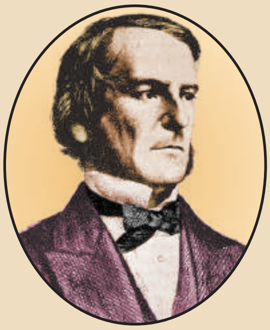
1

SETS AND FUNCTIONS

A set is Many that allows itself to be thought of as a One
- Georg Cantor

- Introduction
- Sets
- Properties of set operations
- De Morgan's Laws
- Functions



GEORGE BOOLE

(1815-1864)
England

Boole believed that there was a close analogy between symbols that represent logical interactions and algebraic symbols.

He used mathematical symbols to express logical relations. Although computers did not exist in his day, Boole would be pleased to know that his Boolean algebra is the basis of all computer arithmetic.

As the inventor of Boolean logic-the basis of modern digital computer logic - Boole is regarded in hindsight as a founder of the field of computer science.

1.1 Introduction

The concept of set is one of the fundamental concepts in mathematics. The notation and terminology of set theory is useful in every part of mathematics. So, we may say that set theory is the language of mathematics. This subject, which originated from the works of George Boole (1815-1864) and Georg Cantor (1845-1918) in the later part of 19th century, has had a profound influence on the development of all branches of mathematics in the 20th century. It has helped in unifying many disconnected ideas and thus facilitated the advancement of mathematics.

In class IX, we have learnt the concept of set, some operations like union, intersection and difference of two sets. Here, we shall learn some more concepts relating to sets and another important concept in mathematics namely, function. First let us recall basic definitions with some examples. We denote all positive integers (natural numbers) by $\mathbb{N}$ and all real numbers by $\mathbb{R}$.

1.2 Sets

Definition

A set is a collection of well-defined objects. The objects in a set are called elements or members of that set.

Here, "well-defined" means that the criteria for deciding if an object belongs to the set or not, should be defined without confusion.

For example, the collection of all "tall people" in Chennai does not form a set, because here, the deciding criteria "tall people" is not clearly defined. Hence this collection does not define a set.

Notation

We generally use capital letters like $A, B, X,$ etc. to denote a set. We shall use small letters like $x, y,$ etc. to denote elements of a set. We write $x \in Y$ to mean x is an element of the set Y. We write $t \notin Y$ to mean t is not an element of the set Y.

Examples

(i) The set of all high school students in Tamil Nadu.

(ii) The set of all students either in high school or in college in Tamil Nadu.

(iii) The set of all positive even integers.

(iv) The set of all integers whose square is negative.

(v) The set of all people who landed on the moon.

Let A, B, C, D and E denote the sets defined in (i), (ii), (iii), (iv), and (v) respectively. Note that square of any integer is an integer that is either zero or positive and so there is no integer whose square is negative. Thus, the set D does not contain any element. Any such set is called an empty set. We denote the empty set by ϕ.

Definition

(i) A set is said to be a finite set if it contains only a finite number of elements in it.

(ii) A set which is not finite is called an infinite set.

Observe that the set A given above is a finite set, whereas the set C is an infinite set. Note that empty set contains no elements in it. That is, the number of elements in an empty set is zero. Thus, empty set is also a finite set.

Definition

(i) If a set X is finite, then we define the cardinality of X to be the number of elements in X. Cardinality of a set X is denoted by $n(X)$.

(ii) If a set X is infinite, then we denote the cardinality of X by a symbol ∞.

Now looking at the sets A, B in the above examples, we see that every element of A is also an element of B. In such cases we say A is a subset of B.

Let us recall some of the definitions that we have learnt in class IX.

Subset Let X and Y be two sets. We say X is a subset of Y if every element of X is also an element of Y. That is, X is a subset of Y if $z \in X$ implies $z \in Y$. It is clear that every set is a subset of itself.

If X is a subset of Y, then we denote this by $X \subseteq Y$.

Set Equality Two sets X and Y are said to be equal if both contain exactly same elements. In such a case, we write $X = Y$. That is, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$.

Equivalent Sets Two finite sets X and Y are said to be equivalent if $n(X) = n(Y)$.

For example, let $P = \{x \mid x^2 - x - 6 = 0\}$ and $Q = \{3, -2\}$. It is easy to see that both P, Q contain same elements and so $P = Q$. If $F = \{3, 2\}$, then F, Q are equivalent sets but $Q \neq F$. Using the concept of function, one can define the equivalent of two infinite sets

Power Set Given a set A, let $P(A)$ denote the collection of all subsets of A. The set $P(A)$ is called the power set of A.

If $n(A) = m$, then the number of elements in $P(A)$ is given by $n(P(A)) = 2^m$.
For example, if $A = \{a,b,c\}$, then $P(A) = \{\phi, \{a\}, \{b\}, \{c\}. \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$ and hence $n(P(A)) = 8$.

Now, given two sets, how can we create new sets using the given sets?

One possibility is to put all the elements together from both sets and create a new set. Another possibility is to create a set containing only common elements from both sets. Also, we may create a set having elements from one set that are not in the other set. Following definitions give a precise way of formalizing these ideas. We include Venn diagram next to each definition to illustrate it.

1.3 Operations on sets

Let X and Y be two sets. We define the following new sets:

(i) **Union** $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$

(read as "X union Y")

Note that $X \cup Y$ contains all the elements of X and all the elements of Y and the Fig. 1.1 illustrates this.

It is clear that $X \subseteq X \cup Y$ and also $Y \subseteq X \cup Y$.

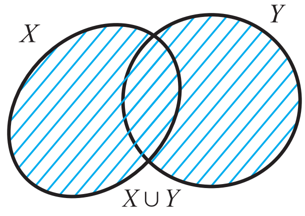


Fig. 1.1

(ii) **Intersection** $X \cap Y = \{z \mid z \in X \text{ and } z \in Y\}$

(read as "X intersection Y")

Note that $X \cap Y$ contains only those elements which belong to both X and Y and the Fig. 1.2 illustrates this.

It is trivial that $X \cap Y \subseteq X$ and also $X \cap Y \subseteq Y$.

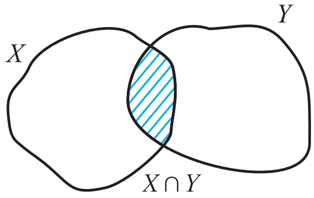


Fig. 1.2

(iii) **Set difference** $X \setminus Y = \{z \mid z \in X \text{ but } z \notin Y\}$

(read as "X difference Y")

Note that $X \setminus Y$ contains only elements of X that are not in Y and the Fig. 1.3 illustrates this. Also, some authors use $A - B$ for $A \setminus B$. We shall use the notation $A \setminus B$ which is widely used in mathematics for set difference.

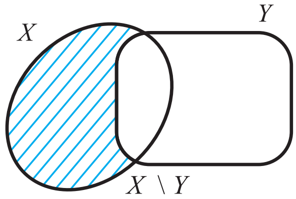


Fig. 1.3

(iv) **Symmetric Difference** $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$

(read as "X symmetric difference Y"). Note that $X \triangle Y$ contains all elements in $X \cup Y$ that are not in $X \cap Y$.

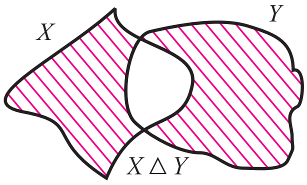


Fig. 1.4

(v) **Complement** If $X \subseteq U$, where U is a universal set, then $U \setminus X$ is called the complement of X with respect to U. If underlying universal set is fixed, then we denote $U \setminus X$ by X' and is called complement of X. The difference set $A \setminus B$ can also be viewed as the complement of B with respect to A.

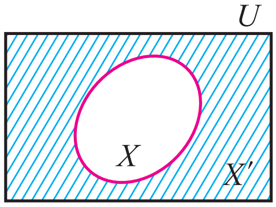


Fig. 1.5

(vi) **Disjoint sets** Two sets X and Y are said to be disjoint if they do not have any common element. That is, X and Y are disjoint if $X \cap Y = \phi$.

It is clear that $n(A \cup B) = n(A) + n(B)$ if A and B are disjoint finite sets.

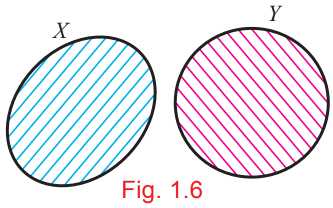


Fig. 1.6

Remarks

Usually circles are used to denote sets in Venn diagrams. However any closed curve may also be used to represent a set in a Venn diagram. While writing the elements of a set, we do not allow repetitions of elements in that set.

Now, we shall see some examples.

Let $A = \{x \mid x$ is a positive integer less than $12\}$, $B = \{1, 2, 4, 6, 7, 8, 12, 15\}$ and $C = \{-2, -1, 0, 1, 3, 5, 7\}$. Now let us find the following:

(i) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

$= \{x \mid x$ is a positive integer less than 12, or $x = 12$, or $15\}$

$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15\}$.

(ii) $C \cap B = \{y \mid y \in C \text{ and } y \in B\} = \{1, 7\}$.

(iii) $A \setminus C = \{x \mid x \in A \text{ but } x \notin C\} = \{2, 4, 6, 8, 9, 10, 11\}$.

(iv) $A \triangle C = (A \setminus C) \cup (C \setminus A)$

$= \{2, 4, 6, 8, 9, 10, 11\} \cup \{-2, -1, 0\} = \{-2, -1, 0, 2, 4, 6, 8, 9, 10, 11\}$.

(v) Let $U = \{x \mid x$ is an integer$\}$ be the universal set.

Note that 0 is neither positive nor negative. Therefore, $0 \notin A$.

Now, $A' = U \setminus A = \{x : x$ is an integer but it should not be in $A\}$

$= \{x \mid x$ is either zero or a negative integer or positive integer greater than or equal to $12\}$

$= \{\cdots, -4, -3, -2, -1, 0\} \cup \{12, 13, 14, 15, \cdots\}$

$= \{\cdots, -4, -3, -2, -1, 0, 12, 13, 14, 15, \cdots\}$.

Let us list out some useful results.

Let U be a universal set and A, B are subsets of U. Then the following hold:

(i) $A \setminus B = A \cap B'$ (ii) $B \setminus A = B \cap A'$

(iii) $A \setminus B = A \Leftrightarrow A \cap B = \phi$ (iv) $(A \setminus B) \cup B = A \cup B$

(v) $(A \setminus B) \cap B = \phi$ (vi) $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

Let us state some properties of set operations.

1.4 Properties of set operations

For any three sets A, B and C, the following hold.

(i) Commutative property

(a) $A \cup B = B \cup A$ (set union is commutative)

(b) $A \cap B = B \cap A$ (set intersection is commutative)

(ii) Associative property

(a) $A \cup (B \cup C) = (A \cup B) \cup C$ (set union is associative)

(b) $A \cap (B \cap C) = (A \cap B) \cap C$ (set intersection is associative)

(iii) Distributive property

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (intersection distributes over union)

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (union distributes over intersection)

Mostly we shall verify these properties with the given sets. Instead of verifying the above properties with examples, it is always better to give a mathematical proof. But this is beyond the scope of this book. However, to understand and appreciate a rigorous mathematical proof, let us take one property and give the proof.

(i) Commutative property of union

In this part we want to prove that for any two sets A and B, the sets $A \cup B$ and $B \cup A$ are equal. Our definition of equality of sets says that two sets are equal only if they contain same elements.

