Lesson Name: Parabola
URL: https://byjus.com/jee/parabola/

Parabola

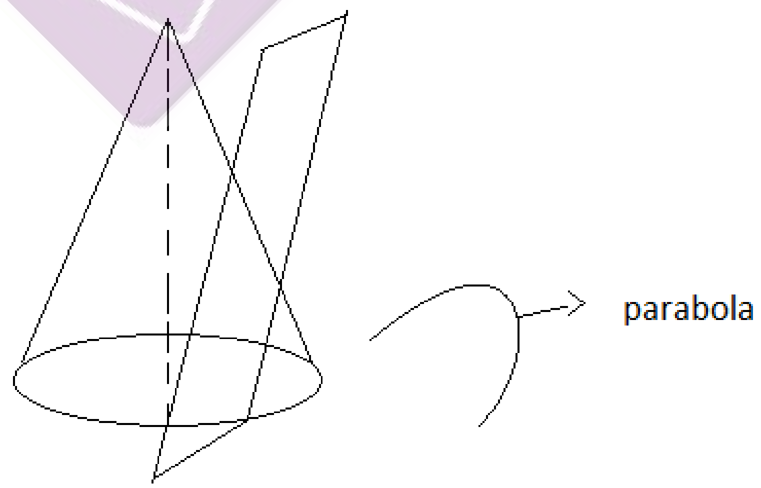
A parabola is a U-shaped plane curve where any point is at an equal distance from a fixed point (known as the focus) and from a fixed straight line which is known as the directrix. Parabola is an integral part of conic section topic and all the concepts related to a parabola are covered here which include the following:

- Parabola Definition
- Standard Equation of Parabola
- Latus Rectum of Parabola
- Parametric co-ordinates of Parabola
- General Equations of Parabola
- Tangent to a Parabola
- Properties of Focal Chord, Tangent and Normal of Parabola
- Forms of a Parabola
- Parabola Questions

What is Parabola?

Section of a right circular cone by a plane parallel to a generator of the cone is a **parabola**. It is a locus of a point, which moves so that distance from a fixed point (focus) is equal to the distance from a fixed line (directrix)

- Fixed point is called focus
- Fixed line is called directrix



Standard Equation of Parabola

The simplest equation of a parabola is $y^2 = x$ when the directrix is parallel to the y-axis. In general, if the directrix is parallel to the y-axis in the standard equation of a parabola is given as:

$$y^2 = 4ax$$

If the parabola is sideways i.e., the directrix is parallel to x-axis, the standard equation of a parabole becomes,

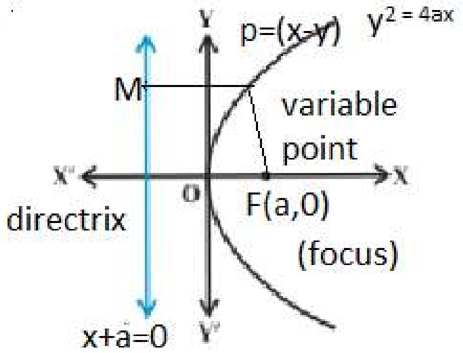
$x^2 = 4ay$

Apart from these two, the equation of a parabola can also be **$y^2 = 4ax$** and **$x^2 = 4ay$** if the parabola is in the negative quadrants. Thus, the four equations of a parabola are given as:

1. $y^2 = 4ax$
2. $y^2 = -4ax$
3. $x^2 = 4ay$
4. $x^2 = -4ay$

Parabola Equation Derivation

In the above equation, "a" is the distance from the origin to the focus. Below is the derivation for the parabola equation. First, refer to the image given below.



