ar + ar^2 + ... + ar^{n-1}$ which is given by

$$S_n = \frac{a(1-r^n)}{1-r}.$$

In this section, we state the formula to find the sum of infinite geometric series $a + ar + ar^2 + ... + ar^{n-1} + ...$ and illustrate the same by examples.

Let us consider the G.P. $1, \frac{2}{3}, \frac{4}{9}, ...$

Here $a = 1$, $r = \frac{2}{3}$. We have

$$S_n = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right] \qquad ... (1)$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger.

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, $\left(\frac{2}{3}\right)^n$ becomes sufficiently small. In other words, as $n \to \infty, \left(\frac{2}{3}\right)^n \to 0$. Consequently, we find that the sum of infinitely many terms is given by S = 3.

Thus, for infinite geometric progression $a, ar, ar^2, ...$, if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case, $r^n \to 0$ as $n \to \infty$ since $|r|<1$ and then $\frac{ar^n}{1-r} \to 0$. Therefore,

$$S_n \to \frac{a}{1-r} \text{ as } n \to \infty.$$

Symbolically, sum to infinity of infinite geometric series is denoted by S. Thus, we have $S = \frac{a}{1-r}$

For example

(i) $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+... = \frac{1}{1-\frac{1}{2}} = 2$

(ii) $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+... = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$

Example 2 Find the sum to infinity of the G.P. ;

$$\frac{-5}{4}, \frac{5}{16}, \frac{-5}{64}, ...$$

Solution Here $a = \frac{-5}{4}$ and $r = -\frac{1}{4}$. Also $|r|<1$.

Hence, the sum to infinity is $\dfrac{\frac{-5}{4}}{1+\frac{1}{4}} = \dfrac{\frac{-5}{4}}{\frac{5}{4}} = -1$.

A.1.4 Exponential Series

Leonhard Euler (1707 – 1783), the great Swiss mathematician introduced the number e in his calculus text in 1748. The number e is useful in calculus as π in the study of the circle.

Consider the following infinite series of numbers

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+... \qquad ... (1)$$

The sum of the series given in (1) is denoted by the number e

Let us estimate the value of the number e.

Since every term of the series (1) is positive, it is clear that its sum is also positive.

Consider the two sums

$$\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...+\frac{1}{n!}+... \qquad ... (2)$$

and $$\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+....+\frac{1}{2^{n-1}}+... \quad ... (3)$$

Observe that

$$\frac{1}{3!}=\frac{1}{6} \text{ and } \frac{1}{2^2}=\frac{1}{4}, \text{ which gives } \frac{1}{3!}<\frac{1}{2^2}$$

$$\frac{1}{4!}=\frac{1}{24} \text{ and } \frac{1}{2^3}=\frac{1}{8}, \text{ which gives } \frac{1}{4!}<\frac{1}{2^3}$$

$$\frac{1}{5!}=\frac{1}{120} \text{ and } \frac{1}{2^4}=\frac{1}{16}, \text{ which gives } \frac{1}{5!}<\frac{1}{2^4}.$$

Therefore, by analogy, we can say that

$$\frac{1}{n!}<\frac{1}{2^{n-1}}, \text{ when } n>2$$

We observe that each term in (2) is less than the corresponding term in (3),

Therefore $$\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...+\frac{1}{n!}\right)<\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right) \quad ... (4)$$

Adding $\left(1+\frac{1}{1!}+\frac{1}{2!}\right)$ on both sides of (4), we get,

$$\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+...+\frac{1}{n!}+...\right)$$

$$<\left\{\left(1+\frac{1}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right)\right\} \quad ... (5)$$

$$=\left\{1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+...+\frac{1}{2^{n-1}}+...\right)\right\}$$

$$=1+\frac{1}{1-\frac{1}{2}}=1+2=3$$

Left hand side of (5) represents the series (1). Therefore $e<3$ and also $e>2$ and hence $2<e<3$.

Remark The exponential series involving variable x can be expressed as

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^n}{n!}+...$$

Example 3 Find the coefficient of x^2 in the expansion of e^{2x+3} as a series in powers of x.

Solution In the exponential series

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+...$$

replacing x by $(2x + 3)$, we get

$$e^{2x+3} = 1 + \frac{(2x+3)}{1!} + \frac{(2x+3)^2}{2!} + ...$$

Here, the general term is $\dfrac{(2x+3)^n}{n!} = \dfrac{(3+2x)^n}{n!}$. This can be expanded by the Binomial Theorem as

$$\frac{1}{n!}\left[3^n + {}^nC_1 3^{n-1}(2x) + {}^nC_2 3^{n-2}(2x)^2 + ... + (2x)^n\right].$$

Here, the coefficient of x^2 is $\dfrac{{}^nC_2 3^{n-2} 2^2}{n!}$. Therefore, the coefficient of x^2 in the whole series is

$$\sum_{n=2}^{\infty} \frac{{}^nC_2 3^{n-2} 2^2}{n!} = 2\sum_{n=2}^{\infty} \frac{n(n-1)3^{n-2}}{n!}$$

$$= 2\sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} \quad \text{[using } n! = n\,(n-1)\,(n-2)!\text{]}$$

$$= 2\left[1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + ...\right]$$

$$= 2e^3.$$

Thus $2e^3$ is the coefficient of x^2 in the expansion of e^{2x+3}.

Alternatively $e^{2x+3} = e^3 \,.\, e^{2x}$

$$= e^3\left[1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + ...\right]$$

Thus, the coefficient of x^2 in the expansion of e^{2x+3} is $e^3 . \dfrac{2^2}{2!} = 2e^3$

Example 4 Find the value of e^2, rounded off to one decimal place.

Solution Using the formula of exponential series involving x, we have $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... + \dfrac{x^n}{n!} + ...$

Putting $x = 2$, we get

$$e^2 = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \frac{2^5}{5!} + \frac{2^6}{6!} + ...$$

$$= 1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} + \frac{4}{15} + \frac{4}{45} + ...$$

$\geq$ the sum of first seven terms ≥ 7.355.

On the other hand, we have

$$e^2 < \left(1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}\right)+\frac{2^5}{5!}\left(1+\frac{2}{6}+\frac{2^2}{6^2}+\frac{2^3}{6^3}+...\right)$$

$$= 7+\frac{4}{15}\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+...\right) = 7+\frac{4}{15}\left(\frac{1}{1-\frac{1}{3}}\right) = 7+\frac{2}{5}=7.4.$$

Thus, e^2 lies between 7.355 and 7.4. Therefore, the value of e^2, rounded off to one decimal place, is 7.4.

A.1.5 Logarithmic Series

Another very important series is logarithmic series which is also in the form of infinite series. We state the following result without proof and illustrate its application with an example.

Theorem If $|x| < 1$, then

$$\log_e(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-...$$

The series on the right hand side of the above is called the *logarithmic series*.

Note The expansion of $\log_e(1+x)$ is valid for $x = 1$. Substituting $x = 1$ in the expansion of $\log_e(1+x)$, we get

$$\log_e 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$$

Example 5 If α, β are the roots of the equation $x^2 - px + q = 0$, prove that

$$\log_e\left(1+px+qx^2\right) = (\alpha+\beta)x - \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 - ...$$

Solution Right hand side $= \left[\alpha x-\frac{\alpha^2x^2}{2}+\frac{\alpha^3x^3}{3}-...\right]+\left[\beta x-\frac{\beta^2x^2}{2}+\frac{\beta^3x^3}{3}-...\right]$

$$= \log_e(1+\alpha x)+\log(1+\beta x)$$

$$= \log_e\left(1+(\alpha+\beta)x+\alpha\beta x^2\right)$$

$$= \log_e\left(1+px+qx^2\right) = \text{Left hand side.}$$

Here, we have used the facts $\alpha+\beta = p$ and $\alpha\beta = q$. We know this from the given roots of the quadratic equation. We have also assumed that both $|\alpha x| < 1$ and $|\beta x| < 1$.

Appendix 2

MATHEMATICAL MODELLING

A.2.1 Introduction

Much of our progress in the last few centuries has made it necessary to apply mathematical methods to real-life problems arising from different fields – be it Science, Finance, Management etc. The use of Mathematics in solving real-world problems has become widespread especially due to the increasing computational power of digital computers and computing methods, both of which have facilitated the handling of lengthy and complicated problems. The process of translation of a real-life problem into a mathematical form can give a better representation and solution of certain problems. The process of translation is called Mathematical Modelling.

Here we shall familiaries you with the steps involved in this process through examples. We shall first talk about what a mathematical model is, then we discuss the steps involved in the process of modelling.

A.2.2 Preliminaries

Mathematical modelling is an essential tool for understanding the world. In olden days the Chinese, Egyptians, Indians, Babylonians and Greeks indulged in understanding and predicting the natural phenomena through their knowledge of mathematics. The architects, artisans and craftsmen based many of their works of art on geometric prinicples.

Suppose a surveyor wants to measure the height of a tower. It is physically very difficult to measure the height using the measuring tape. So, the other option is to find out the factors that are useful to find the height. From his knowledge of trigonometry, he knows that if he has an angle of elevation and the distance of the foot of the tower to the point where he is standing, then he can calculate the height of the tower.

So, his job is now simplified to find the angle of elevation to the top of the tower and the distance from the foot of the tower to the point where he is standing. Both of which are easily measurable. Thus, if he measures the angle of elevation as $40°$ and the distance as 450m, then the problem can be solved as given in Example 1.

Example 1 The angle of elevation of the top of a tower from a point O on the ground, which is 450 m away from the foot of the tower, is $40°$. Find the height of the tower.

Solution We shall solve this in different steps.

Step 1 We first try to understand the real problem. In the problem a tower is given and its height is to be measured. Let h denote the height. It is given that the horizontal distance of the foot of the tower from a particular point O on the ground is 450 m. Let d denotes this distance. Then $d = 450$m. We also know that the angle of elevation, denoted by θ, is $40°$.

The real problem is to find the height h of the tower using the known distance d and the angle of elevation θ.

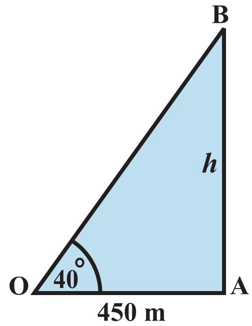


Fig 1

Step 2 The three quantities mentioned in the problem are height, distance and angle of elevation.

So we look for a relation connecting these three quantities. This is obtained by expressing it geometrically in the following way (Fig 1).

AB denotes the tower. OA gives the horizontal distance from the point O to foot of the tower. ∠AOB is the angle of elevation. Then we have

$$\tan\theta = \frac{h}{d} \text{ or } h = d\tan\theta \qquad \text{... (1)}$$

This is an equation connecting θ, h and d.

Step 3 We use Equation (1) to solve h. We have $\theta = 40°$. and $d = 450$m. Then we get $h = \tan 40° \times 450 = 450 \times 0.839 = 377.6$m

Step 4 Thus we got that the height of the tower approximately 378m.

Let us now look at the different steps used in solving the problem. In step 1, we have studied the real problem and found that the problem involves three parameters height, distance and angle of elevation. That means in this step we have *studied the real-life problem and identified the parameters.*

In the Step 2, we used some geometry and found that the problem can be represented geometrically as given in Fig 1. Then we used the trigonometric ratio for the "tangent" function and found the relation as

$$h = d\tan\theta$$

So, in this step we formulated the problem mathematically. That means we found an equation representing the real problem.

In Step 3, we solved the mathematical problem and got that $h = 377.6$m. That is we found

Solution of the problem.

In the last step, we interpreted the solution of the problem and stated that the height of the tower is approximately 378m. We call this as

Interpreting the mathematical solution to the real situation

In fact these are the steps mathematicians and others use to study various real-life situations. We shall consider the question, "why is it necessary to use mathematics to solve different situations."

Here are some of the examples where mathematics is used effectively to study various situations.

1. Proper flow of blood is essential to transmit oxygen and other nutrients to various parts of the body in humanbeings as well as in all other animals. Any constriction in the blood vessel or any change in the characteristics of blood vessels can change the flow and cause damages ranging from minor discomfort to sudden death. The problem is to find the relationship between blood flow and physiological characteristics of blood vessel.
2. In cricket a third umpire takes decision of a LBW by looking at the trajectory of a ball, simulated, assuming that the batsman is not there. Mathematical equations are arrived at, based on the known paths of balls before it hits the batsman's leg. This simulated model is used to take decision of LBW.
3. Meteorology department makes weather predictions based on mathematical models. Some of the parameters which affect change in weather conditions are temperature, air pressure, humidity, wind speed, etc. The instruments are used to measure these parameters which include thermometers to measure temperature, barometers to measure airpressure, hygrometers to measure humidity, anemometers to measure wind speed. Once data are received from many stations around the country and feed into computers for further analysis and interpretation.

4. Department of Agriculture wants to estimate the yield of rice in India from the standing crops. Scientists identify areas of rice cultivation and find the average yield per acre by cutting and weighing crops from some representative fields. Based on some statistical techniques decisions are made on the average yield of rice.

How do mathematicians help in solving such problems? They sit with experts in the area, for example, a physiologist in the first problem and work out a mathematical equivalent of the problem. This equivalent consists of one or more equations or inequalities etc. which are called the mathematical models. Then solve the model and interpret the solution in terms of the original problem. Before we explain the process, we shall discuss what a mathematical model is.

A mathematical model is a representation which comprehends a situation.

An interesting geometric model is illustrated in the following example.

Example 2 (Bridge Problem) Konigsberg is a town on the Pregel River, which in the 18th century was a German town, but now is Russian. Within the town are two river islands that are connected to the banks with seven bridges as shown in (Fig 2).

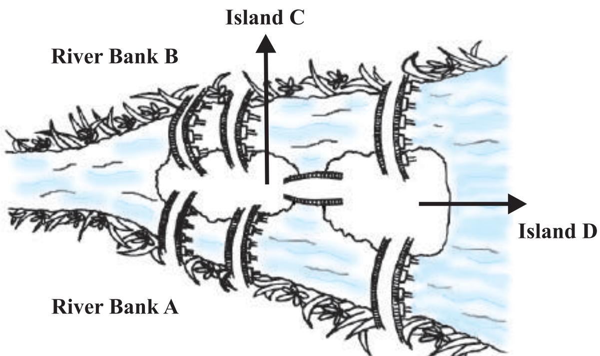


Fig 2

People tried to walk around the town in a way that only crossed each bridge once, but it proved to be difficult problem. Leonhard Euler, a Swiss mathematician in the service of the Russian empire Catherine the Great, heard about the problem. In 1736 Euler proved that the walk was not possible to do. He proved this by inventing a kind of diagram called a network, that is made up of vertices (dots where lines meet) and arcs (lines) (Fig3).

He used four dots (vertices) for the two river banks and the two islands. These have been marked A, B and C, D. The seven lines (arcs) are the seven bridges. You can see that 3 bridges (arcs) join to riverbank, A, and 3 join to riverbank B. 5 bridges (arcs) join to island C, and 3 join to island D. This means that all the vertices have an odd number of arcs, so they are called odd vertices (An even vertex would have to have an even number of arcs joining to it).

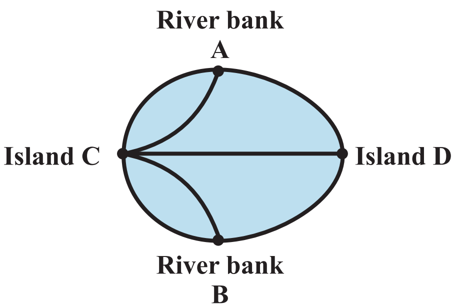


Fig 3

Remember that the problem was to travel around town crossing each bridge only once. On Euler's network this meant tracing over each arc only once, visiting all the vertices. Euler proved it could not be done because he worked out that, to have an odd vertex you would have to begin or end the trip at that vertex. (Think about it). Since there can only be one beginning and one end, there can only be two odd vertices if you are to trace over each arc only once. Since the bridge problem has 4 odd vertices, it just not possible to do!

After Euler proved his Theorem, much water has flown under the bridges in Konigsberg. In 1875, an extra bridge was built in Konigsberg, joining the land areas of river banks A and B (Fig 4). Is it possible now for the Konigsbergians to go round the city, using each bridge only once?

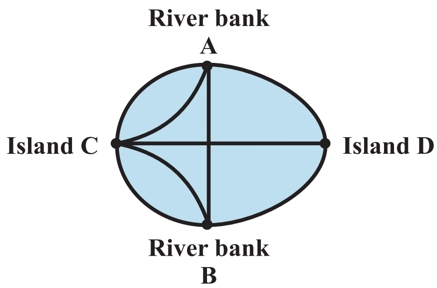


Fig 4

Here the situation will be as in Fig 4. After the addition of the new edge, both the vertices A and B have become even degree vertices. However, D and C still have odd degree. So, it is possible for the Konigsbergians to go around the city using each bridge exactly once.

The invention of networks began a new theory called graph theory which is now used in many ways, including planning and mapping railway networks (Fig 4).

A.2.3 What is Mathematical Modelling?

Here, we shall define what mathematical modelling is and illustrate the different processes involved in this through examples.

