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SEQUENCES AND SERIES OF REAL NUMBERS

Mathematics is the Queen of Sciences, and arithmetic is the Queen of Mathematics - C.F.Gauss

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Leonardo Pisano (Fibonacci)

(1170-1250)

Italy

Fibonacci played an important role in reviving ancient mathematics. His name is known to modern mathematicians mainly because of a number sequence named after him, known as the 'Fibonacci numbers', which he did not discover but used as an example.

2.1 Introduction

In this chapter, we shall learn about sequences and series of real numbers. Sequences are fundamental mathematical objects with a long history in mathematics. They are tools for the development of other concepts as well as tools for mathematization of real life situations.

Let us recall that the letters $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and real numbers respectively.

Let us consider the following real-life situations.

(i) A team of ISRO scientists observes and records the height of a satellite from the sea level at regular intervals over a period of time.

(ii) The Railway Ministry wants to find out the number of people using Central railway station in Chennai on a daily basis and so it records the number of people entering the Central Railway station daily for 180 days.

(iii) A curious 9th standard student is interested in finding out all the digits that appear in the decimal part of the irrational number $\sqrt{5} = 2.236067978\cdots$ and writes down as

$2, 3, 6, 0, 6, 7, 9, 7, 8, \cdots$.

(iv) A student interested in finding all positive fractions with numerator 1, writes $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$.

(v) A mathematics teacher writes down the marks of her class according to alphabetical order of the students' names as 75, 95, 67, 35, 58, 47, 100, 89, 85, 60..

(vi) The same teacher writes down the same data in an ascending order as

35, 47, 58, 60, 67, 75, 85, 89, 95, 100.

In each of the above examples, some sets of real numbers have been listed in a specific order.

Note that in (iii) and (iv) the arrangements have infinite number of terms. In (i), (ii), (v) and (vi) there are only finite number of terms; but in (v) and (vi) the same set of numbers are written in different order.

2.2 Sequences

Definition

A sequence of real numbers is an arrangement or a list of real numbers in a specific order.

(i) If a sequence has only finite number of terms, then it is called a finite sequence.

(ii) If a sequence has infinitely many terms, then it is called an infinite sequence.

We denote a finite sequence as $S: a_1, a_2, a_3, \cdots, a_n$ or $S = \{a_j\}_{j=1}^{n}$ and an infinite sequence as $S: a_1, a_2, a_3, \cdots, a_n, \cdots$ or $S = \{a_j\}_{j=1}^{\infty}$ where a_k denotes the k^{th} term of the sequence. For example, a_1 denotes the first term and a_7 denotes the seventh term in the sequence.

Note that in the above examples, (i), (ii), (v) and (vi) are finite sequences, whereas (iii) and (iv) are infinite sequences

Observe that, when we say that a collection of numbers is listed in a sequence, we mean that the sequence has an identified first member, second member, third member and so on. We have already seen some examples of sequences. Let us consider some more examples below.

(i) 2, 4, 6, 8, $\cdots$, 2010. (finite number of terms)

(ii) 1, -1, 1, -1, 1, -1, 1, $\cdots$. (terms just keep oscillating between 1 and -1)

(iii) π, π, π, π, π. (terms are same; such sequences are constant sequences)

(iv) 2, 3, 5, 7, 11, 13, 17, 19, 23, $\cdots$. (list of all prime numbers)

(v) 0.3, 0.33, 0.333, 0.3333, 0.33333, $\cdots$. (infinite number of terms)

(vi) $S = \{a_n\}_1^{\infty}$ where $a_n = 1$ or 0 according to the outcome head or tail in the n^{th} toss of a coin.

From the above examples, (i) and (iii) are finite sequences and the other sequences are infinite sequences. One can easily see that some of them, i.e., (i) to (v) have a definite pattern or rule in the listing and hence we can find out any term in a particular position in the sequence. But in (vi), we cannot predict what a particular term is, however, we know it must be either 1 or 0. Here, we have used the word "pattern" to mean that the n^{th} term of a sequence is found based on the knowledge of its preceding elements in the sequence. In general, sequences can be viewed as functions.

2.2.1 Sequences viewed as functions

A finite real sequence $a_1, a_2, a_3, \cdots, a_n$ or $S = \{a_j\}_{j=1}^{n}$ can be viewed as a function $f : \{1, 2, 3, 4, \cdots, n\} \to \mathbb{R}$ defined by $f(k) = a_k,\ k = 1, 2, 3, \cdots, n.$

An infinite real sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ or $S = \{a_j\}_{j=1}^{\infty}$ can be viewed as a function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(k) = a_k,\ \forall\, k \in \mathbb{N}.$

The symbol $\forall$ means "for all". If the general term a_k of a sequence $\{a_k\}_1^{\infty}$ is given, we can construct the whole sequence. Thus, a sequence is a function whose domain is the set$\{1, 2, 3, \cdots, \}$of natural numbers, or some subset of the natural numbers and whose range is a subset of real numbers.

Remarks

A function is not necessarily a sequence. For example, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = 2x + 1, \forall\, x \in \mathbb{R}$ is not a sequence since the required listing is not possible. Also, note that the domain of f is not $\mathbb{N}$ or a subset $\{1, 2, \cdots, n\}$ of $\mathbb{N}$.

Example 2.1

Write the first three terms in a sequence whose n^{th} term is given by

$$c_n = \frac{n(n+1)(2n+1)}{6},\ \forall\, n \in \mathbb{N}$$

Solution Here, $$c_n = \frac{n(n+1)(2n+1)}{6},\ \forall\, n \in \mathbb{N}$$

For $n = 1$, $c_1 = \dfrac{1(1+1)(2(1)+1)}{6} = 1.$

For $n = 2$, $c_2 = \dfrac{2(2+1)(4+1)}{6} = \dfrac{2(3)(5)}{6} = 5.$

Finally $n = 3$, $c_3 = \dfrac{3(3+1)(7)}{6} = \dfrac{(3)(4)(7)}{6} = 14.$

Hence, the first three terms of the sequence are 1, 5, and 14.

In the above example, we were given a formula for the general term and were able to find any particular term directly. In the following example, we shall see another way of generating a sequence.

Example 2.2

Write the first five terms of each of the following sequences.

(i) $a_1 = -1,$ $\quad a_n = \dfrac{a_{n-1}}{n+2},\quad n > 1$ and $\forall\, n \in \mathbb{N}$

(ii) $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2},\quad n = 3, 4, \cdots.$

Solution

(i) Given $a_1 = -1$ and $a_n = \frac{a_{n-1}}{n+2}$, $n > 1$

$$a_2 = \frac{a_1}{2+2} = -\frac{1}{4}$$

$$a_3 = \frac{a_2}{3+2} = \frac{-\frac{1}{4}}{5} = -\frac{1}{20}$$

$$a_4 = \frac{a_3}{4+2} = \frac{-\frac{1}{20}}{6} = -\frac{1}{120}$$

$$a_5 = \frac{a_4}{5+2} = \frac{-\frac{1}{120}}{7} = -\frac{1}{840}$$

$\therefore$ The required terms of the sequence are $-1, -\frac{1}{4}, -\frac{1}{20}, -\frac{1}{120}$ and $-\frac{1}{840}$.

(ii) Given that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n = 3, 4, 5, \cdots$.

Now, $F_1 = 1, F_2 = 1$

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$

$$F_4 = F_3 + F_2 = 2 + 1 = 3$$

$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

$\therefore$ The first five terms of the sequence are 1, 1, 2, 3, 5.

Remarks

The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence. Its terms are listed as $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$. The Fibonacci sequence occurs in nature, like the arrangement of seeds in a sunflower. The number of spirals in the opposite directions of the seeds in a sunflower are consecutive numbers of the Fibonacci sequence.

Exercise 2.1

1. Write the first three terms of the following sequences whose n^{th} terms are given by

   (i) $a_n = \frac{n(n-2)}{3}$ (ii) $c_n = (-1)^n 3^{n+2}$ (iii) $z_n = \frac{(-1)^n n(n+2)}{4}$

2. Find the indicated terms in each of the sequences whose n^{th} terms are given by

   (i) $a_n = \frac{n+2}{2n+3}$; a_7, a_9 (ii) $a_n = (-1)^n 2^{n+3}(n+1)$; a_5, a_8

   (iii) $a_n = 2n^2 - 3n + 1$; a_5, a_7. (iv) $a_n = (-1)^n(1 - n + n^2)$; a_5, a_8

3. Find the 18th and 25th terms of the sequence defined by

$$a_n = \begin{cases} n(n+3), & \text{if } n \in \mathbb{N} \text{ and } n \text{ is even} \\ \dfrac{2n}{n^2+1}, & \text{if } n \in \mathbb{N} \text{ and } n \text{ is odd.} \end{cases}$$

4. Find the 13th and 16th terms of the sequence defined by

$$b_n = \begin{cases} n^2, & \text{if } n \in \mathbb{N} \text{ and } n \text{ is even} \\ n(n+2), & \text{if } n \in \mathbb{N} \text{ and } n \text{ is odd.} \end{cases}$$

5. Find the first five terms of the sequence given by

$$a_1 = 2, a_2 = 3 + a_1 \text{ and } a_n = 2a_{n-1} + 5 \quad \text{for } n > 2.$$

6. Find the first six terms of the sequence given by

$$a_1 = a_2 = a_3 = 1 \text{ and } a_n = a_{n-1} + a_{n-2} \quad \text{for } n > 3.$$

2.3 Arithmetic sequence or Arithmetic Progression (A.P.)

In this section we shall see some special types of sequences.