From definition,

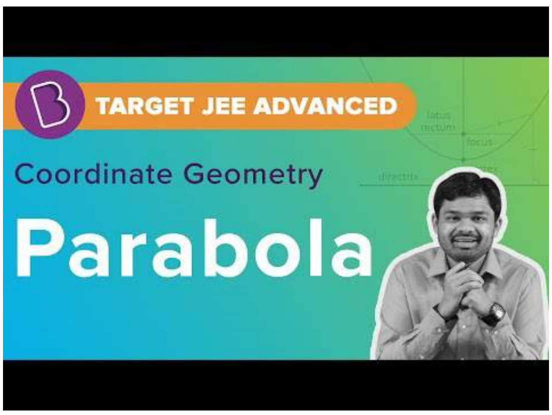
$$\frac{SP}{PM} = 1$$

SP = PM $\qquad \sqrt{(x-a)^2 + y^2} = \left|\frac{x+a}{1}\right|$

$(x - a)^2 + y^2 = (x + a)^2$

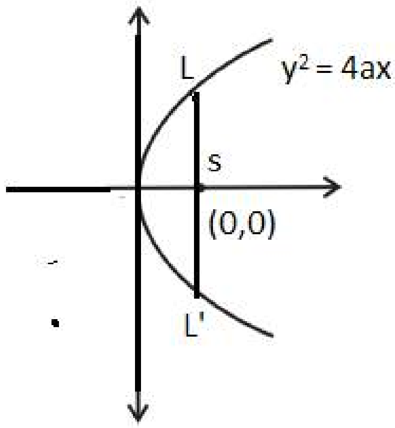
⇒ **Standard equation of Parabola**.

Parabola Video Lesson:



Latus Rectum of Parabola

The latus rectum of a parabola is the chord that passes through the focus and is perpendicular to the axis of the parabola.



LSL' Latus Ractum

= $2\left(\sqrt{4a.a}\right)$

= 4a (length of latus Rectum)

Note: – Two parabola are said to be equal if their latus rectum are equal.

Parametric co-ordinates of Parabola

For a parabola, the equation is $y^2 = -4ax$. Now, to represent the co-ordinates of a point on the parabola, the easiest form will be = at^2 and y = 2at as for any value of "t", the coordinates (at^2, 2at) will always satisfy the parabola equation i.e. $y^2 = 4ax$. So,

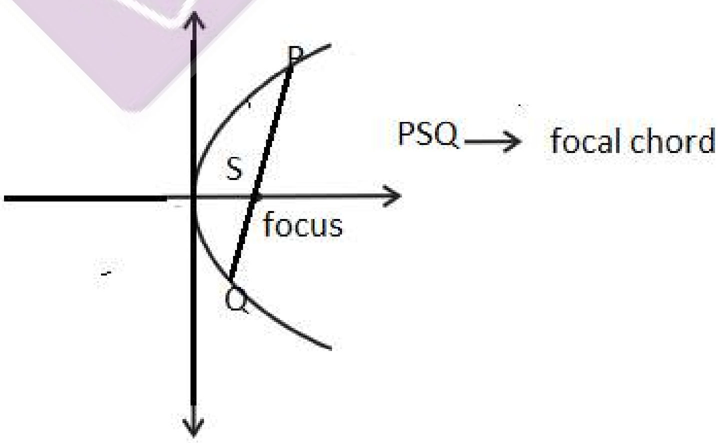
Any point on the parabola

$y^2 = 4ax$ (at^2, 2at)

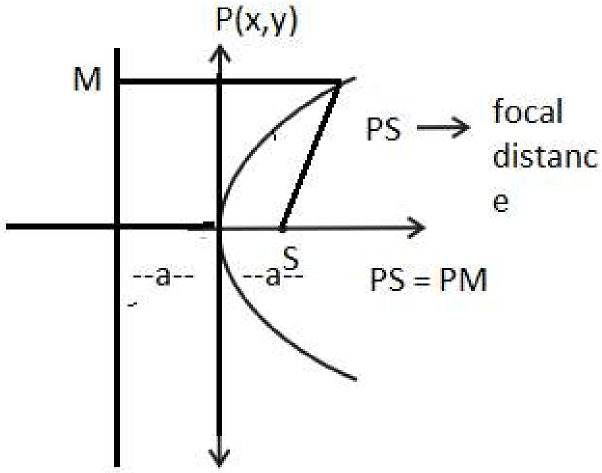
where 't' is a parameter.

Focal Chord and Focal Distance

Focal chord: Any chord passes through the focus of the parabola is a fixed chord of the parabola.