Definition Mathematical modelling is an attempt to study some part (or form) of the real-life problem in mathematical terms.

Conversion of physical situation into mathematics with some suitable conditions is known as mathematical modelling. Mathematical modelling is nothing but a technique and the pedagogy taken from fine arts and not from the basic sciences. Let us now understand the different processes involved in Mathematical Modelling. Four steps are involved in this process. As an illustrative example, we consider the modelling done to study the motion of a simple pendulum.

Understanding the problem

This involves, for example, understanding the process involved in the motion of simple pendulum. All of us are familiar with the simple pendulum. This pendulum is simply a mass (known as bob) attached to one end of a string whose other end is fixed at a point. We have studied that the motion of the simple pendulum is periodic. The period depends upon the length of the string and acceleration due to gravity. So, what we need to find is the period of oscillation. Based on this, we give a precise statement of the problem as

Statement How do we find the period of oscillation of the simple pendulum?
The next step is formulation.

Formulation Consists of two main steps.
1. Identifying the relevant factors In this, we find out what are the factors/ parameters involved in the problem. For example, in the case of pendulum, the factors are period of oscillation (T), the mass of the bob (m), effective length (l) of the pendulum which is the distance between the point of suspension to the centre of mass of the bob. Here, we consider the length of string as effective length of the pendulum and acceleration due to gravity (g), which is assumed to be constant at a place.

So, we have identified four parameters for studying the problem. Now, our purpose is to find T. For this we need to understand what are the parameters that affect the period which can be done by performing a simple experiment.

We take two metal balls of two different masses and conduct experiment with each of them attached to two strings of equal lengths. We measure the period of oscillation. We make the observation that there is no appreciable change of the period with mass. Now, we perform the same experiment on equal mass of balls but take strings of different lengths and observe that there is clear dependence of the period on the length of the pendulum.

This indicates that the mass m is not an *essential parameter* for finding period whereas the length l is an essential parameter.

This process of searching the **essential parameters** is necessary before we go to the next step.

2. Mathematical description This involves finding an equation, inequality or a geometric figure using the parameters already identified.

In the case of simple pendulum, experiments were conducted in which the values of period T were measured for different values of l. These values were plotted on a graph which resulted in a curve that resembled a parabola. It implies that the relation between T and l could be expressed

$$T^2 = kl \qquad \text{... (1)}$$

It was found that $k = \dfrac{4\pi^2}{g}$. This gives the equation

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad \text{... (2)}$$

Equation (2) gives the mathematical formulation of the problem.

Finding the solution The mathematical formulation rarely gives the answer directly. Usually we have to do some operation which involves solving an equation, calculation or applying a theorem etc. In the case of simple pendulums the solution involves applying the formula given in Equation (2).

The period of oscillation calculated for two different pendulums having different lengths is given in Table 1

Table 1

l	225 cm	275cm
T	3.04 sec	3.36 sec

The table shows that for l = 225 cm, T = 3.04 sec and for l = 275 cm, T = 3.36 sec.

Interpretation/Validation

A mathematical model is an attempt to study, the essential characteristic of a real life problem. Many times model equations are obtained by assuming the situation in an idealised context. The model will be useful only if it explains all the facts that we would like it to explain. Otherwise, we will reject it, or else, improve it, then test it again. In other words, ***we measure the effectiveness of the model by comparing the results obtained from the mathematical model, with the known facts about the real problem. This process is called validation of the model.*** In the case of simple pendulum, we conduct some experiments on the pendulum and find out period of oscillation. The results of the experiment are given in Table 2.

Table 2

Periods obtained experimentally for four different pendulums

Mass (gms)	Length (cms)	Time (secs)
385	275	3.371
	225	3.056
230	275	3.352
	225	3.042

Now, we compare the measured values in Table 2 with the calculated values given in Table 1.

The difference in the observed values and calculated values gives the error. For example, for l = 275 cm, and mass m = 385 gm,

$$\text{error} = 3.371 - 3.36 = 0.011$$

which is small and the model is accepted.

Once we accept the model, we have to interpret the model. ***The process of describing the solution in the context of the real situation is called interpretation of the model.*** In this case, we can interpret the solution in the following way:

(a) The period is directly proportional to the square root of the length of the pendulum.

(b) It is inversely proportional to the square root of the acceleration due to gravity.

Our validation and interpretation of this model shows that the mathematical model is in good agreement with the practical (or observed) values. But we found that there is some error in the calculated result and measured result. This is because we have neglected the mass of the string and resistance of the medium. So, in such situation we look for a better model and this process continues.

This leads us to an important observation. The real world is far too complex to understand and describe completely. We just pick one or two main factors to be completely accurate that may influence the situation. Then try to obtain a simplified model which gives some information about the situation. We study the simple situation with this model expecting that we can obtain a better model of the situation.

Now, we summarise the main process involved in the modelling as

(a) Formulation (b) Solution (c) Interpretation/Validation

The next example shows how modelling can be done using the techniques of finding graphical solution of inequality.

Example 3 A farm house uses atleast 800 kg of special food daily. The special food is a mixture of corn and soyabean with the following compositions

Table 3

Material	Nutrients present per Kg Protein	Nutrients present per Kg Fibre	Cost per Kg
Corn	.09	.02	Rs 10
Soyabean	.60	.06	Rs 20

The dietary requirements of the special food stipulate atleast 30% protein and at most 5% fibre. Determine the daily minimum cost of the food mix.

Solution Step 1 Here the objective is to minimise the total daily cost of the food which is made up of corn and soyabean. So the variables (factors) that are to be considered are

x = the amount of corn

y = the amount of soyabean

z = the cost

Step 2 The last column in Table 3 indicates that z, x, y are related by the equation

$$z = 10x + 20y \quad \text{... (1)}$$

The problem is to minimise z with the following constraints:

(a) The farm used atleast 800 kg food consisting of corn and soyabean

$$\text{i.e., } x + y \geq 800 \quad \text{... (2)}$$

(b) The food should have atleast 30% protein dietary requirement in the proportion as given in the first column of Table 3. This gives

$$0.09x + 0.6y \geq 0.3\,(x + y) \quad \text{... (3)}$$

(c) Similarly the food should have atmost 5% fibre in the proportion given in 2nd column of Table 3. This gives

$$0.02x + 0.06\,y \leq 0.05\,(x + y) \quad \text{... (4)}$$

We simplify the constraints given in (2), (3) and (4) by grouping all the coefficients of x, y.

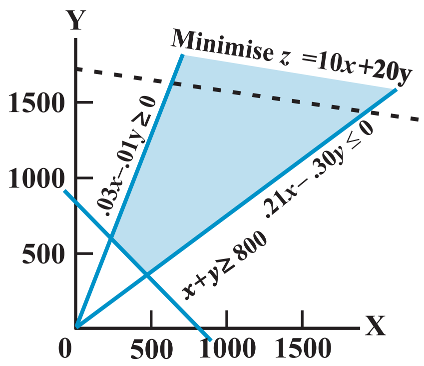
Then the problem can be restated in the following mathematical form.

Statement Minimise z subject to

$$x + y \geq 800$$
$$0.21x - .30y \leq 0$$
$$0.03x - .01y \geq 0$$

This gives the formulation of the model.

Step 3 This can be solved graphically. The shaded region in Fig 5 gives the possible solution of the equations.



Optimum: $x = 470.6$ **kg**
$y = 329.4$ **kg**
$z =$ **Rs 11294**

Fig 5

From the graph it is clear that the minimum value is got at the point (470.6,329.4) i.e., $x = 470.6$ and $y = 329.4$. This gives the value of z as $z = 10 \times 470.6 + 20 \times 329.4 = 11294$

This is the mathematical solution.

Step 4 The solution can be interpreted as saying that, "The minimum cost of the special food with corn and soyabean having the required portion of nutrient contents, protein and fibre is Rs 11294 and we obtain this minimum cost if we use 470.6 kg of corn and 329.4 kg of soyabean."

In the next example, we shall discuss how modelling is used to study the population of a country at a particular time.

Example 4 Suppose a population control unit wants to find out "how many people will be there in a certain country after 10 years"

Step 1 Formulation We first observe that the population changes with time and it increases with birth and decreases with deaths.

We want to find the population at a particular time. Let t denote the time in years. Then t takes values 0, 1, 2, ..., $t = 0$ stands for the present time, $t = 1$ stands for the next year etc. For any time t, let $p(t)$ denote the population in that particular year.

Suppose we want to find the population in a particular year, say $t_0 = 2006$. How will we do that. We find the population by Jan. 1st, 2005. Add the number of births in that year and subtract the number of deaths in that year. Let B(t) denote the number of births in the one year between t and $t + 1$ and D(t) denote the number of deaths between t and $t + 1$. Then we get the relation

$$P(t + 1) = P(t) + B(t) - D(t)$$

Now we make some assumptions and definitions

1. $\dfrac{B(t)}{P(t)}$ is called the *birth rate* for the time interval t to $t+1$.

2. $\dfrac{D(t)}{P(t)}$ is called the *death rate* for the time interval t to $t+1$.

Assumptions

1. The birth rate is the same for all intervals. Likewise, the death rate is the same for all intervals. This means that there is a constant b, called the birth rate, and a constant d, called the death rate so that, for all $t \geq 0$,

$$b = \frac{B(t)}{P(t)} \quad \text{and} \quad d = \frac{D(t)}{P(t)} \qquad \text{... (1)}$$

2. There is no migration into or out of the population; i.e., the only source of population change is birth and death.

As a result of assumptions 1 and 2, we deduce that, for $t \geq 0$,

$$\begin{aligned} P(t+1) &= P(t) + B(t) - D(t) \\ &= P(t) + bP(t) - dP(t) \\ &= (1 + b - d)\, P(t) \qquad \text{... (2)} \end{aligned}$$

Setting $t = 0$ in (2) gives

$$P(1) = (1 + b - d) P(0) \qquad \text{... (3)}$$

Setting $t = 1$ in Equation (2) gives

$$\begin{aligned} P(2) &= (1 + b - d)\, P(1) \\ &= (1 + b - d)\,(1 + b - d)\, P(0) \qquad \text{(Using equation 3)} \\ &= (1 + b - d)^2\, P(0) \end{aligned}$$

Continuing this way, we get

$$P(t) = (1 + b - d)^t\, P(0) \qquad \text{... (4)}$$

for $t = 0, 1, 2, ...$ The constant $1 + b - d$ is often abbreviated by r and called the *growth rate or*, in more high-flown language, the *Malthusian parameter,* in honor of Robert Malthus who first brought this model to popular attention. In terms of r, Equation (4) becomes

$$P(t) = P(0)r^t \quad , \qquad t = 0, 1, 2, ... \qquad \text{... (5)}$$

$P(t)$ is an example of an *exponential function.* Any function of the form cr^t, where c and r are constants, is an exponential function.

Equation (5) gives the mathematical formulation of the problem.

Step 2 – Solution

Suppose the current population is 250,000,000 and the rates are $b = 0.02$ and $d = 0.01$. What will the population be in 10 years? Using the formula, we calculate P(10).

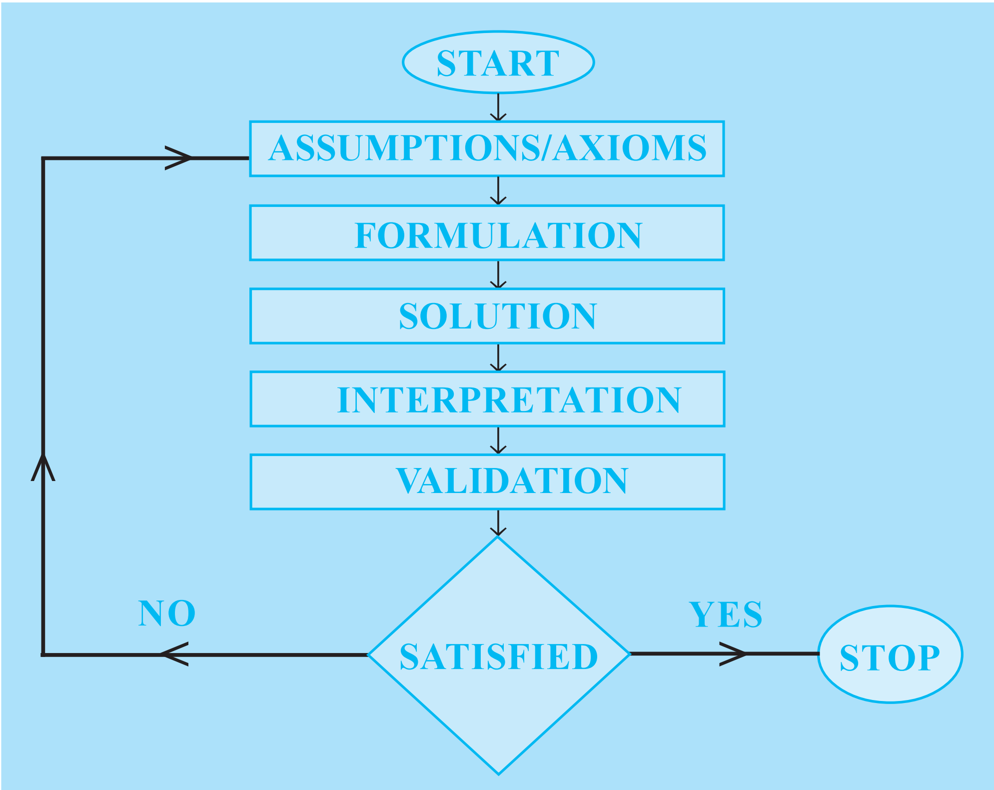
$$\begin{aligned} P(10) &= (1.01)^{10}\,(250{,}000{,}000) \\ &= (1.104622125)\,(250{,}000{,}000) \\ &= 276{,}155{,}531.25 \end{aligned}$$

Step 3 Interpretation and Validation

Naturally, this result is absurd, since one can't have 0.25 of a person.
So, we do some approximation and conclude that the population is 276,155,531 (approximately). Here, we are not getting the exact answer because of the assumptions that we have made in our mathematical model.

The above examples show how modelling is done in variety of situations using different mathematical techniques.

Since a mathematical model is a simplified representation of a real problem, by its very nature, has built-in assumptions and approximations. Obviously, the most important question is to decide whether our model is a good one or not i.e., when the obtained results are interpreted physically whether or not the model gives reasonable answers. If a model is not accurate enough, we try to identify the sources of the shortcomings. It may happen that we need a new formulation, new mathematical manipulation and hence a new evaluation. Thus mathematical modelling can be a cycle of the modelling process as shown in the flowchart given below:



ANSWERS

EXERCISE 1.1

1. (i), (iv), (v), (vi), (vii) and (viii) are sets.