First we shall show that every element of $A \cup B$, is also an element of $B \cup A$.
Let $z \in A \cup B$ be an arbitrary element. Then by the definition of union of A and B we have $z \in A$ or $z \in B$. That is,

for every $z \in A \cup B \implies z \in A$ or $z \in B$

$\implies z \in B$ or $z \in A$

$\implies z \in B \cup A$ by the definition of $B \cup A$. (1)

Since (1) is true for every $z \in A \cup B$, the above work shows that every element of $A \cup B$ is also is an element of $B \cup A$. Hence, by the definition of a subset, we have $(A \cup B) \subseteq (B \cup A)$.

Next, we consider an arbitrary $y \in B \cup A$ and show that this y is also an element of $A \cup B$.
Now, for every $y \in B \cup A \implies y \in B$ or $y \in A$

$\implies y \in A$ or $y \in B$

$\implies y \in A \cup B$ by the definition of $A \cup B$. (2)

Since (2) is true for every $y \in B \cup A$, the above work shows that every element of $B \cup A$ is also an element of $A \cup B$. Hence, by the definition of a subset, we have $(B \cup A) \subseteq (A \cup B)$.

So, we have shown that $(A \cup B) \subseteq (B \cup A)$ and $(B \cup A) \subseteq (A \cup B)$. This can happen only when $(A \cup B) = (B \cup A)$. One could follow above steps to prove other properties listed above by exactly the same method.

About proofs in Mathematics

In mathematics, a statement is called a **true statement** if it is always true. If a statement is not true even in one instance, then the statement is said to be a **false statement**. For example, let us consider a few statements:

(i) Any positive odd integer is a prime number
(ii) Sum of all angles in a triangle is 180°
(iii) Every prime number is an odd integer
(iv) For any two sets A and B, $A \setminus B = B \setminus A$

Now, the statement (i) is false, though very many odd positive integers are prime, because integers like $9, 15, 21, 45$ etc. are positive and odd but not prime.

The statement (ii) is a true statement because no matter which triangle you consider, the sum of its angles equals 180°.

The statement (iii) is false, because 2 is a prime number but it is an even integer. In fact, the statement (iii) is true for every prime number except for 2. So, if we want to prove a statement we have to prove that it is true for all instances. If we want to disprove a statement it is enough to give an example of one instance, where it is false.

The statement (iv) is false. Let us analyze this statement. Basically, when we form $A \setminus B$ we are removing all elements of B from A. Similarly, for $B \setminus A$. So it is highly possible that the above statement is false. Indeed, let us consider a case where $A = \{2, 5, 8\}$ and $B = \{5, 7, -1\}$. In this case, $A \setminus B = \{2, 8\}$ and $B \setminus A = \{7, -1\}$ and we have $A \setminus B \neq B \setminus A$. Hence the statement given in (iv) is false.

Example 1.1

For the given sets $A = \{-10, 0, 1, 9, 2, 4, 5\}$ and $B = \{-1, -2, 5, 6, 2, 3, 4\}$, verify that (i) set union is commutative. Also verify it by using Venn diagram.

(ii) set intersection is commutative. Also verify it by using Venn diagram.

Solution

(i) Now, $A \cup B = \{-10, 0, 1, 9, 2, 4, 5\} \cup \{-1, -2, 5, 6, 2, 3, 4\}$

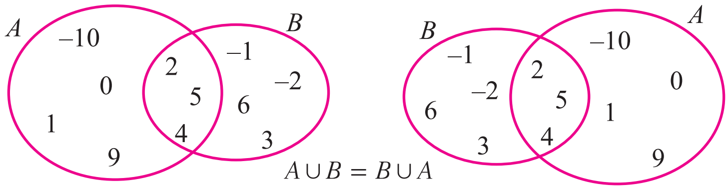
$$= \{-10, -2, -1, 0, 1, 2, 3, 4, 5, 6, 9\} \quad (1)$$

Also, $B \cup A = \{-1, -2, 5, 6, 2, 3, 4\} \cup \{-10, 0, 1, 9, 2, 4, 5\}$

$$= \{-10, -2, -1, 0, 1, 2, 3, 4, 5, 6, 9\} \quad (2)$$

Thus, from (1) and (2) we have verified that $A \cup B = B \cup A$.

By Venn diagram, we have



$A \cup B = B \cup A$

Fig. 1.7

Hence, it is verified that set union is commutative.

(ii) Let us verify that intersection is commutative.

Now, $A \cap B = \{-10, 0, 1, 9, 2, 4, 5\} \cap \{-1, -2, 5, 6, 2, 3, 4\}$

$= \{2, 4, 5\}$. (1)

Also, $B \cap A = \{-1, -2, 5, 6, 2, 3, 4\} \cap \{-10, 0, 1, 9, 2, 4, 5\}$

$= \{2, 4, 5\}$. (2)

From (1) and (2), we have $A \cap B = B \cap A$ for the given sets A and B.

By Venn diagram, we have

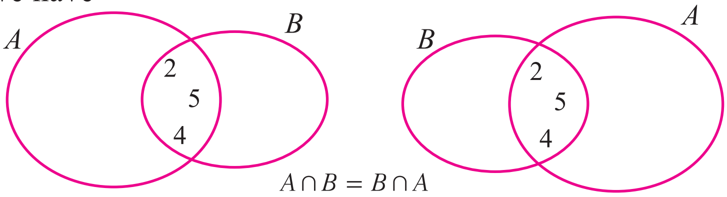


Fig. 1.8

Hence, it is verified.

Example 1.2

Given, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6\}$ and $C = \{5, 6, 7, 8\}$, show that

(i) $A \cup (B \cup C) = (A \cup B) \cup C$. (ii) Verify (i) using Venn diagram.

Solution

(i) Now, $B \cup C = \{3, 4, 5, 6\} \cup \{5, 6, 7, 8\} = \{3, 4, 5, 6, 7, 8\}$

$\therefore \quad A \cup (B \cup C) = \{1, 2, 3, 4, 5\} \cup \{3, 4, 5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (1)

Now, $A \cup B = \{1, 2, 3, 4, 5\} \cup \{3, 4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}$

$\therefore \quad (A \cup B) \cup C = \{1, 2, 3, 4, 5, 6\} \cup \{5, 6, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (2)

From (1) and (2), we have $A \cup (B \cup C) = (A \cup B) \cup C$.

(ii) Using Venn diagram, we have

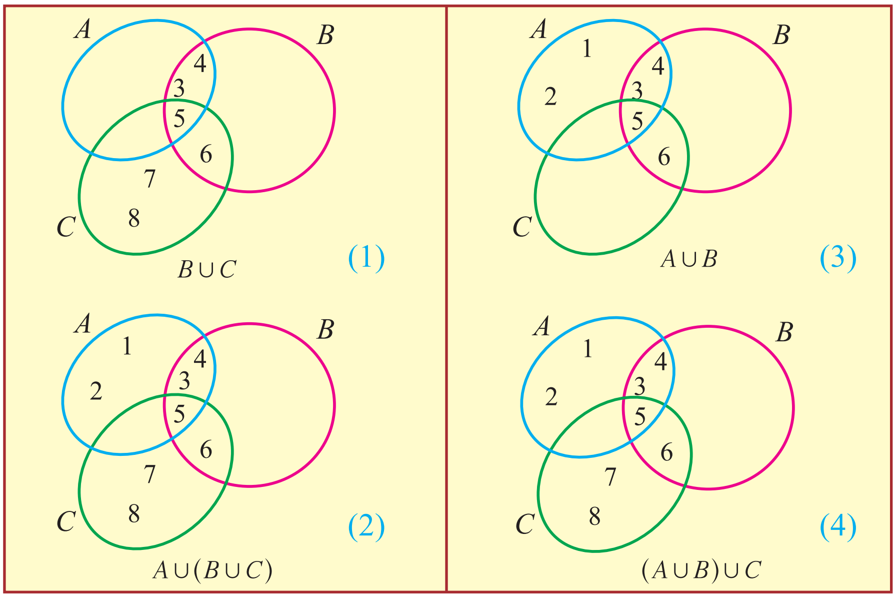


Fig. 1.9

Thus, from (2) and (4), we have verified that the set union is associative.

Example 1.3

Let $A = \{a,b,c,d\}$, $B = \{a,c,e\}$ and $C = \{a,e\}$.

(i) Show that $A \cap (B \cap C) = (A \cap B) \cap C$. (ii) Verify (i) using Venn diagram.

Solution

(i) We are given $A = \{a,b,c,d\}$, $B = \{a,c,e\}$ and $C = \{a,e\}$.

We need to show $A \cap (B \cap C) = (A \cap B) \cap C$. So, we first consider $A \cap (B \cap C)$.

Now, $B \cap C = \{a,c,e\} \cap \{a,e\} = \{a,e\}$; thus,

$$A \cap (B \cap C) = \{a,b,c,d\} \cap \{a,e\} = \{a\}. \quad (1)$$

Next, we shall find $A \cap B = \{a,b,c,d.\} \cap \{a,c,e\} = \{a,c\}$. Hence

$$(A \cap B) \cap C = \{a,c\} \cap \{a,e\} = \{a\} \quad (2)$$

Now (1) and (2) give the desired result.

(ii) Using Venn diagram, we have

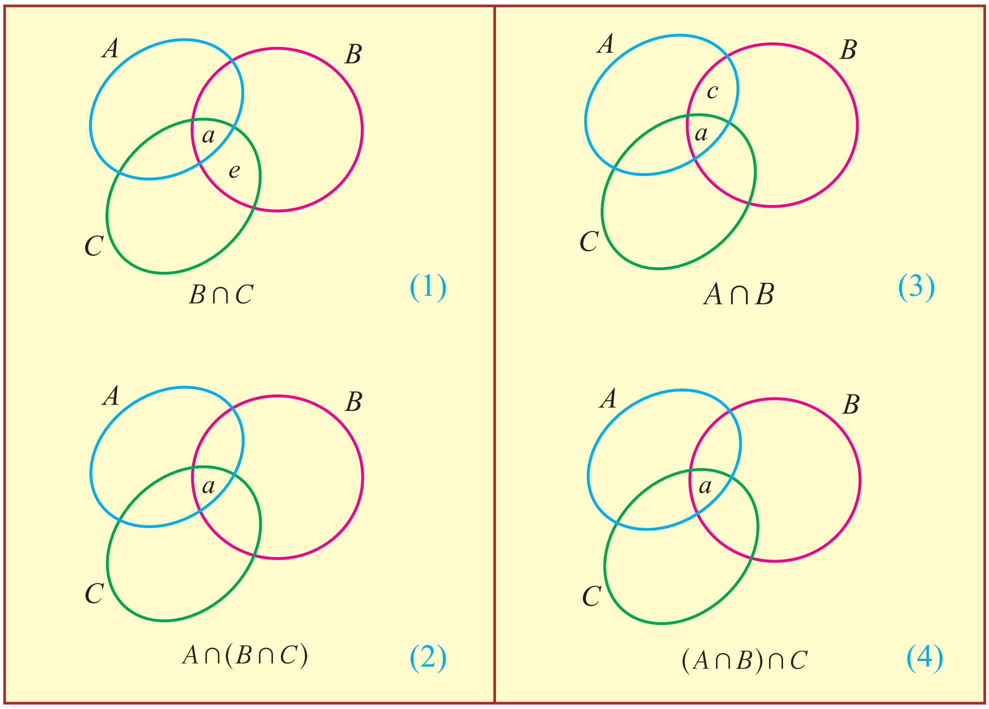


Fig. 1.10

Thus, from (2) and (4), it is verified that $A \cap (B \cap C) = (A \cap B) \cap C$

Example 1.4

Given $A = \{a,b,c,d,e\}$, $B = \{a,e,i,o,u\}$ and $C = \{c,d,e,u\}$.