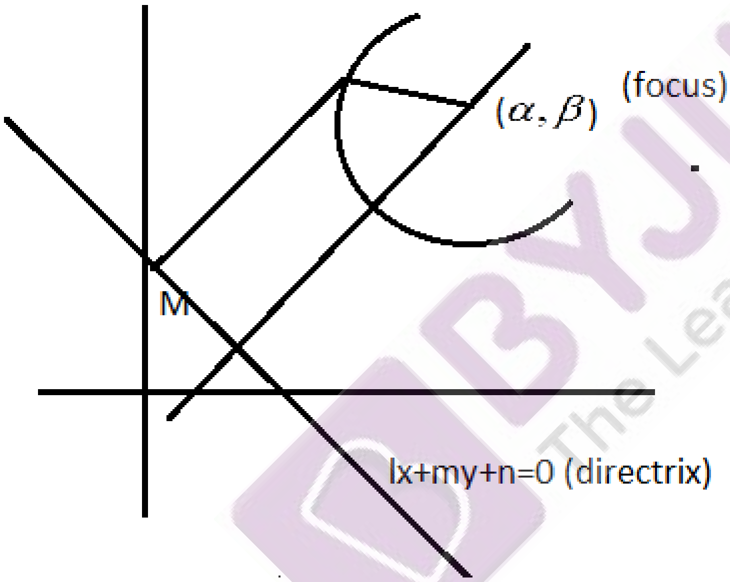
Focal Distance: The focal distance of any point p(x, y) on the parabola $y^2 = 4ax$ is the distance between point 'p' and focus.



PM = a + x

PS = Focal distance = x + a

General Equations of Parabola



Equation of parabola by definition.

SP = PM

$$(x-\alpha)^2 + (y-\beta)^2 = \frac{(\ell x+my+n)^2}{\ell^2+m^2}$$

The general equation of 2nd degree i.e. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ if

$$\Delta \neq 0 \; h^2 = ab$$

Position of a point with respect to parabola

For parabola

$$S \equiv y^2 - 4ax = 0 \qquad , p(x_1, y_1)$$

$$S_1 = {y_1}^2 - 4ax_1$$

$$S_1 < 0 \quad (inside\, curve)$$

$$S_1 = 0 \quad (on\, curve)$$

$S_1 > 0 \quad (outside\ curve)$

Intersection of a straight line (https://byjus.com/jee/straight-lines/) with the parabola $y^2 = 4ax$

Straight line y = mx + c

m slope of straight line

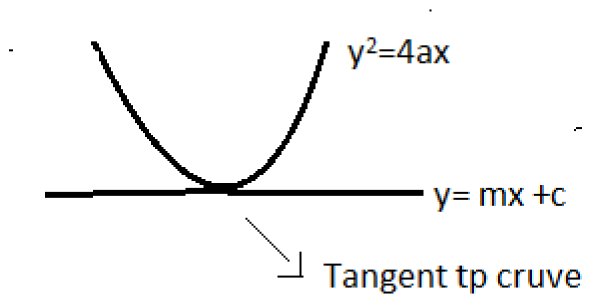
$(mx + c)^2 - 4ax = 0$

$m^2x^2 + 2x(mc - 2a) + c^2 = 0$

$Ax^2\ Bx + c = 0$

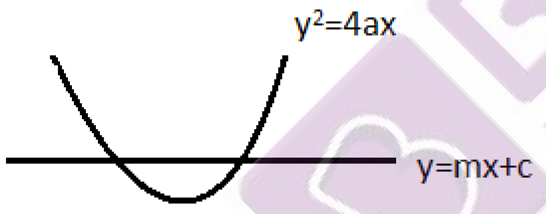
$B^2 - 4AC$ = discriminant D

D = 0



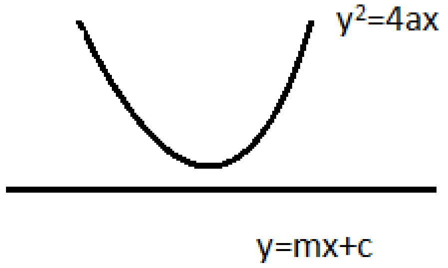
$c = {}^{a}/_{m}$

D > 0



mc − a > 0: Straight line intersect the curve

D < 0 (mc − a) < 0: Straight line not touching the curve



Tangent to a Parabola

Tangent at point (x_1, y_1)

