10

GRAPHS

I think, therefore I am

- Rene Descartes

- Introduction
- Quadratic Graphs
- Special Graphs

Rene Descartes
(1596-1650)
France

Descartes devised the cartesian plane while he was in a hospital bed watching a fly buzzing around a corner of his room.

He created analytical geometry which paved the way of plotting graphs using coordinate axes.

10.1 Introduction

Graphs are diagrams that show information. They show how two different quantities are related to each other like weight is related to height. Sometimes algebra may be hard to visualize. Learning to show relationships between symbolic expressions and their graphs opens avenues to realize algebraic patterns.

Students should acquire the habit of drawing a reasonably accurate graph to illustrate a given problem under consideration. A carefully made graph not only serves to clarify the geometric interpretation of a problem but also may serve as a valuable check on the accuracy of the algebraic work. One should never forget that graphical results are at best only approximations, and of value only in proportion to the accuracy with which the graphs are drawn.

10.2 Quadratic Graphs

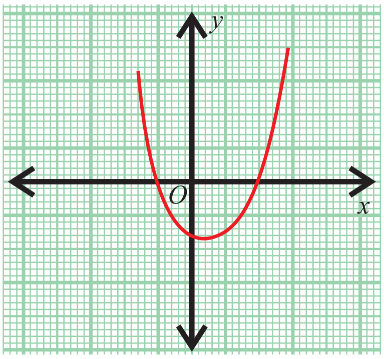
Definition

Let $f: A \to B$ be a function where A and B are subsets of $\mathbb{R}$. The set $\{(x,y) \mid x \in A, y = f(x)\}$ of all such ordered pairs (x, y) is called the graph of f.

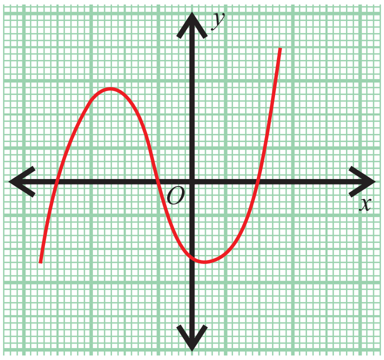
A polynomial function in x can be represented by a graph. The graph of a first degree polynomial $y = f(x) = ax + b, a \neq 0$ is an oblique line with slope a.

The graph of a second degree polynomial $y = f(x) = ax^2 + bx + c, \quad a \neq 0$ is a continuous non-linear curve, known as a parabola.

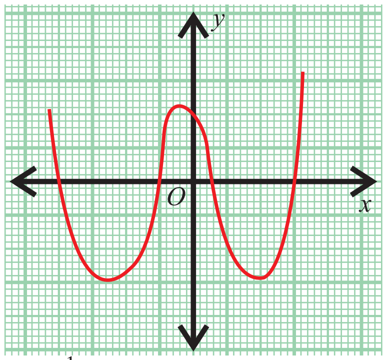
The following graphs represent different polynomials.



$y = (x+1)(x-2)$,
a polynomial of degree 2



$y = (x+4)(x+1)(x-2)$,
a polynomial of degree 3



$y = \frac{1}{14}(x+4)(x+1)(x-3)(x-0.5)$
a polynomial of degree 4

In class IX, we have learnt how to draw the graphs of linear polynomials of the form $y = ax + b, a \neq 0$. Now we shall focus on graphing a quadratic function $y = f(x) = ax^2 + bx + c$, where a, b and c are real constants, $a \neq 0$ and describe the nature of a quadratic graph.

Consider $y = ax^2 + bx + c$

By completing squares, the above polynomial can be rewritten as

$$\left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right).$$

Hence $\frac{1}{a}\left(y + \frac{b^2 - 4ac}{4a}\right) \geq 0$. (square of an expression is always positive)

The vertex of the curve (parabola) is $V\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right)$

If $a > 0$, then the curve is open upward; it lies above or on the line $y = \frac{4ac - b^2}{4a}$ and it is symmetric about $x = -\frac{b}{2a}$.

If $a < 0$, then the curve is open downward; it lies below or on the line $y = \frac{4ac - b^2}{4a}$ and it is symmetric about $x = -\frac{b}{2a}$.