(i) Show that $A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$. (ii) Verify (i) using Venn diagram.

Solution

(i) First let us find $A\setminus(B\setminus C)$. To do so, consider

$$(B\setminus C) = \{a,e,i,o,u\}\setminus\{c,d,e,u\} = \{a,i,o\}.$$

Thus, $A\setminus(B\setminus C) = \{a,b,c,d,e\}\setminus\{a,i,o\} = \{b,c,d,e\}$. (1)

Next, we find $(A\setminus B)\setminus C$.

$$A\setminus B = \{a,b,c,d,e\}\setminus\{a,e,i,o,u\} = \{b,c,d\}.$$

Hence, $(A\setminus B)\setminus C = \{b,c,d\}\setminus\{c,d,e,u\} = \{b\}$. (2)

From (1) and (2) we see that $A\setminus(B\setminus C) \neq (A\setminus B)\setminus C$.

Thus, the set difference is not associative.

(ii) Using Venn diagram, we have

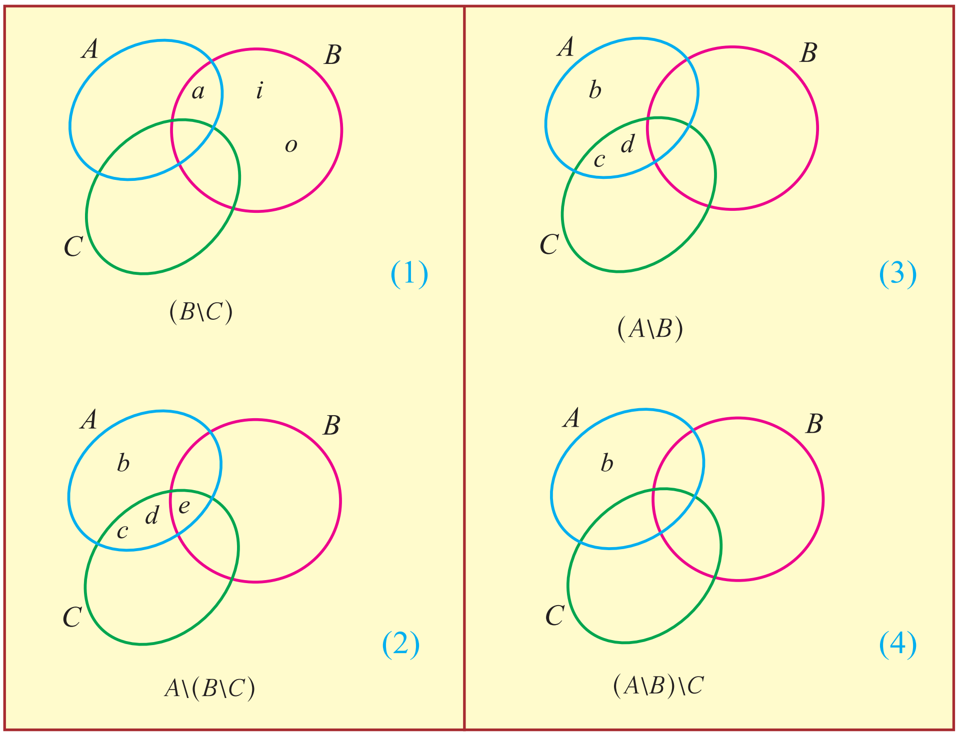


Fig. 1.11

From (2) and (4), it is verified that $A\setminus(B\setminus C) \neq (A\setminus B)\setminus C$.

Remarks

The set difference is not associative. However, if the sets A, B and C are mutually disjoint, then $A\setminus(B\setminus C) = (A\setminus B)\setminus C$. This is very easy to prove; so let us prove it. Since B and C are disjoint we have $B\setminus C = B$. Since A, B are disjoint we have $A\setminus B = A$. Thus, we have $A\setminus(B\setminus C) = A$. Again, $A\setminus B = A$ and A, C are disjoint and so we have $A\setminus C = A$. Hence, $(A\setminus B)\setminus C = A$. So we have $A\setminus(B\setminus C) = (A\setminus B)\setminus C$ as desired. Thus, for sets which are mutually disjoint, the set difference is associative.

Example 1.5

Let $A = \{0,1,2,3,4\}, B = \{1, -2, 3, 4, 5, 6\}$ and $C = \{2, 4, 6, 7\}$.

(i) Show that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (ii) Verify using Venn diagram.

Solution

(i) First, we find $A \cup (B \cap C)$.

Consider $B \cap C = \{1, -2, 3, 4, 5, 6\} \cap \{2, 4, 6, 7\} = \{4, 6\}$;

$A \cup (B \cap C) = \{0, 1, 2, 3, 4\} \cup \{4, 6\} = \{0, 1, 2, 3, 4, 6\}$. (1)

Next, consider $A \cup B = \{0,1,2,3,4\} \cup \{1, -2, 3, 4, 5, 6\}$

$= \{-2, 0, 1, 2, 3, 4, 5, 6\}$,

$A \cup C = \{0,1,2,3,4\} \cup \{2,4,6,7\} = \{0, 1, 2, 3, 4, 6, 7\}$.

Thus, $(A \cup B) \cap (A \cup C) = \{-2, 0, 1, 2, 3, 4, 5, 6\} \cap \{0, 1, 2, 3, 4, 6, 7\}$

$= \{0, 1, 2, 3, 4, 6\}$. (2)

From (1) and (2), we get $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) Using Venn diagram, we have

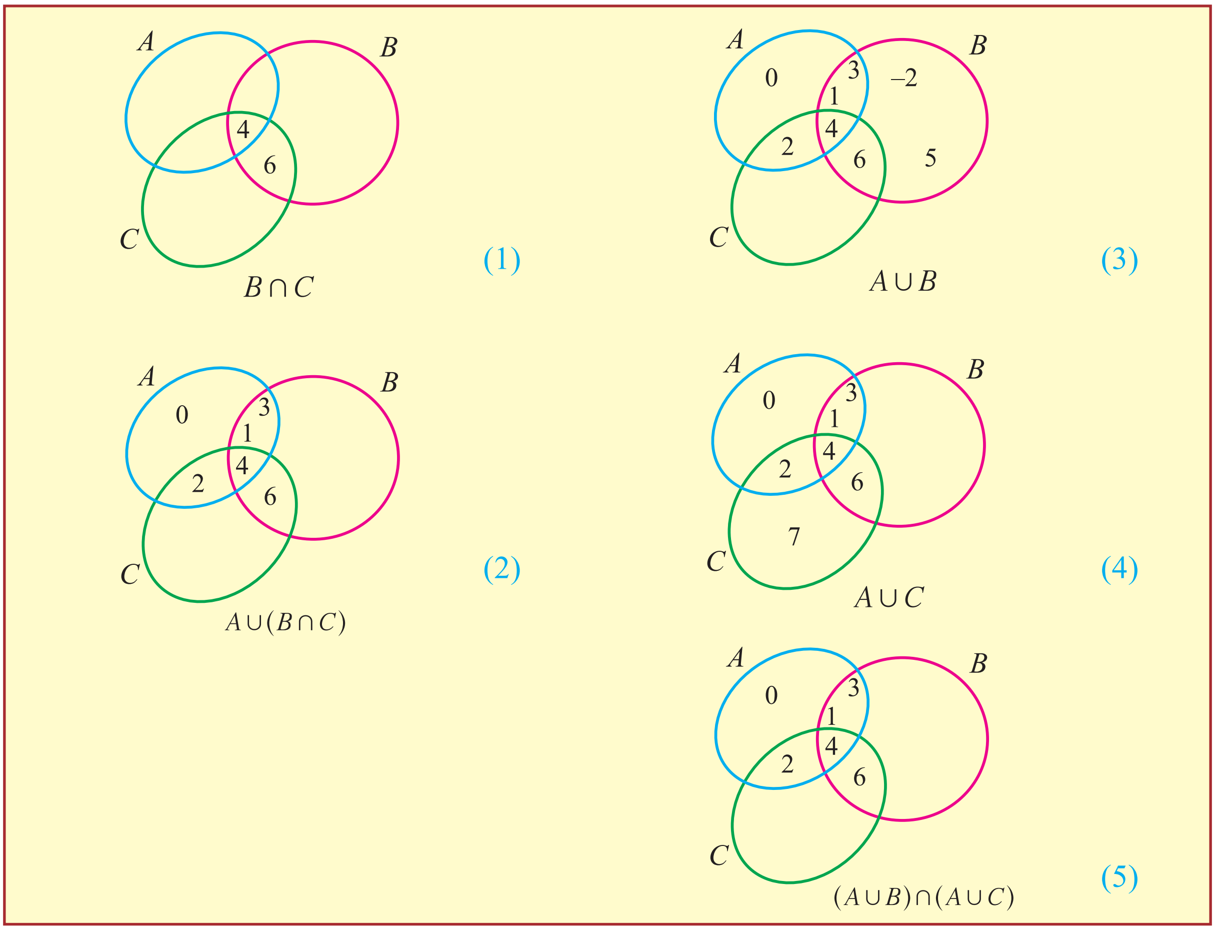


Fig. 1.12

From (2) and (5) it is verified that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example 1.6

For $A = \{x \mid -3 \leq x < 4, x \in \mathbb{R}\}, B = \{x \mid x < 5, x \in \mathbb{N}\}$ and

$C = \{-5, -3, -1, 0, 1, 3\}$, Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution First note that the set A contains all the real numbers (not just integers) that are greater than or equal to -3 and less than 4.

On the other hand the set B contains all the positive integers that are less than 5. So,

$A = \{x \mid -3 \leq x < 4, x \in \mathbb{R}\}$; that is, A consists of all real numbers from – 3 upto 4 but 4 is not included.

Also, $B = \{x \mid x < 5, x \in \mathbb{N}\} = \{1, 2, 3, 4\}$. Now, we find

$$B \cup C = \{1, 2, 3, 4\} \cup \{-5, -3, -1, 0, 1, 3\}$$

$$= \{1, 2, 3, 4, -5, -3, -1, 0\}; \text{ thus}$$

$$A \cap (B \cup C) = A \cap \{1, 2, 3, 4, -5, -3, -1, 0\}$$

$$= \{-3, -1, 0, 1, 2, 3\}. \quad (1)$$

Next, to find $(A \cap B) \cup (A \cap C)$, we consider

$$A \cap B = \{x \mid -3 \leq x < 4, x \in \mathbb{R}\} \cap \{1, 2, 3, 4\} = \{1, 2, 3\};$$

and

$$A \cap C = \{x \mid -3 \leq x < 4, x \in \mathbb{R}\} \cap \{-5, -3, -1, 0, 1, 3\}$$

$$= \{-3, -1, 0, 1, 3\}.$$

Hence,

$$(A \cap B) \cup (A \cap C) = \{1, 2, 3,\} \cup \{-3, -1, 0, 1, 3\}$$

$$= \{-3, -1, 0, 1, 2, 3\}. \quad (2)$$

Now (1) and (2) imply $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Exercise 1.1

1. If $A \subset B$, then show that $A \cup B = B$ (use Venn diagram).
2. If $A \subset B$, then find $A \cap B$ and $A \setminus B$ (use Venn diagram).
3. Let $P = \{a, b, c\}, Q = \{g, h, x, y\}$ and $R = \{a, e, f, s\}$. Find the following:

   (i) $P \setminus R$ (ii) $Q \cap R$ (iii) $R \setminus (P \cap Q)$.
4. If $A = \{4, 6, 7, 8, 9\}$, $B = \{2, 4, 6\}$ and $C = \{1, 2, 3, 4, 5, 6\}$, then find

   (i) $A \cup (B \cap C)$ (ii) $A \cap (B \cup C)$ (iii) $A \setminus (C \setminus B)$
5. Given $A = \{a, x, y, r, s\}, B = \{1, 3, 5, 7, -10\}$, verify the commutative property of set union.