$y^2 = 4ax$ (parabola)

equation of Tangent

$$y - y_1 = m(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{dy}{dx}$$

$$\frac{2y\,dy}{dx}\Big|_{x_1 y_1} = 4a$$

$$\frac{dy}{dx} = \frac{2a}{y_1}$$

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$yy_1 - {y_1}^2 = 2a(x - x_1)$$

$$yy_1 - 4ax_1 = 2a(x - x_1)$$

$$yy_1 = 2a(x + x_1)$$

⇒ Point $(x_1\, y_1)$

Tangent in slope (m) form:

$y^2 = 4ax$

Let equation of Tangent y = mx + c

From the previous illustration

y = mx + c touches curve at a point

so , $c = {}^{9}/_{m}$

equation of Tangent :- y = mx + ${}^{a}/_{m}$

so, point of Tangency is $\left({}^{a}/_{m^2}, \frac{2a}{m}\right)$

Tangent in parameter form (at^2, 2at)

$ty = x + at^2$ where 't' is

parameter

Pair of Tangents from (x_1, y_1) external points

Let y^2 = 4 are, (parabola)

P(x_1, y_1) external point then equation of Tangents is given by

$SS_1 = T^2$

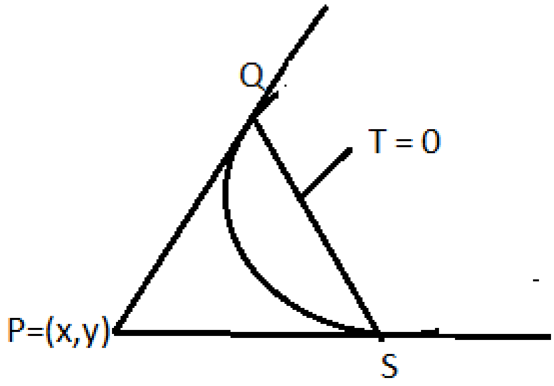
$$S \equiv y^2 - 4\,are, \qquad S_1 \equiv {y_1}^2 - 4ax_1$$

$$T \equiv yy_1 - 2a(x - x_1)$$

Chord of contact:

Equation of chord of contact of Tangents from a point p(x_1, y_1) to the parabola $y^2 = 4ax$ is given by T = 0

i.e., $yy_1 - 2a(x + x_1) = 0$



Equation of QS T = 0

Normal to the parabola:

Normal to the point p(x_1, y_1) since normal is perpendicular to Tangent so slope of normal be will

${}^{-1}/_{Slope\ of\ Tangent}$

slope of normal at 'p' (x_1 y_1) is $\frac{-y_1}{2a}$

equation of normal$y - y_1 = \frac{-y_1}{2a}(x - x_1)$

Normal in term of 'm':

$(slope\ of\ normal) \Rightarrow m = -\frac{dx}{dy}$ $\quad y^2 = 4ax$

$y_1 = -2am$

$m = \frac{-y_1}{2a}$ $\quad x_1 = am^2$

$y = mx - 2am - am^3$

$m = \frac{-dx}{dy}$

Equation of normal at point (am^2, − 2am)

$$y = mx - 2am - am^3$$

$$m = \frac{-dx}{dy}$$

equation of normal at point (am^2, −2 am)

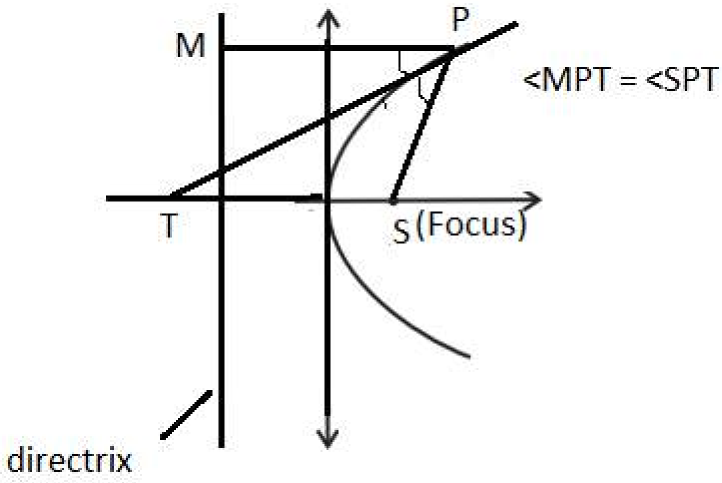
Normal at point (at^2, 2at)