Let us give some examples of quadratic polynomials and the nature of their graphs in the following table.

S.No.	Polynomial ($y = ax^2 + bx + c$)	Vertex	Sign of a	Nature of curve
1	$y = 2x^2$ $a = 2,\ b = 0,\ c = 0$	(0, 0)	positive	(i) open upward (ii) lies above and on the line $y = 0$ (iii) symmetric about $x = 0$, i.e., y-axis
2	$y = -3x^2$ $a = -3,\ b = 0,\ c = 0$	(0, 0)	negative	(i) open downward (ii) lies below and on the line $y = 0$ (iii) symmetric about $x = 0$ i.e., y-axis
3	$y = x^2 - 2x - 3$ $a = 1, b = -2, c = -3$	(1, –4)	positive	(i) open upward (ii) lies above and on the line $y = -4$ (iii) symmetric about $x = 1$

Procedures to draw the quadratic graph $y = ax^2 + bx + c$

(i) Construct a table with the values of x and y using $y = ax^2 + bx + c$.

(ii) Choose a suitable scale.

The scale used on the x-axis does not have to be the same as the scale on the y-axis. The scale chosen should allow for the largest possible graph to be drawn. The bigger the graph, the more accurate will be the results obtained from it.

(iii) Plot the points on the graph paper and join these points by a smooth curve, as the graph of $y = ax^2 + bx + c$ does not contain line segments.

Example 10.1

Draw the graph of $y = 2x^2$.

Solution

First let us assign the integer values from -3 to 3 for x and form the following table.

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = 2x^2$	18	8	2	0	2	8	18

Plot the points $(-3, 18)$, $(-2, 8)$, $(-1, 2)$, $(0, 0)$, $(1, 2)$, $(2, 8)$, $(3, 18)$.

Join the points by a smooth curve.

The curve, thus obtained is the graph of $y = 2x^2$.

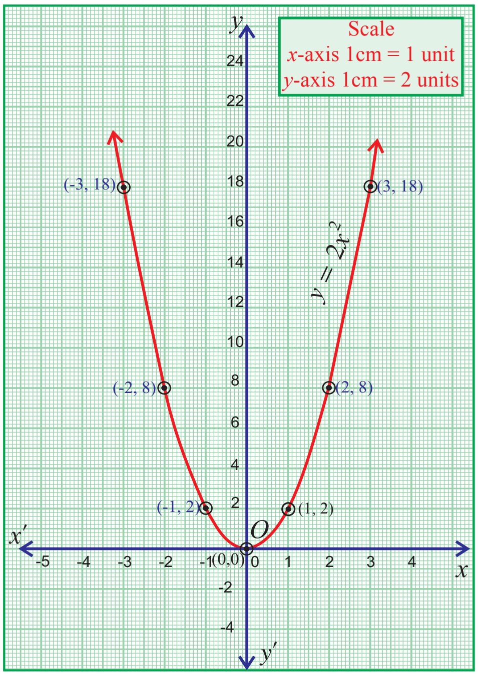


Fig. 10.1

Note

(i) It is symmetrical about y-axis. That is, the part of the graph to the left side of y-axis is the mirror image of the part to the right side of y-axis.

(ii) The graph does not lie below the x-axis as the values of y are non-negative.

Example 10.2

Draw the graph of $y = -3x^2$

Solution

Let us assign the integer values from -3 to 3 for x and form the following table.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = -3x^2$	–27	–12	–3	0	–3	–12	–27

Plot the points (–3, –27), (–2, –12), (–1, –3), (0, 0), (1,–3), (2, –12) and (3, –27).

Join the points by a smooth curve.

The curve thus obtained, is the graph of $y = -3x^2$

Note

(i) The graph of $y = -3x^2$ does not lie above the x-axis as y is always negative.

(ii) The graph is symmetrical about y-axis.