Definition

A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference. An arithmetic sequence is also called an Arithmetic Progression (A.P.).

Examples

(i) $2, 5, 8, 11, 14, \cdots$ is an A.P. because $a_1 = 2$ and the common difference $d = 3$.

(ii) $-4, -4, -4, -4, \cdots$ is an A.P. because $a_1 = -4$ and $d = 0$.

(iii) $2, 1.5, 1, 0.5, 0, -0.5, -1.0, -1.5, \cdots$ is an A.P. because $a_1 = 2$ and $d = -0.5$.

The general form of an A.P.

Let us understand the general form of an A.P. Suppose that a is the first term and d is the common difference of an arithmetic sequence $\{a_k\}_{k=1}^{\infty}$. Then, we have

$$a_1 = a \text{ and } a_{n+1} = a_n + d, \ \forall\, n \in \mathbb{N}.$$

For $n = 1, 2, 3$ we get,

$$a_2 = a_1 + d = a + d = a + (2-1)d$$
$$a_3 = a_2 + d = (a+d) + d = a + 2d = a + (3-1)d$$
$$a_4 = a_3 + d = (a+2d) + d = a + 3d = a + (4-1)d$$

Following the pattern, we see that the n^{th} term a_n as

$$a_n = a_{n-1} + d = [a + (n-2)d] + d = a + (n-1)d.$$

Thus, we have $a_n = a + (n-1)d$ for every $n \in \mathbb{N}$.

So, a typical arithmetic sequence or A.P. looks like

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ a+(n-1)d,\ a+nd, \cdots$$

Also, the formula for the general term of an Arithmetic sequence is of the form

$$t_n = a + (n-1)d \quad \text{for every } n \in \mathbb{N}.$$

Note

(i) Remember a sequence may also be a finite sequence. So, if an A.P. has only n terms, then the last term l is given by $l = a + (n-1)d$

(ii) $l = a + (n-1)d$ can also be rewritten as $n = \left(\frac{l-a}{d}\right) + 1$. This helps us to find the number of terms when the first, the last term and the common difference are given.

(iii) Three consecutive terms of an A.P. may be taken as $m-d,\ m,\ m+d$

(iv) Four consecutive terms of an A.P. may be taken as $m-3d,\ m-d,\ m+d,\ m+3d$ with common difference $2d$.

(v) An A.P. remains an A.P. if each of its terms is added or subtracted by a same constant.

(vi) An A.P. remains an A.P. if each of its terms is multiplied or divided by a non-zero constant.

Example 2.3

Which of the following sequences are in an A.P.?

(i) $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \cdots$. (ii) $3m-1, 3m-3, 3m-5, \cdots$.

Solution

(i) Let t_n, $n \in \mathbb{N}$ be the n^{th} term of the given sequence.

$\therefore$ $t_1 = \frac{2}{3},\ t_2 = \frac{4}{5},\ t_3 = \frac{6}{7}$

So $t_2 - t_1 = \frac{4}{5} - \frac{2}{3} = \frac{12-10}{15} = \frac{2}{15}$

$t_3 - t_2 = \frac{6}{7} - \frac{4}{5} = \frac{30-28}{35} = \frac{2}{35}$

Since $t_2 - t_1 \neq t_3 - t_2$, the given sequence is not an A.P.

(ii) Given $3m-1, 3m-3, 3m-5, \cdots$.

Here $t_1 = 3m-1,\ t_2 = 3m-3,\ t_3 = 3m-5, \cdots$.

$\therefore$ $t_2 - t_1 = (3m-3) - (3m-1) = -2$

Also, $t_3 - t_2 = (3m-5) - (3m-3) = -2$

Hence, the given sequence is an A.P. with first term $3m-1$ and the common difference -2.

Example 2.4

Find the first term and common difference of the A.P.

(i) $5, 2, -1, -4, \cdots$. (ii) $\frac{1}{2}, \frac{5}{6}, \frac{7}{6}, \frac{3}{2}, \cdots, \frac{17}{6}$

Solution

(i) First term $a = 5$, and the common difference $d = 2 - 5 = -3$.

(ii) $a = \frac{1}{2}$ and the common difference $d = \frac{5}{6} - \frac{1}{2} = \frac{5-3}{6} = \frac{1}{3}$.

Example 2.5

Find the smallest positive integer n such that t_n of the arithmetic sequence

$20, 19\frac{1}{4}, 18\frac{1}{2}, \cdots$ is negative.?

Solution Here we have $a = 20$, $d = 19\frac{1}{4} - 20 = -\frac{3}{4}$.

We want to find the first positive integer n such that $t_n < 0$.

This is same as solving $a + (n-1)d < 0$ for smallest $n \in \mathbb{N}$.

That is solving $20 + (n-1)\left(-\frac{3}{4}\right) < 0$ for smallest $n \in \mathbb{N}$.

Now, $(n-1)\left(-\frac{3}{4}\right) < -20$

$\Longrightarrow (n-1) \times \frac{3}{4} > 20$ (The inequality is reversed on multiplying both sides by -1)

$\therefore \quad n - 1 > 20 \times \frac{4}{3} = \frac{80}{3} = 26\frac{2}{3}$.

This implies $n > 26\frac{2}{3} + 1$. That is, $n > 27\frac{2}{3} = 27.66$

Thus, the smallest positive integer $n \in \mathbb{N}$ satisfying the inequality is $n = 28$.

Hence, the 28^{th} term, t_{28} is the first negative term of the A.P.

Example 2.6

In a flower garden, there are 23 rose plants in the first row, 21 in the second row, 19 in the third row and so on. There are 5 rose plants in the last row. How many rows are there in the flower garden?

Solution Let n be the number of rows in the flower garden .

The number of rose plants in the $1^{st}, 2^{nd}, 3^{rd}, \cdots, n^{th}$ rows are $23, 21, 19, \cdots, 5$ respectively.

Now, $t_k - t_{k-1} = -2$ for $k = 2, \cdots, n$.

Thus, the sequence $23, 21, 19, \cdots, 5$ is in an A.P.

We have $a = 23$, $d = -2$, and $l = 5$.

$$\therefore \quad n = \frac{l-a}{d} + 1 = \frac{5-23}{-2} + 1 = 10.$$

So, there are 10 rows in the flower garden.

Example 2.7

If a person joins his work in 2010 with an annual salary of ₹30,000 and receives an annual increment of ₹600 every year, in which year, will his annual salary be ₹39,000?

Solution Suppose that the person's annual salary reaches ₹39,000 in the n^{th} year.

Annual salary of the person in 2010, 2011, 2012, $\cdots$, $[2010 + (n-1)]$ will be

₹30,000, ₹30,600, ₹31,200, $\cdots$, ₹39000 respectively.

First note that the sequence of salaries form an A.P.

To find the required number of terms, let us divide each term of the sequence by a fixed constant 100. Now, we get the new sequence 300, 306, 312, $\cdots$, 390.

Here $a = 300$, $d = 6$, $l = 390$.

$$\text{So,} \quad n = \frac{l-a}{d} + 1$$

$$= \frac{390-300}{6} + 1 = \frac{90}{6} + 1 = 16$$

Thus, 16^{th} annual salary of the person will be ₹39,000.

$\therefore$ His annual salary will reach ₹39,000 in the year 2025.

Example 2.8

Three numbers are in the ratio $2:5:7$. If the first number, the resulting number on the substraction of 7 from the second number and the third number form an arithmetic sequence, then find the numbers.

Solution Let the numbers be $2x, 5x$ and $7x$ for some unknown $x, (x \neq 0)$

By the given information, we have that $2x, 5x-7, 7x$ are in A.P.

$\therefore \ (5x-7) - 2x = 7x - (5x-7) \implies 3x - 7 = 2x + 7$ and so $x = 14$.

Thus, the required numbers are 28, 70, 98.

Exercise 2.2

1. The first term of an A.P. is 6 and the common difference is 5. Find the A.P. and its general term.
2. Find the common difference and 15^{th} term of the A.P. $125, 120, 115, 110, \cdots$.
3. Which term of the arithmetic sequence $24, 23\frac{1}{4}, 22\frac{1}{2}, 21\frac{3}{4}, \cdots$. is 3?

4. Find the 12th term of the A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \cdots$.

5. Find the 17th term of the A.P. 4, 9, 14, $\cdots$.

6. How many terms are there in the following Arithmetic Progressions?

   (i) $-1, -\frac{5}{6}, -\frac{2}{3}, \cdots, \frac{10}{3}$. (ii) 7, 13, 19, $\cdots$, 205.

7. If 9th term of an A.P. is zero, prove that its 29th term is double (twice) the 19th term.

8. The 10th and 18th terms of an A.P. are 41 and 73 respectively. Find the 27th term.

9. Find n so that the n^{th} terms of the following two A.P.'s are the same.

   1, 7, 13, 19,$\cdots$ and 100, 95, 90, $\cdots$.

10. How many two digit numbers are divisible by 13?

11. A TV manufacturer has produced 1000 TVs in the seventh year and 1450 TVs in the tenth year. Assuming that the production increases uniformly by a fixed number every year, find the number of TVs produced in the first year and in the 15th year.

