Lesson Name: Determinants Properties
URL: https://byjus.com/jee/properties-of-determinants/

Properties of Determinants

There are 10 main properties of determinants which include reflection property, all-zero property, proportionality or repetition property, switching property, scalar multiple property, sum property, invariance property, factor property, triangle property, and co-factor matrix property. All the determinant properties have been covered below in a detailed way along with solved examples.

All Topics in Determinants

- Introduction to Determinants (https://byjus.com/jee/determinants/)
- Minors and Cofactors (https://byjus.com/jee/minors-and-cofactors/)
- Properties of Determinants
- System of Linear Equations Using Determinants (https://byjus.com/jee/system-of-linear-equations-using-determinants/)
- Differentiation and Integration of Determinants (https://byjus.com/jee/differentiation-integration-of-determinants/)
- Standard Determinants (https://byjus.com/jee/standard-determinants/)

Determinants have some properties that are useful as they permit us to generate the same results with different and simpler configurations of entries (elements).

Important Properties of Determinants

1. Reflection Property:

The determinant remains unaltered if its rows are changed into columns and the columns into rows. This is known as the property of reflection.

2. All-zero Property:

If all the elements of a row (or column) are zero, then the determinant is zero.

3. Proportionality (Repetition) Property:

If the all elements of a row (or column) are proportional (identical) to the elements of some other row (or column), then the determinant is zero.

4. Switching Property:

The interchange of any two rows (or columns) of the determinant changes its sign.

5. Scalar Multiple Property:

If all the elements of a row (or column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

6. Sum Property:

$$\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

7. Property of Invariance:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \alpha b_1 + \beta c_1 & b_1 & c_1 \\ a_2 + \alpha b_2 + \beta c_2 & b_2 & c_2 \\ a_3 + \alpha b_3 + \beta c_3 & b_3 & c_3 \end{vmatrix}$$

That is, a determinant remains unaltered under an operation of the form $C_i \to C_i + \alpha C_j + \beta C_k,$ where $j, k \neq i,$ or an operation of the form $R_i \to R_i + \alpha R_j + \beta R_k,$ where $j, k \neq i$

8. Factor Property:

If a determinant Δ becomes zero when we put $x = \alpha,$ then $(x - \alpha)$ is a factor of Δ.

9. Triangle Property:

If all the elements of a determinant above or below the main diagonal consist of zeros, then the determinant is equal to the product of diagonal elements. That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

10. Determinant of cofactor matrix:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \ then\ \Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2 \ where\ C_{ij}\ denotes\ the\ cofactor\ of\ the\ element\ a_{ij}\ in\ \Delta.$$

Example Problems on Properties of Determinants

Question 1: Using properties of determinants, prove that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right)$

Solution:

By using invariance and scalar multiple property of determinant we can prove the given problem.

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} [Operating C_1 \to C_1 + C_2 + C_3]$$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} [Operating\,(R_2 \to R_2 - R_1\, and\, R_3 \to R_3 - R_1)]$$

= (a + b + c) [(c – b) (b – c) – (a – b) (a – c)]

$= (a+b+c)\left(ab+bc+ca-a^2-b^2-c^2\right)$

Question 2: Prove the following identity $\begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix} = 4\alpha^2\beta^2\gamma^2$

Solution:

Take α, β, γ common from the L.H.S. and then by using scalar multiple property and invariance property of determinant we can prove the given problem.

$$\Delta = \begin{vmatrix} -\alpha^2 & \beta\alpha & \gamma\alpha \\ \alpha\beta & -\beta^2 & \gamma\beta \\ \alpha\gamma & \beta\gamma & -\gamma^2 \end{vmatrix}$$

$$Taking\ \alpha, \beta, \gamma\ common\ from\ C_1, C_2, C_3\ \ respectively\ \ \Delta = \alpha\beta\gamma \begin{vmatrix} -\alpha & \alpha & \alpha \\ \beta & -\beta & \beta \\ \gamma & \gamma & -\gamma \end{vmatrix}$$

$$Now\ taking\ [\alpha, \beta, \gamma]\ common\ from\ R_1, R_2, R_3\ respectively$$

$$\Delta = \alpha^2\beta^2\gamma^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Now applying and $R_3 \to R_3 + R_1\ we\ have\ \Delta = \alpha^2\beta^2\gamma^2 \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$

$$Now\ expanding\ along\ C_1, \Delta\alpha^2 \times \beta^2(-1) \times \gamma^2(-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = \ \alpha^2\beta^2(-1)\gamma^2(0-4)$$

$$= 4\alpha^2\beta^2\gamma^2$$

Hence proved.