2. (i) $\in$ (ii) $\notin$ (iii) $\notin$ (vi) $\in$ (v) $\in$ (vi) $\notin$

3. (i) A = {–3, –2, –1, 0, 1, 2, 3, 4, 5, 6 } (ii) B = {1, 2, 3, 4, 5}
(iii) C = {17, 26, 35, 44, 53, 62, 71, 80} (iv) D = {2, 3, 5}
(v) E = {T, R, I, G, O, N, M, E, Y} (vi) F = {B, E, T, R}

4. (i) $\{ x : x = 3n, n \in \mathrm{N} \text{ and } 1 \le n \le 4 \}$ (ii) $\{ x : x = 2^n, n \in \mathrm{N} \text{ and } 1 \le n \le 5 \}$
(iii) $\{ x : x = 5^n, n \in \mathrm{N} \text{ and } 1 \le n \le 4 \}$ (iv) $\{ x : x$ is an even natural number$\}$
(v) $\{ x : x = n^2, n \in \mathrm{N} \text{ and } 1 \le n \le 10 \}$

5. (i) A = {1, 3, 5, . . . } (ii) B = {0, 1, 2, 3, 4 }
(iii) C = { –2, –1, 0, 1, 2 } (iv) D = { L, O, Y, A }
(v) E = { February, April, June, September, November }
(vi) F = {*b*, *c*, *d*, *f*, *g*, *h*, *j* }

6. (i) $\leftrightarrow$ (c) (ii) $\leftrightarrow$ (a) (iii) $\leftrightarrow$ (d) (iv) $\leftrightarrow$ (b)

EXERCISE 1.2

1. (i), (iii), (iv)

2. (i) Finite (ii) Infinite (iii) Finite (iv) Infinite (v) Finite

3. (i) Infinite (ii) Finite (iii) Infinite (iv) Finite (v) Infinite

4. (i) Yes (ii) No (iii) Yes (iv) No

5. (i) No (ii) Yes **6.** B= D, E = G

EXERCISE 1.3

1. (i) $\subset$ (ii) $\not\subset$ (iii) $\subset$ (iv) $\not\subset$ (v) $\not\subset$ (vi) $\subset$
(vii) $\subset$

2. (i) False (ii) True (iii) False (iv) True (v) False (vi) True

3. (i), (v), (vii), (viii), (ix), (xi)

4. (i) ϕ, { *a* } (ii) ϕ, { *a* }, { *b* }, { *a*, *b* }
(iii) ϕ, { 1 }, { 2 }, { 3 }, { 1, 2 }, { 1, 3 }, { 2, 3 }, { 1, 2, 3 } (iv) ϕ

5. 1

6. (i) (– 4, 6] (ii) (– 12, –10) (iii) [0, 7)
(iv) [3, 4]

7. (i) $\{ x : x \in \mathrm{R}, -3 < x < 0 \}$ (ii) $\{ x : x \in \mathrm{R}, 6 \le x \le 12 \}$
(iii) $\{ x : x \in \mathrm{R}, 6 < x \le 12 \}$ (iv) $\{ x \ \mathrm{R} : -23 \le x < 5 \}$ **9.** (iii)

EXERCISE 1.4

1. (i) X $\cup$ Y = {1, 2, 3, 5 } (ii) A $\cup$ B = { *a*, *b*, *c*, *e*, *i*, *o*, *u* }
(iii) A $\cup$ B = $\{x : x = 1, 2, 4, 5$ or a multiple of 3$\}$
(iv) A $\cup$ B = $\{x : 1 < x < 10, x \in \mathrm{N}\}$ (v) A $\cup$ B = {1, 2, 3}

2. Yes, $A \cup B = \{ a, b, c \}$ **3.** B

4. (i) $\{ 1, 2, 3, 4, 5, 6 \}$ (ii) $\{1, 2, 3, 4, 5, 6, 7,8 \}$ (iii) $\{3, 4, 5, 6, 7, 8 \}$

(iv) $\{3, 4, 5, 6, 7, 8, 9, 10\}$ (v) $\{1, 2, 3, 4, 5, 6, 7, 8 \}$

(vi) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ (vii) $\{ 3, 4, 5, 6, 7, 8, 9, 10 \}$

5. (i) $X \cap Y = \{ 1, 3 \}$ (ii) $A \cap B = \{ a \}$ (iii) $\{ 3 \}$ (iv) ϕ (v) ϕ

6. (i) $\{ 7, 9, 11 \}$ (ii) $\{ 11, 13 \}$ (iii) ϕ (iv) $\{ 11 \}$

(v) ϕ (vi) $\{ 7, 9, 11 \}$ (vii) ϕ

(viii) $\{ 7, 9, 11 \}$ (ix) $\{7, 9, 11 \}$ (x) $\{ 7, 9, 11, 15 \}$

7. (i) B (ii) C (iii) D (iv) ϕ

(v) $\{ 2 \}$ (vi) $\{ x : x$ is an odd prime number $\}$ **8.** (iii)

9. (i) $\{3, 6, 9, 15, 18, 21\}$ (ii) $\{3, 9, 15, 18, 21 \}$ (iii) $\{3, 6, 9, 12, 18, 21\}$

(iv) $\{4, 8, 16, 20)$ (v) $\{ 2, 4, 8, 10, 14, 16 \}$ (vi) $\{ 5, 10, 20 \}$

(vii) $\{ 20 \}$ (viii) $\{ 4, 8, 12, 16 \}$ (ix) $\{ 2, 6, 10, 14\}$

(x) $\{ 5, 10, 15 \}$ (xi) $\{2, 4, 6, 8, 12, 14, 16\}$ (xii) $\{5, 15, 20\}$

10. (i) $\{ a, c \}$ (ii) $\{f, g \}$ (iii) $\{ b , d \}$

11. Set of irrational numbers **12.** (i) F (ii) F (iii) T (iv) T

EXERCISE 1.5

1. (i) $\{ 5, 6, 7, 8, 9\}$ (ii) $\{1, 3, 5, 7, 9 \}$ (iii) $\{7, 8, 9 \}$

(iv) $\{ 5, 7, 9 \}$ (v) $\{ 1, 2, 3, 4 \}$ (vi) $\{ 1, 3, 4, 5, 6, 7, 9 \}$

2. (i) $\{ d, e, f, g, h\}$ (ii) $\{ a, b, c, h \}$ (iii) $\{ b, d , f, h \}$

(iv) $\{ b, c, d, e)$

3. (i) $\{ x : x$ is an odd natural number $\}$

(ii) $\{ x : x$ is an even natural number $\}$

(iii) $\{ x : x \in \mathbf{N}$ and x is not a multiple of 3 $\}$

(iv) $\{ x : x$ is a positive composite number and $x = 1]$

(v) $\{ x : x$ is a positive integer which is not divisible by 3 or not divisible by 5$\}$

(vi) $\{ x : x \in \mathbf{N}$ and x is not a perfect square $\}$

(vii) $\{ x : x \in \mathbf{N}$ and x is not a perfect cube $\}$

(viii) $\{ x : x \in \mathbf{N}$ and $x \neq 3 \}$ (ix) $\{ x : x \in \mathbf{N}$ and $x \neq 2 \}$

(x) $\{ x : x \in \mathbf{N}$ and $x < 7 \}$ (xi) $\{ x : x \in \mathbf{N}$ and $x \leq \frac{9}{2} \}$

6. A' is the set of all equilateral triangles.

7. (i) U (ii) A (iii) ϕ (iv) ϕ

EXERCISE 1.6

1. 2 **2.** 5 **3.** 50 **4.** 42

5. 30 **6.** 19 **7.** 25, 35 **8.** 60

Miscellaneous Exercise on Chapter 1

1. $A \subset B$, $A \subset C$, $B \subset C$, $D \subset A$, $D \subset B$, $D \subset C$

2. (i) False (ii) False (iii) True (iv) False (v) False
(vi) True

7. False **12.** We may take A = { 1, 2 }, B = { 1, 3 }, C = { 2 , 3 }

13. 325 **14.** 125 **15.** (i) 52, (ii) 30 **16.** 11

EXERCISE 2.1

1. $x = 2$ and $y = 1$ **2.** The number of elements in A × B is 9.

3. $G \times H = \{(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)\}$
$H \times G = \{(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)\}$

4. (i) False
$P \times Q = \{(m, n), (m, m), (n, n), (n, m)\}$
(ii) True
(iii) True

5. $A \times A = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$
$A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$

6. $A = \{a, b\}$, $B = \{x, y\}$

8. $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$
A × B will have $2^4 = 16$ subsets.

9. $A = \{x, y, z\}$ and $B = \{1,2\}$

10. $A = \{-1, 0, 1\}$, remaining elements of
$A \times A$ are $(-1, -1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0), (1, 1)$

EXERCISE 2.2

1. $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$
Domain of $R = \{1, 2, 3, 4\}$
Range of $R = \{3, 6, 9, 12\}$
Co domain of $R = \{1, 2, ..., 14\}$

2. $R = \{(1, 6), (2, 7), (3, 8)\}$
Domain of $R = \{1, 2, 3\}$
Range of $R = \{6, 7, 8\}$

3. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

4. (i) $R = \{(x, y) : y = x - 2 \text{ for } x = 5, 6, 7\}$
(ii) $R = \{(5,3), (6,4), (7,5)\}$. Domain of $R = \{5, 6, 7\}$, Range of $R = \{3, 4, 5\}$

5. (i) $R = \{(1, 1), (1,2), (1, 3), (1, 4), (1, 6), (2\ 4), (2, 6), (2, 2), (4, 4), (6, 6), (3, 3), (3, 6)\}$
(ii) Domain of $R = \{1, 2, 3, 4, 6\}$
(iii) Range of $R = \{1, 2, 3, 4, 6\}$

6. Domain of $R = \{0, 1, 2, 3, 4, 5,\}$
Range of $R = \{5, 6, 7, 8, 9, 10\}$

7. $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

8. No. of relations from A into $B = 2^6$

9. Domain of $R = \mathbf{Z}$
Range of $R = \mathbf{Z}$

EXERCISE 2.3

1. (i) yes, Domain = {2, 5, 8, 11, 14, 17}, Range = {1}
(ii) yes, Domain = (2, 4, 6, 8, 10, 12, 14}, Range = {1, 2, 3, 4, 5, 6, 7}
(iii) No.

2. (i) Domain = **R**, Range = $(-\infty, 0]$
(ii) Domain of function = $\{x : -3 \le x \le 3\}$
Range of function = $\{x : 0 \le x \le 3\}$

3. (i) $f(0) = -5$ (ii) $f(7) = 9$ (iii) $f(-3) = -11$

4. (i) $t(0) = 32$ (ii) $t(28) = \frac{412}{5}$ (iii) $t(-10) = 14$ (iv) 100

5. (i) Range = $(-\infty, 2)$ (ii) Range = $[2, \infty)$ (iii) Range = **R**

Miscellaneous Exercise on Chapter 2

2. 2.1 **3.** Domain of function is set of real numbers except 6 and 2.
4. Domain = $[1, \infty)$, Range = $[0, \infty)$
5. Domain = **R**, Range = non-negative real numbers
6. Range = Any positive real number x such that $0 \le x < 1$
7. $(f + g)\, x = 3x - 2$ **8.** $a = 2,\ b = -1$ **9.** (i) No (ii) No (iii) No
$(f - g)\, x = -x + 4$

$$\left(\frac{f}{g}\right)x = \frac{x+1}{2x-3},\quad x \ne \frac{3}{2}$$

10. (i) Yes, (ii) No **11.** No **12.** Range of $f = \{3, 5, 11, 13\}$

EXERCISE 3.1

1. (i) $\frac{5\pi}{36}$ (ii) $-\frac{19\pi}{72}$ (iii) $\frac{4\pi}{3}$ (iv) $\frac{26\pi}{9}$

2. (i) 39° 22′ 30″ (ii) –229° 5′ 29″ (iii) 300° (iv) 210°

3. 12π **4.** 12° 36′ **5.** $\frac{20\pi}{3}$ **6.** 5 : 4

7. (i) $\frac{2}{15}$ (ii) $\frac{1}{5}$ (iii) $\frac{7}{25}$

EXERCISE 3.2

1. $\sin x = -\frac{\sqrt{3}}{2}, \operatorname{cosec} x = -\frac{2}{\sqrt{3}}, \sec x = -2, \tan x = \sqrt{3}, \cot x = \frac{1}{\sqrt{3}}$

2. $\operatorname{cosec} x = \frac{5}{3}, \cos x = -\frac{4}{5}, \sec x = -\frac{5}{4}, \tan x = -\frac{3}{4}, \cot x = -\frac{4}{3}$

3. $\sin x = -\frac{4}{5}$, $\text{cosec } x = -\frac{5}{4}$, $\cos x = -\frac{3}{5}$, $\sec x = -\frac{5}{3}$, $\tan x = \frac{4}{3}$

4. $\sin x = -\frac{12}{13}$, $\text{cosec } x = -\frac{13}{12}$, $\cos x = \frac{5}{13}$, $\tan x = -\frac{12}{5}$, $\cot x = -\frac{5}{12}$

5. $\sin x = \frac{5}{13}$, $\text{cosec } x = \frac{13}{5}$, $\cos x = -\frac{12}{13}$, $\sec x = -\frac{13}{12}$, $\cot x = -\frac{12}{5}$

6. $\frac{1}{\sqrt{2}}$ 7. 2 8. $\sqrt{3}$ 9. $\frac{\sqrt{3}}{2}$ 10. 1

EXERCISE 3.3

5. (i) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (ii) $2 - \sqrt{3}$

EXERCISE 3.4

1. $\frac{\pi}{3}, \frac{4\pi}{3}, n\pi + \frac{\pi}{3}, n \in \mathbf{Z}$

2. $\frac{\pi}{3}, \frac{5\pi}{3}, 2n\pi \pm \frac{\pi}{3}, n \in \mathbf{Z}$

3. $\frac{5\pi}{6}, \frac{11\pi}{6}, n\pi + \frac{5\pi}{6}, n \in \mathbf{Z}$

4. $\frac{7\pi}{6}, \frac{11\pi}{6}, n\pi + (-1)^n \frac{7\pi}{6}, n \in \mathbf{Z}$

5. $x = \frac{n\pi}{3}$ or $x = n\pi$, $n \in \mathbf{Z}$

6. $x = (2n+1)\frac{\pi}{4}$, or $2n\pi \pm \frac{\pi}{3}$, $n \in \mathbf{Z}$

7. $x = n\pi + (-1)^n \frac{7\pi}{6}$ or $(2n+1)\frac{\pi}{2}$, $n \in \mathbf{Z}$

8. $x = \frac{n\pi}{2}$, or $\frac{n\pi}{2} + \frac{3\pi}{8}$, $n \in \mathbf{Z}$

9. $x = \frac{n\pi}{3}$, or $n\pi \pm \frac{\pi}{3}$, $n \in \mathbf{Z}$

Miscellaneous Exercise on Chapter 3

8. $\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, 2$

9. $\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}, -\sqrt{2}$

10. $\frac{\sqrt{8+2\sqrt{15}}}{4}, \frac{\sqrt{8-2\sqrt{15}}}{4}, 4+\sqrt{15}$

EXERCISE 5.1

1. 3 **2.** 0 **3.** i **4.** $14 + 28\,i$

5. $2 - 7\,i$ **6.** $-\frac{19}{5} - \frac{21i}{10}$ **7.** $\frac{17}{3} + i\frac{5}{3}$ **8.** -4

9. $-\frac{242}{27} - 26i$ **10.** $\frac{-22}{3} - i\frac{107}{27}$ **11.** $\frac{4}{25} + i\frac{3}{25}$ **12.** $\frac{\sqrt{5}}{14} - i\frac{3}{14}$

13. i **14.** $\frac{-7\sqrt{2}}{2}\,i$

EXERCISE 5.2

1. $2, \frac{-2\pi}{3}$ **2.** $2, \frac{5\pi}{6}$ **3.** $\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

4. $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ **5.** $\sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$

6. $3\,(\cos\pi + i\sin\pi)$ **7.** $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ **8.** $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

EXERCISE 5.3

1. $\pm\sqrt{3}\,i$ **2.** $\frac{-1\pm\sqrt{7}\,i}{4}$ **3.** $\frac{-3\pm 3\sqrt{3}\,i}{2}$ **4.** $\frac{-1\pm\sqrt{7}i}{-2}$

5. $\frac{-3\pm\sqrt{11}\,i}{2}$ **6.** $\frac{1\pm\sqrt{7}\,i}{2}$ **7.** $\frac{-1\pm\sqrt{7}\,i}{2\sqrt{2}}$ **8.** $\frac{\sqrt{2}\pm\sqrt{34}\,i}{2\sqrt{3}}$

9. $\frac{-1\pm\sqrt{\left(2\sqrt{2}-1\right)}\,i}{2}$ **10.** $\frac{-1\pm\sqrt{7}\,i}{2\sqrt{2}}$

Miscellaneous Exercise on Chapter 5

1. $2 - 2\,i$ **3.** $\frac{307+599i}{442}$

5. (i) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, (ii) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

6. $\frac{2}{3} \pm \frac{4}{3}i$ 7. $1 \pm \frac{\sqrt{2}}{2}i$ 8. $\frac{5}{27} \pm \frac{\sqrt{2}}{27}i$ 9. $\frac{2}{3} \pm \frac{\sqrt{14}}{21}i$

10. $\sqrt{2}$ 12. (i) $\frac{-2}{5}$, (ii) 0 13. $\frac{1}{\sqrt{2}}, \frac{3\pi}{4}$ 14. $x = 3, y = -3$ 15. 2

17. 1 18. 0 20. 4

EXERCISE 6.1

1. (i) {1, 2, 3, 4} (ii) {... – 3, – 2, –1, 0, 1,2,3,4,}
2. (i) No Solution (ii) {... – 4, – 3}
3. (i) {... – 2, – 1, 0, 1} (ii) (–∞, 2)
4. (i) {–1, 0, 1, 2, 3, ...} (ii) (–2, ∞)
5. (–4, ∞) 6. (– ∞, –3) 7. (–∞, –3] 8. (–∞, 4]
9. (– ∞, 6) 10. (–∞, –6) 11. (–∞, 2] 12. (– ∞, 120]
13. (4, ∞) 14. (–∞, 2] 15. (4, ∞) 16. (–∞, 2]

17. $x < 3$,

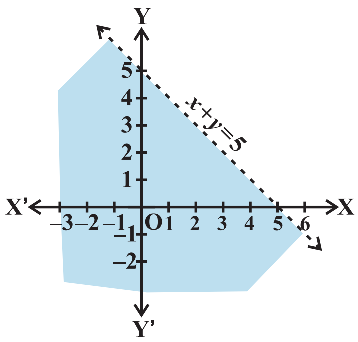

18. $x \geq -1$,

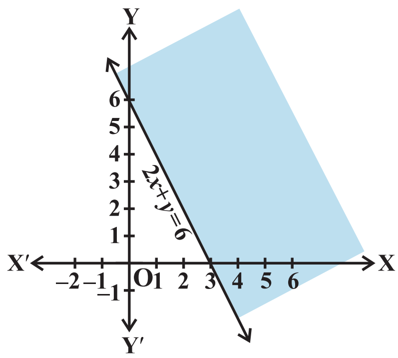

19. $x > -1$,


20. $x \geq -\frac{2}{7}$,

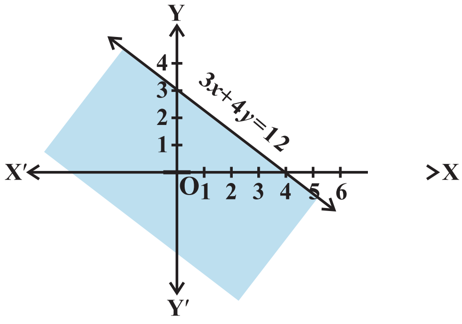

21. Greater than or equal to 35 22. Greater than or equal to 82
23. (5,7), (7,9) 24. (6,8), (8,10), (10,12)
25. 9 cm 26. Greater than or equal to 8 but less than or equal to 22

EXERCISE 6.2

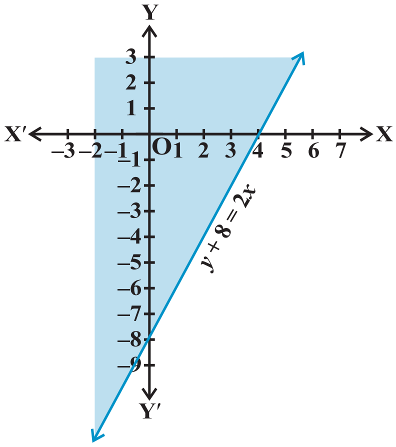
1.