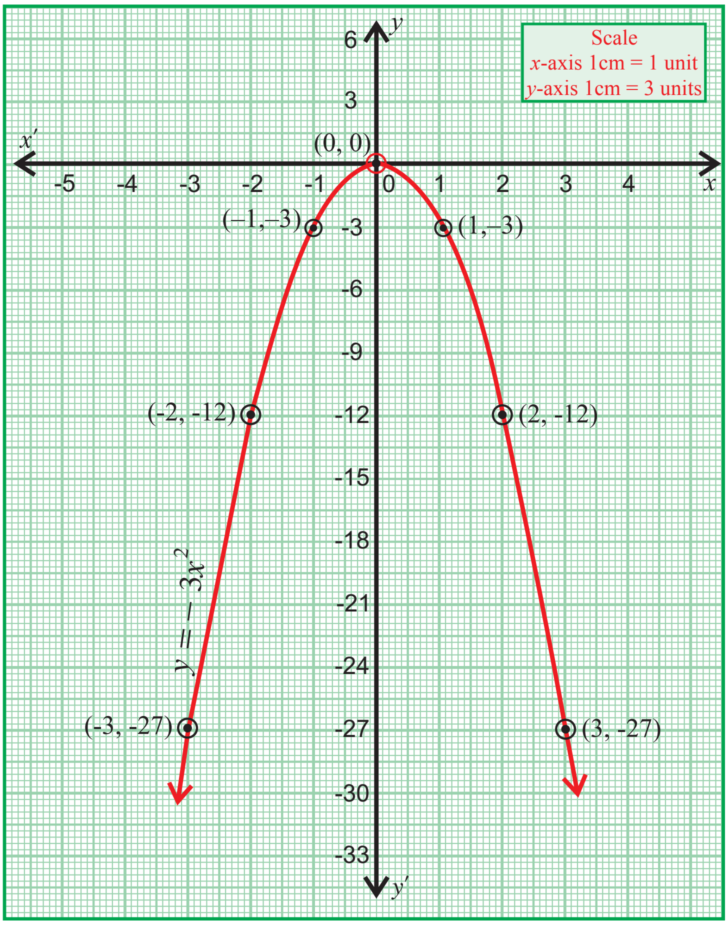


Fig. 10.2

10.2.1 To solve the quadratic equation $ax^2 + bx + c = 0$ graphically.

To find the roots of the quadratic equation $ax^2 + bx + c = 0$ graphically, let us draw the graph of $y = ax^2 + bx + c$.The x- coordinates of the points of intersection of the curve with the x-axis are the roots of the given equation, provided they intersect.

Example 10.3

Solve the equation $x^2 - 2x - 3 = 0$ graphically.

Solution

Let us draw the graph of $y = x^2 - 2x - 3$.

Now, form the following table by assigning integer values from –3 to 4 for x and finding the corresponding values of $y = x^2 - 2x - 3$.

x	–3	–2	–1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
$-2x$	6	4	2	0	–2	–4	–6	–8
–3	–3	–3	–3	–3	–3	–3	–3	–3
y	12	5	0	–3	–4	–3	0	5

Plot the points (–3, 12), (–2, 5), (–1, 0), (0, –3), (1, –4), (2, –3), (3, 0), (4, 5) and join the points by a smooth curve.

The curve intersects the x-axis at the points (–1, 0) and (3, 0).

The x-coordinates of the above points are –1 and 3.

Hence, the solution set is {–1, 3}.

Note

(i) On the x-axis, $y = 0$ always.

(ii) The values of y are both positive and negative. Thus, the curve lies below and above the x-axis.

(iii) The curve is symmetric about the line $x = 1$. (It is not symmetric about the y-axis.)