Question 3: Show that $\begin{vmatrix} \alpha & \beta & \gamma \\ \theta & \phi & \psi \\ \lambda & \mu & v \end{vmatrix} = \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & v & \psi \end{vmatrix}$

Solution:

Interchange the rows and columns across the diagonal using reflection property and then using the switching property of determinant we can obtain the required result.

L.H.S. = $\begin{vmatrix} \alpha & \beta & \lambda \\ \theta & \phi & \psi \\ \lambda & \mu & v \end{vmatrix} = \begin{vmatrix} \alpha & \theta & \lambda \\ \beta & \phi & \mu \\ \gamma & \psi & v \end{vmatrix}$

(Interchanging rows and columns across the diagonal)

$$= (-1) \begin{vmatrix} \alpha & \lambda & \theta \\ \beta & \mu & \phi \\ \gamma & v & \psi \end{vmatrix} = (-1)^2 \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & v & \psi \end{vmatrix} = \begin{vmatrix} \beta & \mu & \phi \\ \alpha & \lambda & \theta \\ \gamma & v & \psi \end{vmatrix} =$$

R.H.S.

Question 4: If a, b, c are all different and if $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0,$

prove that abc = –1.

Solution:

Split the given determinant using sum property. Then by using scalar multiple, switching and invariance properties of determinants, we can prove the given equation.

$$D = \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^3 & 1+c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (-1)^1 \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \ [C_1 \leftrightarrow C_3 \ in \ 1st \ \text{det.}]$$

$$= (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \ [C_2 \leftrightarrow C_3 \ in \ 1st \ \text{det.}]$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (1+abc) \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \ [R_2 \to R_2 - R_1 \ and \ R_3 \to R_3 - R_1]$$

$$= (1+abc) \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} (expanding \ along \ 1st \ row)$$

$$= (1+abc)(b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (1+abc)(b-c)(c-a)(c+a-b-a) = (1+abc)(b-a)(c-a)(c-b)$$

$$\Rightarrow D = (1+abc)(a-b)(b-c)(c-a)\,; \ But \ given \ D = 0$$

$$\Rightarrow (1+abc)(a-b)(b-c)(c-a) = 0$$

∴ (1 + abc) = 0

[since a, b, c are different $a \neq b, b \neq c, c \neq a$ Hence, abc = -1

Question 5: Prove that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

Solution:

Simply by using switching and scalar multiple property we can expand the L.H.S.

Given determinant $= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + (C_2 + C_3)$, we obtain

$$\begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$R_1 \to R_2 - R_1 \;\; and \;\; R_3 \to R_3 - R_1 \;\; (given)$

$$2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} = 2(a+b+c.1)\{(b+c+a)(c+a+b) - (0 \times 0)\}$$

$= 2(a+b+c)^3$

Hence proved.

Question 6: Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$

Solution:

Expand the determinant $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

by using scalar multiple and invariance property.

L.H.S.= $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$; Multiplying C_1, C_2, C_3 by a, b, c respectively

$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$; Now taking a, b, c common from R_1, R_2, R_3 respectively

$$= \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix} = \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix} \quad [C_1 \to C_1 + C_2 + C_3]$$

$$= \left(1+a^2+b^2+c^2\right)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix} = \left(1+a^2+b^2+c^2\right)\begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$[R_2 \to R_2 - R_1 \, and \; R_3 \to R_3 - R_1]$

$= \left(1+a^2+b^2+c^2\right)(1.1.1) = 1 + a^2 + b^2 + c^2 = R.H.S.$

Hence proved