6. Verify the commutative property of set intersection for $A = \{l, m, n, o, 2, 3, 4, 7\}$ and $B = \{2, 5, 3, -2, m, n, o, p\}$.

7. For $A = \{x \mid x$ is a prime factor of $42\}$, $B = \{x \mid 5 < x \leq 12, x \in \mathbb{N}\}$ and $C = \{1, 4, 5, 6\}$, verify $A \cup (B \cup C) = (A \cup B) \cup C$.

8. Given $P = \{a, b, c, d, e\}$, $Q = \{a, e, i, o, u\}$ and $R = \{a, c, e, g\}$. Verify the associative property of set intersection.

9. For $A = \{5, 10, 15, 20\}$; $B = \{6, 10, 12, 18, 24\}$ and $C = \{7, 10, 12, 14, 21, 28\}$, verify whether $A \setminus (B \setminus C) = (A \setminus B) \setminus C$. Justify your answer.

10. Let $A = \{-5, -3, -2, -1\}$, $B = \{-2, -1, 0\}$, and $C = \{-6, -4, -2\}$. Find $A \setminus (B \setminus C)$ and $(A \setminus B) \setminus C$. What can we conclude about set difference operation?

11. For $A = \{-3, -1, 0, 4, 6, 8, 10\}$, $B = \{-1, -2, 3, 4, 5, 6\}$ and $C = \{-1, 2, 3, 4, 5, 7\}$, show that (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(iii) Verify (i) using Venn diagram (iv) Verify (ii) using Venn diagram.

1.5 De Morgan's laws

De Morgan's father (a British national) was in the service of East India Company, India. Augustus De Morgan (1806-1871) was born in Madurai, Tamilnadu, India. His family moved to England when he was seven months old. He had his education at Trinity college, Cambridge, England. De Morgan's laws relate the three basic set operations Union, Intersection and Complementation.

De Morgan's laws for set difference

For any three sets A, B and C, we have
(i) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ (ii) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

De Morgan's laws for complementation

Let U be the universal set containing sets A and B. Then
(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

Observe that proof of the laws for complementation follows from that of the set difference because for any set D, we have $D' = U \setminus D$. Again we shall not attempt to prove these; but we shall learn how to apply these laws in problem solving.

Example 1.7

Use Venn diagrams to verify $(A \cap B)' = A' \cup B'$.

Solution

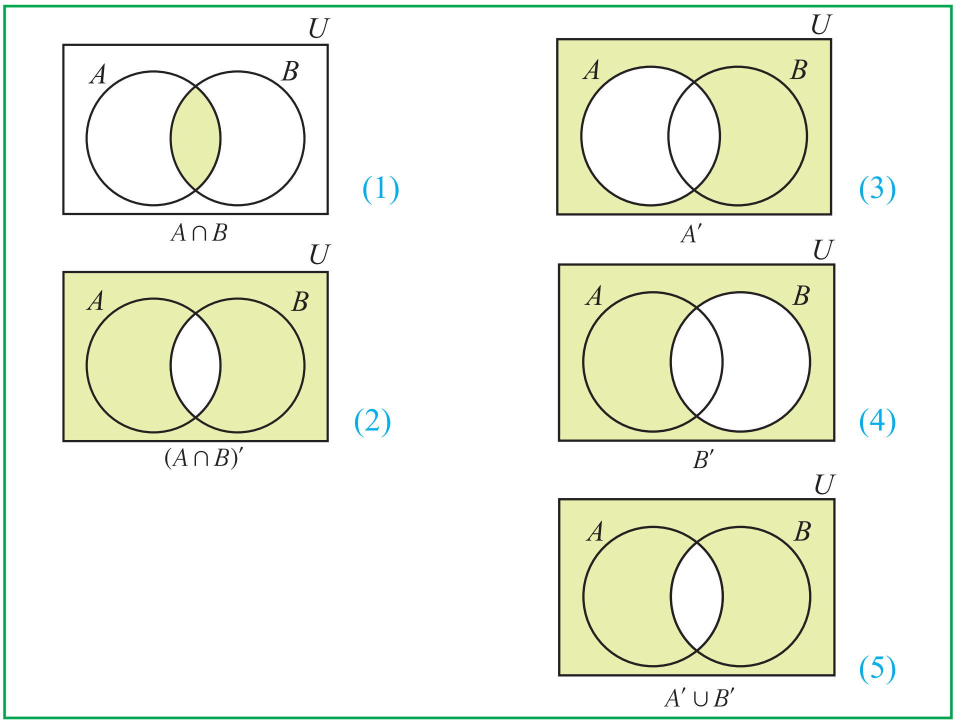


Fig. 1.13

From (2) and (5) it follows that $(A\cap B)' = A'\cup B'$.

Example 1.8

Use Venn diagrams to verify De Morgan's law for set difference

$$A\setminus(B\cap C) = (A\setminus B)\cup(A\setminus C).$$

Solution

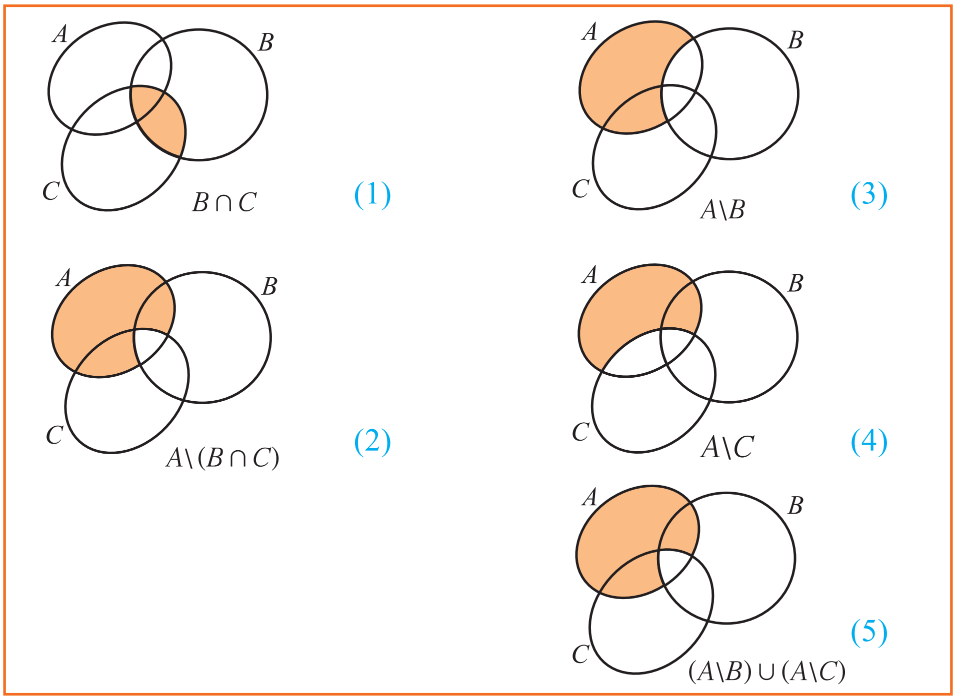


Fig. 1.14

From (2) and (5) we have $A\setminus(B\cap C) = (A\setminus B)\cup(A\setminus C)$.

Example 1.9

Let $U = \{-2, -1, 0, 1, 2, 3, \cdots, 10\}$, $A = \{-2, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 8, 9\}$.

Verify De Morgan's laws of complementation.

Solution First we shall verify $(A \cup B)' = A' \cap B'$. To do this we consider

$$A \cup B = \{-2, 2, 3, 4, 5\} \cup \{1, 3, 5, 8, 9\} = \{-2, 1, 2, 3, 4, 5, 8, 9\};$$

which implies

$$(A \cup B)' = U \setminus \{-2, 1, 2, 3, 4, 5, 8, 9\} = \{-1, 0, 6, 7, 10\}. \quad (1)$$

Next, we find $A' = U \setminus A = \{-1, 0, 1, 6, 7, 8, 9, 10\}$

$$B' = U \setminus B = \{-2, -1, 0, 2, 4, 6, 7, 10\}.$$

Thus, we have $A' \cap B' = \{-1, 0, 1, 6, 7, 8, 9, 10\} \cap \{-2, -1, 0, 2, 4, 6, 7, 10\}$

$$= \{-1, 0, 6, 7, 10\}. \quad (2)$$

From (1) and (2) it follows that $(A \cup B)' = A' \cap B'$.

Similarly, one can verify $(A \cap B)' = A' \cup B'$ for the given sets above. We leave the details as an exercise.

Example 1.10

Let $A = \{a, b, c, d, e, f, g, x, y, z\}$, $B = \{1, 2, c, d, e\}$ and $C = \{d, e, f, g, 2, y\}$.

Verify $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Solution First, we find $B \cup C = \{1, 2, c, d, e\} \cup \{d, e, f, g, 2, y\}$

$$= \{1, 2, c, d, e, f, g, y\}.$$

Then $A \setminus (B \cup C) = \{a, b, c, d, e, f, g, x, y, z\} \setminus \{1, 2, c, d, e, f, g, y\}$

$$= \{a, b, x, z\}. \quad (1)$$

Next, we have $A \setminus B = \{a, b, f, g, x, y, z\}$ and $A \setminus C = \{a, b, c, x, z\}$

and so $(A \setminus B) \cap (A \setminus C) = \{a, b, x, z\}$. (2)

Hence, from (1) and (2) it follows that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Exercise 1.2

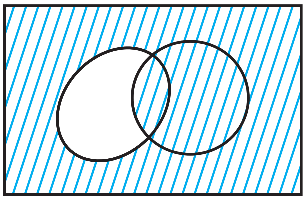
1. Represent the following using Venn diagrams

   (i) $U = \{5, 6, 7, 8, ...13\}$, $A = \{5, 8, 10, 11\}$, and $B = \{5, 6, 7, 9, 10\}$

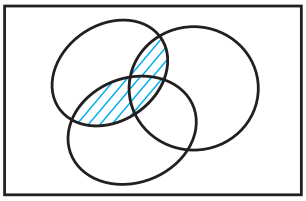
   (ii) $U = \{a, b, c, d, e, f, g, h\}$, $M = \{b, d, f, g\}$, and $N = \{a, b, d, e, g\}$

2. Write a description of each shaded area. Use symbols U, $A, B, C, \cup, \cap, '$ and $\setminus$ as necessary.

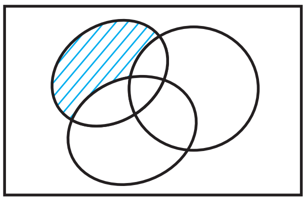
(i)



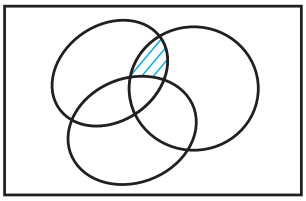
(ii)



(iii)



(iv)



3. Draw Venn diagram of three sets A, B and C illustrating the following:

   (i) $A \cap B \cap C$
   (ii) A and B are disjoint but both are subsets of C
   (iii) $A \cap (B \setminus C)$
   (iv) $(B \cup C) \setminus A$
   (v) $A \cup (B \cap C)$
   (vi) $C \cap (B \setminus A)$
   (vii) $C \cap (B \cup A)$

4. Use Venn diagram to verify $(A \cap B) \cup (A \setminus B) = A$.

5. Let $U = \{4, 8, 12, 16, 20, 24, 28\}$, $A = \{8, 16, 24\}$ and $B = \{4, 16, 20, 28\}$. Find $(A \cup B)'$ and $(A \cap B)'$.