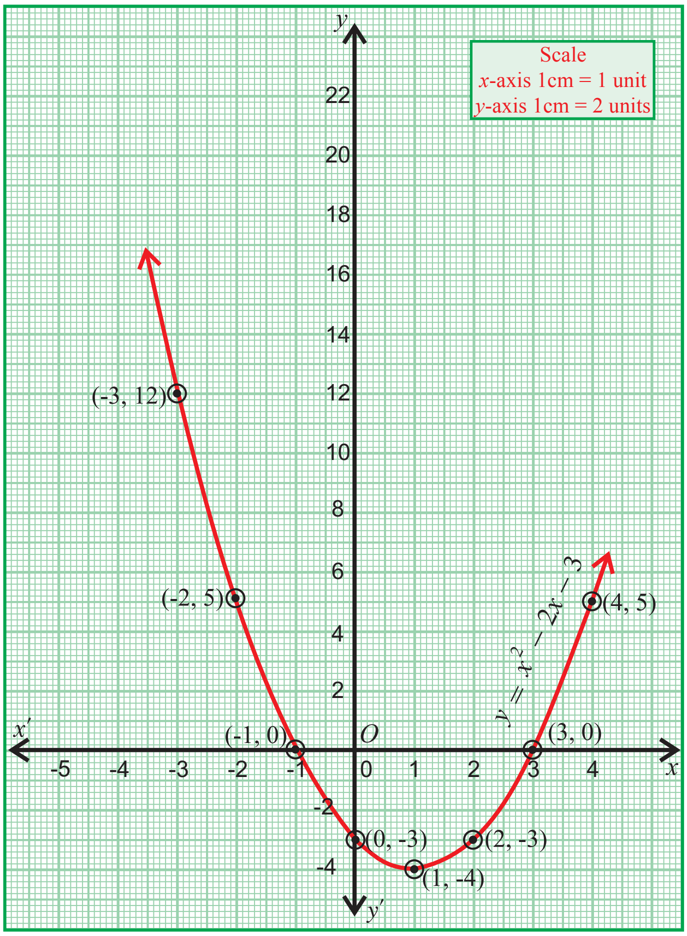


Fig. 10.3

Example 10.4

Solve graphically $2x^2 + x - 6 = 0$

Solution

First, let us form the following table by assigning integer values for x from –3 to 3 and finding the corresponding values of $y = 2x^2 + x - 6$.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$2x^2$	18	8	2	0	2	8	18
x	–3	–2	–1	0	1	2	3
–6	–6	–6	–6	–6	–6	–6	–6
y	9	0	–5	–6	–3	4	15

Plot the points (–3, 9), (–2, 0), (–1, –5), (0, –6), (1, –3), (2, 4) and (3, 15) on the graph.

Join the points by a smooth curve. The curve, thus obtained, is the graph of $y = 2x^2 + x - 6$.

The curve cuts the x-axis at the points (–2 , 0) and (1.5 , 0).

The x-coordinates of the above points are –2 and 1.5.

Hence, the solution set is {–2 , 1.5 }.

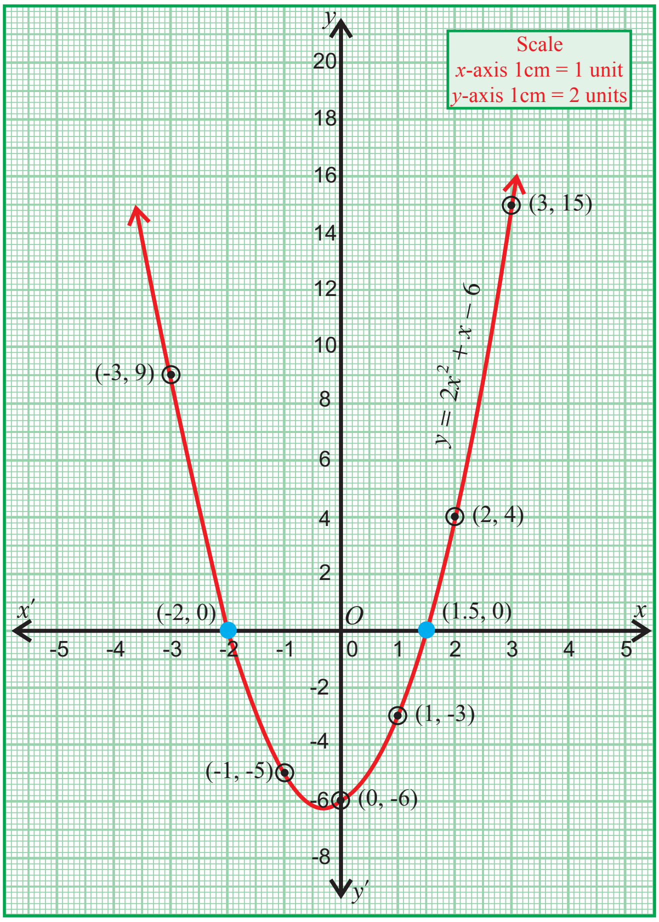


Fig. 10.4

Remarks

To solve $y = 2x^2 + x - 6$ graphically, one can proceed as follows.