12. A man has saved ₹640 during the first month, ₹720 in the second month and ₹800 in the third month. If he continues his savings in this sequence, what will be his savings in the 25th month?

13. The sum of three consecutive terms in an A.P. is 6 and their product is –120. Find the three numbers.

14. Find the three consecutive terms in an A. P. whose sum is 18 and the sum of their squares is 140.

15. If m times the m^{th} term of an A.P. is equal to n times its n^{th} term, then show that the $(m+n)^{th}$ term of the A.P. is zero.

16. A person has deposited ₹25,000 in an investment which yields 14% simple interest annually. Do these amounts (principal + interest) form an A.P.? If so, determine the amount of investment after 20 years.

17. If a, b, c are in A.P. then prove that $(a-c)^2 = 4(b^2 - ac)$.

18. If a, b, c are in A.P. then prove that $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are also in A.P.

19. If a^2, b^2, c^2 are in A.P. then show that $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are also in A.P.

20. If $a^x = b^y = c^z, x \neq 0, y \neq 0, z \neq 0$ and $b^2 = ac$, then show that $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

2.4 Geometric Sequence or Geometric Progression (G.P.)

Definition

A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called a geometric sequence if $a_{n+1} = a_n r$, $n \in \mathbb{N}$, where r is a non-zero constant. Here, a_1 is the first term and the constant r is called the common ratio. A geometric sequence is also called a Geometric Progression (G.P.).

Let us consider some examples of geometric sequences.

(i) $3, 6, 12, 24, \cdots$.

A sequence $\{a_n\}_1^{\infty}$ is a geometric sequence if $\frac{a_{n+1}}{a_n} = r \neq 0$, $n \in \mathbb{N}$.

Now, $\frac{6}{3} = \frac{12}{6} = \frac{24}{12} = 2 \neq 0$. So the given sequence is a geometric sequence.

(ii) $\frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}, \cdots$.

Here, we have $\frac{-\frac{1}{27}}{\frac{1}{9}} = \frac{\frac{1}{81}}{-\frac{1}{27}} = \frac{-\frac{1}{243}}{\frac{1}{81}} = -\frac{1}{3} \neq 0$.

Thus, the given sequence is a geometric sequence.

The general form of a G.P.

Let us derive the general form of a G.P. Suppose that a is the first term and r is the common ratio of a geometric sequence $\{a_k\}_{k=1}^{\infty}$. Then, we have

$$a_1 = a \text{ and } \frac{a_{n+1}}{a_n} = r \quad \text{for } n \in \mathbb{N}.$$

Thus, $a_{n+1} = r\, a_n$ for $n \in \mathbb{N}$.

For $n = 1, 2, 3$ we get,

$$a_2 = a_1 r = ar = ar^{2-1}$$

$$a_3 = a_2 r = (ar)r = ar^2 = ar^{3-1}$$

$$a_4 = a_3 r = (ar^2)r = ar^3 = ar^{4-1}$$

Following the pattern, we have

$$a_n = a_{n-1} r = (ar^{n-2})r = ar^{n-1}.$$

Thus, $a_n = ar^{n-1}$ for every $n \in \mathbb{N}$, gives n^{th} term of the G.P.

So, a typical geometric sequence or G.P. looks like

$$a, ar, ar^2, ar^3, \cdots, ar^{n-1}, ar^n, \cdots.$$

Thus , the formula for the general term of a geometric sequence is

$$t_n = ar^{n-1}, \; n = 1, 2, 3, \cdots.$$

Suppose we are given the first few terms of a sequence, how can we determine if the given sequence is a geometric sequence or not?

If $\frac{t_{n+1}}{t_n} = r, \forall\, n \in \mathbb{N}$, where r is a non-zero constant, then $\{t_n\}_1^{\infty}$ is in G.P.

Note

(i) If the ratio of any term other than the first term to its preceding term of a sequence is a non-zero constant, then it is a geometric sequence.

(ii) A geometric sequence remains a geometric sequence if each term is multiplied or divided by a non zero constant.

(iii) Three consecutive terms in a G.P may be taken as $\frac{a}{r}, a, ar$ with common ratio r.

(iv) Four consecutive terms in a G.P may be taken as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.
(here, the common ratio is r^2 not r as above)

Example 2.9

Which of the following sequences are geometric sequences

(i) $5, 10, 15, 20, \cdots$. (ii) $0.15, 0.015, 0.0015, \cdots$. (iii) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, 3\sqrt{21}, \cdots$.

Solution

(i) Considering the ratios of the consecutive terms, we see that $\frac{10}{5} \neq \frac{15}{10}$.

Thus, there is no common ratio. Hence it is not a geometric sequence.

(ii) We see that $\frac{0.015}{0.15} = \frac{0.0015}{0.015} = \cdots = \frac{1}{10}$.

Since the common ratio is $\frac{1}{10}$, the given sequence is a geometric sequence.

(iii) Now, $\frac{\sqrt{21}}{\sqrt{7}} = \frac{3\sqrt{7}}{\sqrt{21}} = \frac{3\sqrt{21}}{3\sqrt{7}} = \cdots = \sqrt{3}$. Thus, the common ratio is $\sqrt{3}$.

Therefore, the given sequence is a geometric sequence.

Example 2.10

Find the common ratio and the general term of the following geometric sequences.

(i) $\frac{2}{5}, \frac{6}{25}, \frac{18}{125}, \cdots$. (ii) $0.02, 0.006, 0.0018, \cdots$.

Solution

(i) Given sequence is a geometric sequence.

The common ratio is given by $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \cdots$.

Thus, $$r = \frac{\frac{6}{25}}{\frac{2}{5}} = \frac{3}{5}.$$

The first term of the sequence is $\frac{2}{5}$. So, the general term of the sequence is

$$t_n = ar^{n-1},\ n = 1, 2, 3, \cdots.$$

$$\Longrightarrow \qquad t_n = \frac{2}{5}\left(\frac{3}{5}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$$

(ii) The common ratio of the given geometric sequence is

$$r = \frac{0.006}{0.02} = 0.3 = \frac{3}{10}.$$

The first term of the geometric sequence is 0.02

So, the sequence can be represented by

$$t_n = (0.02)\left(\frac{3}{10}\right)^{n-1}, \quad n = 1, 2, 3, \cdots$$

Example 2.11

The 4th term of a geometric sequence is $\frac{2}{3}$ and the seventh term is $\frac{16}{81}$. Find the geometric sequence.

Solution Given that $t_4 = \frac{2}{3}$ and $t_7 = \frac{16}{81}$.

Using the formula $t_n = ar^{n-1},\ n = 1, 2, 3, \cdots$. for the general term we have,

$$t_4 = ar^3 = \frac{2}{3} \quad \text{and} \quad t_7 = ar^6 = \frac{16}{81}.$$

Note that in order to find the geometric sequence, we need to find a and r.

By dividing t_7 by t_4 we obtain,

$$\frac{t_7}{t_4} = \frac{ar^6}{ar^3} = \frac{\frac{16}{81}}{\frac{2}{3}} = \frac{8}{27}.$$

Thus, $r^3 = \frac{8}{27} = \left(\frac{2}{3}\right)^3$ which implies $r = \frac{2}{3}$.

Now, $t_4 = \frac{2}{3} \Longrightarrow ar^3 = \left(\frac{2}{3}\right)$.

$$\Longrightarrow \quad a\left(\frac{8}{27}\right) = \frac{2}{3}. \qquad \therefore \quad a = \frac{9}{4}.$$

Hence, the required geometric sequence is $a,\ ar,\ ar^2,\ ar^3,\ \cdots,\ ar^{n-1},\ ar^n, \cdots$

That is, $\frac{9}{4},\ \frac{9}{4}\left(\frac{2}{3}\right),\ \frac{9}{4}\left(\frac{2}{3}\right)^2,\ \cdots$.

Example 2.12

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture initially, how many bacteria will be present at the end of 14th hour?

Solution Note that the number of bacteria present in the culture doubles at the end of successive hours.

Number of bacteria present initially in the culture $= 30$

Number of bacteria present at the end of first hour $= 2(30)$

Number of bacteria present at the end of second hour $= 2(2(30)) = 30(2^2)$

Continuing in this way, we see that the number of bacteria present at the end of every hour forms a G.P. with the common ratio $r = 2$.

Thus, if t_n denotes the number of bacteria after n hours,

$$t_n = 30\,(2^n) \text{ is the general term of the G.P.}$$

Hence, the number of bacteria at the end of 14th hour is given by $t_{14} = 30\,(2^{14})$.

Example 2.13

An amount ₹500 is deposited in a bank which pays annual interest at the rate of 10% compounded annually. What will be the value of this deposit at the end of 10th year?

Solution

The principal is ₹500. So, the interest for this principal for one year is $500\left(\frac{10}{100}\right) = 50$.

Thus, the principal for the 2nd year = Principal for 1st year + Interest

$$= 500 + 500\left(\frac{10}{100}\right) = 500\left(1 + \frac{10}{100}\right)$$

Now, the interest for the second year $= \left(500\left(1 + \frac{10}{100}\right)\right)\left(\frac{10}{100}\right)$.