6. Given that $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, b, f, g\}$, and $B = \{a, b, c\}$, verify De Morgan's laws of complementation.

7. Verify De Morgan's laws for set difference using the sets given below:

   $A = \{1, 3, 5, 7, 9, 11, 13, 15\}$, $B = \{1, 2, 5, 7\}$ and $C = \{3, 9, 10, 12, 13\}$.

8. Let $A = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$, $B = \{1, 5, 10, 15, 20, 30\}$ and $C = \{7, 8, 15, 20, 35, 45, 48\}$. Verify $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

9. Using Venn diagram, verify whether the following are true:

   (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (iii) $(A \cup B)' = A' \cap B'$
   (iv) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

1.6 Cardinality of sets

In class IX, we have learnt to solve problems involving two sets, using the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. This formula helps us in calculating the cardinality of the set $A \cup B$ when the cardinalities of A, B and $A \cap B$ are known. Suppose we have three sets A, B and C and we want to find the cardinality of $A \cup B \cup C$, what will be the corresponding formula? The formula in this case is given by

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$$

Following example illustrates the usage of the above formula.

Example 1.11

In a group of students, 65 play foot ball, 45 play hockey, 42 play cricket, 20 play foot ball and hockey, 25 play foot ball and cricket, 15 play hockey and cricket and 8 play all the three games. Find the number of students in the group.
(Assume that each student in the group plays atleast one game.)

Solution Let F, H and C represent the set of students who play foot ball, hockey and cricket respectively. Then $n(F) = 65$, $n(H) = 45$, and $n(C) = 42$.

Also, $n(F \cap H) = 20$, $n(F \cap C) = 25$, $n(H \cap C) = 15$ and $n(F \cap H \cap C) = 8$.

We want to find the number of students in the whole group; that is $n(F \cup H \cup C)$.

By the formula, we have

$$\begin{aligned} n(F \cup H \cup C) &= n(F) + n(H) + n(C) - n(F \cap H) \\ &\quad - n(H \cap C) - n(F \cap C) + n(F \cap H \cap C) \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100. \end{aligned}$$

Hence, the number of students in the group = 100.

Alternate method

The same problem can also be solved using Venn diagram. Nowadays, it is possible to solve some of the problems that we come across in daily life using Venn diagrams and logic. The Venn diagram will have three intersecting sets, each representing a game. Look at the diagram and try to find the number of players in the group by working carefully through the statements and fill in as you go along.

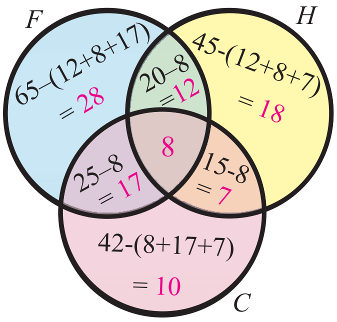


Fig. 1.15

Number of students in the group

$= 28 + 12 + 18 + 7 + 10 + 17 + 8 = 100.$

Example 1.12

In a survey of university students, 64 had taken mathematics course, 94 had taken computer science course, 58 had taken physics course, 28 had taken mathematics and physics, 26 had taken mathematics and computer science, 22 had taken computer science and physics course, and 14 had taken all the three courses. Find the number of students who were surveyed. Find how many had taken one course only.

Solution Let us represent the given data in a Venn diagram.

Let M, C, P represent sets of students who had taken mathematics, computer science and physics respectively. The given details are filled in the Venn diagram

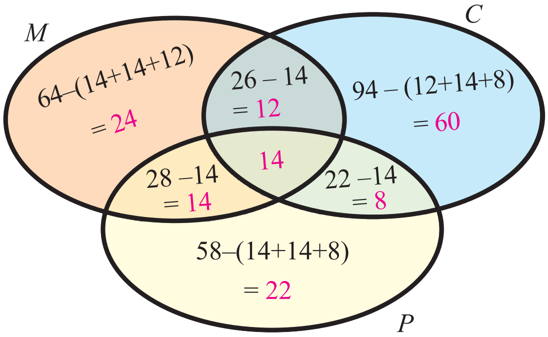


Fig. 1.16

$n(M \cap C \cap P) = 14$

$n(M \cap C \cap P') = 26 - 14 = 12$

$n(M \cap P \cap C') = 28 - 14 = 14$

$n(C \cap P \cap M') = 22 - 14 = 8$

Number of students surveyed

$= 24 + 12 + 60 + 8 + 22 + 14 + 14 = 154$

The number of students who had taken only mathematics = 64–(14+14+12) = 24

The number of students who had taken only computer science = 94 – (12+14+8) = 60

The number of students who had taken only physics = 58 – (14+14+8) = 22

The number of students who had taken one course only = 24+60+22 = 106.

Example 1.13

A radio station surveyed 190 students to determine the types of music they liked. The survey revealed that 114 liked rock music, 50 liked folk music, and 41 liked classical music, 14 liked rock music and folk music, 15 liked rock music and classical music, 11 liked classical music and folk music. 5 liked all the three types of music.

Find (i) how many did not like any of the 3 types?

(ii) how many liked any two types only?

(iii) how many liked folk music but not rock music?

Solution Let R, F and C represent the sets of students who liked rock music, folk music and classical music respectively. Let us fill in the given details in the Venn diagram. Thus, we have

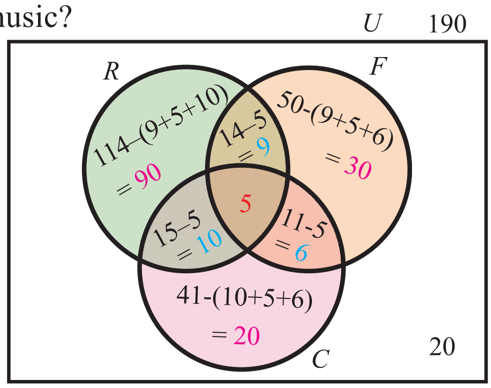


Fig. 1.17

$n(R \cap F \cap C') = 14 - 5 = 9$

$n(R \cap C \cap F') = 15 - 5 = 10$

$n(F \cap C \cap R') = 11 - 5 = 6.$

From the Venn diagram, the number of students who liked any one of the three types of music equals 90 + 9 + 30 + 6 + 20 + 10 + 5 = 170.

Number of students surveyed = 190.

Number of students who did not like any of the three types = $190 - 170 = 20$.

Number of students who liked any two types only = 9 + 6 + 10 = 25.

Number of students who liked folk music but not rock music = 30 + 6 = 36.

Exercise 1.3

1. If A and B are two sets and U is the universal set such that $n(U) = 700$, $n(A) = 200, n(B) = 300$ and $n(A \cap B) = 100$, find $n(A' \cap B')$.

2. Given $n(A) = 285, n(B) = 195, n(U) = 500, n(A \cup B) = 410$, find $n(A' \cup B')$.

3. For any three sets A, B and C if $n(A) = 17$ $n(B) = 17$, $n(C) = 17$, $n(A \cap B) = 7$ $n(B \cap C) = 6$, $n(A \cap C) = 5$ and $n(A \cap B \cap C) = 2$, find $n(A \cup B \cup C)$.

4. Verify $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$ for the sets given below:
   (i) $A = \{4,5,6\}$, $B = \{5,6,7,8\}$ and $C = \{6,7,8,9\}$
   (ii) $A = \{a,b,c,d,e\}$, $B = \{x,y,z\}$ and $C = \{a,e,x\}$.

5. In a college, 60 students enrolled in chemistry, 40 in physics, 30 in biology, 15 in chemistry and physics, 10 in physics and biology, 5 in biology and chemistry. No one enrolled in all the three. Find how many are enrolled in at least one of the subjects.

6. In a town 85% of the people speak Tamil, 40% speak English and 20% speak Hindi. Also, 32% speak English and Tamil, 13% speak Tamil and Hindi and 10% speak English and Hindi, find the percentage of people who can speak all the three languages.

7. An advertising agency finds that, of its 170 clients, 115 use Television, 110 use Radio and 130 use Magazines. Also, 85 use Television and Magazines, 75 use Television and Radio, 95 use Radio and Magazines, 70 use all the three. Draw Venn diagram to represent these data. Find
   (i) how many use only Radio? (ii) how many use only Television?
   (iii) how many use Television and magazine but not radio?

8. In a school of 4000 students, 2000 know French, 3000 know Tamil and 500 know Hindi, 1500 know French and Tamil, 300 know French and Hindi, 200 know Tamil and Hindi and 50 know all the three languages.
   (i) How many do not know any of the three languages?
   (ii) How many know at least one language?
   (iii) How many know only two languages?

9. In a village of 120 families, 93 families use firewood for cooking, 63 families use kerosene, 45 families use cooking gas, 45 families use firewood and kerosene, 24 families use kerosene and cooking gas, 27 families use cooking gas and firewood. Find how many use firewood, kerosene and cooking gas.

1.7 Relations

In the previous section, we have seen the concept of Set. We have also seen how to create new sets from the given sets by taking union, intersection and complementation. Here we shall see yet another way of creating a new set from the given two sets A and B. This new set is important in defining other important concepts of mathematics “relation, function”.

Given two non empty sets A and B, we can form a new set $A \times B$, read as‘A cross B’, called the cartesian product of A with B. It is defined as

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

Similarly, the set B cross A is defined as

$$B \times A = \{(b,a) \mid b \in B \text{ and } a \in A\}.$$

Note

(i) The order in the pair (a, b) is important. That is, $(a, b) \neq (b, a)$ if $a \neq b$.

(ii) It is possible that the sets A and B are equal in the cartesian product $A \times B$.

Let us look at an example.

Suppose that a cell phone store sells three different types of cell phones and we call them C_1, C_2, C_3. Let us also suppose that the price of C_1 is ₹ 1200, price of C_2 is ₹ 2500 and price of C_3 is ₹ 2500.

We take $A = \{ C_1, C_2, C_3 \}$ and $B = \{ 1200, 2500 \}$.

In this case, $A \times B$={(C_1, 1200), (C_1, 2500), (C_2, 1200), (C_2, 2500), (C_3, 1200), (C_3,2500)}

but $B \times A$= {(1200, C_1), (2500, C_1), (1200, C_2), (2500, C_2,),(1200, C_3), (2500, C_3).