(i) Draw the graph of $y = 2x^2$

(ii) Draw the graph of $y = 6 - x$

(iii) The x-coordinates of the points of intersection of the two graphs are the solutions of $2x^2 + x - 6 = 0$.

Example 10.5

Draw the graph of $y = 2x^2$ and hence solve $2x^2 + x - 6 = 0$.

Solution

First, let us draw the graph of $y = 2x^2$. Form the following table.

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$y = 2x^2$	18	8	2	0	2	8	18

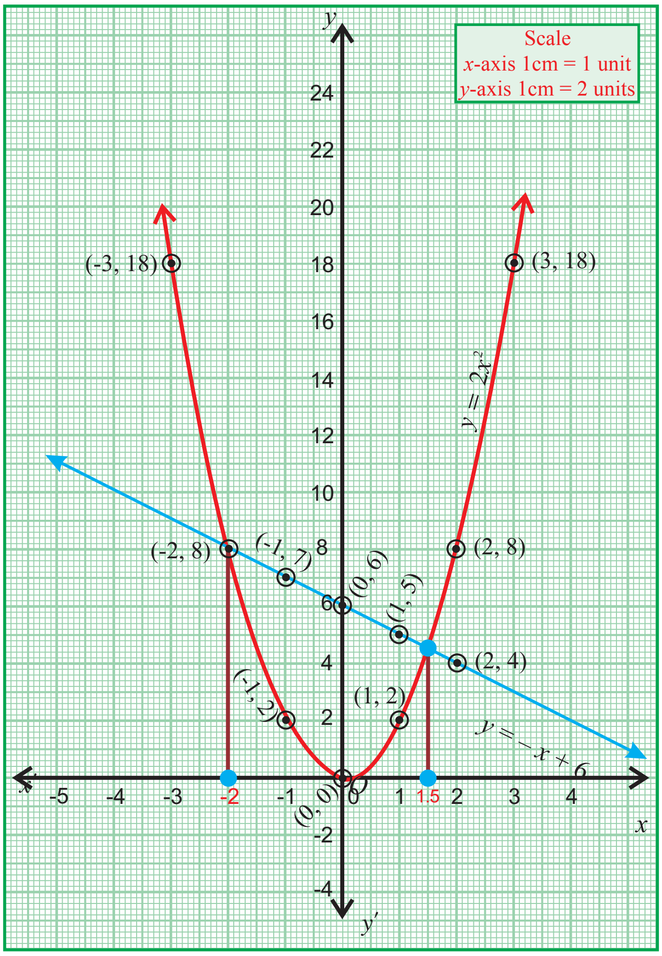


Fig. 10.5

Plot the points (–3, 18), (–2, 8), (–1,2), (0, 0), (1, 2), (2, 8), (3, 18).

Draw the graph by joining the points by a smooth curve.

To find the roots of $2x^2 + x - 6 = 0$, solve the two equations

$y = 2x^2$ and $2x^2 + x - 6 = 0$.

Now, $2x^2 + x - 6 = 0$.

$\Longrightarrow \quad y + x - 6 = 0$, since $y = 2x^2$

Thus, $y = -x + 6$

Hence, the roots of $2x^2 + x - 6 = 0$ are nothing but the x-coordinates of the points of intersection of

$y = 2x^2$ and $y = -x + 6$.

Now, for the straight line $y = -x + 6$, form the following table.

x	–1	0	1	2
$y = -x + 6$	7	6	5	4

Draw the straight line by joining the above points.

The points of intersection of the line and the parabola are (–2 , 8) and (1.5 , 4.5). The x-coordinates of the points are –2 and 1.5.

Thus, the solution set for the equation $2x^2 + x - 6 = 0$ is {–2 , 1.5}.

Example 10.6

Draw the graph of $y = x^2 + 3x + 2$ and use it to solve the equation $x^2 + 2x + 4 = 0$.