T parameter

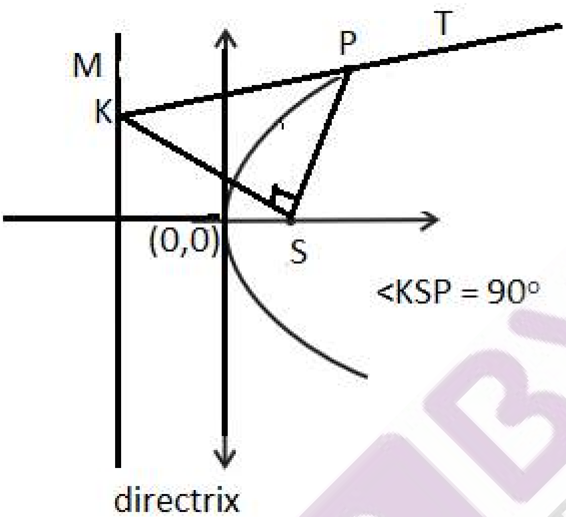
y = tx + 2at + at^3

Important properties of focal chord, Tangent and normal of Parabola

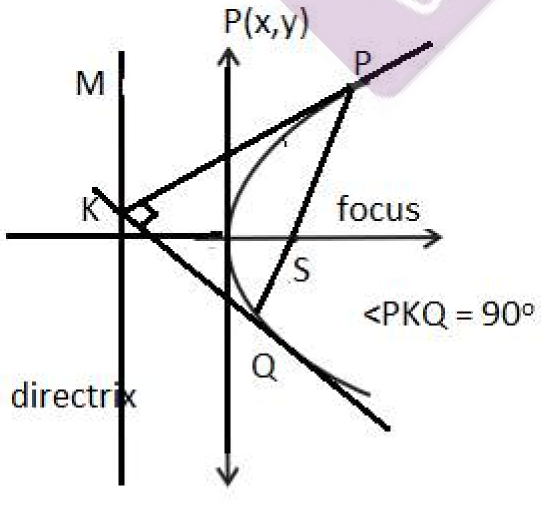
- The tangent at any point P on a parabola bisects the angle between the focal chord through P and the perpendicular from P on the directix.



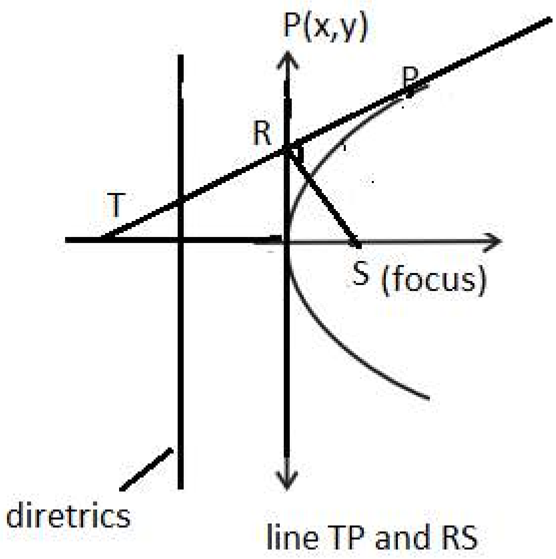
- The portion of a tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.



(iii) Tangents at the extremities of any focal chord intersect at right angles on the directrix.



(iv) Any Tangent to a parabola and perpendicular on it from the focus meet on the Tangent at its vertex.