2.


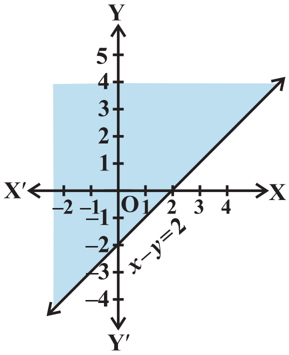
3.



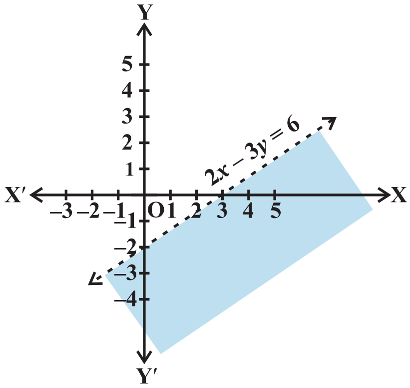
4.



5.



6.



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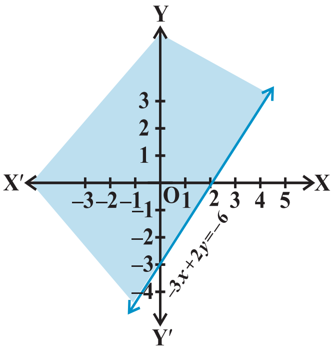

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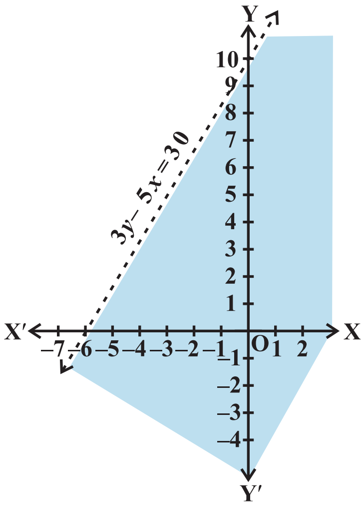
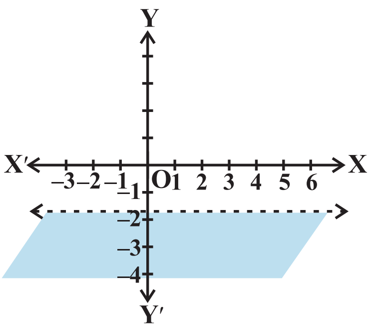
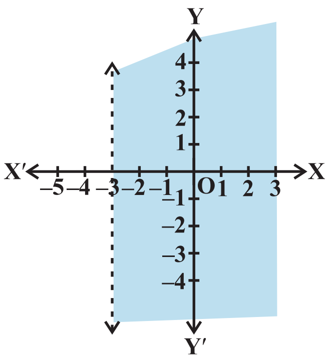


9.

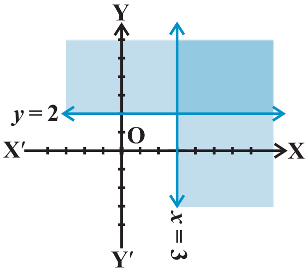


10.

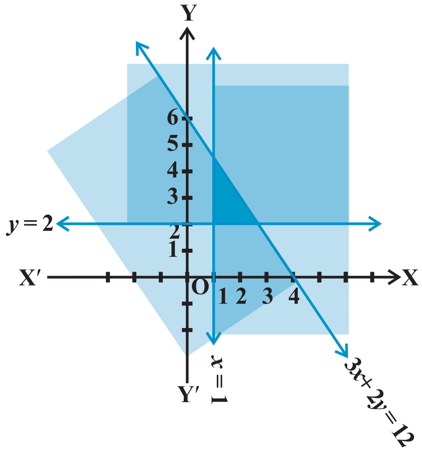


EXERCISE 6.3

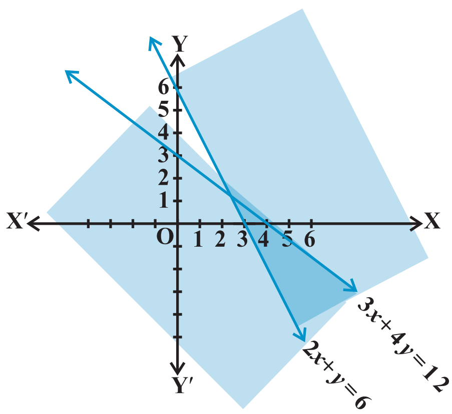
1.



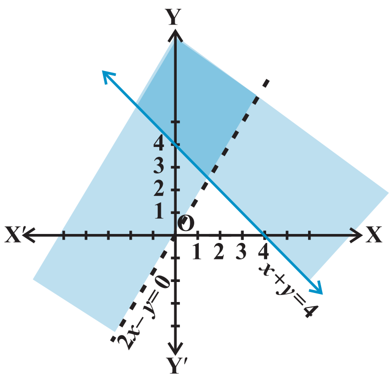
2.



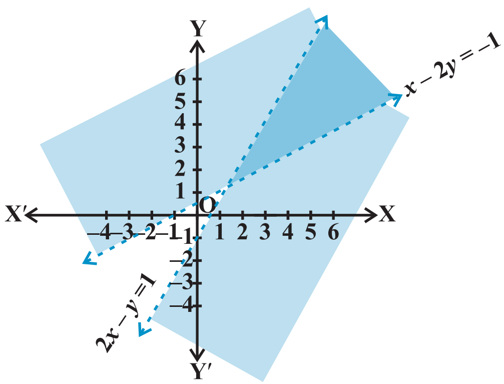
3.



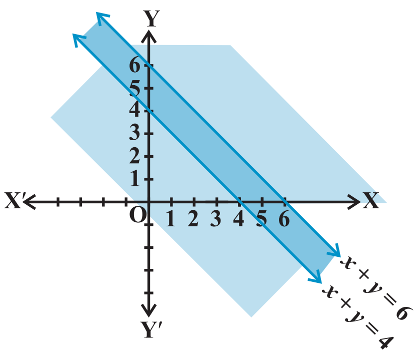
4.



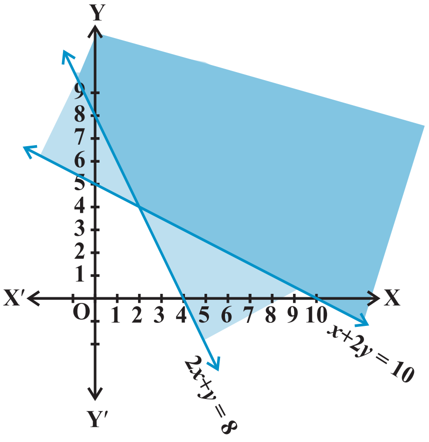
5.



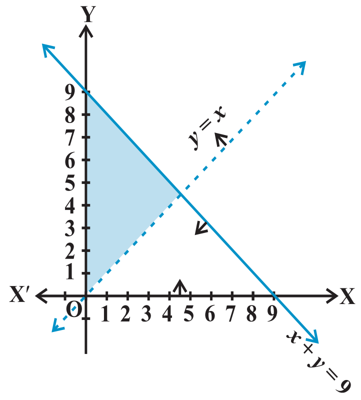
6.



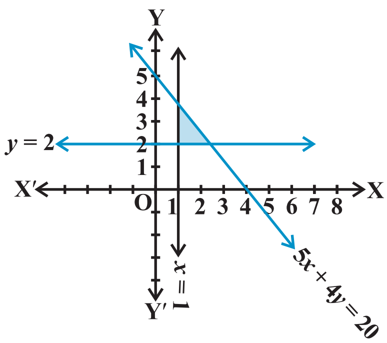
7.


Y
X′
X
Y′
O
x+2y = 10
2x+y = 8

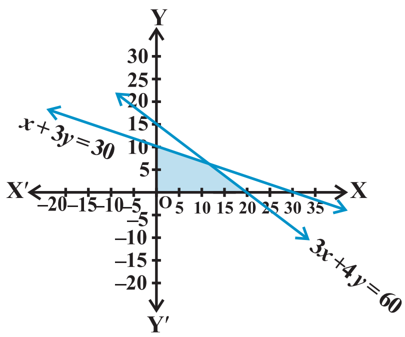
8.


Y
X′
X
O
y = x
x + y = 9

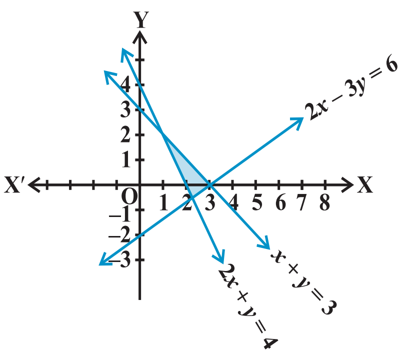
9.


Y
X′
X
Y′
O
y = 2
x = 1
5x + 4y = 20

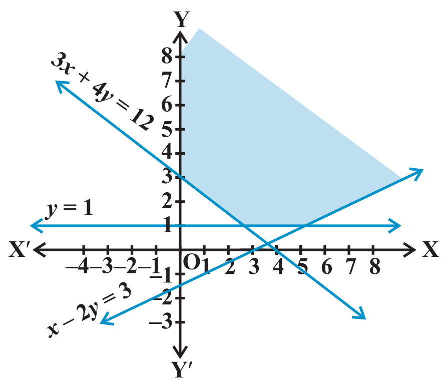
10.


Y
X′
X
Y′
O
x + 3y = 30
3x + 4y = 60

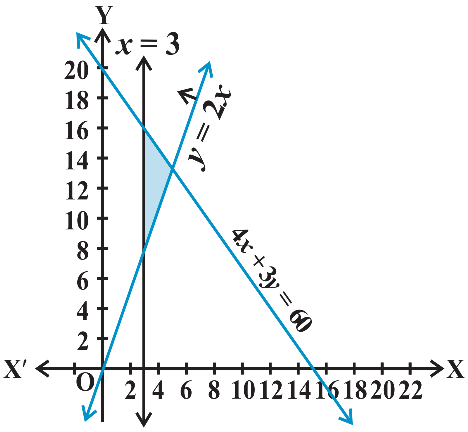
11.


Y
X′
X
O
2x – 3y = 6
x + y = 3
2x + y = 4

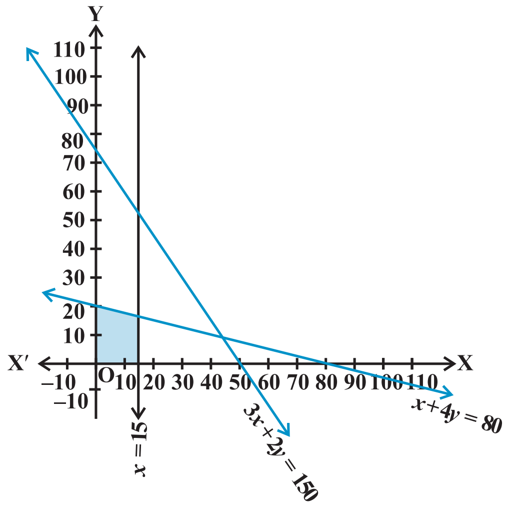
12.


Y
X′
X
Y′
O
3x + 4y = 12
y = 1
x – 2y = 3

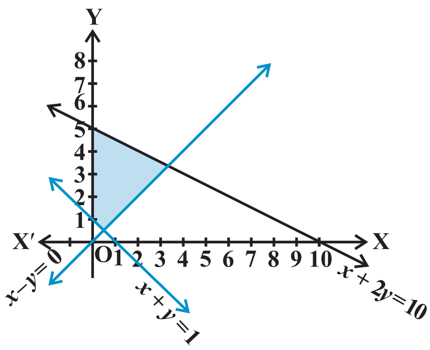
13.



14.



15.

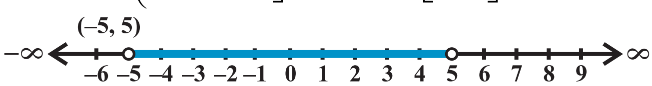


Miscellaneous Exercise on Chapter 6

1. [2, 3] **2.** (0, 1] **3.** [– 4, 2]

4. (– 23, 2] **5.** $\left(\frac{-80}{3}, \frac{-10}{3}\right]$ **6.** $\left[1, \frac{11}{3}\right]$

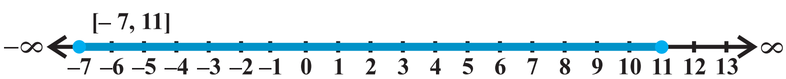
7. (–5, 5)



8. (–1, 7)

9. (5, ∞)

10. [– 7, 11]



11. Between 20°C and 25°C

12. More than 320 litres but less than 1280 litres.

13. More than 562.5 litres but less than 900 litres.

14. $9.6 \le MA \le 16.8$

EXERCISE 7.1

1. (i) 125, (ii) 60. **2.** 108 **3.** 5040 **4.** 336 **5.** 8 **6.** 20

EXERCISE 7.2

1. (i) 40320, (ii) 18 **2.** 30, No **3.** 28 **4.** 64 **5.** (i) 30, (ii) 15120

EXERCISE 7.3

1. 504 **2.** 4536 **3.** 60 **4.** 120, 48 **5.** 56 **6.** 9
7. (i) 3, (ii) 4 **8.** 40320 **9.** (i) 360, (ii) 720, (iii) 240 **10.** 33810
11. (i) 1814400, (ii) 2419200, (iii) 25401600

EXERCISE 7.4

1. 45 **2.** (i) 5, (ii) 6 **3.** 210 **4.** 40 **5.** 2000
6. 778320 **7.** 3960 **8.** 200 **9.** 35

Miscellaneous Exercise on Chapter 7

1. 3600 **2.** 1440 **3.** (i) 504, (ii) 588, (iii) 1632
4. 907200 **5.** 120 **6.** 50400 **7.** 420
8. ${}^4C_1 \times {}^{48}C_4$ **9.** 2880 **10.** ${}^{22}C_7 + {}^{22}C_{10}$ **11.** 151200

EXERCISE 8.1

1. $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

2. $\dfrac{32}{x^5} - \dfrac{40}{x^3} + \dfrac{20}{x} - 5x + \dfrac{5}{8}x^3 - \dfrac{x^5}{32}$

3. $64\,x^6 - 576\,x^5 + 2160\,x^4 - 4320\,x^3 + 4860\,x^2 - 2916\,x + 729$

4. $\dfrac{x^5}{243} + \dfrac{5x^3}{81} + \dfrac{10}{27}x + \dfrac{10}{9x} + \dfrac{5}{3x^3} + \dfrac{1}{x^5}$

5. $x^6 + 6x^4 + 15x^2 + 20 + \dfrac{15}{x^2} + \dfrac{6}{x^4} + \dfrac{1}{x^6}$

6. 884736 **7.** 11040808032 **8.** 104060401

9. 9509900499 **10.** $(1.1)^{10000} > 1000$ **11.** $8(a^3b + ab^3)$; $40\sqrt{6}$

12. $2(x^6 + 15x^4 + 15x^2 + 1)$, 198

EXERCISE 8.2

1. 1512 **2.** $-\,101376$ **3.** $(-1)^r\, {}^6C_r . x^{12-2r} . y^r$

4. $(-1)^r\, {}^{12}C_r . x^{24-r} . y^r$ **5.** $-\,1760\, x^9 y^3$ **6.** 18564

7. $\dfrac{-105}{8}x^9;\ \dfrac{35}{48}x^{12}$ **8.** $61236\, x^5 y^5$ **10.** $n = 7$; $r = 3$

12. $m = 4$

Miscellaneous Exercise on Chapter 8

1. $a = 3;\ b = 5;\ n = 6$ **2.** $a = \frac{9}{7}$ **3.** 171

5. $396\sqrt{6}$ **6.** $2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

7. 0.9510 **8.** $n = 10$

9. $\frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$

10. $27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

EXERCISE 9.1

1. 3, 8, 15, 24, 35 **2.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ **3.** 2, 4, 8, 16 and 32

4. $-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$ and $\frac{7}{6}$ **5.** 25, –125, 625, –3125, 15625

6. $\frac{3}{2}, \frac{9}{2}, \frac{21}{2}$, 21 and $\frac{75}{2}$ **7.** 65, 93 **8.** $\frac{49}{128}$

9. 729 **10.** $\frac{360}{23}$

11. 3, 11, 35, 107, 323; 3 + 11 + 35 + 107 + 323 + ...