It is easy to see that $A \times B \neq B \times A$ if $A \neq B$.

Let us consider a subset F = {(C_1, 1200), (C_2, 2500), (C_3, 2500)} of $A \times B$.

Every first component in the above ordered pairs is associated with a unique element. That is no element in the first place is paired with more than one element in the second place.

For every element in F, basically the second component indicates the price of the first component. Next, consider a subset E= {(1200, C_1), (2500, C_2), (2500, C_3)} of $B \times A$

Here, the first component 2500 is associated with two different elements C_2 and C_3.

Definition

Let A and B be any two non empty sets. A relation R from A to B is a non-empty subset of $A \times B$. That is, $R \subseteq A \times B$.

Domain of $R = \{x \in A \mid (x,y) \in R$ for some $y \in B\}$

Range of $R = \{y \in B \mid (x,y) \in R$ for some $x \in A\}$.

1.8 Functions

Peter Dirichlet

(1805-1859)
Germany

Dirichlet made major contributions in the fields of number theory, analysis and mechanics.

In 1837 he introduced the modern concept of a function with notation $y = f(x)$. He also formulated the well known Pigeonhole principle.

Let A and B be any two non empty sets. A function from A to B is a relation

$f \subseteq A \times B$ such that the following hold:

(i) Domain of f is A.

(ii) For each $x \in A$, there is only one $y \in B$ such that $(x, y) \in f$.

Note that a function from A to B is a special kind of relation that satisfies (i) and (ii). A function is also called as a mapping or a transformation.

A function from A to B is denoted by $f : A \to B$, and if $(x,y) \in f$, then we write $y = f(x)$.

We can reformulate the definition of a function without using the idea of relation as follows: In fact, most of the time this formulation is used as a working definition of a function,

Definition

Let A and B be any two non empty sets. A function f from A to B is a rule of correspondence that assigns each element $x \in A$ to a unique element $y \in B$. We denote $y = f(x)$ to mean y is a function of x.

The set A is called the **domain** of the function and set B is called the **co-domain** of the function. Also, y is called **the image** of x under f and x is called **a preimage** of y. The set of all images of elements of A under f is called the **range** of f. Note that the range of a function is a subset of its co-domain.

This modern definition of a function, given above, was given by Nikolai Labachevsky and **Peter Dirichlet** independently around 1837. Prior to this, there was no clear definition of a function.

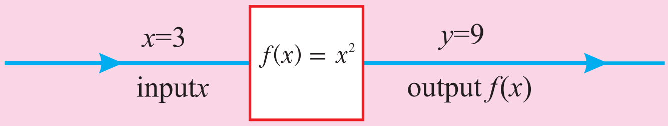
In the example we considered in section 1.7, prior to the above definitions, the set

$F = \{(C_1, 1200), (C_2, 2500), (C_3, 2500)\}$ represents a function; because $F \subseteq A \times B$ is a relation satisfying conditions (i) and (ii) given above.

But $E = \{(1200, C_1), (2500, C_2), (2500, C_3)\}$ does not represent a function, because condition (ii) given above is not satisfied as $(2500, C_2), (2500, C_3) \in E$.

Remarks

(i) A function f may be thought of as a machine which yields a unique output y for every input value of x.



(ii) In defining a function we need a domain, co-domain and a rule that assigns each element of the domain to a unique element in the co-domain.

Example 1.14

Let $A = \{1, 2, 3, 4\}$ and $B = \{-1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12\}$.

Let $R = \{(1, 3), (2, 6), (3, 10), (4, 9)\} \subseteq A \times B$ be a relation. Show that R is a function and find its domain, co-domain and the range of R.

Solution The domain of $R = \{1, 2, 3, 4\} = A$.

Also, for each $x \in A$ there is only one $y \in B$ such that $y = R(x)$.

So, given R is a function. The co-domain is obviously B. Since

$R(1) = 3, R(2) = 6, R(3) = 10$ and $R(4) = 9$, the range of R is given by $\{3, 6, 10, 9\}$.

Example 1.15

Does each of the following arrow diagrams represent a function? Explain.

(i)

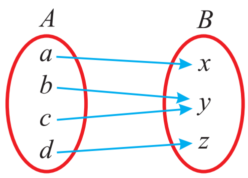


Fig. 1.18

(ii)

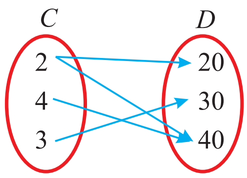


Fig. 1.19

Solution In arrow diagram (i), every element in A has a unique image. Hence it is a function. In arrow diagram (ii), the element 2 in C has two images namely 20 and 40. Hence, it is not a function.

Example 1.16

Let $X = \{ 1, 2, 3, 4 \}$. Examine whether each of the relations given below is a function from X to X or not. Explain.

(i) $f = \{ (2, 3), (1, 4), (2, 1), (3, 2), (4, 4) \}$

(ii) $g = \{ (3, 1), (4, 2), (2, 1) \}$ (iii) $h = \{ (2, 1), (3, 4), (1, 4), (4, 3) \}$

Solution

(i) Now, f = { (2, 3), (1, 4), (2, 1), (3, 2), (4, 4) }

f is not a function because 2 is associated with two different elements 3 and 1.

(ii) The relation g = { (3, 1), (4, 2), (2, 1)} is not a function because the element 1 does not have a image. That is, domain of $g = \{2, 3, 4\} \neq X$.

(iii) Next, we consider h = { (2, 1), (3, 4), (1, 4), (4, 3) }.

Each element in X is associated with a unique element in X.

Thus, h is a function.

Example 1.17

Which of the following relations are functions from A = { 1, 4, 9, 16 } to B = { –1, 2, –3, –4, 5, 6 }? In case of a function, write down its range.

(i) f_1 = { (1, –1), (4, 2), (9, –3), (16, –4) }

(ii) f_2 = { (1, –4), (1, –1), (9, –3), (16, 2) }

(iii) f_3 = { (4, 2), (1, 2), (9, 2), (16, 2) }

(iv) f_4 = { (1, 2), (4, 5), (9, –4), (16, 5) }

Solution (i) We have f_1 = { (1, –1), (4, 2), (9, – 3), (16,– 4) }.

Each element in A is associated with a unique element in B.

Thus, f_1 is a function.

Range of f_1 is $\{-1, 2, -3, -4\}$.

(ii) Here, we have f_2 = { (1, – 4), (1, –1), (9, – 3), (16, 2) }.

f_2 is not a function because 1 is associated with two different image elements -4 and -1. Also, note that f_2 is not a function since 4 has no image.

(iii) Consider f_3 = { (4, 2), (1, 2), (9, 2), (16, 2) }.

Each element in A is associated with a unique element in B.

Thus, f_3 is a function.

Range of f_3 = { 2 }.

(iv) We have f_4 = { (1, 2), (4, 5), (9, – 4), (16, 5) }.

Each element in A is associated with a unique element in B.

Hence, f_4 is a function.

Range of f_4 = { 2, 5, – 4}.

Example 1.18

Let $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, where $x \in \mathbb{R}$. Does the relation $\{ (x,y) \mid y = |x|, x \in \mathbb{R} \}$ define a function? Find its range.

Solution For every value of x, there exists a unique value $y = |x|$. Therefore, the given relation defines a function.

The domain of the function is the set $\mathbb{R}$ of all real numbers.

Since $|x|$ is always either zero or positive for every real number x, and every positive real number can be obtained as an image under this function, the range will be the set of non-negative real numbers (either positive or zero).

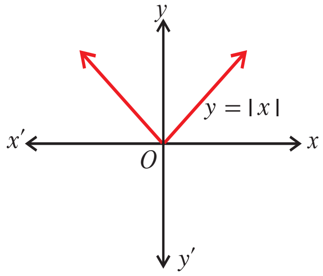


Fig. 1.20

Remarks

The function $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, where $x \in \mathbb{R}$, is known as **modulus** or **absolute value** function.

Thus, for example, $|-8| = -(-8) = 8$ and also $|8| = 8$.

1.8.1 Representation of functions

A function may be represented by

(i) a set of ordered pairs, (ii) a table, (iii) an arrow diagram, (iv) a graph

Let $f : A \to B$ be a function.

(i) The set $f = \{(x, y) : y = f(x), x \in A\}$ of all ordered pairs represents the function.

(ii) The values of x and the values of their respective images under f can be given in the form of a table.

(iii) An arrow diagram indicates the elements of the domain of f and their respective images by means of arrows.

(iv) The ordered pairs in the collection $f = \{(x, y) : y = f(x), x \in A\}$ are plotted as points in the x-y plane. The graph of f is the totality of all such points.

Let us illustrate the representation of functions in different forms through some examples.

For many functions we can obtain its graph. But not every graph will represent a function. Following test helps us in determining if the given graph is a function or not.

1.8.2 Vertical line test

A graph represents a function only if every vertical line intersects the graph in at most one point.

Note

It is possible that some vertical lines may not intersect the graph, which is alright. If there is even one vertical line that meets the graph in more than one point, then that graph cannot represent a function, because in this case, we shall have at least two y-values for the same x-value. For example, the graph of $y^2 = x$ is not a function.

Example 1.19

Use the vertical line test to determine which of the following graphs represent a function.

(i)

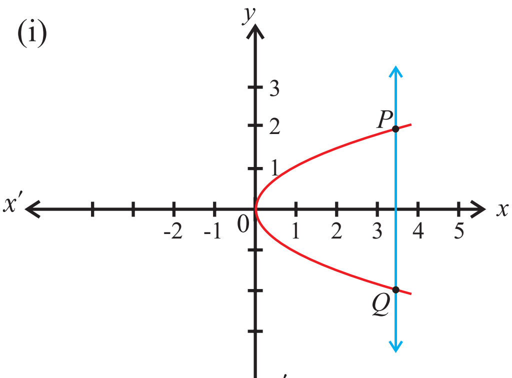


Fig. 1.21

(ii)

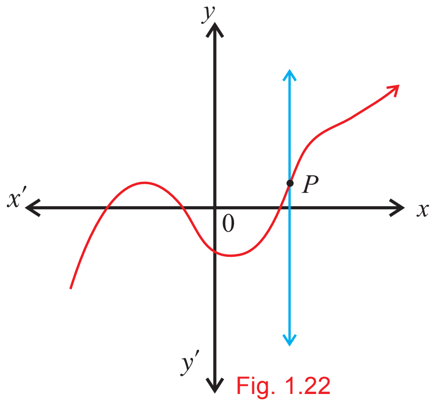


Fig. 1.22

(iii)

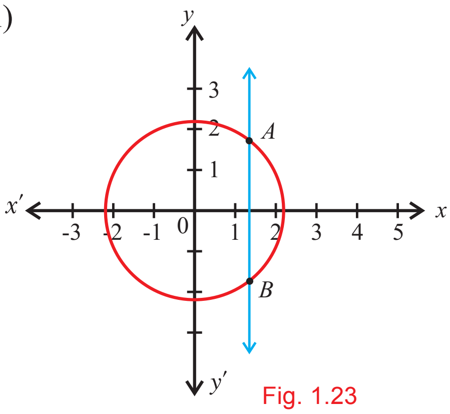


Fig. 1.23

(iv)

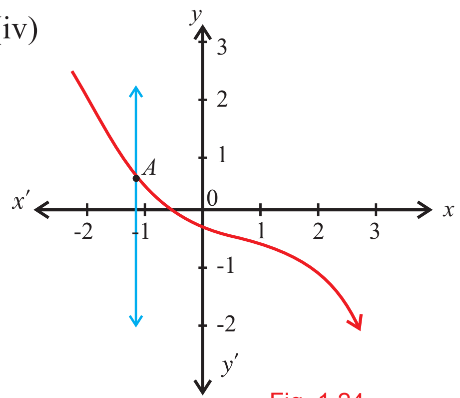


Fig. 1.24

Solution

(i) The given graph does not represent a function as a vertical line cuts the graph at two points P and Q.