Solution

First, let us form a table for $y = x^2 + 3x + 2$.

x	–4	–3	–2	–1	0	1	2	3
x^2	16	9	4	1	0	1	4	9
$3x$	–12	–9	–6	–3	0	3	6	9
2	2	2	2	2	2	2	2	2
y	6	2	0	0	2	6	12	20

Plot the points (–4, 6), (–3, 2), (–2, 0), (–1, 0), (0, 2), (1, 6), (2, 12) and (3, 20).

Now, join the points by a smooth curve. The curve so obtained, is the graph of $y = x^2 + 3x + 2$.

Now, $x^2 + 2x + 4 = 0$

$\Longrightarrow$ $x^2 + 3x + 2 - x + 2 = 0$

$\Longrightarrow$ $y = x - 2$ $\because y = x^2 + 3x + 2$

Thus, the roots of $x^2 + 2x + 4 = 0$ are obtained form the points of intersection of

$y = x - 2$ and $y = x^2 + 3x + 2$.

Let us draw the graph of the straight line $y = x - 2$.

Now, form the table for the line $y = x - 2$

x	–2	0	1	2
$y = x - 2$	–4	–2	–1	0

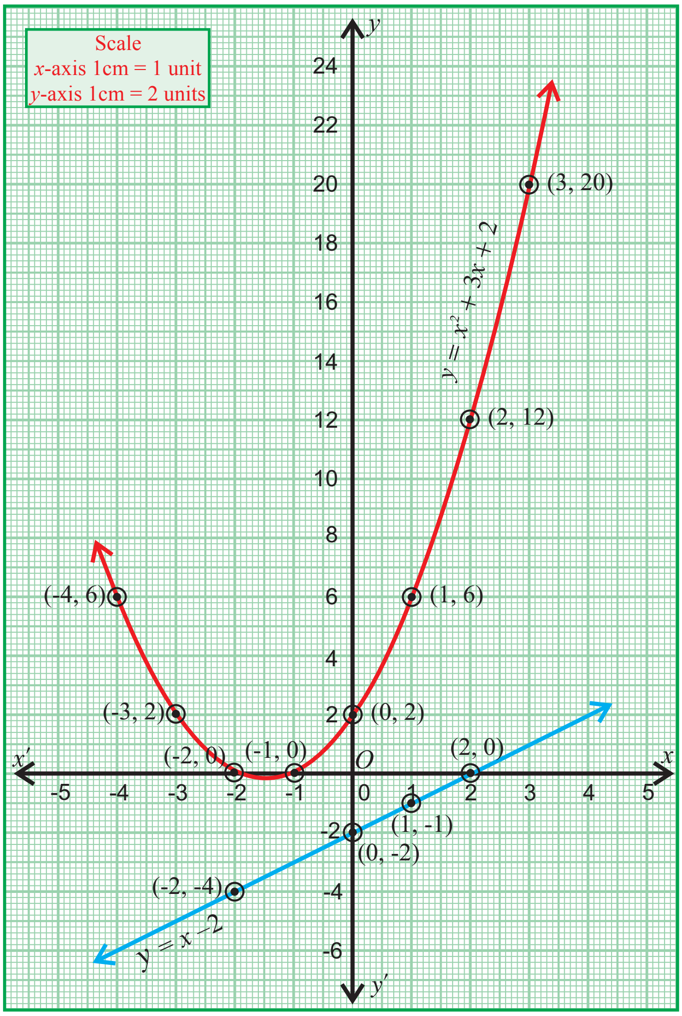


Fig. 10.6

The straight line $y = x - 2$ does not intersect the curve $y = x^2 + 3x + 2$.

Thus, $x^2 + 2x + 4 = 0$ has no real roots.

Exercise 10.1

1. Draw the graph of the following functions.

   (i) $y = 3x^2$

   (ii) $y = -4x^2$

   (iii) $y = (x+2)(x+4)$

   (iv) $y = 2x^2 - x + 3$

2. Solve the following equations graphically

   (i) $x^2 - 4 = 0$

   (ii) $x^2 - 3x - 10 = 0$

   (iii) $(x-5)(x-1) = 0$

   (iv) $(2x+1)(x-3) = 0$

3. Draw the graph of $y = x^2$ and hence solve $x^2 - 4x - 5 = 0$.
4. Draw the graph of $y = x^2 + 2x - 3$ and hence find the roots of $x^2 - x - 6 = 0$.
5. Draw the graph of $y = 2x^2 + x - 6$ and hence solve $2x^2 + x - 10 = 0$.
6. Draw the graph of $y = x^2 - x - 8$ and hence find the roots of $x^2 - 2x - 15 = 0$.
7. Draw the graph of $y = x^2 + x - 12$ and hence solve $x^2 + 2x + 2 = 0$.

