3

ALGEBRA

The human mind has never invented a labour-saving machine equal to algebra - Author unknown

Al-Khwarizmi

(780-850)

Arab

Al-Khwarizmi's contribution to Mathematics and Geography established the basis for innovation in Algebra and Trigonometry. He presented the first systematic solution of linear and quadratic equations.

He is considered the founder of algebra. His work on arithmetic was responsible for introducing the Arabic numerals based on the Hindu-Arabic numeral system developed in Indian Mathematics, to the Western world.

3.1 Introduction

Algebra is an important and a very old branch of mathematics which deals with solving algebraic equations. In third century, the Greek mathematician Diophantus wrote a book "Arithmetic" which contained a large number of practical problems. In the sixth and seventh centuries, Indian mathematicians like Aryabhatta and Brahmagupta have worked on linear equations and quadratic equations and developed general methods of solving them.

The next major development in algebra took place in ninth century by Arab mathematicians. In particular, Al-Khwarizmi's book entitled "Compendium on calculation by completion and balancing" was an important milestone. There he used the word aljabra - which was latinized into algebra - translates as competition or restoration. In the 13th century, Leonardo Fibonacci's books on algebra was important and influential. Other highly influential works on algebra were those of the Italian mathematician Luca Pacioli (1445-1517), and of the English mathematician Robert Recorde (1510-1558).

In later centuries Algebra blossomed into more abstract and in 19th century British mathematicians took the lead in this effort. Peacock (Britain, 1791-1858) was the founder of axiomatic thinking in arithmetic and algebra. For this reason he is sometimes called the "Euclid of Algebra". DeMorgan (Britain, 1806-1871) extended Peacock's work to consider operations defined on abstract symbols.

In this chapter, we shall focus on learning techniques of solving linear system of equations and quadratic equations.

3.2 System of linear equations in two unknowns

In class IX, we have studied the linear equation $ax + b = 0$, $a \neq 0$, in one unknown x.

Let us consider a linear equation $ax + by = c$, where at least one of a and b is non-zero, in two unknowns x and y. An ordered pair (x_0, y_0) is called a solution to the linear equation if the values $x = x_0$, $y = y_0$ satisfy the equation.

Geometrically, the graph of the linear equation $ax + by = c$ is a straight line in a plane. So each point (x, y) on this line corresponds to a solution of the equation $ax + by = c$. Conversely, every solution (x, y) of the equation is a point on this straight line. Thus, the equation $ax + by = c$ has infinitely many solutions.

A set of finite number of linear equations in two unknowns x and y that are to be treated together, is called a system of linear equations in x and y. Such a system of equations is also called simultaneous equations.

Definition

An ordered pair (x_0, y_0) is called a solution to a linear system in two variables if the values $x = x_0$, $y = y_0$ satisfy all the equations in the system.

A system of linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

in two variables is said to be

(i) consistent if at least one pair of values of x and y satisfies both equations and

(ii) inconsistent if there are no values of x and y that satisfy both equations.

In this section, we shall discuss only a pair of linear equations in two variables.

Remarks

(i) An equation of the form $ax + by = c$ is called linear because the variables are only to the first power, and there are no products of variables in the equation.

(ii) It is also possible to consider linear systems in more than two variables. You will learn this in higher classes.

Let us consider a linear system

$$a_1x + b_1y = c_1 \qquad (1)$$
$$a_2x + b_2y = c_2 \qquad (2)$$

in two variables x and y, where any of the constants a_1, b_1, a_2 and b_2 can be zero with the exception that each equation must have at least one variable in it or simply,

$$a_1^2 + b_1^2 \neq 0 \ , \ a_2^2 + b_2^2 \neq 0.$$

Geometrically the following situations occur. The two straight lines represented by (1) and (2)

(i) may intersect at exactly one point

(ii) may not intersect at any point

(iii) may coincide.

If (i) happens, then the intersecting point gives the unique solution of the system. If (ii) happens, then the system does not have a solution. If (iii) happens, then every point on the line corresponds to a solution to the system. Thus, the system will have infinitely many solutions in this case.

Now, we will solve a system of linear equations in two unknowns using the following algebraic methods (i) the method of elimination (ii) the method of cross multiplication.

3.2.1 Elimination method

In this method, we may combine equations of a system in such a manner as to get rid of one of the unknowns. The elimination of one unknown can be achieved in the following ways.

(i) Multiply or divide the members of the equations by such numbers as to make the coefficients of the unknown to be eliminated numerically equal.

(ii) Then, eliminate by addition if the resulting coefficients have unlike signs and by subtraction if they have like signs.