(ii) The given graph represents a function as any vertical line will intersect the graph at most one point P.

(iii) The given graph does not represent a function as a vertical line cuts the graph at two points A and B.

(iv) The given graph represents a function as the graph satisfies the vertical line test.

Example 1.20

Let $A = \{ 0, 1, 2, 3 \}$ and $B = \{ 1, 3, 5, 7, 9 \}$ be two sets. Let $f : A \rightarrow B$ be a function given by $f(x) = 2x + 1$. Represent this function as (i) a set of ordered pairs (ii) a table (iii) an arrow diagram and (iv) a graph.

Solution A = $\{ 0, 1, 2, 3 \}$, $B = \{ 1, 3, 5, 7, 9 \}$, $f(x) = 2x + 1$

$f(0) = 2(0) + 1 = 1, f(1) = 2(1)+1 = 3\,, \quad f(2) = 2(2) + 1 = 5, f(3) = 2(3) + 1 = 7$

(i) Set of ordered pairs

The given function f can be represented as a set of ordered pairs as

$$f = \{ (0, 1), (1, 3), (2, 5), (3, 7) \}$$

(ii) Table form

Let us represent f using a table as shown below.

x	0	1	2	3
$f(x)$	1	3	5	7

(iii) Arrow Diagram

Let us represent f by an arrow diagram.

We draw two closed curves to represent the sets A and B.

Here each element of A and its unique image element in B are related with an arrow.

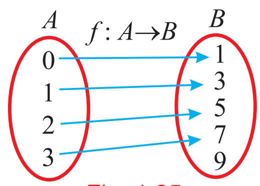


Fig. 1.25

(iv) Graph

We are given that

$$f = \{(x, f(x)) \mid x \in A\} = \{(0,1), (1,3), (2,5), (3,7)\}.$$

Now, the points (0, 1), (1, 3), (2, 5) and (3, 7) are plotted on the plane as shown below.

The totality of all points represent the graph of the function.

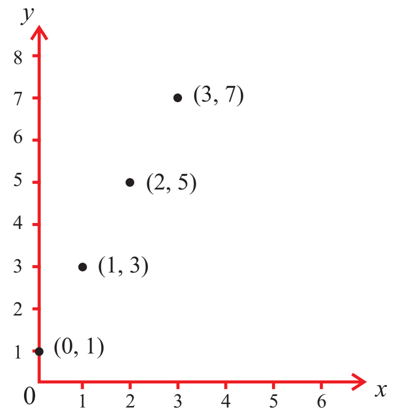


Fig. 1.26

1.8.3 Types of functions

Based on some properties of a function, we divide functions into certain types.

(i) One-One function

Let $f : A \rightarrow B$ be a function. The function f is called an **one-one** function if it takes different elements of A into different elements of B. That is, we say f is one-one if $u \neq v$ in A always imply $f(u) \neq f(v)$. In other words f is one-one if no element in B is associated with more than one element in A.

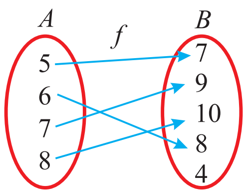


Fig. 1.27

A one-one function is also called an injective function. The above figure represents a one-one function.

(ii) Onto function

A function $f: A \to B$ is said to be an onto function if every element in B has a pre-image in A. That is, a function f is onto if for each $b \in B$, there is atleast one element $a \in A$, such that $f(a) = b$. This is same as saying that B is the range of f. An onto function is also called a surjective function. In the above figure, f is an onto function.

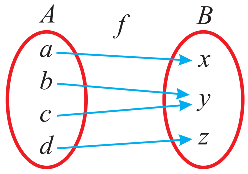


Fig. 1.28

(iii) One-One and onto function

A function $f: A \to B$ is called a one-one and onto or a bijective function if f is both a one-one and an onto function. Thus $f: A \to B$ is one-one and onto if f maps distinct elements of A into distinct images in B and every element in B is an image of some element in A.

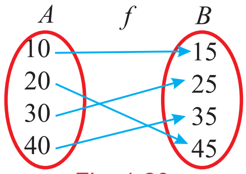


Fig. 1.29

Note

(i) A function $f: A \to B$ is onto if and only if B = range of f.

(ii) $f: A \to B$ is one-one and onto, if and only if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ in A and every element in B has exactly one pre-image in A.

(iii) If $f: A \to B$ is a bijective function and if A and B are finite sets, then the cardinalities of A and B are same. In Fig.1.29, the function f is one - one and onto.

(iv) If $f: A \to B$ is a bijective function, then A and B are equivalent sets

(v) A one-one and onto function is also called a one-one correspondence.

(iv) Constant function

A function $f: A \to B$ is said to be a constant function if every element of A has the same image in B.

Range of a constant function is a singleton set.

Let $A = \{x, y, u, v, 1\}$, $B = \{3, 5, 7, 8, 10, 15\}$.

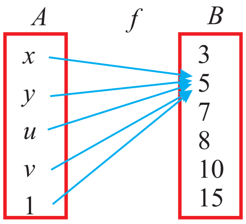


Fig. 1.30

The function $f: A \to B$ defined by $f(x) = 5$ for every $x \in A$ is a constant function.

The given figure represents a constant function.

(v) Identity function

Let A be a non-empty set. A function $f: A \to A$ is called an identity function of A if $f(a) = a$ for all $a \in A$. That is, an identity function maps each element of A into itself.

For example, let $A = \mathbb{R}$. The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x$ for all $x \in \mathbb{R}$ is the identity function on $\mathbb{R}$. Fig.1.31 represents the graph of the identity function on $\mathbb{R}$.

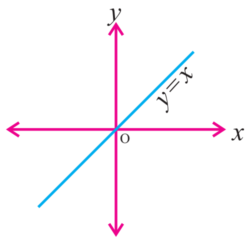


Fig. 1.31

Example 1.21

Let $A = \{1, 2, 3, 4, 5\}$, $B = \mathbb{N}$ and $f : A \to B$ be defined by $f(x) = x^2$. Find the range of f. Identify the type of function.

Solution Now, $A = \{1, 2, 3, 4, 5\}$; $B = \{1, 2, 3, 4, \cdots\}$

Given $f : A \to B$ and $f(x) = x^2$

$\therefore$ $f(1) = 1^2 = 1$; $f(2) = 4$; $f(3) = 9$; $f(4) = 16$; $f(5) = 25$.

Range of $f = \{1, 4, 9, 16, 25\}$

Since distinct elements are mapped into distinct images, it is a one-one function. However, the function is not onto, since $3 \in B$ but there is no $x \in A$ such that

$f(x) = x^2 = 3$.

Remarks

However, a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(x) = x^2$ is not one-one because, if $u = 1$ and $v = -1$ then $u \neq v$ but $g(u) = g(1) = 1 = g(-1) = g(v)$. So, just formula alone does not make a function one-one or onto. We need to consider the rule, its domain and co-domain in deciding one-to-one and onto.

Example 1.22

A function $f : [1, 6) \longrightarrow \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} 1 + x, & 1 \le x < 2 \\ 2x - 1, & 2 \le x < 4 \\ 3x^2 - 10, & 4 \le x < 6 \end{cases} \qquad (\text{ Here, } [1, 6) = \{x \in \mathbb{R} : 1 \le x < 6\})$$

Find the value of (i) $f(5)$ (ii) $f(3)$ (iii) $f(1)$

(iv) $f(2) - f(4)$ (v) $2f(5) - 3f(1)$

Solution

(i) Let us find $f(5)$. Since 5 lies between 4 and 6, we have to use $f(x) = 3x^2 - 10$.

Thus, $f(5) = 3(5^2) - 10 = 65$.

(ii) To find $f(3)$, note that 3 lies between 2 and 4.

So, we use $f(x) = 2x - 1$ to calculate $f(3)$.

Thus, $f(3) = 2(3) - 1 = 5$.

(iii) Let us find $f(1)$.

Now, 1 is in the interval $1 \le x < 2$

Thus, we have to use $f(x) = 1 + x$ to obtain $f(1) = 1 + 1 = 2$.

(iv) $f(2) - f(4)$

Now, 2 is in the interval $2 \le x < 4$ and so, we use $f(x) = 2x - 1$.

Thus, $f(2) = 2(2) - 1 = 3$.

Also, 4 is in the interval $4 \le x < 6$. Thus, we use $f(x) = 3x^2 - 10$.

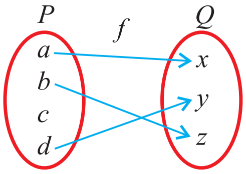
Therefore, $f(4) = 3(4^2) - 10 = 3(16) - 10 = 48 - 10 = 38$.

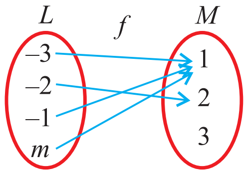
Hence, $f(2) - f(4) = 3 - 38 = -35$.

(v) To calculate $2f(5) - 3f(1)$, we shall make use of the values that we have already calculated in (i) and (iii). Thus, $2f(5) - 3f(1) = 2(65) - 3(2) = 130 - 6 = 124$.

Exercise 1.4

1. State whether each of the following arrow diagrams define a function or not. Justify your answer.

(i)


(ii)


2. For the given function $F = \{ (1, 3), (2, 5), (4, 7), (5, 9), (3, 1) \}$, write the domain and range.

3. Let $A = \{ 10, 11, 12, 13, 14 \}$; $B = \{ 0, 1, 2, 3, 5 \}$ and $f_i : A \to B$, $i = 1,2,3$. State the type of function for the following (give reason):

(i) $f_1 = \{ (10, 1), (11, 2), (12, 3), (13, 5), (14, 3) \}$

(ii) $f_2 = \{ (10, 1), (11, 1), (12, 1), (13, 1), (14, 1) \}$

(iii) $f_3 = \{ (10, 0), (11, 1), (12, 2), (13, 3), (14, 5) \}$

4. If $X = \{ 1, 2, 3, 4, 5 \}$, $Y = \{ 1, 3, 5, 7, 9 \}$ determine which of the following relations from X to Y are functions? Give reason for your answer. If it is a function, state its type.

(i) $R_1 = \{ (x, y) | y = x + 2, x \in X, y \in Y \}$

(ii) $R_2 = \{ (1, 1), (2, 1), (3, 3), (4, 3), (5, 5) \}$

(iii) $R_3 = \{ (1, 1), (1, 3), (3, 5), (3, 7), (5, 7) \}$

(iv) $R_4 = \{ (1, 3), (2, 5), (4, 7), (5, 9), (3, 1) \}$

5. If $R = \{(a, -2), (-5, b), (8, c), (d, -1)\}$ represents the identity function, find the values of a, b, c and d.

6. $A = \{ -2, -1, 1, 2 \}$ and $f = \left\{\left(x, \frac{1}{x}\right) : x \in A\right\}$. Write down the range of f. Is f a function from A to A ?