So, the principal for the third year $= 500\left(1 + \frac{10}{100}\right) + 500\left(1 + \frac{10}{100}\right)\frac{10}{100}$

$$= 500\left(1 + \frac{10}{100}\right)^2$$

Continuing in this way we see that the principal for the n^{th} year $\Big\} = 500\left(1 + \frac{10}{100}\right)^{n-1}$.

The amount at the end of $(n-1)^{th}$ year = Principal for the n^{th} year.

Thus, the amount in the account at the end of n^{th} year.

$$= 500\left(1 + \frac{10}{100}\right)^{n-1} + 500\left(1 + \frac{10}{100}\right)^{n-1}\left(\frac{10}{100}\right) = 500\left(\frac{11}{10}\right)^n.$$

The amount in the account at the end of 10th year

$$= ₹\,500\left(1 + \frac{10}{100}\right)^{10} = ₹\,500\left(\frac{11}{10}\right)^{10}.$$

Remarks

By using the above method, one can derive a formula for finding the total amount for compound interest problems. Derive the formula:

$$A = P(1 + i)^n$$

where A is the amount, P is the principal, $i = \frac{r}{100}$, r is the annual interest rate and n is the number of years.

Example 2.14

The sum of first three terms of a geometric sequence is $\frac{13}{12}$ and their product is -1. Find the common ratio and the terms.

Solution We may take the first three terms of the geometric sequence as $\frac{a}{r}, a, ar$.

Then,
$$\frac{a}{r} + a + ar = \frac{13}{12}$$
$$a\left(\frac{1}{r} + 1 + r\right) = \frac{13}{12} \implies a\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12} \qquad (1)$$

Also,
$$\left(\frac{a}{r}\right)(a)(ar) = -1$$
$$\implies a^3 = -1 \qquad \therefore \quad a = -1$$

Substituting $a = -1$ in (1) we obtain,
$$(-1)\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12}$$
$$\implies 12r^2 + 12r + 12 = -13r$$
$$12r^2 + 25r + 12 = 0$$
$$(3r + 4)(4r + 3) = 0$$

Thus, $r = -\frac{4}{3}$ or $-\frac{3}{4}$

When $r = -\frac{4}{3}$ and $a = -1$, the terms are $\frac{3}{4}, -1, \frac{4}{3}$.

When $r = -\frac{3}{4}$ and $a = -1$, we get $\frac{4}{3}, -1, \frac{3}{4}$, which is in the reverse order.

Example 2.15

If a, b, c, d are in geometric sequence, then prove that

$(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$

Solution Given a, b, c, d are in a geometric sequence.

Let r be the common ratio of the given sequence. Here, the first term is a.

Thus, $b = ar, \quad c = ar^2, \quad d = ar^3$

Now, $(b-c)^2 + (c-a)^2 + (d-b)^2$
$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$
$$= a^2[(r - r^2)^2 + (r^2 - 1)^2 + (r^3 - r)^2]$$
$$= a^2[r^2 - 2r^3 + r^4 + r^4 - 2r^2 + 1 + r^6 - 2r^4 + r^2]$$
$$= a^2[r^6 - 2r^3 + 1] = a^2[r^3 - 1]^2$$
$$= (ar^3 - a)^2 = (a - ar^3)^2 = (a - d)^2$$

Exercise 2.3

1. Find out which of the following sequences are geometric sequences. For those geometric sequences, find the common ratio.

   (i) $0.12,\ 0.24,\ 0.48,\cdots.$ (ii) $0.004,\ 0.02,\ 0.1,\cdots.$ (iii) $\frac{1}{2},\ \frac{1}{3},\ \frac{2}{9},\ \frac{4}{27},\ \cdots.$

   (iv) $12,\ 1,\ \frac{1}{12},\cdots.$ (v) $\sqrt{2},\ \frac{1}{\sqrt{2}},\ \frac{1}{2\sqrt{2}},\cdots.$ (vi) $4,-2,\ -1,-\frac{1}{2},\cdots.$

2. Find the 10^{th} term and common ratio of the geometric sequence $\frac{1}{4},-\frac{1}{2},1,-2,\cdots.$

3. If the 4^{th} and 7^{th} terms of a G.P. are 54 and 1458 respectively, find the G.P.

4. In a geometric sequence, the first term is $\frac{1}{3}$ and the sixth term is $\frac{1}{729}$, find the G.P.

5. Which term of the geometric sequence,

   (i) $5,\ 2,\ \frac{4}{5},\ \frac{8}{25},\cdots$, is $\frac{128}{15625}$? (ii) $1,\ 2,\ 4,\ 8,\cdots$, is 1024 ?

6. If the geometric sequences $162,\ 54,\ 18,\cdots.$ and $\frac{2}{81},\ \frac{2}{27},\ \frac{2}{9},\cdots$ have their n^{th} term equal, find the value of n.

7. The fifth term of a G.P. is 1875. If the first term is 3, find the common ratio.

8. The sum of three terms of a geometric sequence is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

9. If the product of three consecutive terms in G.P. is 216 and sum of their products in pairs is 156, find them.

10. Find the first three consecutive terms in G.P. whose sum is 7 and the sum of their reciprocals is $\frac{7}{4}$

11. The sum of the first three terms of a G.P. is 13 and sum of their squares is 91. Determine the G.P.

12. If ₹1000 is deposited in a bank which pays annual interest at the rate of 5% compounded annually, find the maturity amount at the end of 12 years .

13. A company purchases an office copier machine for ₹50,000. It is estimated that the copier depreciates in its value at a rate of 15% per year. What will be the value of the copier after 15 years?

14. If a, b, c, d are in a geometric sequence, then show that

    $$(a-b+c)(b+c+d)=ab+bc+cd.$$

15. If a, b, c, d are in a G.P., then prove that $a+b,\quad b+c,\quad c+d,$ are also in G.P.

2.5 Series

Let us consider the following problem:

A person joined a job on January 1, 1990 at an annual salary of ₹25,000 and received an annual increment of ₹500 each year. What is the total salary he has received upto January 1, 2010?

First of all note that his annual salary forms an arithmetic sequence

$$25000,\ 25500,\ 26000,\ 26500, \cdots,\ (25000 + 19(500)).$$

To answer the above question, we need to add all of his twenty years salary. That is,

$$25000 + 25500 + 26000 + 26500 + \cdots + (25000 + 19(500)).$$

So, we need to develop an idea of summing terms of a sequence.

Definition

An expression of addition of terms of a sequence is called a series.
If a series consists only a finite number of terms, it is called a finite series.
If a series consists of infinite number of terms of a sequence, it is called an infinite series.

Consider a sequence $S = \{a_n\}_{n=1}^{\infty}$ of real numbers. For each $n \in \mathbb{N}$ we define the partial sums by $S_n = a_1 + a_2 + , \cdots + a_n,\ n = 1, 2, 3, \cdots$. Then $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the given sequence $\{a_n\}_{n=1}^{\infty}$.

The ordered pair $(\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty})$ is called an infinite series of terms of the sequence $\{a_n\}_1^{\infty}$. The infinite series is denoted by $a_1 + a_2 + a_3 + \cdots$, or simply $\sum_{n=1}^{\infty} a_n$ where the symbol $\sum$ stands for summation and is pronounced as sigma.

Well, we can easily understand finite series (adding finite number of terms). It is impossible to add all the terms of an infinite sequence by the ordinary addition, since one could never complete the task. How can we understand (or assign a meaning to) adding infinitely many terms of a sequence? We will learn about this in higher classes in mathematics. For now we shall focus mostly on finite series.

In this section , we shall study Arithmetic series and Geometric series.

2.5.1 Arithmetic series

An arithmetic series is a series whose terms form an arithmetic sequence.

Sum of first n terms of an arithmetic sequence

Consider an arithmetic sequence with first term a and common difference d given by $a,\ a + d,\ a + 2d, ...,\ a + (n-1)d, \cdots$.

Let S_n be the sum of first n terms of the arithmetic sequence.

Thus, $S_n = a + (a + d) + (a + 2d) + \cdots\ (a + (n-1)d)$

$\implies S_n = na + (d + 2d + 3d + \cdots + (n-1)d)$

$= na + d(1 + 2 + 3 + \cdots + (n-1))$

So, we can simplify this formula if we can find the sum $1 + 2 + \cdots + (n-1)$.

This is nothing but the sum of the arithmetic sequence $1, 2, 3, \cdots, (n-1)$.

So, first we find the sum $1 + 2 + \cdots + (n-1)$ below.

Now, let us find the sum of the first n positive integers.

Let $S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$. (1)

We shall use a small trick to find the above sum. Note that we can write S_n also as

$$S_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1. \quad (2)$$

Adding (1) and (2) we obtain,

$$2S_n = (n+1) + (n+1) + \cdots + (n+1) + (n+1). \quad (3)$$

Now, how many $(n+1)$ are there on the right hand side of (3)?

There are n terms in each of (1) and (2). We merely added corresponding terms from (1) and (2).

Thus, there must be exactly n such $(n+1)$'s.

Therefore, (3) simplifies to $2S_n = n(n+1)$.

Hence, the sum of the first n positive integers is given by

$$S_n = \frac{n(n+1)}{2}. \qquad \text{So, } 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \quad (4)$$

This is a useful formula in finding the sums.