10.3 Some Special Graphs

In this section, we will know how to draw graphs when the variables are in

(i) Direct variation (ii) Indirect variation.

If y is directly proportional to x, then we have $y = kx$, for some positive k. In this case the variables are said to be in direct variation and the graph is a straight line.

If y is inversely proportional to x, then we have $y = \frac{k}{x}$, for some positive k. In this case, the variables are said to be in indirect variation and the graph is a smooth curve , known as a Rectangular Hyperbola. (The equation of a rectangular hyperbola is of the form $xy = k,\ k > 0$.)

Example 10.7

Draw a graph for the following table and identify the variation.

x	2	3	5	8	10
y	8	12	20	32	40

Hence, find the value of y when $x = 4$.

Solution

From the table, we found that as x increases, y also increases. Thus, the variation is a direct variation.

Let $y = kx$.

$\Longrightarrow \quad \frac{y}{x} = k$

where k is the constant of proportionality.

From the given values, we have

$$k = \frac{8}{2} = \frac{12}{3} = \cdots = \frac{40}{10}. \quad \therefore \; k = 4$$

The relation $y = 4x$ forms a straight line graph.

Plot the points (2, 8), (3, 12), (5, 20), (8, 32) and (10, 40) and join these points to get the straight line.

Clearly, $y = 4x = 16$ when $x=4$.

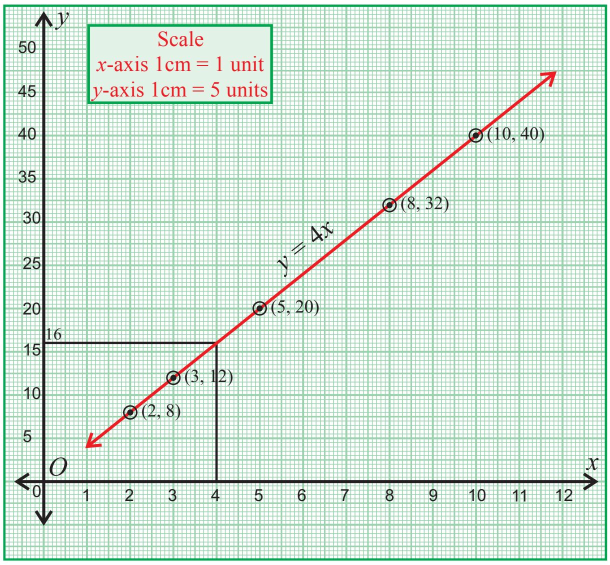


Fig. 10.7

Example 10.8

A cyclist travels from a place A to a place B along the same route at a uniform speed on different days. The following table gives the speed of his travel and the corresponding time he took to cover the distance.

Speed in km / hr x	2	4	6	10	12
Time in hrs y	60	30	20	12	10

Draw the speed-time graph and use it to find

(i) the number of hours he will take if he travels at a speed of 5 km / hr

(ii) the speed with which he should travel if he has to cover the distance in 40 hrs.

Solution

From the table, we observe that as x increases, y decreases.

This type of variation is called indirect variation.

Here, $xy = 120$.

Thus, $y = \dfrac{120}{x}$.

Plot the points (2 , 60), (4 , 30), (6 , 20), (10 , 12) and (12 , 10).

Join these points by a smooth curve.

From the graph, we have

(i) The number of hours he needed to travel at a speed of 5 km/hr is 24 hrs.

(ii) The required speed to cover the distance in 40 hrs, is 3 km / hr.