12. $-1, \frac{-1}{2}, \frac{-1}{6}, \frac{-1}{24}, \frac{-1}{120};\ -1 + \left(\frac{-1}{2}\right) + \left(\frac{-1}{6}\right) + \left(\frac{-1}{24}\right) + \left(\frac{-1}{120}\right) + \ldots$

13. 2, 2, 1, 0, –1; 2 + 2 + 1 + 0 + (–1) + ... **14.** $1, 2, \frac{3}{2}, \frac{5}{3}$ and $\frac{8}{5}$

EXERCISE 9.2

1. 1002001 **2.** 98450 **4.** 5 or 20 **6.** 4

7. $\frac{n}{2}(5n+7)$ **8.** $2q$ **9.** $\frac{179}{321}$ **10.** 0

13. 27 **14.** 11, 14, 17, 20 and 23 **15.** 1

16. 14 **17.** Rs 245 **18.** 9

EXERCISE 9.3

1. $\frac{5}{2^{20}}, \frac{5}{2^n}$ **2.** 3072 **4.** – 2187

5. (a) 13$^{\text{th}}$, (b) 12$^{\text{th}}$, (c) 9$^{\text{th}}$ **6.** ± 1 **7.** $\frac{1}{6}\left[1 - (0.1)^{20}\right]$

8. $\frac{\sqrt{7}}{2}\left(\sqrt{3}+1\right)\left(3^{\frac{n}{2}}-1\right)$ **9.** $\frac{\left[1-(-a)^n\right]}{1+a}$ **10.** $\frac{x^3\left(1-x^{2n}\right)}{1-x^2}$

11. $22+\frac{3}{2}\left(3^{11}-1\right)$ **12.** $r=\frac{5}{2}\text{or}\frac{2}{5}$; Terms are $\frac{2}{5},1,\frac{5}{2}\text{or}\frac{5}{2},1,\frac{2}{5}$

13. 4 **14.** $\frac{16}{7};2;\frac{16}{7}\left(2^n-1\right)$ **15.** 2059

16. $\frac{-4}{3},\frac{-8}{3},\frac{-16}{3},\ldots\text{or } 4,-8,16,-32,64,..$ **18.** $\frac{80}{81}\left(10^n-1\right)-\frac{8}{9}n$

19. 496 **20.** rR **21.** 3, –6, 12, –24 **26.** 9 and 27

27. $n=\frac{-1}{2}$ **30.** 120, 480, 30 (2^n) **31.** Rs 500 $(1.1)^{10}$ **32.** $x^2-16x+25=0$

EXERCISE 9.4

1. $\frac{n}{3}(n+1)(n+2)$ **2.** $\frac{n(n+1)(n+2)(n+3)}{4}$

3. $\frac{n}{6}(n+1)\left(3n^2+5n+1\right)$ **4.** $\frac{n}{n+1}$ **5.** 2840

6. $3n\ (n+1)\ (n+3)$ **7.** $\frac{n(n+1)^2(n+2)}{12}$

8. $\frac{n(n+1)}{12}\left(3n^2+23n+34\right)$

9. $\frac{n}{6}(n+1)(2n+1)+2\left(2^n-1\right)$ **10.** $\frac{n}{3}(2n+1)(2n-1)$

Miscellaneous Exercise on Chapter 9

2. 5, 8, 11 **4.** 8729 **5.** 3050 **6.** 1210
7. 4 **8.** 160; 6 **9.** ± 3 **10.** 8, 16, 32
11. 4 **12.** 11

21. (i) $\frac{50}{81}\left(10^n-1\right)-\frac{5n}{9}$, (ii) $\frac{2n}{3}-\frac{2}{27}\left(1-10^{-n}\right)$ **22.** 1680

23. $\frac{n}{3}\left(n^2+3n+5\right)$ **25.** $\frac{n}{24}\left(2n^2+9n+13\right)$

27. Rs 16680 **28.** Rs 39100 **29.** Rs 43690 **30.** Rs 17000; 20,000
31. Rs 5120 **32.** 25 days

EXERCISE 10.1

1. $\frac{121}{2}$ square unit.

2. $(0, a)$, $(0, -a)$ and $\left(-\sqrt{3}a, 0\right)$ or $(0, a)$, $(0, -a)$, and $\left(\sqrt{3}a, 0\right)$

3. (i) $|y_2 - y_1|$, (ii) $|x_2 - x_1|$ **4.** $\left(\frac{15}{2}, 0\right)$ **5.** $-\frac{1}{2}$

7. $-\sqrt{3}$ **8.** $x = 1$ **10.** 135°

11. 1 and 2, or $\frac{1}{2}$ and 1, or – 1 and –2, or $-\frac{1}{2}$ and – 1 **14.** $\frac{1}{2}$, 104.5 Crores

EXERCISE 10.2

1. $y = 0$ and $x = 0$ **2.** $x - 2y + 10 = 0$ **3.** $y = mx$

4. $\left(\sqrt{3}+1\right)x - \left(\sqrt{3}-1\right)y = 4\left(\sqrt{3}-1\right)$ **5.** $2x + y + 6 = 0$

6. $x - \sqrt{3}y + 2\sqrt{3} = 0$ **7.** $5x + 3y + 2 = 0$

8. $\sqrt{3}x + y = 10$ **9.** $3x - 4y + 8 = 0$ **10.** $5x - y + 20 = 0$

11. $(1 + n)x + 3(1 + n)y = n + 11$ **12.** $x + y = 5$

13. $x + 2y - 6 = 0$, $2x + y - 6 = 0$

14. $\sqrt{3}x + y - 2 = 0$ and $\sqrt{3}x + y + 2 = 0$ **15.** $2x - 9y + 85 = 0$

16. $L = \frac{.192}{90}(C - 20) + 124.942$ **17.** 1340 litres. **19.** $2kx + hy = 3kh.$

EXERCISE 10.3

1. (i) $y = -\frac{1}{7}x + 0, -\frac{1}{7}, 0$; (ii) $y = -2x + \frac{5}{3}, -2, \frac{5}{3}$; (iii) $y = 0x + 0, 0, 0$

2. (i) $\frac{x}{4} + \frac{y}{6} = 1, 4, 6;$ (ii) $\frac{x}{\frac{3}{2}} + \frac{y}{-2} = 1, \frac{3}{2}, -2;$

(iii) $y = -\frac{2}{3}$, intercept with y-axis $= -\frac{2}{3}$ and no intercept with x-axis.

3. (i) $x \cos 120° + y \sin 120° = 4$, 4, 120° (ii) $x \cos 90° + y \sin 90° = 2$, 2, 90°;

(iii) $x \cos 315° + y \sin 315° = 2\sqrt{2}$, $2\sqrt{2}$, 315° **4.** 5 units

5. $(-2, 0)$ and $(8, 0)$ **6.** (i) $\frac{65}{17}$ units, (ii) $\frac{1}{\sqrt{2}}\left|\frac{p+r}{l}\right|$ units.

7. $3x - 4y + 18 = 0$ **8.** $y + 7x = 21$ **9.** $30°$ and $150°$ **10.** $\frac{22}{9}$

12. $\left(\sqrt{3}+2\right)x+\left(2\sqrt{3}-1\right)y=8\sqrt{3}+1$ or $\left(\sqrt{3}-2\right)x+\left(1+2\sqrt{3}\right)y=-1+8\sqrt{3}$

13. $2x + y = 5$ **14.** $\left(\frac{68}{25},-\frac{49}{25}\right)$ **15.** $m=\frac{1}{2}, c=\frac{5}{2}$ **17.** $y-x=1, \sqrt{2}$

Miscellaneous Exercise on Chapter 10

1. (a) 3, (b) ± 2, (c) 6 or 1 **2.** $\frac{7\pi}{6}, 1$

3. $2x-3y=6, -3x+2y=6$ **4.** $\left(0,-\frac{8}{3}\right), \left(0,\frac{32}{3}\right)$

5. $\frac{\left|\sin(\phi-\theta)\right|}{2\left|\sin\frac{\phi-\theta}{2}\right|}$ **6.** $x=-\frac{5}{22}$ **7.** $2x - 3y + 18 = 0$

8. k^2 square units **9.** 5 **11.** $3x - y = 7,\quad x + 3y = 9$

12. $13x + 13y = 6$ **14.** 1 : 2 **15.** $\frac{23\sqrt{5}}{18}$ units

16. The line is parallel to x - axis or parallel to y-axis

17. $x = 1,\quad y = 1.$ **18.** $(-1, -4)$. **19.** $\frac{1\pm 5\sqrt{2}}{7}$

21. $18x + 12y + 11 = 0$ **22.** $\left(\frac{13}{5},0\right)$ **24.** $119x + 102y = 125$

EXERCISE 11.1

1. $x^2 + y^2 - 4y = 0$ **2.** $x^2 + y^2 + 4x - 6y - 3 = 0$

3. $36x^2 + 36y^2 - 36x - 18y + 11 = 0$ **4.** $x^2 + y^2 - 2x - 2y = 0$

5. $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$ **6.** $c(-5, 3), r = 6$

7. $c(2, 4), r = \sqrt{65}$ **8.** $c(4, -5), r = \sqrt{53}$ **9.** $c\left(\frac{1}{4}, 0\right)$; $r = \frac{1}{4}$

10. $x^2 + y^2 - 6x - 8y + 15 = 0$ **11.** $x^2 + y^2 - 7x + 5y - 14 = 0$

12. $x^2 + y^2 + 4x - 21 = 0$ & $x^2 + y^2 - 12x + 11 = 0$

13. $x^2 + y^2 - ax - by = 0$ **14.** $x^2 + y^2 - 4x - 4y = 5$

15. Inside the circle; since the distance of the point to the centre of the circle is less than the radius of the circle.

EXERCISE 11.2

1. F (3, 0), axis - x - axis, directrix $x = -3$, length of the Latus rectum = 12
2. F $(0, \frac{3}{2})$, axis - y - axis, directrix $y = -\frac{3}{2}$, length of the Latus rectum = 6
3. F (–2, 0), axis - x - axis, directrix $x = 2$, length of the Latus rectum = 8
4. F (0, –4), axis - y - axis, directrix $y = 4$, length of the Latus rectum = 16
5. F $(\frac{5}{2}, 0)$ axis - x - axis, directrix $x = -\frac{5}{2}$, length of the Latus rectum = 10
6. F $(0, \frac{-9}{4})$, axis - y - axis, directrix $y = \frac{9}{4}$, length of the Latus rectum = 9
7. $y^2 = 24x$
8. $x^2 = -12y$
9. $y^2 = 12x$
10. $y^2 = -8x$
11. $2y^2 = 9x$
12. $2x^2 = 25y$

EXERCISE 11.3

1. F $(\pm\sqrt{20}, 0)$; V (± 6, 0); Major axis = 12; Minor axis = 8, $e = \frac{\sqrt{20}}{6}$, Latus rectum = $\frac{16}{3}$
2. F $(0, \pm\sqrt{21})$; V (0, ± 5); Major axis = 10; Minor axis = 4, $e = \frac{\sqrt{21}}{5}$; Latus rectum = $\frac{8}{5}$
3. F $(\pm\sqrt{7}, 0)$; V (± 4, 0); Major axis = 8; Minor axis = 6, $e = \frac{\sqrt{7}}{4}$; Latus rectum = $\frac{9}{2}$
4. F $(0, \pm\sqrt{75})$; V (0,± 10); Major axis = 20; Minor axis = 10, $e = \frac{\sqrt{3}}{2}$; Latus rectum = 5
5. F $(\pm\sqrt{13}, 0)$; V (± 7, 0); Major axis =14 ; Minor axis = 12, $e = \frac{\sqrt{13}}{7}$; Latus rectum = $\frac{72}{7}$
6. F $(0, \pm 10\sqrt{3})$; V (0,± 20); Major axis =40 ; Minor axis = 20, $e = \frac{\sqrt{3}}{2}$; Latus rectum = 10
7. F $(0, \pm 4\sqrt{2})$; V (0,± 6); Major axis =12 ; Minor axis = 4, $e = \frac{2\sqrt{2}}{3}$; Latus rectum = $\frac{4}{3}$
8. F $(0, \pm\sqrt{15})$; V (0,± 4); Major axis = 8 ; Minor axis = 2, $e = \frac{\sqrt{15}}{4}$; Latus rectum = $\frac{1}{2}$
9. F $(\pm\sqrt{5}, 0)$; V (± 3, 0); Major axis = 6 ; Minor axis = 4, $e = \frac{\sqrt{5}}{3}$; Latus rectum = $\frac{8}{3}$
10. $\frac{x^2}{25} + \frac{y^2}{9} = 1$
11. $\frac{x^2}{144} + \frac{y^2}{169} = 1$
12. $\frac{x^2}{36} + \frac{y^2}{20} = 1$

13. $\frac{x^2}{9}+\frac{y^2}{4}=1$ **14.** $\frac{x^2}{1}+\frac{y^2}{5}=1$ **15.** $\frac{x^2}{169}+\frac{y^2}{144}=1$

16. $\frac{x^2}{64}+\frac{y^2}{100}=1$ **17.** $\frac{x^2}{16}+\frac{y^2}{7}=1$ **18.** $\frac{x^2}{25}+\frac{y^2}{9}=1$

19. $\frac{x^2}{10}+\frac{y^2}{40}=1$ **20.** $x^2+4y^2=52$ or $\frac{x^2}{52}+\frac{y^2}{13}=1$

EXERCISE 11.4

1. Foci (± 5, 0), Vertices (± 4, 0); $e=\frac{5}{4}$; Latus rectum $=\frac{9}{2}$

2. Foci (0 ± 6), Vertices (0, ± 3); $e=2$; Latus rectum = 18

3. Foci $(0,\ \pm\sqrt{13})$, Vertices (0, ± 2); $e=\frac{\sqrt{13}}{2}$; Latus rectum $=9$

4. Foci (± 10, 0), Vertices (± 6, 0); $e=\frac{5}{3}$; Latus rectum $=\frac{64}{3}$

5. Foci $(0,\pm\frac{2\sqrt{14}}{\sqrt{5}})$, Vertices $(0,\pm\frac{6}{\sqrt{5}})$; $e=\frac{\sqrt{14}}{3}$; Latus rectum $=\frac{4\sqrt{5}}{3}$

6. Foci $(0,\pm\sqrt{65})$, Vertices (0, ± 4); $e=\frac{\sqrt{65}}{4}$; Latus rectum $=\frac{49}{2}$

7. $\frac{x^2}{4}-\frac{y^2}{5}=1$ **8.** $\frac{y^2}{25}-\frac{x^2}{39}=1$ **9.** $\frac{y^2}{9}-\frac{x^2}{16}=1$

10. $\frac{x^2}{16}-\frac{y^2}{9}=1$ **11.** $\frac{y^2}{25}-\frac{x^2}{144}=1$ **12.** $\frac{x^2}{25}-\frac{y^2}{20}=1$

13. $\frac{x^2}{4}-\frac{y^2}{12}=1$ **14.** $\frac{x^2}{49}-\frac{9y^2}{343}=1$ **15.** $\frac{y^2}{5}-\frac{x^2}{5}=1$

Miscellaneous Exercise on Chapter 11

1. Focus is at the mid-point of the given diameter.

2. 2.23 m (approx.) **3.** 9.11 m (approx.) **4.** 1.56m (approx.) **5.** $\frac{x^2}{81}+\frac{y^2}{9}=1$

6. 18 sq units **7.** $\frac{x^2}{25}+\frac{y^2}{9}=1$ **8.** $8\sqrt{3}a$

EXERCISE 12.1

1. y and z - coordinates are zero
2. y - coordinate is zero
3. I, IV, VIII, V, VI, II, III, VII
4. (i) XY - plane (ii) $(x, y, 0)$ (iii) Eight

EXERCISE 12.2

1. (i) $2\sqrt{5}$ (ii) $\sqrt{43}$ (iii) $2\sqrt{26}$ (iv) $2\sqrt{5}$
4. $x - 2z = 0$
5. $9x^2 + 25y^2 + 25z^2 - 225 = 0$

EXERCISE 12.3

1. (i) $\left(\frac{-4}{5}, \frac{1}{5}, \frac{27}{5}\right)$ (ii) $(-8, 17, 3)$
2. $1 : 2$
3. $2 : 3$
5. $(6, -4, -2), (8, -10, 2)$

Miscellaneous Exercise on Chapter 12

1. $(1, -2, 8)$
2. $7, \sqrt{34}, 7$
3. $a = -2,\ b = -\frac{16}{3},\ c = 2$
4. $(0, 2, 0)$ and $(0, -6, 0)$
5. $(4, -2, 6)$
6. $x^2 + y^2 + z^2 - 2x - 7y + 2z = \frac{k^2 - 109}{2}$

EXERCISE 13.1

1. 6
2. $\left(\pi - \frac{22}{7}\right)$
3. π
4. $\frac{19}{2}$
5. $-\frac{1}{2}$
6. 5
7. $\frac{11}{4}$
8. $\frac{108}{7}$
9. b
10. 2
11. 1
12. $-\frac{1}{4}$
13. $\frac{a}{b}$
14. $\frac{a}{b}$
15. $\frac{1}{\pi}$
16. $\frac{1}{\pi}$
17. 4
18. $\frac{a+1}{b}$
19. 0
20. 1
21. 0
22. 2
23. 3, 6
24. Limit does not exist at $x = 1$
25. Limit does not exist at $x = 0$
26. Limit does not exist at $x = 0$
27. 0
28. $a = 0, b = 4$
29. $\lim_{x \to a_1} f(x) = 0$ and $\lim_{x \to a} f(x) = (a - a_1)(a - a_2) \ldots (a - a_x)$

30. $\lim_{x \to a} f(x)$ exists for all $a \neq 0$. **31.** 2

32. For $\lim_{x \to 0} f(x)$ to exists, we need $m = n$; $\lim_{x \to 1} f(x)$ exists for any integral value of m and n.