Remarks

The above method was first used by the famous German mathematician **Carl Fredrick Gauss,** known as **Prince of Mathematics,** to find the sum of positive integers upto 100. This problem was given to him by his school teacher when he was just five years old. When you go to higher studies in mathematics, you will learn other methods to arrive at the above formula.

Carl Fredrick Gauss
(1777 – 1855)

Now, let us go back to summing first n terms of a general arithmetic sequence. We have already seen that

$$S_n = na + [d + 2d + 3d + \cdots + (n-1)d]$$
$$= na + d[1 + 2 + 3 + \cdots + (n-1)]$$
$$= na + d\frac{n(n-1)}{2} \quad \text{using (4)}$$
$$= \frac{n}{2}[2a + (n-1)d] \quad (5)$$

Hence, we have

$$S_n = \frac{n}{2}[a + (a + (n - 1)d)] = \frac{n}{2}\text{(first term + last term)}$$
$$= \frac{n}{2}(a + l).$$

The sum S_n of the first n terms of an arithmetic sequence with first term a is given by

(i) $S_n = \frac{n}{2}[2a + (n - 1)d]$ if the common difference d is given.

(ii) $S_n = \frac{n}{2}(a + l)$, if the last term l is given.

Example 2.16

Find the sum of the arithmetic series $5 + 11 + 17 + \cdots + 95$.

Solution Given that the series $5 + 11 + 17 + \cdots + 95$ is an arithmetic series.

Note that $a = 5$, $d = 11 - 5 = 6$, $l = 95$.

Now, $$n = \frac{l - a}{d} + 1$$
$$= \frac{95 - 5}{6} + 1 = \frac{90}{6} + 1 = 16.$$

Hence, the sum $$S_n = \frac{n}{2}[l + a]$$
$$S_{16} = \frac{16}{2}[95 + 5] = 8(100) = 800.$$

Example 2.17

Find the sum of the first $2n$ terms of the following series.

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots.$$

Solution We want to find $1^2 - 2^2 + 3^2 - 4^2 + \cdots$ to $2n$ terms

$= 1 - 4 + 9 - 16 + 25 - \cdots$ to $2n$ terms

$= (1 - 4) + (9 - 16) + (25 - 36) + \cdots$ to n terms. (after grouping)

$= -3 + (-7) + (-11) + \cdots$ n terms

Now, the above series is in an A.P. with first term $a = -3$ and common difference $d = -4$

Therefore, the required sum $$= \frac{n}{2}[2a + (n - 1)d]$$
$$= \frac{n}{2}[2(-3) + (n - 1)(-4)]$$
$$= \frac{n}{2}[-6 - 4n + 4] = \frac{n}{2}[-4n - 2]$$
$$= \frac{-2n}{2}(2n + 1) = -n(2n + 1).$$

Example 2.18

In an arithmetic series, the sum of first 14 terms is -203 and the sum of the next 11 terms is –572. Find the arithmetic series.

Solution Given that

$$S_{14} = -203$$

$$\implies \frac{14}{2}[2a + 13d] = -203$$

$$\implies 7[2a + 13d] = -203$$

$$\implies 2a + 13d = -29. \qquad (1)$$

Also, the sum of the next 11 terms $= -572$.

Now, $S_{25} = S_{14} + (-572)$

That is, $S_{25} = -203 - 572 = -775$.

$$\implies \frac{25}{2}[2a + 24d] = -775$$

$$\implies 2a + 24d = -31 \times 2$$

$$\implies a + 12d = -31 \qquad (2)$$

Solving (1) and (2) we get, $a = 5$ and $d = -3$.

Thus, the required arithmetic series is $5 + (5 - 3) + (5 + 2(-3)) + \cdots$.

That is, the series is $5 + 2 - 1 - 4 - 7 - \cdots$.

Example 2.19

How many terms of the arithmetic series $24 + 21 + 18 + 15 + \cdots$, be taken continuously so that their sum is -351.

Solution In the given arithmetic series, $a = 24$, $d = -3$.

Let us find n such that $S_n = -351$

Now, $S_n = \frac{n}{2}[2a + (n-1)d] = -351$

That is, $\frac{n}{2}[2(24) + (n-1)(-3)] = -351$

$$\implies \frac{n}{2}[48 - 3n + 3] = -351$$

$$\implies n(51 - 3n) = -702$$

$$\implies n^2 - 17n - 234 = 0$$

$$(n - 26)(n + 9) = 0$$

$$\therefore n = 26 \text{ or } n = -9$$

Here n, being the number of terms needed, cannot be negative.

Thus, 26 terms are needed to get the sum -351.

Example 2.20

Find the sum of all 3 digit natural numbers, which are divisible by 8.

Solution

The three digit natural numbers divisible by 8 are 104, 112, 120, $\cdots$, 992.

Let S_n denote their sum. That is, $S_n = 104 + 112 + 120 + 128 + , \cdots + 992$.

Now, the sequence 104, 112, 120, $\cdots$, 992 forms an A.P.

Here, $a = 104, \quad d = 8 \quad \text{and} \quad l = 992.$

$$\therefore \quad n = \frac{l-a}{d} + 1 = \frac{992 - 104}{8} + 1$$

$$= \frac{888}{8} + 1 = 112.$$

Thus, $S_{112} = \frac{n}{2}[a + l] = \frac{112}{2}[104 + 992] = 56(1096) = 61376$.

Hence, the sum of all three digit numbers, which are divisible by 8 is equal to 61376.

Example 2.21

The measures of the interior angles taken in order of a polygon form an arithmetic sequence. The least measurement in the sequence is 85°. The greatest measurement is 215°. Find the number of sides in the given polygon.

Solution Let n denote the number of sides of the polygon.

Now, the measures of interior angles form an arithmetic sequence.

Let the sum of the interior angles of the polygon be

$$S_n = a + (a + d) + (a + 2d) + \cdots + l \text{, where } a = 85 \text{ and } l = 215.$$

We have, $S_n = \frac{n}{2}[l + a]$ (1)

We know that the sum of the interior angles of a polygon is $(n-2)\times 180^0$.

Thus, $S_n = (n-2)\times 180$

From (1), we have $\frac{n}{2}[l + a] = (n-2)\times 180$

$$\Longrightarrow \quad \frac{n}{2}[215 + 85] = (n-2)\times 180$$

$$150n = 180(n-2) \quad \Longrightarrow \quad n = 12..$$

Hence, the number of sides of the polygon is 12.

Exercise 2.4

1. Find the sum of the first (i) 75 positive integers (ii) 125 natural numbers.
2. Find the sum of the first 30 terms of an A.P. whose n^{th} term is $3 + 2n$.
3. Find the sum of each arithmetic series

(i) $38 + 35 + 32 + \cdots + 2$. (ii) $6 + 5\frac{1}{4} + 4\frac{1}{2} + \cdots$ 25 terms.

4. Find the S_n for the following arithmetic series described.

(i) $a = 5, \quad n = 30, \quad l = 121$ (ii) $a = 50, \quad n = 25, \quad d = -4$

5. Find the sum of the first 40 terms of the series $1^2 - 2^2 + 3^2 - 4^2 + \cdots$.

6. In an arithmetic series, the sum of first 11 terms is 44 and that of the next 11 terms is 55. Find the arithmetic series.

7. In the arithmetic sequence $60, 56, 52, 48, \cdots$, starting from the first term, how many terms are needed so that their sum is 368?

8. Find the sum of all 3 digit natural numbers, which are divisible by 9.

9. Find the sum of first 20 terms of the arithmetic series in which 3rd term is 7 and 7th term is 2 more than three times its 3rd term.

10. Find the sum of all natural numbers between 300 and 500 which are divisible by 11.

11. Solve: $1 + 6 + 11 + 16 + \cdots + x = 148$.

12. Find the sum of all numbers between 100 and 200 which are not divisible by 5.

13. A construction company will be penalised each day for delay in construction of a bridge. The penalty will be ₹4000 for the first day and will increase by ₹1000 for each following day. Based on its budget, the company can afford to pay a maximum of ₹1,65,000 towards penalty. Find the maximum number of days by which the completion of work can be delayed

14. A sum of ₹1000 is deposited every year at 8% simple interest. Calculate the interest at the end of each year. Do these interest amounts form an A.P.? If so, find the total interest at the end of 30 years.

15. The sum of first n terms of a certain series is given as $3n^2 - 2n$. Show that the series is an arithmetic series.

16. If a clock strikes once at 1 o'clock, twice at 2 o'clock and so on, how many times will it strike in a day?

17. Show that the sum of an arithmetic series whose first term is a, second term b and the last term is c is equal to $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$.

18. If there are $(2n+1)$ terms in an arithmetic series, then prove that the ratio of the sum of odd terms to the sum of even terms is $(n+1):n$.

19. The ratio of the sums of first m and first n terms of an arithmetic series is $m^2:n^2$ show that the ratio of the m^{th} and n^{th} terms is $(2m-1):(2n-1)$

20. A gardener plans to construct a trapezoidal shaped structure in his garden. The longer side of trapezoid needs to start with a row of 97 bricks. Each row must be decreased by 2 bricks on each end and the construction should stop at 25th row. How many bricks does he need to buy?

2.5.2 Geometric series

A series is a geometric series if the terms of the series form a geometric sequence.

Let $a, ar, ar^2, \cdots, ar^{n-1}, ar^n, \cdots$ be a geometric sequence where $r \neq 0$ is the common ratio. We want to find the sum of the first n terms of this sequence.

Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ (1)

If $r = 1$, then from (1) it follows that $S_n = na$.

For $r \neq 1$, using (1) we have

$$rS_n = r(a + ar + ar^2 + \cdots + ar^{n-1}) = ar + ar^2 + ar^3 + \cdots + ar^n. \quad (2)$$

Now subtracting (2) from (1), we get

$$S_n - rS_n = (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + \cdots + ar^n)$$

$$\Longrightarrow \quad S_n(1 - r) = a(1 - r^n)$$

Hence, we have $S_n = \dfrac{a(1 - r^n)}{1 - r}$, since $r \neq 1$.

The sum of the first n terms of a geometric series is given by

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

where a is the first term and r is the common ratio.

Remarks

Actually, if $-1 < r < 1$, then the following formula holds:

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1 - r}.$$

Note that the sum of infinite number of positive numbers may give a finite value.

Example 2.22

Find the sum of the first 25 terms of the geometric series

$16 - 48 + 144 - 432 + \cdots$.

Solution Here, $a = 16$, $r = -\dfrac{48}{16} = -3 \neq 1$. Now, $S_n = \dfrac{a(1 - r^n)}{1 - r}$, $r \neq 1$.

So, we have $S_{25} = \dfrac{16(1 - (-3)^{25})}{1 - (-3)} = \dfrac{16(1 + 3^{25})}{4} = 4(1 + 3^{25})$.

Example 2.23

Find S_n for each of the geometric series described below:

(i) $a = 2,\ t_6 = 486,\ n = 6$ (ii) $a = 2400,\ r = -3,\ n = 5$

Solution

(i) Here $a = 2,\ t_6 = 486,\ n = 6$

Now $t_6 = 2(r)^5 = 486$

$\Longrightarrow \quad r^5 = 243 \quad \therefore \quad r = 3.$

Now, $S_n = \dfrac{a(r^n - 1)}{r - 1}$ if $r \neq 1$

Thus, $S_6 = \dfrac{2(3^6 - 1)}{3 - 1} = 3^6 - 1 = 728.$

(ii) Here $a = 2400,\ r = -3,\ n = 5$

Thus, $S_5 = \dfrac{a(r^5 - 1)}{r - 1}$ if $r \neq 1$

$= \dfrac{2400[(-3)^5 - 1]}{(-3) - 1}$

Hence, $S_5 = \dfrac{2400}{4}(1 + 3^5) = 600(1 + 243) = 146400.$

Example 2.24

In the geometric series $2 + 4 + 8 + \cdots$, starting from the first term how many consecutive terms are needed to yield the sum 1022?

Solution Given the geometric series is $2 + 4 + 8 + \cdots$.

Let n be the number of terms required to get the sum.

Here $a = 2,\ r = 2,\ S_n = 1022$.

To find n, let us consider

$$S_n = \frac{a[r^n - 1]}{r - 1} \text{ if } r \neq 1$$

$$= (2)\left[\frac{2^n - 1}{2 - 1}\right] = 2(2^n - 1).$$

But $S_n = 1022$ and hence $2(2^n - 1) = 1022$

$\Longrightarrow \quad 2^n - 1 = 511$

$\Longrightarrow \quad 2^n = 512 = 2^9.$ Thus, $n = 9$.

Example 2.25

The first term of a geometric series is 375 and the fourth term is 192. Find the common ratio and the sum of the first 14 terms.

Solution Let a be the first term and r be the common ratio of the given G.P.

Given that $a = 375,\ t_4 = 192$.

Now, $t_n = ar^{n-1}$

$\therefore\ t_4 = 375r^3 \implies 375r^3 = 192$

$r^3 = \frac{192}{375} \implies r^3 = \frac{64}{125}$

$r^3 = \left(\frac{4}{5}\right)^3 \implies r = \frac{4}{5}$, which is the required common ratio.

Now, $S_n = a\left[\frac{r^n - 1}{r - 1}\right]$ if $r \neq 1$

Thus, $S_{14} = \dfrac{375\left[\left(\frac{4}{5}\right)^{14} - 1\right]}{\frac{4}{5} - 1} = (-1)\times 5\times 375\left[\left(\frac{4}{5}\right)^{14} - 1\right]$

$= (375)(5)\left[1 - \left(\frac{4}{5}\right)^{14}\right] = 1875\left[1 - \left(\frac{4}{5}\right)^{14}\right]$.

Note

In the above example, one can use $S_n = a\left[\frac{1 - r^n}{1 - r}\right]$ if $r \neq 1$ instead of $S_n = a\left[\frac{r^n - 1}{r - 1}\right]$ if $r \neq 1$.

Example 2.26

A geometric series consists of four terms and has a positive common ratio. The sum of the first two terms is 8 and the sum of the last two terms is 72. Find the series.

Solution Let the sum of the four terms of the geometric series be $a + ar + ar^2 + ar^3$ and $r > 0$

Given that $a + ar = 8$ and $ar^2 + ar^3 = 72$

Now, $ar^2 + ar^3 = r^2(a + ar) = 72$

$\implies r^2(8) = 72 \quad \therefore\ r = \pm 3$

Since $r > 0$, we have $r = 3$.

Now, $a + ar = 8 \implies a = 2$

Thus, the geometric series is $2 + 6 + 18 + 54$.

Example 2.27

Find the sum to n terms of the series $6 + 66 + 666 + \cdots$

Solution Note that the given series is not a geometric series.

We need to find $S_n = 6 + 66 + 666 + \cdots$ to n terms

$S_n = 6(1 + 11 + 111 + \cdots$ to n terms$)$

$= \frac{6}{9}(9 + 99 + 999 + \cdots$ to n terms$)$ (Multiply and divide by 9)

$= \frac{2}{3}[(10 - 1) + (100 - 1) + (1000 - 1) + \cdots$ to n terms$]$

$= \frac{2}{3}[(10 + 10^2 + 10^3 + \cdots\ n$ terms$) - n]$

Thus, $S_n = \frac{2}{3}\left[\frac{10(10^n - 1)}{9} - n\right]$.

Example 2.28

An organisation plans to plant saplings in 25 streets in a town in such a way that one sapling for the first street, two for the second, four for the third, eight for the fourth street and so on. How many saplings are needed to complete the work?

Solution The number of saplings to be planted for each of the 25 streets in the town forms a G.P. Let S_n be the total number of saplings needed.

Then, $S_n = 1 + 2 + 4 + 8 + 16 + \cdots$ to 25 terms.

Here, $a = 1,\quad r = 2,\quad n = 25$

$$S_n = a\left[\frac{r^n - 1}{r - 1}\right]$$

$$S_{25} = (1)\,\frac{[2^{25} - 1]}{2 - 1}$$

$$= 2^{25} - 1$$

Thus, the number of saplings to be needed is $2^{25} - 1$.

Exercise 2.5

1. Find the sum of the first 20 terms of the geometric series $\frac{5}{2} + \frac{5}{6} + \frac{5}{18} + \cdots$.
2. Find the sum of the first 27 terms of the geometric series $\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots$.
3. Find S_n for each of the geometric series described below.

   (i) $a = 3,\ t_8 = 384,\ n = 8.$ (ii) $a = 5,\ r = 3,\ n = 12.$
4. Find the sum of the following finite series

   (i) $1 + 0.1 + 0.01 + 0.001 + \cdots + (0.1)^9$ (ii) $1 + 11 + 111 + \cdots$ to 20 terms.
5. How many consecutive terms starting from the first term of the series

   (i) $3 + 9 + 27 + \cdots$ would sum to 1092? (ii) $2 + 6 + 18 + \cdots$ would sum to 728?
6. The second term of a geometric series is 3 and the common ratio is $\frac{4}{5}$. Find the sum of first 23 consecutive terms in the given geometric series.
7. A geometric series consists of four terms and has a positive common ratio. The sum of the first two terms is 9 and sum of the last two terms is 36. Find the series.
8. Find the sum of first n terms of the series

   (i) $7 + 77 + 777 + \cdots$. (ii) $0.4 + 0.94 + 0.994 + \cdots$.
9. Suppose that five people are ill during the first week of an epidemic and each sick person spreads the contagious disease to four other people by the end of the second week and so on. By the end of 15^{th} week, how many people will be affected by the epidemic?

10. A gardener wanted to reward a boy for his good deeds by giving some mangoes. He gave the boy two choices. He could either have 1000 mangoes at once or he could get 1 mango on the first day, 2 on the second day, 4 on the third day, 8 mangoes on the fourth day and so on for ten days. Which option should the boy choose to get the maximum number of mangoes?

11. A geometric series consists of even number of terms. The sum of all terms is 3 times the sum of odd terms. Find the common ratio.

12. If S_1, S_2 and S_3 are the sum of first n, $2n$ and $3n$ terms of a geometric series respectively, then prove that $S_1(S_3 - S_2) = (S_2 - S_1)^2$.

Remarks

The sum of the first n terms of a geometric series with $a = 1$ and common ratio $x \neq 1$, is given by $1 + x + x^2 + \cdots + x^{n-1} = \dfrac{x^n - 1}{x - 1}, \quad x \neq 1$.

Note that the left hand side of the above equation is a special polynomial in x of degree $n - 1$. This formula will be useful in finding the sum of some series.