7. Let $f = \{ (2, 7), (3, 4), (7, 9), (-1, 6), (0, 2), (5, 3) \}$ be a function from

   $A = \{ -1, 0, 2, 3, 5, 7 \}$ to $B = \{ 2, 3, 4, 6, 7, 9 \}$. Is this (i) an one-one function (ii) an onto function (iii) both one-one and onto function?

8. Write the pre-images of 2 and 3 in the function

   $f = \{ (12, 2), (13, 3), (15, 3), (14, 2), (17, 17) \}$.

9. The following table represents a function from $A = \{ 5, 6, 8, 10 \}$ to

   $B = \{ 19, 15, 9, 11 \}$ where $f(x) = 2x - 1$. Find the values of a and b.

x	5	6	8	10
$f(x)$	a	11	b	19

10. Let $A = \{ 5, 6, 7, 8 \}$; $B = \{ -11, 4, 7, -10, -7, -9, -13 \}$ and

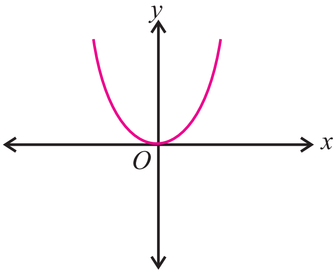
    $f = \{(x, y) : y = 3 - 2x,\ x \in A,\ y \in B \}$

    (i) Write down the elements of f. (ii) What is the co-domain?

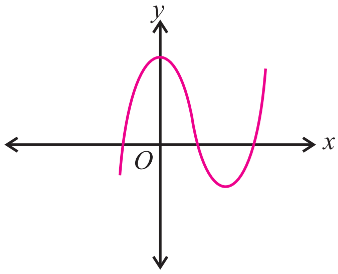
    (iii) What is the range ? (iv) Identify the type of function.

11. State whether the following graphs represent a function. Give reason for your answer.

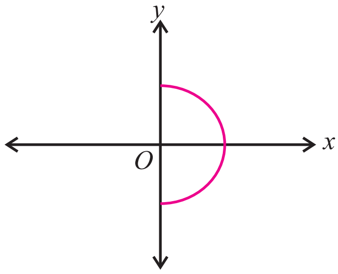
(i)



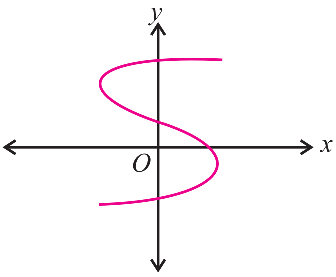
(ii)



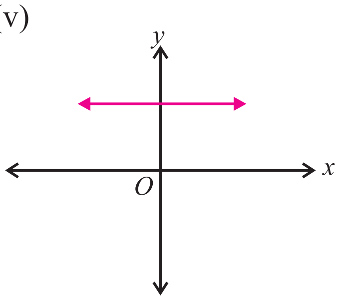
(iii)



(iv)



(v)



12. Represent the function $f = \{ (-1, 2), (-3, 1), (-5, 6), (-4, 3) \}$ as

(i) a table (ii) an arrow diagram

13. Let $A = \{ 6, 9, 15, 18, 21 \}$; $B = \{ 1, 2, 4, 5, 6 \}$ and $f : A \to B$ be defined by $f(x) = \frac{x-3}{3}$. Represent f by

(i) an arrow diagram (ii) a set of ordered pairs

(iii) a table (iv) a graph .

14. Let $A = \{4, 6, 8, 10\}$ and $B = \{3, 4, 5, 6, 7\}$. If $f : A \to B$ is defined by $f(x) = \frac{1}{2}x + 1$ then represent f by (i) an arrow diagram (ii) a set of ordered pairs and (iii) a table.

15. A function $f : [-3, 7) \to \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} 4x^2 - 1; & -3 \le x < 2 \\ 3x - 2; & 2 \le x \le 4. \\ 2x - 3; & 4 < x < 7 \end{cases}$$

Find (i) $f(5) + f(6)$ (ii) $f(1) - f(-3)$

(iii) $f(-2) - f(4)$ (iv) $\frac{f(3) + f(-1)}{2f(6) - f(1)}$.

16. A function $f : [-7, 6) \to \mathbb{R}$ is defined as follows

$$f(x) = \begin{cases} x^2 + 2x + 1; & -7 \le x < -5 \\ x + 5; & -5 \le x \le 2 \\ x - 1; & 2 < x < 6 . \end{cases}$$

Find (i) $2f(-4) + 3f(2)$ (ii) $f(-7) - f(-3)$ (iii) $\frac{4f(-3) + 2f(4)}{f(-6) - 3f(1)}$.

Exercise 1.5

Choose the correct answer

1. For two sets A and B, $A \cup B = A$ if and only if

(A) $B \subseteq A$ (B) $A \subseteq B$ (C) $A \neq B$ (D) $A \cap B = \phi$

2. If $A \subset B$, then $A \cap B$ is

(A) B (B) $A \backslash B$ (C) A (D) $B \backslash A$

3. For any two sets P and Q, $P \cap Q$ is

(A) $\{x : x \in P \text{ or } x \in Q\}$ (B) $\{x : x \in P \text{ and } x \notin Q\}$

(C) $\{x : x \in P \text{ and } x \in Q\}$ (D) $\{x : x \notin P \text{ and } x \in Q\}$

4. If $A = \{ p, q, r, s \}$, $B = \{ r, s, t, u \}$, then $A \backslash B$ is

(A) $\{ p, q \}$ (B) $\{ t, u \}$ (C) $\{ r, s \}$ (D) $\{p, q, r, s \}$

5. If $n[p(A)] = 64$, then $n(A)$ is

(A) 6 (B) 8 (C) 4 (D) 5

6. For any three sets A, B and C, $A \cap (B \cup C)$ is

(A) $(A \cup B) \cup (B \cap C)$ (B) $(A \cap B) \cup (A \cap C)$

(C) $A \cup (B \cap C)$ (D) $(A \cup B) \cap (B \cup C)$

7. For any two sets A and B, $\{(A \setminus B) \cup (B \setminus A)\} \cap (A \cap B)$ is

(A) ϕ (B) $A \cup B$ (C) $A \cap B$ (D) $A' \cap B'$

8. Which one of the following is not true ?

(A) $A \setminus B = A \cap B'$ (B) $A \setminus B = A \cap B$

(C) $A \setminus B = (A \cup B) \cap B'$ (D) $A \setminus B = (A \cup B) \setminus B$

9. For any three sets A, B and C, $B \setminus (A \cup C)$ is

(A) $(A \setminus B) \cap (A \setminus C)$ (B) $(B \setminus A) \cap (B \setminus C)$

(C) $(B \setminus A) \cap (A \setminus C)$ (D) $(A \setminus B) \cap (B \setminus C)$

10. If $n(A) = 20$, $n(B) = 30$ and $n(A \cup B) = 40$, then $n(A \cap B)$ is equal to

(A) 50 (B) 10 (C) 40 (D) 70.

11. If $\{ (x, 2), (4, y) \}$ represents an identity function, then (x, y) is

(A) (2, 4) (B) (4, 2) (C) (2, 2) (D) (4, 4)

12. If $\{ (7, 11), (5, a) \}$ represents a constant function, then the value of 'a' is

(A) 7 (B) 11 (C) 5 (D) 9

13. Given $f(x) = (-1)^x$ is a function from $\mathbb{N}$ to $\mathbb{Z}$. Then the range of f is

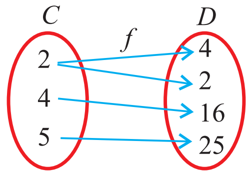
(A) $\{ 1\}$ (B) $\mathbb{N}$ (C) $\{ 1, -1 \}$ (D) $\mathbb{Z}$

14. If $f = \{ (6, 3), (8, 9), (5, 3), (-1, 6) \}$, then the pre-images of 3 are

(A) 5 and –1 (B) 6 and 8 (C) 8 and –1 (D) 6 and 5.

15. Let $A = \{ 1, 3, 4, 7, 11 \}$, $B = \{-1, 1, 2, 5, 7, 9 \}$ and $f : A \rightarrow B$ be given by $f = \{ (1, -1), (3, 2), (4, 1), (7, 5), (11, 9) \}$. Then f is

(A) one-one (B) onto (C) bijective (D) not a function

16. 

The given diagram represents

(A) an onto function (B) a constant function

(C) an one-one function (D) not a function

17. If $A = \{ 5, 6, 7 \}$, $B = \{ 1, 2, 3, 4, 5 \}$ and $f : A \to B$ is defined by $f(x) = x - 2$, then the range of f is

(A) $\{ 1, 4, 5 \}$ (B) $\{ 1, 2, 3, 4, 5 \}$ (C) $\{ 2, 3, 4 \}$ (D) $\{ 3, 4, 5 \}$

18. If $f(x) = x^2 + 5$, then $f(-4) =$

(A) 26 (B) 21 (C) 20 (D) –20

19. If the range of a function is a singleton set, then it is

(A) a constant function (B) an identity function

(C) a bijective function (D) an one-one function

20. If $f : A \to B$ is a bijective function and if $n(A) = 5$, then $n(B)$ is equal to

(A) 10 (B) 4 (C) 5 (D) 25

Points to Remember

SETS

- A set is a collection of well defined objects.
  - Set union is commutative and associative.
  - Set intersection is commutative and associative.
  - Set difference is not commutative.
  - Set difference is associative only when the sets are mutually disjoint.
- Distributive Laws
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's Laws for set difference
  - $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
  - $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
- De Morgan's Laws for complementation.
  - $(A \cup B)' = A' \cap B'$
  - $(A \cap B)' = A' \cup B'$
- Formulae for the cardinality of union of sets
  - $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
  - $n(A \cup B \cup C)$
    $= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

FUNCTIONS

- The cartesian product of A with B is defined as

  $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

- A relation R from A to B is a non-empty subset of $A \times B$. That is, $R \subseteq A \times B$.
- A function $f : X \to Y$ is defined if the following condition hold:

  Every $x \in X$ is associated with only one $y \in Y$.

- Every function can be represented by a graph. However, the converse is not true in general.
- If every vertical line intersects a graph in at most one point, then the graph represents a function.
- A function can be described by

  ➢ a set of ordered pairs ➢ an arrow diagram ➢ a table and ➢ a graph.

- The modulus or absolute value function $y = |x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- Some types of functions:
  - One-One function (injective function) — (distinct elements have distinct images)
  - Onto function (surjective function) — (the range and the co-domain are equal)
  - Bijective function — (both one-one and onto)
  - Constant function — (range is a singleton set)
  - Identity function — (which leaves each input as it is)

Do you know?

The Millennium Prize problems are seven problems in Mathematics that were stated by the Clay Mathematics Institute in USA in 2000. As of August 2010, six of the problems remain unsolved. A correct solution to any of the problems results in a US $1000,000 being awarded by the institute. Only Poincare conjecture has been solved by a Russian Mathematician Girigori Perelman in 2010. However, he declined the Millinnium Prize award. (Here, the word conjecture means a mathematical problem is to be proved or disproved)