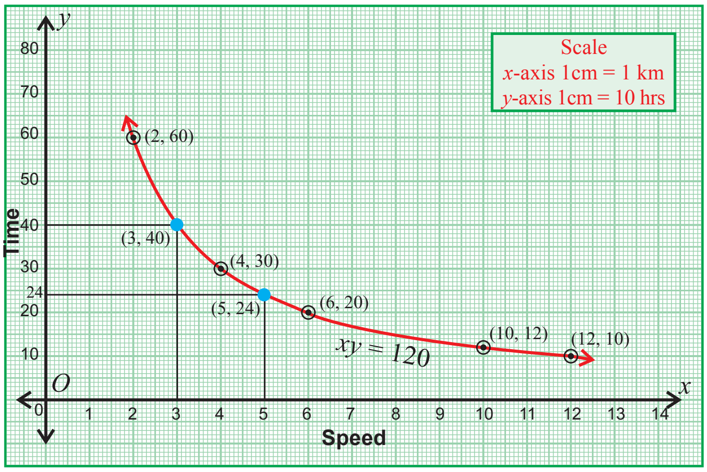


Fig. 10.8

Example 10.9

A bank gives 10% S.I on deposits for senior citizens. Draw the graph for the relation between the sum deposited and the interest earned for one year. Hence find

(i) the interest on the deposit of ₹650

(ii) the amount to be deposited to earn an interest of ₹45.

Solution

Let us form the following table.

Deposit ₹ x	100	200	300	400	500	600	700
S.I. earned ₹ y	10	20	30	40	50	60	70

Clearly $y = \dfrac{1}{10}x$ and the graph is a straight line.

Draw the graph using the points given in the table. From the graph, we see that

(i) The interest for the deposit of ₹650 is ₹65.

(ii) The amount to be deposited to earn an interest of ₹45 is ₹450.

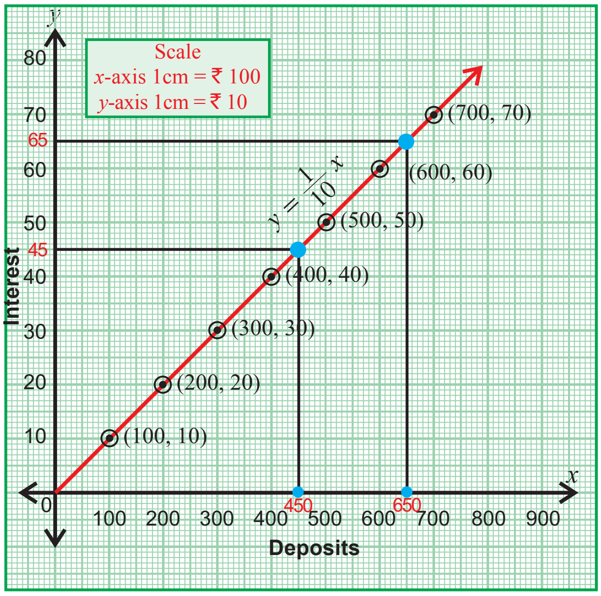


Fig. 10.9

Exercise 10.2

1. A bus travels at a speed of 40 km / hr. Write the distance-time formula and draw the graph of it. Hence, find the distance travelled in 3 hours.

2. The following table gives the cost and number of notebooks bought.

No. of note books x	2	4	6	8	10	12
Cost ₹ y	30	60	90	120	150	180

Draw the graph and hence (i) Find the cost of seven note books.
(ii) How many note books can be bought for ₹ 165.

3.

x	1	3	5	7	8
y	2	6	10	14	16

Draw the graph for the above table and hence find
(i) the value of y if $x = 4$
(ii) the value of x if $y = 12$

4. The cost of the milk per litre is ₹ 15. Draw the graph for the relation between the quantity and cost . Hence find
(i) the proportionality constant.
(ii) the cost of 3 litres of milk.

5. Draw the Graph of $xy = 20$, $x, y > 0$. Use the graph to find y when $x = 5$, and to find x when $y = 10$.

6.

No. of workers x	3	4	6	8	9	16
No of days y	96	72	48	36	32	18

Draw graph for the data given in the table. Hence find the number of days taken by 12 workers to complete the work.

Notable Quotes
1. In mathematics the art of proposing a question must be held of higher than solving it -Georg Cantor.
2. One reason why mathematics enjoys special esteem, above all other sciences, is that its laws are absolutely certain and indisputable, while those of other sciences are to some extent debatable and in constant danger of being overthrown by newly discovered facts - Albert Einstein