EXERCISE 13.2

1. 20 **2.** 99 **3.** 1

4. (i) $3x^2$ (ii) $2x - 3$ (iii) $\frac{-2}{x^3}$ (iv) $\frac{-2}{(x-1)^2}$

6. $nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + ... + a^{n-1}$

7. (i) $2x - a - b$ (ii) $4ax\left(ax^2 + b\right)$ (iii) $\frac{a-b}{(x-b)^2}$

8. $\frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$

9. (i) 2 (ii) $20x^3 - 15x^2 + 6x - 4$ (iii) $\frac{-3}{x^4}(5+2x)$ (iv) $15x^4 + \frac{24}{x^5}$

(v) $\frac{-12}{x^5} + \frac{36}{x^{10}}$ (vi) $\frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$ **10.** $-\sin x$

11. (i) $\cos 2x$ (ii) $\sec x \tan x$
(iii) $5\sec x \tan x - 4\sin x$ (iv) $-\operatorname{cosec} x \cot x$
(v) $-3\operatorname{cosec}^2 x - 5 \operatorname{cosec} x \cot x$ (vi) $5\cos x + 6\sin x$
(vii) $2\sec^2 x - 7\sec x \tan x$

Miscellaneous Exercise on Chapter 13

1. (i) -1 (ii) $\frac{1}{x^2}$ (iii) $\cos(x+1)$ (iv) $-\sin\left(x - \frac{\pi}{8}\right)$ **2.** 1

3. $\frac{-qr}{x^2} + ps$ **4.** $2c(ax+b)(cx+d) + a(cx+d)^2$

5. $\frac{ad - bc}{(cx+d)^2}$ **6.** $\frac{-2}{(x-1)^2},\ x \neq 0, 1$ **7.** $\frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$

8. $\frac{-apx^2 - 2bpx + ar - bq}{\left(px^2+qx+r\right)^2}$ **9.** $\frac{apx^2 + 2bpx + bq - ar}{(ax+b)^2}$ **10.** $\frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x$

11. $\frac{2}{\sqrt{x}}$ **12.** $na(ax+b)^{n-1}$

13. $(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right]$

14. $\cos(x+a)$

15. $-\operatorname{cosec}^3 x - \operatorname{cosec} x \cot^2 x$

16. $\dfrac{-1}{1+\sin x}$

17. $\dfrac{-2}{(\sin x-\cos x)^2}$

18. $\dfrac{2\sec x \tan x}{(\sec x+1)^2}$

19. $n \sin^{n-1} x \cos x$

20. $\dfrac{bc\cos x + ad\sin x + bd}{(c+d\cos x)^2}$

21. $\dfrac{\cos a}{\cos^2 x}$

22. $x^3(5x\cos x + 3x\sin x + 20\sin x - 12\cos x)$

23. $-x^2\sin x - \sin x + 2x\cos x$

24. $-q\sin x(ax^2+\sin x)+(p+q\cos x)(2a\,x+\cos x)$

25. $-\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)$

26. $\dfrac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$

27. $\dfrac{x\cos\dfrac{\pi}{4}(2\sin x - x\cos x)}{\sin^2 x}$

28. $\dfrac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$

29. $(x+\sec x)(1-\sec^2 x)+(x-\tan x).(1+\sec x\tan x)$

30. $\dfrac{\sin x - n\,x\cos x}{\sin^{n+1} x}$

EXERCISE 14.1

1. (i) This sentence is always false because the maximum number of days in a month is 31. Therefore, it is a statement.

(ii) This is not a statement because for some people mathematics can be easy and for some others it can be difficult.

(iii) This sentence is always true because the sum is 12 and it is greater than 10. Therefore, it is a statement.

(iv) This sentence is sometimes true and sometimes not true. For example the square of 2 is even number and the square of 3 is an odd number. Therefore, it is not a statement.

(v) This sentence is sometimes true and sometimes false. For example, squares and rhombus have equal length whereas rectangles and trapezium have unequal length. Therefore, it is not a statement.

(vi) It is an order and therefore, is not a statement.

(vii) This sentence is false as the product is (–8). Therefore, it is a statement.

(viii) This sentence is always true and therefore, it is a statement.

(ix) It is not clear from the context which day is referred and therefore, it is not a statement.

(x) This is a true statement because all real numbers can be written in the form $a + i \times 0$.

2. The three examples can be:

(i) Everyone in this room is bold. This is not a statement because from the context it is not clear which room is referred here and the term bold is not precisely defined.

(ii) She is an engineering student. This is also not a statement because who 'she' is.

(iii) "$\cos^2\theta$ is always greater than 1/2". Unless, we know what θ is, we cannot say whether the sentence is true or not.

EXERCISES 14.2

1. (i) Chennai is not the capital of Tamil Nadu.

(ii) $\sqrt{2}$ is a complex number.

(iii) All triangles are equilateral triangles.

(iv) The number 2 is not greater than 7.

(v) Every natural number is not an integer.

2. (i) The negation of the first statement is "the number x is a rational number." which is the same as the second statement" This is because when a number is not irrational, it is a rational. Therefore, the given pairs are negations of each other.

(ii) The negation of the first statement is "x is an irrational number" which is the same as the second statement. Therefore, the pairs are negations of each other.

3. (i) Number 3 is prime; number 3 is odd (True).

(ii) All integers are positive; all integers are negative (False).

(iii) 100 is divisible by 3,100 is divisible by 11 and 100 is divisible by 5 (False).

EXERCISE 14.3

1. (i) "And". The component statements are:
All rational numbers are real.
All real numbers are not complex.

(ii) "Or". The component statements are:
Square of an integer is positive.
Square of an integer is negative.

(iii) "And". the component statements are:
The sand heats up quickily in the sun.
The sand does not cool down fast at night.

(iv) "And". The component statements are:
$x = 2$ is a root of the equation $3x^2 - x - 10 = 0$
$x = 3$ is a root of the equation $3x^2 - x - 10 = 0$

2. (i) "There exists". The negation is
There does not exist a number which is equal to its square.

(ii) "For every". The negation is
There exists a real number x such that x is not less than $x + 1$.

(iii) "There exists". The negation is
There exists a state in India which does not have a capital.

3. No. The negation of the statement in (i) is "There exists real number x and y for which $x + y \neq y + x$", instead of the statement given in (ii).

4. (i) Exclusive

(ii) Inclusive

(iii) Exclusive

EXERCISE 14.4

1. (i) A natural number is odd implies that its square is odd.
   (ii) A natural number is odd only if its square is odd.
   (iii) For a natural number to be odd it is necessary that its square is odd.
   (iv) For the square of a natural number to be odd, it is sufficient that the number is odd
   (v) If the square of a natural number is not odd, then the natural number is not odd.
2. (i) The contrapositive is
   If a number x is not odd, then x is not a prime number.
   The converse is
   If a number x in odd, then it is a prime number.
   (ii) The contrapositive is
   If two lines intersect in the same plane, then they are not parallel
   The converse is
   If two lines do not interesect in the same plane, then they are parallel
   (iii) The contrapositive is
   If something is not at low temperature, then it is not cold
   The converse is
   If something is at low temperature, then it is cold
   (iv) The contrapositive is
   If you know how to reason deductively, then you can comprehend geometry.
   The converse is
   If you do not know how to reason deductively, then you can not comprehend geometry.
   (v) This statement can be written as "If x is an even number, then x is divisible by 4".
   The contrapositive is, If x is not divisible by 4, then x is not an even number.
   The converse is, If x is divisible by 4, then x is an even number.
3. (i) If you get a job, then your credentials are good.
   (ii) If the banana tree stays warm for a month, then it will bloom.
   (iii) If diagonals of a quadrilateral bisect each other, then it is a parallelogram.
   (iv) If you get A^+ in the class, then you do all the exercises in the book.
4. a (i) Contrapositive
   (ii) Converse
   b (i) Contrapositive
   (ii) Converse

EXERCISE 14.5

5. (i) False. By definition of the chord, it should intersect the circle in two points.
   (ii) False. This can be shown by giving a counter example. A chord which is not a dimaeter gives the counter example.
   (iii) True. In the equation of an ellipse if we put $a = b$, then it is a circle (Direct Method)
   (iv) True, by the rule of inequality
   (v) False. Since 11 is a prime number, therefore $\sqrt{11}$ is irrational.

Miscellaneous Exercise on Chapter 14

1. (i) There exists a positive real number x such that $x-1$ is not positive.
(ii) There exists a cat which does not scratch.
(iii) There exists a real number x such that neither $x > 1$ nor $x < 1$.
(iv) There does not exist a number x such that $0 < x < 1$.

2. (i) The statement can be written as "If a positive integer is prime, then it has no divisors other than 1 and itself.
The converse of the statement is
If a positive integer has no divisors other than 1 and itself, then it is a prime.
The contrapositive of the statement is
If positive integer has divisors other than 1 and itself then it is not prime.
(ii) The given statement can be written as "If it is a sunny day, then I go to a beach.
The converse of the statement is
If I go to beach, then it is a sunny day.
The contrapositive is
If I do not go to a beach, then it is not a sunny day.
(iii) The converse is
If you feel thirsty, then it is hot outside.
The contrapositive is
If you do not feel thirsty, then it is not hot outside.

3. (i) If you log on to the server, then you have a password.
(ii) If it rains, then there is traffic jam.
(iii) If you can access the website, then you pay a subscription fee.

4. (i) You watch television if and only if your mind in free.
(ii) You get an A grade if and only if you do all the homework regularly.
(iii) A quadrilateral is equiangular if and only if it is a rectangle.

5. The compound statement with "And" is 25 is a multiple of 5 and 8
This is a false statement.
The compound statement with "Or" is 25 is a multiple of 5 or 8
This is true statement.

7. Same as Q1 in Exercise 14.4

EXERCISE 15.1

1. 3 **2.** 8.4 **3.** 2.33 **4.** 7 **5.** 6.32
6. 16 **7.** 3.23 **8.** 5.1 **9.** 157.92 **10.** 11.28
11. 10.34 **12.** 7.35

EXERCISE 15.2

1. 9, 9.25 **2.** $\frac{n+1}{2}, \frac{n^2-1}{12}$ **3.** 16.5, 74.25 **4.** 19, 43.4
5. 100, 29.09 **6.** 64, 1.69 **7.** 107, 2276 **8.** 27, 132
9. 93, 105.52, 10.27
10. 5.55, 43.5

EXERCISE 15.3

1. B **2.** Y **3.** (i) B, (ii) B
4. A **5.** Weight

Miscellaneous Exercise on Chapter 15

1. 4, 8 **2.** 6, 8 **3.** 24, 12
5. (i) 10.1, 1.99 (ii) 10.2, 1.98
6. Highest Chemistry and lowest Mathematics **7.** 20, 3.036

EXERCISE 16.1

1. {HHH, HHT, HTH, THH, TTH, HTT, THT, TTT}
2. $\{(x, y) : x, y = 1,2,3,4,5,6\}$
or {(1,1), (1,2), (1,3), ..., (1,6), (2,1), (2,2), ..., (2,6), ..., (6, 1), (6, 2), ..., (6,6)}
3. {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}
4. {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}
5. {H1, H2, H3, H4, H5, H6, T}
6. $\{XB_1, XB_2, XG_1, XG_2, YB_3, YG_3, YG_4, YG_5\}$
7. {R1, R2, R3, R4, R5, R6, W1, W2, W3, W4, W5, W6, B1, B2, B3, B4, B5, B6}
8. (i) {BB, BG, GB, GG} (ii) {0, 1, 2}
9. {RW, WR, WW}
10. [HH, HT, T1, T2, T3, T4, T5, T6}
11. {DDD, DDN, DND, NDD, DNN, NDN, NND, NNN}
12. {T, H1, H3, H5, H21, H22, H23, H24, H25, H26, H41, H42, H43, H44, H45, H46, H61, H62, H63, H64, H65, H66}
13. {(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)}
14. {1HH, 1HT, 1TH, 1TT, 2H, 2T, 3HH, 3HT, 3TH, 3TT, 4H, 4T, 5HH, 5HT, 5TH, 5TT, 6H, 6T}
15. $\{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H2, H3, H4, H5, H6\}$
16. {6, (1,6), (2,6), (3,6), (4,6), (5,6), (1,1,6), (1,2,6), ..., (1,5,6), (2,1,6). (2,2,6), ..., (2,5,6), ..., (5,1,6), (5,2,6), ... }

EXERCISE 16.2

1. No.
2. (i) {1, 2, 3, 4, 5, 6} (ii) ϕ (iii) {3, 6} (iv) {1, 2, 3} (v) {6}
(vi) {3, 4, 5, 6}, $A\cup B = \{1, 2, 3, 4, 5, 6\}$, $A\cap B = \phi$, $B\cup C = \{3, 6\}$, $E\cap F = \{6\}$, $D\cap E = \phi$,
$A - C = \{1, 2,4,5\}$, $D - E = \{1,2,3\}$, $E\cap F' = \phi$, $F' = \{1, 2\}$
3. A = {(3,6), (4,5), (5, 4), (6,3), (4,6), (5,5), (6,4), (5,6), (6,5), (6,6)}
B = {(1,2), (2,2), (3, 2), (4,2), (5,2), (6,2), (2,1), (2,3), (2,4), (2,5), (2,6)}
C ={(3,6), (6,3), (5, 4), (4,5), (6,6)}
A and B, B and C are mutually exclusive.
4. (i) A and B; A and C; B and C; C and D (ii) A and C (iii) B and D
5. (i) "Getting at least two heads", and "getting at least two tails"
(ii) "Getting no heads", "getting exactly one head" and "getting at least two heads"

(iii) "Getting at most two tails", and "getting exactly two tails"
(iv) "Getting exactly one head" and "getting exactly two heads"
(v) "Getting exactly one tail", "getting exactly two tails", and getting exactly three tails"

Note There may be other events also as answer to the above question.

6. A = {(2, 1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

B = {(1, 1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)}

C = {(1, 1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (4,1)}

(i) A′ = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)} = B

(ii) B′ = {(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} = A

(iii) $A \cup B$ = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (2,1), (2,2), (2,3), (2,5), (2,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)} = S

(iv) $A \cap B = \phi$

(v) $A - C$ = {(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

(vi) $B \cup C$ = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6)}

(vii) $B \cap C$ = {(1,1), (1,2), (1,3), (1,4), (3,1), (3,2)}

(viii) $A \cap B' \cap C'$ = {(2,4), (2,5), (2,6), (4,2), (4,3), (4,4), (4,5), (4,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}

7. (i) True (ii) True (iii) True (iv) False (v) False (vi) False

EXERCISE 16.3

1. (a) Yes (b) Yes (c) No (d) No (e) No **2.** $\frac{3}{4}$

3. (i) $\frac{1}{2}$ (ii) $\frac{2}{3}$ (iii) $\frac{1}{6}$ (iv) 0 (v) $\frac{5}{6}$ **4.** (a) 52 (b) $\frac{1}{52}$ (c) (i) $\frac{1}{13}$ (ii) $\frac{1}{2}$

5. (i) $\frac{1}{12}$ (ii) $\frac{1}{12}$ **6.** $\frac{3}{5}$

7. Rs 4.00 gain, Rs 1.50 gain, Re 1.00 loss, Rs 3.50 loss, Rs 6.00 loss.

P (Winning Rs 4.00) $= \frac{1}{16}$, P(Winning Rs 1.50) $= \frac{1}{4}$, P (Losing Re. 1.00) $= \frac{3}{8}$

P (Losing Rs 3.50) $= \frac{1}{4}$, P (Losing Rs 6.00) $= \frac{1}{16}$.