2.5.3 Special series $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ and $\sum_{k=1}^{n} k^3$

We have already used the symbol Σ for summation.

Let us list out some examples of finite series represented by sigma notation.

Sl. No.	Notation	Expansion
1.	$\sum_{k=1}^{n} k$ or $\sum_{j=1}^{n} j$	$1 + 2 + 3 + \cdots + n$
2.	$\sum_{n=2}^{6} (n-1)$	$1 + 2 + 3 + 4 + 5$
3.	$\sum_{d=0}^{5} (d+5)$	$5 + 6 + 7 + 8 + 9 + 10$
4.	$\sum_{k=1}^{n} k^2$	$1^2 + 2^2 + 3^3 + \cdots + n^2$
5.	$\sum_{k=1}^{10} 3 = 3\sum_{k=1}^{10} 1$	$3[1 + 1 + \cdots 10 \text{ terms}] = 30.$

We have derived that $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$. This can also be obtained using A.P. with $a = 1$, $d = 1$ and $l = n$ as $S_n = \frac{n}{2}(a + l) = \frac{n}{2}(1 + n)$.

Hence, using sigma notation we write it as $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Let us derive the formulae for

(i) $\sum_{k=1}^{n}(2k-1)$, (ii) $\sum_{k=1}^{n} k^2$ and (iii) $\sum_{k=1}^{n} k^3$.

Proof:

(i) Let us find $\sum_{k=1}^{n}(2k-1) = 1 + 3 + 5 + \cdots + (2n-1)$.

This is an A.P. consisting of n terms with $a = 1$, $d = 2$, $l = (2n-1)$.

$\therefore \quad S_n = \frac{n}{2}(1 + 2n - 1) = n^2 \qquad\qquad (S_n = \frac{n}{2}(a+l))$

Thus, $\sum_{k=1}^{n}(2k-1) = n^2$ (1)

Remarks

1. The formula (1) can also be obtained by the following method

$$\sum_{k=1}^{n}(2k-1) = \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 = 2\left(\sum_{k=1}^{n} k\right) - n = \frac{2(n)(n+1)}{2} - n = n^2.$$

2. From (1), $1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2$, since $l = 2n-1 \Longrightarrow n = \frac{l+1}{2}$.

(ii) We know that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

$\therefore \quad k^3 - (k-1)^3 = k^2 + k(k-1) + (k-1)^2$ (take $a = k$ and $b = k-1$)

$\Longrightarrow \quad k^3 - (k-1)^3 = 3k^2 - 3k + 1$ (2)

When $k = 1$, $1^3 - 0^3 = 3(1)^2 - 3(1) + 1$

When $k = 2$, $2^3 - 1^3 = 3(2)^2 - 3(2) + 1$

When $k = 3$, $3^3 - 2^3 = 3(3)^2 - 3(3) + 1$. Continuing this, we have

when $k = n$, $n^3 - (n-1)^3 = 3(n)^2 - 3(n) + 1$.

Adding the above equations corresponding to $k = 1, 2, \cdots, n$ column-wise, we obtain

$$n^3 = 3[1^2 + 2^2 + \cdots + n^2] - 3[1 + 2 + \cdots + n] + n$$

Thus, $3[1^2 + 2^2 + \cdots + n^2] = n^3 + 3[1 + 2 + \cdots + n] - n$

$$3\left[\sum_{k=1}^{n} k^2\right] = n^3 + \frac{3n(n+1)}{2} - n$$

Hence, $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (3)$$

(iii) $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots n^3$

Let us observe the following pattern.

$$1^3 = 1 = (1)^2$$
$$1^3 + 2^3 = 9 = (1+2)^2$$
$$1^3 + 2^3 + 3^3 = 36 = (1+2+3)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 100 = (1+2+3+4)^2.$$

Extending this pattern to n terms, we get

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = [1 + 2 + 3 + \cdots + n]^2$$
$$= \left[\frac{n(n+1)}{2}\right]^2$$

Thus, $$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2 = \left[\frac{n(n+1)}{2}\right]^2. \quad (4)$$

(i) The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

(ii) The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k-1) = n^2$.

(iii) The sum of first n odd natural numbers (when the last term l is given) is

$$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$

(iv) The sum of squares of first n natural numbers,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

(v) The sum of cubes of the first n natural numbers,

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Example 2.29

Find the sum of the following series

(i) $26 + 27 + 28 + \cdots + 60$ (ii) $1 + 3 + 5 + \cdots$ to 25 terms (iii) $31 + 33 + \cdots + 53$.

Solution

(i) We have $26 + 27 + 28 + \cdots + 60 = (1 + 2 + 3 + \cdots + 60) - (1 + 2 + 3 + \cdots + 25)$

$$= \sum_{1}^{60} n - \sum_{1}^{25} n$$
$$= \frac{60(60+1)}{2} - \frac{25(25+1)}{2}$$
$$= (30 \times 61) - (25 \times 13) = 1830 - 325 = 1505.$$

(ii) Here, $n = 25$

$\therefore$ $1 + 3 + 5 + \cdots$ to 25 terms $= 25^2$ $\quad\left(\sum_{k=1}^{n}(2k-1) = n^2\right)$

$= 625.$

(iii) $31 + 33 + \cdots + 53$

$= (1 + 3 + 5 + \cdots + 53) - (1 + 3 + 5 + \cdots + 29)$

$= \left(\frac{53+1}{2}\right)^2 - \left(\frac{29+1}{2}\right)^2$ $\quad\left(1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2\right)$

$= 27^2 - 15^2 = 504.$

Example 2.30

Find the sum of the following series

(i) $1^2 + 2^2 + 3^2 + \cdots + 25^2$ (ii) $12^2 + 13^2 + 14^2 + \cdots + 35^2$

(iii) $1^2 + 3^2 + 5^2 + \cdots + 51^2$.

Solution

(i) Now, $1^2 + 2^2 + 3^2 + \cdots + 25^2 = \sum_{1}^{25} n^2$

$= \frac{25(25+1)(50+1)}{6}$ $\quad\left(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{(25)(26)(51)}{6}$

$\therefore$ $1^2 + 2^2 + 3^2 + \cdots + 25^2 = 5525.$

(ii) Now, $12^2 + 13^2 + 14^2 + \cdots + 35^2$

$= (1^2 + 2^2 + 3^2 + \cdots + 35^2) - (1^2 + 2^2 + 3^2 + \cdots + 11^2)$

$= \sum_{1}^{35} n^2 - \sum_{1}^{11} n^2$

$= \frac{35(35+1)(70+1)}{6} - \frac{11(12)(23)}{6}$

$= \frac{(35)(36)(71)}{6} - \frac{(11)(12)(23)}{6}$

$= 14910 - 506 = 14404.$

(iii) Now, $1^2 + 3^2 + 5^2 + \cdots + 51^2$

$= (1^2 + 2^2 + 3^2 + \cdots + 51^2) - (2^2 + 4^2 + 6^2 + \cdots 50^2)$

$= \sum_{1}^{51} n^2 - 2^2[1^2 + 2^2 + 3^2 + \cdots + 25^2]$

$$= \sum_{1}^{51} n^2 - 4\sum_{1}^{25} n^2$$

$$= \frac{51(51+1)(102+1)}{6} - 4 \times \frac{25(25+1)(50+1)}{6}$$

$$= \frac{(51)(52)(103)}{6} - 4 \times \frac{25(26)(51)}{6}$$

$$= 45526 - 22100 = 23426.$$

Example 2.31

Find the sum of the series.

(i) $1^3 + 2^3 + 3^3 + \cdots + 20^3$ (ii) $11^3 + 12^3 + 13^3 + \cdots + 28^3$

Solution

(i) $1^3 + 2^3 + 3^3 + \cdots 20^3 = \sum_{1}^{20} n^3$

$$= \left(\frac{20(20+1)}{2}\right)^2 \qquad \text{using } \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

$$= \left(\frac{20 \times 21}{2}\right)^2 = (210)^2 = 44100.$$

(ii) Next we consider $11^3 + 12^3 + \cdots + 28^3$

$$= (1^3 + 2^3 + 3^3 + \cdots + 28^3) - (1^3 + 2^3 + \cdots + 10^3)$$

$$= \sum_{1}^{28} n^3 - \sum_{1}^{10} n^3$$

$$= \left[\frac{28(28+1)}{2}\right]^2 - \left[\frac{10(10+1)}{2}\right]^2$$

$$= 406^2 - 55^2 = (406+55)(406-55)$$

$$= (461)(351) = 161811.$$

Example 2.32

Find the value of k, if $1^3 + 2^3 + 3^3 + \cdots + k^3 = 4356$

Solution Note that k is a positive integer.

Given that $1^3 + 2^3 + 3^3 + \cdots + k^3 = 4356$

$$\Longrightarrow \left(\frac{k(k+1)}{2}\right)^2 = 4356 = 6 \times 6 \times 11 \times 11$$

Taking square root, we get $\frac{k(k+1)}{2} = 66$

$$\Longrightarrow k^2 + k - 132 = 0 \Longrightarrow (k+12)(k-11) = 0$$

Thus, $k = 11$, since k is positive.

Example 2.33

(i) If $1 + 2 + 3 + \cdots + n = 120$, find $1^3 + 2^3 + 3^3 + \cdots n^3$.