Intersect at y-axis, at u = 0

Four common forms of a Parabola:

Form:	**y^2 = − 4ax**	**y^2 = − 4ax**	**x^2 = 4ay**	**x^2 = − 4ay**
Vertex:	(0, 0)	(0,0)	(0, 0)	(0, 0)
Focus:	(a, 0)	(-a, 0)	(0, a)	(0, -a)
Equation of the directrix:	x = − a	x =	y = − a	y = a
Equation of the axis:	y = 0	y = 0	x = 0	x = 0
Tangent at the vertex:	x = 0	x = 0	y = 0	y = 0

Practice Problems on Parabola

Illustration 1: Find the vertex, axis, directrix, tangent at the vertex and the length of the latus rectum of the parabola $2y^2+3y-4x-3=0$.

Solution: The given equation can be re-written as $\left(y+\frac{3}{4}\right)^2=2\left(x+\frac{33}{32}\right)$

which is of the form $Y^2=4aX$where $Y=y+\frac{3}{4},\ X=x+\frac{33}{32},\ 4a=2$.

Hence the vertex is $X=0, Y=0$ i.e. $\left(-\frac{33}{32},-\frac{3}{4}\right)$.

The axis is $y+\frac{3}{4}=0 \Rightarrow y=-\frac{3}{4}$.

The directix is $X=a=0$

$\Rightarrow x+\frac{33}{32}+\frac{1}{2}=0 \Rightarrow x=-\frac{49}{32}$

The tangent at the vertex is $X=0\, or\, x+\frac{33}{32}=0 \Rightarrow x=-\frac{33}{32}$.

Length of the latus rectum = 4a = 2.

Illustration 2: Find the equation of the parabola whose focus is (3, -4) and directix x − y + 5 = 0.

Solution: Let P(x, y) be any point on the parabola. Then

$\sqrt{(x-3)^3+(y+4)^2} = \frac{|x-y+5|}{\sqrt{1+1}}$

$\Rightarrow (x-3)^2+(y+4)^2 = \frac{(x-y+5)^2}{2}$

$\Rightarrow x^2+y^2+2xy-22x+26y+25=0$

$\Rightarrow (x+y)^2 = 22x-26y-25$

Illustration 3: Find the equation of the parabola having focus (-6, -6) and vertex (-2, 2).

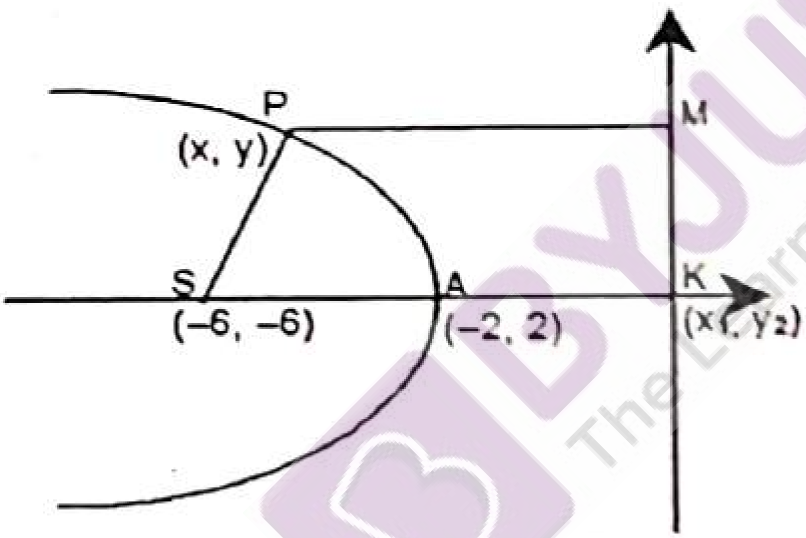
Solution: Let S(6, -6) be the focus and A(-2, 2) the vertex of the parabola. On SA take a point K (x_1 , y_1) such that SA = AK. Draw KM perpendicular on SK. Then KM is the directix of the parabola.

Since A bisects SK, $\left(\frac{-6+x_1}{2}, \frac{-6+y_1}{2}\right) = (-2, 2)$

$\Rightarrow -6+x_1 = -4\, and\, -6+y_1 = 4\, or\, (x_1, y_1) = (2, 10).$

Hence the equation of the directrix KM is y − 10 = m(x+2) ……(1)

Also gradient of $SK = \frac{10-(-6)}{2-(-6)} = \frac{16}{8} = 2;\ m = \frac{-1}{2}$



So that equation (1) becomes

$y-10 = \frac{1}{2}(x-2)$ or $x+2y-22=0$ is the directrix.