8. (i) $\frac{1}{8}$ (ii) $\frac{3}{8}$ (iii) $\frac{1}{2}$ (iv) $\frac{7}{8}$ (v) $\frac{1}{8}$ (vi) $\frac{1}{8}$ (vii) $\frac{3}{8}$ (viii) $\frac{1}{8}$ (ix) $\frac{7}{8}$

9. $\frac{9}{11}$ **10.** (i) $\frac{6}{13}$ (ii) $\frac{7}{13}$ **11.** $\frac{1}{38760}$

12. (i) No, because P(A∩B) must be less than or equal to P(A) and P(B), (ii) Yes

13. (i) $\frac{7}{15}$ (ii) 0.5 (iii) 0.15

14. $\frac{4}{5}$

15. (i) $\frac{5}{8}$ (ii) $\frac{3}{8}$

16. No

17. (i) 0.58 (ii) 0.52 (iii) 0.74

18. 0.6

19. 0.55

20. 0.65

21. (i) $\frac{19}{30}$ (ii) $\frac{11}{30}$ (iii) $\frac{2}{15}$

Miscellaneous Exercise on Chapter 16

1. (i) $\frac{{}^{20}C_5}{{}^{60}C_5}$ (ii) $1-\frac{{}^{30}C_5}{{}^{60}C_5}$

2. $\frac{{}^{13}C_3 \cdot {}^{13}C_1}{{}^{52}C_4}$

3. (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$ (iii) $\frac{5}{6}$

4. (a) $\frac{999}{1000}$ (b) $\frac{{}^{9990}C_2}{{}^{10000}C_2}$ (c) $\frac{{}^{9990}C_{10}}{{}^{10000}C_{10}}$

5. (a) $\frac{17}{33}$ (b) $\frac{16}{33}$

6. $\frac{2}{3}$

7. (i) 0.88 (ii) 0.12 (iii) 0.19 (iv) 0.34

8. $\frac{4}{5}$

9. (i) $\frac{33}{83}$ (ii) $\frac{3}{8}$

10. $\frac{1}{5040}$

SUPPLEMENTARY MATERIAL

CHAPTER 3

3.6 Proofs and Simple Applications of Sine and Cosine Formulae

Let ABC be a triangle. By angle A, we mean the angle between the sides AB and AC which lies between 0° and 180°. The angles B and C are similarly defined. The sides AB, BC and CA opposite to the vertices C, A and B will be denoted by c, a and b respectively (see Fig. 3.15).

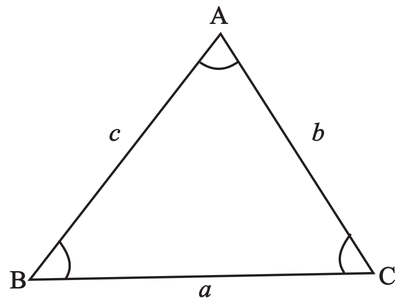
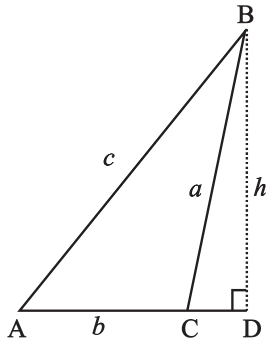


Fig. 3.15

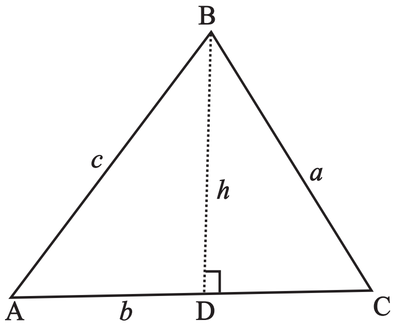
Theorem 1 (Sine formulae) In any triangle, sides are proportional to the sines of the opposite angles. That is, in a triangle ABC

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Proof Let ABC be either of the triangles as shown in Fig. 3.16 (i) and (ii).



(i)



(ii)

Fig. 3.16

The altitude h is drawn from the vertex B to meet the side AC in point D [in (i) AC is produced to meet the altitude in D]. From the right angled triangle ABD in Fig. 3.16(i), we have

$$\sin A = \frac{h}{c}, \text{ i.e., } h = c \sin A \tag{1}$$

and $$\sin (180° - C) = \frac{h}{a} \Rightarrow h = a \sin C \tag{2}$$

From (1) and (2), we get

$$c \sin A = a \sin C, \text{ i.e., } \frac{\sin A}{a} = \frac{\sin C}{c} \qquad (3)$$

Similarly, we can prove that

$$\frac{\sin A}{a} = \frac{\sin B}{b} \qquad (4)$$

From (3) and (4), we get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

For triangle ABC in Fig. 3.16 (ii), equations (3) and (4) follow similarly.

Theorem 2 (Cosine formulae) Let A, B and C be angles of a triangle and a, b and c be lengths of sides opposite to angles A, B and C respectively, then

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = c^2 + a^2 - 2ca \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Proof Let ABC be triangle as given in Fig. 3.17 (i) and (ii)

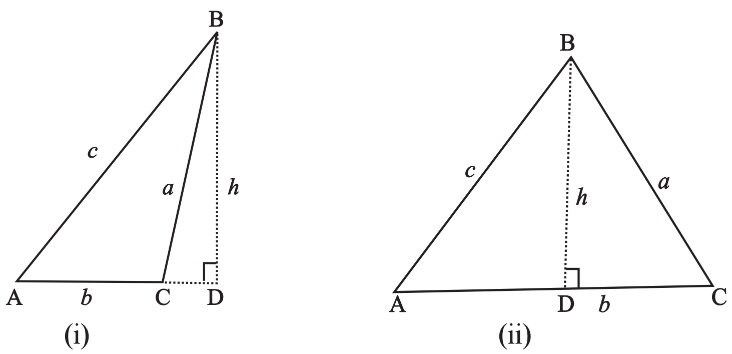


Fig. 3.17

Referring to Fig. 3.17 (ii), we have

$$BC^2 = BD^2 + DC^2 = BD^2 + (AC - AD)^2$$
$$= BD^2 + AD^2 + AC^2 - 2AC.AD$$
$$= AB^2 + AC^2 - 2AC\,AB \cos A$$

or $\quad a^2 = b^2 + c^2 - 2bc \cos A$

Similarly, we can obtain

$$b^2 = c^2 + a^2 - 2ca \cos B$$

and $$c^2 = a^2 + b^2 - 2ab \cos C$$

Same equations can be obtained for Fig. 3.17 (i), where C is obtuse.

A convenient form of the cosine formulae, when angles are to be found are as follows:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Example 25 In triangle ABC, prove that

$$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$$

$$\tan \frac{C - A}{2} = \frac{c - a}{c + a} \cot \frac{B}{2}$$

$$\tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2}$$

Proof By sine formulae, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k\,(say).$$

Therefore, $$\frac{b - c}{b + c} = \frac{k(\sin B - \sin C)}{k(\sin B + \sin C)}$$

$$= \frac{2 \cos \frac{B + C}{2} \sin \frac{B - C}{2}}{2 \sin \frac{B + C}{2} \cos \frac{B - C}{2}}$$

$$= \cot \frac{(B + C)}{2} \tan \frac{(B - C)}{2}$$

$$= \cot \left(\frac{\pi}{2} - \frac{A}{2} \right) \tan \left(\frac{B - C}{2} \right)$$

$$= \frac{\tan \frac{B - C}{2}}{\cot \frac{A}{2}}$$

Therefore, $$\tan \frac{B - C}{2} = \frac{b - c}{b + c} \cot \frac{A}{2}$$

Similarly, we can prove other results. These results are well known as Napier's Analogies.

Example 26 In any triangle ABC, prove that

$$a \sin (B - C) + b \sin (C - A) + C \sin (A - B) = 0$$

Solution Consider

$$a \sin (B - C) = a [\sin B \cos C - \cos B \sin C] \qquad (1)$$

Now $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k \text{ (say)}$$

Therefore, $\sin A = ak$, $\sin B = bk$, $\sin C = ck$

Substituting the values of sin B and sin C in (1) and using cosine formulae, we get

$$a \sin(B - C) = a\left[bk\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - ck\left(\frac{c^2 + a^2 - b^2}{2ac}\right)\right]$$

$$= \frac{k}{2}\left(a^2 + b^2 - c^2 - c^2 - a^2 + b^2\right)$$

$$= k(b^2 - c^2)$$

Similarly, $b \sin (C - A) = k (c^2 - a^2)$

and $c \sin (A - B) = k (a^2 - b^2)$

Hence L.H.S $= k (b^2 - c^2 + c^2 - a^2 + a^2 - b^2)$

$= 0 =$ R.H.S.

Example 27 The angle of elevation of the top point P of the vertical tower PQ of height h from a point A is 45° and from a point B, the angle of elevation is 60°, where B is a point at a distance d from the point A measured along the line AB which makes an angle 30° with AQ. Prove that $d = h(\sqrt{3} - 1)$

Proof From the Fig. 3.18, we have $\angle PAQ = 45°$, $\angle BAQ = 30°$, $\angle PBH = 60°$

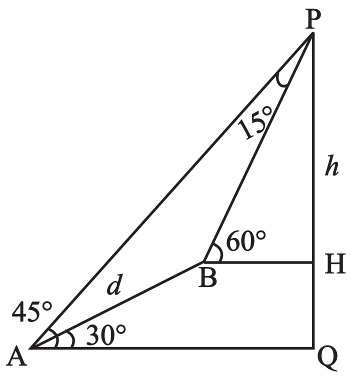


Fig. 3.18

Clearly $\angle APQ = 45°, \angle BPH = 30°$, giving $\angle APB = 15°$

Again $\angle PAB = 15° \Rightarrow \angle ABP = 150°$

From triangle APQ, we have $AP^2 = h^2 + h^2 = 2h^2$ (Why?)

or $$AP = \sqrt{2}h$$

Applying sine formulae in Δ ABP, we get

$$\frac{AB}{\sin 15^\circ} = \frac{AP}{\sin 150^\circ} \Rightarrow \frac{d}{\sin 15^\circ} = \frac{\sqrt{2}h}{\sin 150^\circ}$$

i.e., $$d = \frac{\sqrt{2}h \sin 15^\circ}{\sin 30^\circ}$$

$$= h(\sqrt{3} - 1) \quad \text{(why?)}$$

Example 28 A lamp post is situated at the middle point M of the side AC of a triangular plot ABC with BC = 7 m, CA = 8 m and AB = 9 m. Lamp post subtends an angle 15° at the point B. Determine the height of the lamp post.

Solution From the Fig. 3.19, we have AB = 9 = c, BC = 7 m = a and AC = 8 m = b.

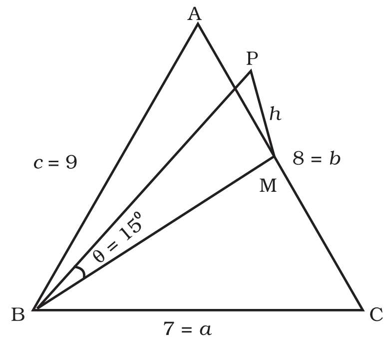


Fig. 3.19

M is the mid-point of the side AC at which lamp post MP of height h (say) is located. Again, it is given that lamp post subtends an angle θ (say) at B which is 15°.

Applying cosine formulae in ΔABC, we have

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{2 \times 7 \times 8} = \frac{2}{7} \qquad (1)$$

Similarly using cosine formulae in ΔBMC, we get

$$BM^2 = BC^2 + CM^2 - 2\ BC \times CM \cos C.$$

Here $CM = \frac{1}{2} CA = 4$, since M is the mid-point of AC.

Therefore, using (1), we get

$$BM^2 = 49 + 16 - 2 \times 7 \times 4 \times \frac{2}{7}$$

$$= 49$$

or BM = 7

Thus, from ΔBMP right angled at M, we have

$$\tan\theta = \frac{PM}{BM} = \frac{h}{7}$$

or $\frac{h}{7} = \tan(15°) = 2 - \sqrt{3}$ (why?)

or $h = 7(2 - \sqrt{3})$ m.

Exercise 3.5

In any triangle ABC, if $a = 18$, $b = 24$, $c = 30$, find

1. cos A, cos B, cos C (Ans. $\frac{4}{5}, \frac{3}{5}, 0$)
2. sin A, sin B, sin C (Ans. $\frac{3}{5}, \frac{4}{5}, 1$)

For any triangle ABC, prove that

3. $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$
4. $\frac{a-b}{c} = \frac{\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$
5. $\sin\frac{B-C}{2} = \frac{b-c}{a}\cos\frac{A}{2}$
6. $a(b\cos C - c\cos B) = b^2 - c^2$
7. $a(\cos C - \cos B) = 2(b-c)\cos^2\frac{A}{2}$
8. $\frac{\sin(B-C)}{\sin(B+C)} = \frac{b^2-c^2}{a^2}$
9. $(b+c)\cos\frac{B+C}{2} = a\cos\frac{B-C}{2}$
10. $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

11. $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}$

12. $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$

13. $\dfrac{b^2-c^2}{a^2}\sin 2A+\dfrac{c^2-a^2}{b^2}\sin 2B+\dfrac{a^2-b^2}{c^2}\sin 2C=0$

14. A tree stands vertically on a hill side which makes an angle of 15° with the horizontal. From a point on the ground 35 m down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°. Find the height of the tree. (Ans. $35\sqrt{2}m$)

15. Two ships leave a port at the same time. One goes 24 km per hour in the direction N45°E and other travels 32 km per hour in the direction S75°E. Find the distance between the ships at the end of 3 hours. (Ans. 86.4 km (approx.))

16. Two trees, A and B are on the same side of a river. From a point C in the river the distance of the trees A and B is 250 m and 300 m, respectively. If the angle C is 45°, find the distance between the trees (use $\sqrt{2}=1.44$). (Ans. 215.5 m)

CHAPTER 5

5.7 Square-root of a Complex Number

We have discussed solving of quadratic equations involving complex roots on page 108-109 of the textbook. Here we explain the particular procedure for finding square root of a complex number expressed in the standard form. We illustrate the same by an example.

Example 12 Find the square root of $-7-24i$

Solution Let $x+iy=\sqrt{-7-24i}$

Then $(x+iy)^2=-7-24i$

or $x^2-y^2+2xyi=-7-24i$

Equating real and imaginary parts, we have

$$x^2-y^2=-7 \quad (1)$$

$$2xy=-24$$

We know the identity

$$\left(x^2+y^2\right)^2=\left(x^2-y^2\right)^2+(2xy)^2$$
$$=49+576$$
$$=625$$

Thus, $x^2+y^2=25$ (2)

From (1) and (2), $x^2=9$ and $y^2=16$

or $\quad x = \pm 3$ and $y = \pm 4$

Since the product xy is negative, we have

$$x = 3,\ y = -4 \text{ or, } x = -3,\ y = 4$$

Thus, the square roots of $-7 - 24i$ are $3 - 4i$ and $-3 + 4i$.

Exercise 5.4

Find the square roots of the following:

1. $-15 - 8i$ $\quad$ (Ans. $1 - 4i$, $-1 + 4i$)
2. $-8 - 6i$ $\quad$ (Ans. $1 - 3i$, $-1 + 3i$)
3. $1 - i$ $\quad$ (Ans. $\left(\pm\sqrt{\frac{\sqrt{2}+1}{2}} \mp \sqrt{\frac{\sqrt{2}-1}{2}}\,i\right)$)
4. $-i$ $\quad$ (Ans. $\left(\pm\frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}}\,i\right)$)
5. i $\quad$ (Ans. $\left(\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\,i\right)$)
6. $1 + i$ $\quad$ (Ans. $\left(\pm\sqrt{\frac{\sqrt{2}+1}{2}} \pm \sqrt{\frac{\sqrt{2}-1}{2}}\,i\right)$)

CHAPTER 9

9.7 Infinite G.P. and its Sum

G.P. of the form $a, ar, ar^2, ar^3, \ldots$ is called infinite G.P. Now, to find the formulae for finding sum to infinity of a G.P., we begin with an example.

Let us consider the G.P.,

$$1, \frac{2}{3}, \frac{4}{9}, \ldots$$

Here $a = 1$, $r = \frac{2}{3}$. We have

$$S_n = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger:

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to zero. Mathematically, we say that as n becomes sufficiently large, $\left(\frac{2}{3}\right)^n$ becomes sufficiently small. In other words as $n \to \infty, \left(\frac{2}{3}\right)^n \to 0$.

Consequently, we find that the sum of infinitely many terms is given by $S_\infty = 3$.

Now, for a geometric progression, a, ar, ar^2, ..., if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a(1-r^n)}{(1-r)} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case as $n \to \infty$, $r^n \to 0$ since $|r| < 1$. Therefore

$$S_n \to \frac{a}{1-r}$$

Symbolically sum to infinity is denoted by S_∞ or S.

Thus, we have $S = \frac{a}{1-r}$.

For examples,

(i) $$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1}{1-\frac{1}{2}} = 2.$$

(ii) $$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots = \frac{1}{1-\left(\frac{-1}{2}\right)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

Exercise 9.4

Find the sum to infinity in each of the following Geometric Progression.