(ii) If $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$, then find $1 + 2 + 3 + \cdots + n$.

Solution

(i) Given $1 + 2 + 3 + \cdots + n = 120$ i.e. $\frac{n(n+1)}{2} = 120$

$$\therefore \quad 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 = 120^2 = 14400$$

(ii) Given $1^3 + 2^3 + 3^3 + \cdots + n^3 = 36100$

$$\Longrightarrow \quad \left(\frac{n(n+1)}{2}\right)^2 = 36100 = 19 \times 19 \times 10 \times 10$$

$$\Longrightarrow \quad \frac{n(n+1)}{2} = 190$$

Thus, $1 + 2 + 3 + \cdots + n = 190$.

Example 2.34

Find the total area of 14 squares whose sides are 11 cm, 12 cm, $\cdots$, 24 cm, respectively.

Solution The areas of the squares form the series $11^2 + 12^2 + \cdots + 24^2$

$$\begin{aligned}
\text{Total area of 14 squares} &= 11^2 + 12^2 + 13^2 + \cdots + 24^2 \\
&= (1^2 + 2^2 + 3^2 + \cdots + 24^2) - (1^2 + 2^2 + 3^2 + \cdots + 10^2) \\
&= \sum_1^{24} n^2 - \sum_1^{10} n^2 \\
&= \frac{24(24+1)(48+1)}{6} - \frac{10(10+1)(20+1)}{6} \\
&= \frac{(24)(25)(49)}{6} - \frac{(10)(11)(21)}{6} \\
&= 4900 - 385 \\
&= 4515 \text{ sq. cm.}
\end{aligned}$$

Exercise 2.6

1. Find the sum of the following series.

(i) $1 + 2 + 3 + \cdots + 45$ (ii) $16^2 + 17^2 + 18^2 + \cdots + 25^2$

(iii) $2 + 4 + 6 + \cdots + 100$ (iv) $7 + 14 + 21 \cdots + 490$

(v) $5^2 + 7^2 + 9^2 + \cdots + 39^2$ (vi) $16^3 + 17^3 + \cdots + 35^3$

2. Find the value of k if

(i) $1^3 + 2^3 + 3^3 + \cdots + k^3 = 6084$ (ii) $1^3 + 2^3 + 3^3 + \cdots + k^3 = 2025$

3. If $1 + 2 + 3 + \cdots + p = 171$, then find $1^3 + 2^3 + 3^3 + \cdots + p^3$.

4. If $1^3 + 2^3 + 3^3 + \cdots + k^3 = 8281$, then find $1 + 2 + 3 + \cdots + k$.

5. Find the total area of 12 squares whose sides are 12cm, 13cm, $\cdots$, 23cm. respectively.

6. Find the total volume of 15 cubes whose edges are 16cm, 17cm, 18cm, $\cdots$, 30cm respectively.

Exercise 2.7

Choose the correct answer.

1. Which one of the following is not true?

(A) A sequence is a real valued function defined on $\mathbb{N}$.

(B) Every function represents a sequence.

(C) A sequence may have infinitely many terms.

(D) A sequence may have a finite number of terms.

2. The 8^{th} term of the sequence 1, 1, 2, 3, 5, 8, $\cdots$ is

(A) 25 (B) 24 (C) 23 (D) 21

3. The next term of $\frac{1}{20}$ in the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \cdots$ is

(A) $\frac{1}{24}$ (B) $\frac{1}{22}$ (C) $\frac{1}{30}$ (D) $\frac{1}{18}$

4. If a, b, c, l, m are in A.P, then the value of $a - 4b + 6c - 4l + m$ is

(A) 1 (B) 2 (C) 3 (D) 0

5. If a, b, c are in A.P. then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{c}$ (C) $\frac{a}{c}$ (D) 1

6. If the n^{th} term of a sequence is $100n+10$, then the sequence is

(A) an A.P. (B) a G.P. (C) a constant sequence (D) neither A.P. nor G.P.

7. If $a_1, a_2, a_3, \cdots$ are in A.P. such that $\frac{a_4}{a_7} = \frac{3}{2}$, then the 13^{th} term of the A.P. is

(A) $\frac{3}{2}$ (B) 0 (C) $12a_1$ (D) $14a_1$

8. If the sequence $a_1, a_2, a_3, \cdots$ is in A.P. , then the sequence $a_5, a_{10}, a_{15}, \cdots$ is

(A) a G.P. (B) an A.P. (C) neither A.P nor G.P. (D) a constant sequence

9. If $k+2$, $4k-6$, $3k-2$ are the three consecutive terms of an A.P, then the value of k is

(A) 2 (B) 3 (C) 4 (D) 5

10. If $a, b, c, l, m. n$ are in A.P., then $3a+7, 3b+7, 3c+7, 3l+7, 3m+7, 3n+7$ form

(A) a G.P. (B) an A.P. (C) a constant sequence (D) neither A.P. nor G.P

11. If the third term of a G.P is 2, then the product of first 5 terms is

(A) 5^2 (B) 2^5 (C) 10 (D) 15

12. If a, b, c are in G.P, then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{a}{c}$ (D) $\frac{c}{b}$

13. If $x,\ 2x+2,\ 3x+3$ are in G.P, then $5x,\ 10x+10,\ 15x+15$ form

(A) an A.P. (B) a G.P. (C) a constant sequence (D) neither A.P. nor a G.P.

14. The sequence –3, –3, –3,$\cdots$ is

(A) an A.P. only (B) a G.P. only (C) neither A.P. nor G.P (D) both A.P. and G.P.

15. If the product of the first four consecutive terms of a G.P is 256 and if the common ratio is 4 and the first term is positive, then its 3rd term is

(A) 8 (B) $\frac{1}{16}$ (C) $\frac{1}{32}$ (D) 16

16. In a G.P, $t_2 = \frac{3}{5}$ and $t_3 = \frac{1}{5}$. Then the common ratio is

(A) $\frac{1}{5}$ (B) $\frac{1}{3}$ (C) 1 (D) 5

17. If $x \neq 0$, then $1+\sec x+\sec^2 x+\sec^3 x+\sec^4 x+\sec^5 x$ is equal to

(A) $(1+\sec x)(\sec^2 x+\sec^3 x+\sec^4 x)$ (B) $(1+\sec x)(1+\sec^2 x+\sec^4 x)$

(C) $(1-\sec x)(\sec x+\sec^3 x+\sec^5 x)$ (D) $(1+\sec x)(1+\sec^3 x+\sec^4 x)$

18. If the n^{th} term of an A.P. is $t_n = 3-5n$, then the sum of the first n terms is

(A) $\frac{n}{2}[1-5n]$ (B) $n(1-5n)$ (C) $\frac{n}{2}(1+5n)$ (D) $\frac{n}{2}(1+n)$

19. The common ratio of the G.P. a^{m-n}, a^m, a^{m+n} is

(A) a^m (B) a^{-m} (C) a^n (D) a^{-n}

20. If $1+2+3+\ldots+n=k$ then $1^3+2^3+\cdots+n^3$ is equal to

(A) k^2 (B) k^3 (C) $\frac{k(k+1)}{2}$ (D) $(k+1)^3$

Points to Remember

- A sequence of real numbers is an arrangement or a list of real numbers in a specific order.
- The sequence given by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, \cdots$ is called the Fibonacci sequence which is nothing but $1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots$
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called an arithmetic sequence if $a_{n+1} = a_n + d$, $n \in \mathbb{N}$ where d is a constant. Here a_1 is called the first term and the constant d is called the common difference.

  The formula for the general term of an A.P. is $t_n = a + (n-1)d \quad \forall\ n \in \mathbb{N}$.
- A sequence $a_1, a_2, a_3, \cdots, a_n, \cdots$ is called a geometric sequence if $a_{n+1} = a_n r$, where $r \neq 0$, $n \in \mathbb{N}$ where r is a constant. Here, a_1 is the first term and the constant r is called the common ratio. The formula for the general term of a G.P. is $t_n = ar^{n-1}$, $n = 1, 2, 3, \cdots$.
- An expression of addition of terms of a sequence is called a series. If the sum consists only finite number of terms, then it is called a finite series. If the sum consists of infinite number of terms of a sequence, then it is called an infinite series.
- The sum S_n of the first n terms of an arithmetic sequence with first term a and common difference d is given by $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}(a + l)$, where l is the last term.
- The sum of the first n terms of a geometric series is given by

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} = \dfrac{a(1 - r^n)}{1 - r}, & \text{if } r \neq 1 \\ na & \text{if } r = 1. \end{cases}$$

  where a is the first term and r is the common ratio.
- The sum of the first n natural numbers, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.
- The sum of the first n odd natural numbers, $\sum_{k=1}^{n} (2k - 1) = n^2$
- The sum of first n odd natural numbers (when the last term l is given) is

$$1 + 3 + 5 + \cdots + l = \left(\frac{l+1}{2}\right)^2.$$

- The sum of squares of first n natural numbers, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
- The sum of cubes of the first n natural numbers, $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Do you know?

A Mersenne number, named after Marin Mersenne, is a positive integer of the form $M = 2^p - 1$, where p is a positive integer. If M is a prime, then it is called a Mersenne prime.Interestingly, if $2^p - 1$ is prime, then p is prime.The largest known prime number $2^{43,112,609} - 1$ is a Mersenne prime.