Next, let PM be a perpendicular on the directrix KM from any point P(x, y) on the parabola.

From SP = PM, the equation of the parabola is

$\sqrt{\left\{(x+6)^2+(y+6)^2\right\}} = \frac{x+2y-22}{\sqrt{(1^2+2^2)}}$

Illustration 4: Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $y^2 = 12x$.

Solution: The given equation is $y^2 = 12x$.

Here, the coefficient of x is positive. Hence, the parabola opens towards the right.

On comparing this equation with $y^2 = 4ax$, we get $4a = 12a$ or $a = 3$.

Coordinates of the focus are given by (a, 0) i.e., (3, 0).

Since the given equation involves y^2, the axis of the parabola is the y-axis.

Equation of directrix is $x = -a$, i.e., $x = -3$.

Length of latus rectum = 4a = 4 x 3 = 12.

Illustration 5: Find the coordinates of the focus, axis of the parabola, the equation of directrix and the length of the latus rectum for $x^2 = -16y$.

Solution: The given equation is $x^2 = -16y$.

Here, the coefficient of y is negative. Hence, the parabola opens downwards.

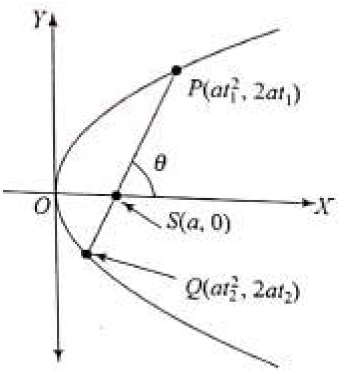
On comparing this equation with $x^2 = -4ay$, we get $-4a = -16$ or $a = 4$.

Coordinates of the focus = (0, -a) = (0, -4).

Since the given equation involves x^2, the axis of the parabola is the y-axis.

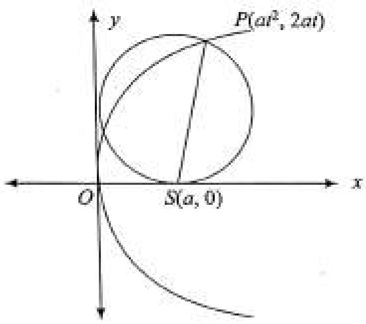
Equation of directrix, y = a i.e. = 4.

Length of latus rectum = 4a = 16.



Important Properties of Focal Chord

1. If chord joining $P = (at_1^2, 2at_1)$ and $Q = (at_2^2, 2at_2)$is focal chord of parabola $y^2 = 4ax$ then $t_1 t_2 = -1$.
2. If one extremity of a focal chord is $(at_1^2, 2at_1)$ then the other extremity $(at_1^2, 2at_2)$ becomes $\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$.
3. If point $P(at^2, 2at)$ lies on parabola $y^2 = 4ax$, then the length of focal chord *PQ* is $a(t + 1/t)^2$.
4. The length of the focal chord which makes an angle θ with positive x-axis is $4a \cos ec^2 \theta$.
5. Semi latus rectum is harmonic mean of SP and SQ, where P and Q are extremities of latus rectum. i.e., $2a = \frac{2SP \times SQ}{SP + SQ}\ or \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$
6. Circle described on focal length as diameter touches tangent at vertex.



7. Circle described on focal chord as diameter touches directrix.

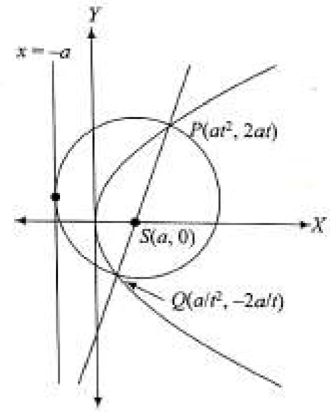


Illustration 6: If the parabola $y^2 = 4x$ and $x^2 = 32y$ intersect at (16, 8) at an angle θ, then find the value of θ.