1. $1, \frac{1}{3}, \frac{1}{9}, \ldots$ (Ans. 1.5)
2. 6, 1.2, .24, ... (Ans. 7.5)
3. $5, \frac{20}{7}, \frac{80}{49}, \ldots$ (Ans. $\frac{35}{3}$)
4. $\frac{-3}{4}, \frac{3}{16}, \frac{-3}{64}, \ldots$ (Ans. $\frac{-3}{5}$)
5. Prove that $3^{\frac{1}{2}} \times 3^{\frac{1}{4}} \times 3^{\frac{1}{8}} \ldots = 3$
6. Let $x = 1 + a + a^2 + \ldots$ and $y = 1 + b + b^2 + \ldots$, where $|a| < 1$ and $|b| < 1$. Prove that

$$1 + ab + a^2b^2 + \ldots = \frac{xy}{x+y-1}$$

CHAPTER 10

10.6 Equation of Family of Lines Passing Through the Point of Intersection of Two Lines

Let the two intersecting lines l_1 and l_2 be given by

$$A_1x + B_1y + C_1 = 0 \qquad (1)$$

and

$$A_2x + B_2y + C_2 = 0 \qquad (2)$$

From the equations (1) and (2), we can form an equation

$$A_1x + B_1y + C_1 + k\left(A_2x + B_2y + C_2\right) = 0 \qquad (3)$$

where k is an arbitrary constant called parameter. For any value of k, the equation (3) is of first degree in x and y. Hence it represents a family of lines. A particular member of this family can be obtained for some value of k. This value of k may be obtained from other conditions.

Example 20 Find the equation of line parallel to the y-axis and drawn through the point of intersection of $x - 7y + 5 = 0$ and $3x + y - 7 = 0$

Solution The equation of any line through the point of intersection of the given lines is of the form

$$x - 7y + 5 + k\ (3x + y - 7) = 0$$

i.e., $(1 + 3k)\ x + (k - 7)\ y + 5 - 7k = 0$ (1)

If this line is parallel to y-axis, then the coefficient of y should be zero, i.e.,

$k - 7 = 0$ which gives $k = 7$.

Substituting this value of k in the equation (1), we get

$22x - 44 = 0$, i.e., $x - 2 = 0$, which is the required equation.

Exercise 10.4

1. Find the equation of the line through the intersection of lines $3x + 4y = 7$ and $x - y + 2 = 0$ and whose slope is 5. (Ans. $35x - 7y + 18 = 0$)
2. Find the equation of the line through the intersection of lines $x + 2y - 3 = 0$ and $4x - y + 7 = 0$ and which is parallel to $5x + 4y - 20 = 0$
   (Ans. $15x + 12y - 7 = 0$)
3. Find the equation of the line through the intersection of the lines $2x + 3y - 4 = 0$ and $x - 5y = 7$ that has its x-intercept equal to -4.
   (Ans. $10x + 93y + 40 = 0$.)
4. Find the equation of the line through the intersection of $5x - 3y = 1$ and $2x + 3y - 23 = 0$ and perpendicular to the line $5x - 3y - 1 = 0$.
   (Ans. $63x + 105y - 781 = 0$)

10.7 Shifting of Origin

An equation corresponding to a set of points with reference to a system of coordinate axes may be simplified by taking the set of points in some other suitable coordinate system such that all geometric properties remain unchanged. One such transformation is that in which the new axes are transformed parallel to the original axes and origin is shifted to a new point. A transformation of this kind is called a *translation of axes*.

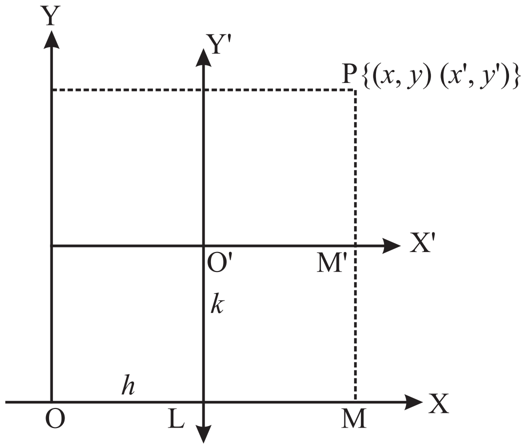


Fig. 10.21

The coordinates of each point of the plane are changed under a translation of axes. By knowing the relationship between the old coordinates and the new coordinates of points, we can study the analytical problem in terms of new system of coordinate axes.

To see how the coordinates of a point of the plane changed under a translation of axes, let us take a point P (x, y) referred to the axes OX and OY. Let O′X′ and O′Y′ be new axes parallel to OX and OY respectively, where O′ is the new origin. Let (h, k) be the coordinates of O′ referred to the old axes, i.e., OL = h and LO′ = k. Also, OM = x and MP = y (see Fig.10.21)

Let O′ M′ = x' and M′P = y' be respectively, the abscissa and ordinates of a point P referred to the new axes O′ X′ and O′ Y′. From Fig.10.21, it is easily seen that

OM = OL + LM, i.e., $x = h + x'$

and MP = MM′ + M′ P, i.e., $y = k + y'$

Hence, $x = x' + h,\ y = y' + k$

These formulae give the relations between the old and new coordinates.

Example 21 Find the new coordinates of point (3, –4) if the origin is shifted to (1, 2) by a translation.

Solution The coordinates of the new origin are $h = 1$, $k = 2$, and the original coordinates are given to be $x = 3$, $y = -4$.

The transformation relation between the old coordinates (x, y) and the new coordinates (x', y') are given by

$x = x' + h$ i.e., $x' = x - h$

and $y = y' + k$ i.e., $y' = y - k$

Substituting the values, we have

$x' = 3 - 1 = 2$ and $y' = -4 - 2 = -6$

Hence, the coordinates of the point (3, –4) in the new system are (2, –6).

Example 22 Find the transformed equation of the straight line $2x - 3y + 5 = 0$, when the origin is shifted to the point (3, –1) after translation of axes.

Solution Let coordinates of a point P changes from (x, y) to (x', y') in new coordinate axes whose origin has the coordinates $h = 3$, $k = -1$. Therefore, we can write the transformation formulae as $x = x' + 3$ and

$y = y' - 1$. Substituting, these values in the given equation of the straight line, we get

$$2(x' + 3) - 3(y' - 1) + 5 = 0$$

or $$2x' - 3y' + 14 = 0$$

Therefore, the equation of the straight line in new system is $2x - 3y + 14 = 0$

Exercise 10.5

1. Find the new coordinates of the points in each of the following cases if the origin is shifted to the point (–3, –2) by a translation of axes.

   (i) (1, 1) (Ans (4, 3)) (ii) (0, 1) (Ans. (3, 3))
   (iii) (5, 0) (Ans. (8, 2)) (iv) (–1, –2) (Ans. (2, 0))
   (v) (3, –5) (Ans. (6, –3))

2. Find what the following equations become when the origin is shifted to the point (1, 1)

   (i) $x^2 + xy - 3y^2 - y + 2 = 0$ (Ans. $x^2 - 3y^2 + xy + 3x - 6y + 1 = 0$)
   (ii) $xy - y^2 - x + y = 0$ (Ans. $xy - y^2 = 0$)
   (iii) $xy - x - y + 1 = 0$ (Ans. $xy = 0$)

CHAPTER 13

13.5 Limits Involving Exponential and Logarithmic Functions

Before discussing evaluation of limits of the expressions involving exponential and logarithmic functions, we introduce these two functions stating their domain, range and also sketch their graphs roughly.

Leonhard Euler (1707–1783), the great Swiss mathematician introduced the number e whose value lies between 2 and 3. This number is useful in defining exponential function and is defined as $f(x) = e^x$, $x \in \mathbf{R}$. Its domain is $\mathbf{R}$, range is the set of positive real numbers. The graph of exponential function, i.e., $y = e^x$ is as given in Fig.13.11.

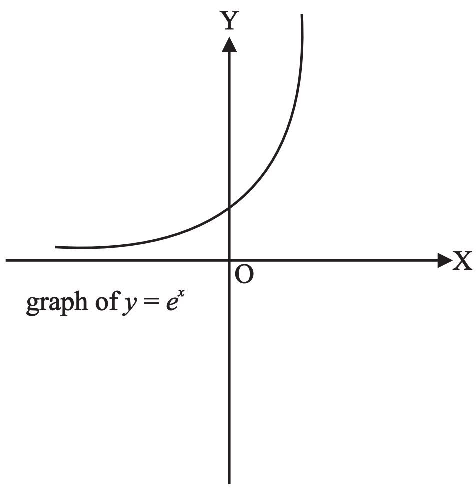


Fig. 13.11

Similarly, the logarithmic function expressed as $\log_e \mathbf{R}^+ \to \mathbf{R}$ is given by $\log_e x = y$, if and only if $e^y = x$. Its domain is $\mathbf{R}^+$ which is the set of all positive real numbers and range is $\mathbf{R}$. The graph of logarithmic function $y = \log_e x$ is shown in Fig.13.12.

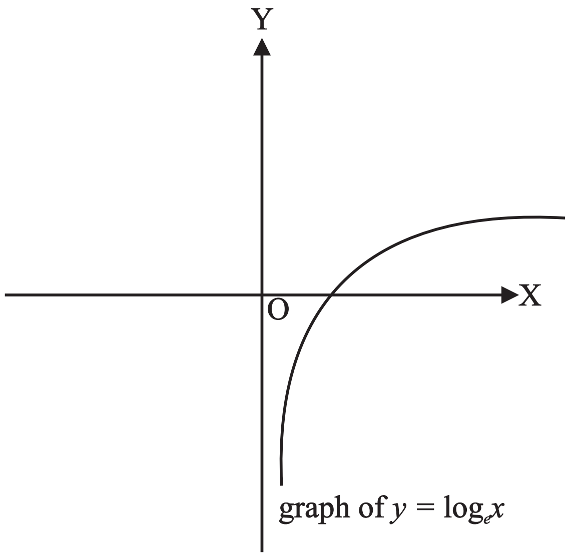


Fig. 13.12

In order to prove the result $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$, we make use of an inequality involving the expression $\dfrac{e^x-1}{x}$ which runs as follows:

$\dfrac{1}{1+|x|} \le \dfrac{e^x-1}{x} \le 1+(e-2)\,|x|$ holds for all x in $[-1, 1] \sim \{0\}$.

Theorem 6 Prove that $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$

Proof Using above inequality, we get

$$\frac{1}{1+|x|} \le \frac{e^x-1}{x} \le 1+|x|\,(e-2),\ x \in [-1, 1] \sim \{0\}$$

Also $\quad \lim\limits_{x\to 0}\dfrac{1}{1+|x|} = \dfrac{1}{1+\lim\limits_{x\to 0}|x|} = \dfrac{1}{1+0} = 1$

and $\quad \lim\limits_{x\to 0}\left[1+(e-2)|x|\right] = 1+(e-2)\lim\limits_{x\to 0}|x| = 1+(e-2)0 = 1$

Therefore, by Sandwich theorem, we get

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

Theorem 7 Prove that $\lim\limits_{x\to 0}\dfrac{\log_e(1+x)}{x}=1$

Proof Let $\dfrac{\log_e(1+x)}{x} = y$. Then

$$\log_e(1+x) = xy$$

$$\Rightarrow 1 + x = e^{xy}$$

$$\Rightarrow \frac{e^{xy}-1}{x} = 1$$

or $$\frac{e^{xy}-1}{xy}.y = 1$$

$$\Rightarrow \lim_{xy\to 0}\frac{e^{xy}-1}{xy}\lim_{x\to 0} y = 1 \text{ (since } x \to 0 \text{ gives } xy \to 0)$$

$$\Rightarrow \lim_{x\to 0} y = 1 \left(\text{as } \lim_{xy\to 0}\frac{e^{xy}-1}{xy} = 1\right)$$

$$\Rightarrow \lim_{x\to 0}\frac{\log_e(1+x)}{x} = 1$$

Example 5 Compute $\lim_{x\to 0}\dfrac{e^{3x}-1}{x}$

Solution We have

$$\lim_{x\to 0}\frac{e^{3x}-1}{x} = \lim_{3x\to 0}\frac{e^{3x}-1}{3x}\cdot 3$$

$$= 3\left(\lim_{y\to 0}\frac{e^{y}-1}{y}\right), \quad \text{where } y = 3x$$

$$= 3.1 = 3$$

Example 6 Compute $\lim_{x\to 0}\dfrac{e^{x}-\sin x-1}{x}$

Solution We have $\lim_{x\to 0}\dfrac{e^{x}-\sin x-1}{x} = \lim_{x\to 0}\left[\dfrac{e^{x}-1}{x}-\dfrac{\sin x}{x}\right]$

$$= \lim_{x\to 0}\frac{e^{x}-1}{x} - \lim_{x\to 0}\frac{\sin x}{x} = 1 - 1 = 0$$

Example 7 Evaluate $\lim_{x\to 1}\dfrac{\log_e x}{x-1}$

Solution Put $x = 1 + h$, then as $x \to 1 \Rightarrow h \to 0$. Therefore,

$$\lim_{x\to 1}\frac{\log_e x}{x-1} = \lim_{h\to 0}\frac{\log_e(1+h)}{h} = 1\left(\text{since } \lim_{x\to 0}\frac{\log_e(1+x)}{x} = 1\right).$$

Exercise 13.2

Evaluate the following limits, if exist

1. $\lim_{x \to 0} \dfrac{e^{4x} - 1}{x}$ (Ans. 4)

2. $\lim_{x \to 0} \dfrac{e^{2+x} - e^2}{x}$ (Ans. e^2)

3. $\lim_{x \to 5} \dfrac{e^x - e^5}{x - 5}$ (Ans. e^5)

4. $\lim_{x \to 0} \dfrac{e^{\sin x} - 1}{x}$ (Ans. 1)

5. $\lim_{x \to 3} \dfrac{e^x - e^3}{x - 3}$ (Ans. e^3)

6. $\lim_{x \to 0} \dfrac{x(e^x - 1)}{1 - \cos x}$ (Ans. 2)

7. $\lim_{x \to 0} \dfrac{\log_e (1 + 2x)}{x}$ (Ans. 2)

8. $\lim_{x \to 0} \dfrac{\log (1 + x^3)}{\sin^3 x}$ (Ans. 1)

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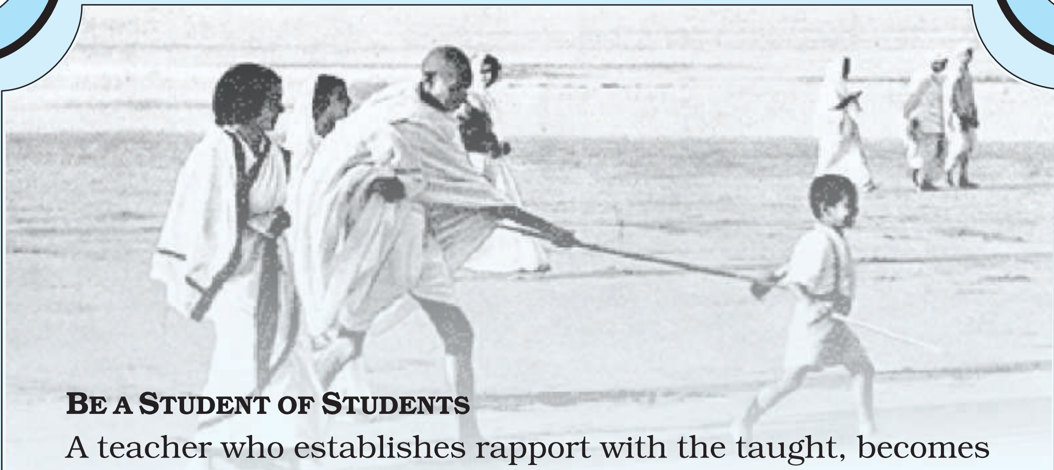

BE A STUDENT OF STUDENTS

A teacher who establishes rapport with the taught, becomes one with them, learns more from them than he teaches them. He who learns nothing from his disciples is, in my opinion, worthless. Whenever I talk with someone I learn from him. I take from him more than I give him. In this way, a true teacher regards himself as a student of his students. If you will teach your pupils with this attitude, you will benefit much from them.

Talk to Khadi Vidyalaya Students, Sevagram
Harijan Seva, 15 February 1942 (CW 75, p. 269)

USE ALL RESOURCES TO BE CONSTRUCTIVE AND CREATIVE

What we need is educationists with originality, fired with true zeal, who will think out from day to day what they are going to teach their pupils. The teacher cannot get this knowledge through musty volumes. He has to use his own faculties of observation and thinking and impart his knowledge to the children through his lips, with the help of a craft. This means a revolution in the method of teaching, a revolution in the teachers' outlook. Up till now you have been guided by inspector's reports. You wanted to do what the inspector might like, so that you might get more money yet for your institutions or higher salaries for yourselves. But the new teacher will not care for all that. He will say, '1 have done my duty to my pupil if I have made him a better man and in doing so I have used all my resources. That is enough for me'.

Harijan, 18 February 1939 (CW 68, pp. 374-75)