Solution: The slope of the tangent to $y^2 = 4x$ at (16, 8) is given by

$$m_1 = \left(\frac{dy}{dx}\right)_{(16,8)} = \left(\frac{4}{2y}\right)_{(16,8)} = \frac{2}{8} = \frac{1}{4}$$

The slope of the tangent to $x^2 = 32y$ at (16, 8) is given by

$$m_2 = \left(\frac{dy}{dx}\right)_{(16,8)} = \left(\frac{2x}{32}\right)_{(16,8)} = 1$$

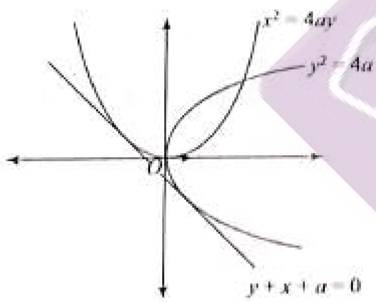
$$\therefore Tan\ \theta = \frac{1-(1/4)}{1+(1/4)} = \frac{3}{5}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{5}\right)$$

Illustration 7: Find the equation of common tangent of $y^2 = 4ax$ and $x^2 – 4ay$.

Solution: Equation of tangent to $y^2 = 4ax$ having slope m is $y = mx + \frac{a}{m}$.

It will touch $x^2 – 4ay$, if $x^2 = 4a\left(mx + \frac{a}{m}\right)$ has a equal roots.



Thus, $16a^2m^2 = \ -16\frac{a^2}{m} \ \Rightarrow\ m = -1$

Thus, common tangent is y + x + a = 0.

Illustration 8: Find the equation of normal to the parabola $y^2 = 4x$ passing through the point (15, 12).

Solution: Equation of the normal having slope m is

$$y = mx - 2m - m^3$$

If it passes through the point (15, 12) then

$$12 = 15m - 2m - m^3$$

$$\Rightarrow \quad m^3 - 13m + 12 = 0$$

$$\Rightarrow \quad (m-1)(m-3)(m+4) = 0$$

$\Rightarrow \quad m = 1,\ 3,\ -4$

Hence, equations of normal are:

$y = x - 3,\ y = 3x - 33\, and\, y + 4x = 72$

Illustration 9: Find the point on y^2 = 8x where line x + y = 6 is a normal.

Solution: Slope m of the normal x + y = 6 is -1 and a = 2

Normal to parabola at point (am^2, -2am) is

$y = mx - 2am - am^3$

$\Rightarrow \quad y = -x + 4 + 2\, at\, (2, 4)$

$\Rightarrow \quad x + y = 6\, is\, normal\, at\, (2, 4)$

Illustration 10: Tangents are drawn to y^2 = 4ax at point where the line lx + my + n = 0 meets this parabola. Find the intersection of these tangents.

Solution: Let the tangents intersects at P (h, k). Then lx + my + n = 0 will be the chord of contact. That means lx + my + n = 0 and yk – 2ax – 2ah = 0 which is chord of contact, will represent the same line.

Comparing the ratios of coefficients, we get

$frackm = \frac{-2a}{l} = \frac{-2ah}{n}$

$\Rightarrow \quad h = \frac{n}{l},\ k = -\frac{2am}{l}$

Illustration 11: If the chord of contact of tangents from a point P to the parabola If the chord of contact of tangents from a point P to the parabola y^2 = 4ax touches the parabola x^2=4by, then find the locus of P.

Solution: Chord of contact of parabola y^2 = 4ax w.r.t. point P(x_1 , y_1)

$yy_1 = 2a(x + x_1)$ ……(1)

This line touches the parabola x^2 = 4by.

Solving line (1) with parabola, we have

$x^2 = 4b\left[(2a/y_1)\,(x + x_1)\right]$

or $y_1 x^2 - 8abx - 8abx_1 = 0$

According to the question, this equation must have equal roots.

$\Rightarrow \quad D = 0$

$\Rightarrow \quad 64a^2b^2 + 32abx_1y_1 = 0$

$\Rightarrow \quad x_1y_1 = -2ab$ or $xy = -2ab$, which is the required locus.