Example 3.1

Solve $3x - 5y = -16$, $2x + 5y = 31$

Solution The given equations are

$$3x - 5y = -16 \quad (1)$$

$$2x + 5y = 31 \quad (2)$$

Note that the coefficients of y in both equations are numerically equal.

So, we can eliminate y easily.

Adding (1) and (2), we obtain an equation

$$5x = 15 \quad (3)$$

That is, $x = 3$.

Now, we substitute $x = 3$ in (1) or (2) to solve for y.

Substituting $x = 3$ in (1) we obtain, $3(3) - 5y = -16$

$$\implies y = 5.$$

Now, (3, 5) a is solution to the given system because (1) and (2) are true when $x = 3$ and $y = 5$ as from (1) and (2) we get, $3(3) - 5(5) = -16$ and $2(3) + 5(5) = 31$.

Note

Obtaining equation (3) in only one variable is an important step in finding the solution. We obtained equation (3) in one variable x by eliminating the variable y. So this method of solving a system by eliminating one of the variables first, is called "method of elimination".

Example 3.2

The cost of 11 pencils and 3 erasers is ₹ 50 and the cost of 8 pencils and 3 erasers is ₹ 38. Find the cost of each pencil and each eraser.

Solution Let x denote the cost of a pencil in rupees and y denote the cost of an eraser in rupees.

Then according to the given information we have

$$11x + 3y = 50 \qquad (1)$$

$$8x + 3y = 38 \qquad (2)$$

Subtracting (2) from (1) we get, $3x = 12$ which gives $x = 4$.

Now substitute $x = 4$ in (1) to find the value of y. We get,

$$11(4) + 3y = 50 \qquad \text{i.e.,} \quad y = 2.$$

Therefore, $x = 4$ and $y = 2$ is the solution of the given pair of equations.

Thus, the cost of a pencil is ₹ 4 and that of an eraser is ₹ 2.

Note

It is always better to check that the obtained values satisfy the both equations.

Example 3.3

Solve by elimination method $3x + 4y = -25$, $2x - 3y = 6$

Solution The given system is

$$3x + 4y = -25 \qquad (1)$$

$$2x - 3y = 6 \qquad (2)$$

To eliminate the variable x, let us multiply (1) by 2 and (2) by –3 to obtain

$$(1) \times 2 \implies 6x + 8y = -50 \qquad (3)$$

$$(2) \times -3 \implies -6x + 9y = -18 \qquad (4)$$

Now, adding (3) and (4) we get, $17y = -68$ which gives $y = -4$

Next, substitute $y = -4$ in (1) to obtain

$$3x + 4(-4) = -25$$

That is, $x = -3$

Hence, the solution is $(-3, -4)$.

Remarks

In Example 3.3, it is not possible to eliminate one of the variables by simply adding or subtracting the given equations as we did in Example 3.1. Thus, first we shall do some manipulations so that coefficients of either x or y are equal except for sign. Then we do the elimination.

Example 3.4

Using elimination method, solve $101x + 99y = 499, \quad 99x + 101y = 501$

Solution The given system of equations is

$$101x + 99y = 499 \qquad (1)$$

$$99x + 101y = 501 \qquad (2)$$

Here, of course we could multiply equations by appropriate numbers to eliminate one of the variables.

However, note that the coefficient of x in one equation is equal to the coefficient of y in the other equation. In such a case, we add and subtract the two equations to get a new system of very simple equations having the same solution.

Adding (1) and (2), we get $200x + 200y = 1000$.

Dividing by 200 we get, $x + y = 5 \qquad (3)$

Subtracting (2) from (1), we get $2x - 2y = -2$ which is same as

$$x - y = -1 \qquad (4)$$

Solving (3) and (4), we get $x = 2, \; y = 3$.

Thus, the required solution is (2, 3).

Example 3.5

Solve $3(2x + y) = 7xy; \quad 3(x + 3y) = 11xy$ using elimination method

Solution The given system of equations is

$$3(2x + y) = 7xy \qquad (1)$$

$$3(x + 3y) = 11xy \qquad (2)$$

Observe that the given system is not linear because of the occurrence of xy term.

Also, note that if $x = 0$, then $y = 0$ and vice versa. So, (0, 0) is a solution for the system and any other solution would have both $x \neq 0$ and $y \neq 0$.

Thus, we consider the case where $x \neq 0, \; y \neq 0$.

Dividing both sides of each equation by xy, we get

$$\frac{6}{y} + \frac{3}{x} = 7, \quad \text{i.e.,} \quad \frac{3}{x} + \frac{6}{y} = 7 \qquad (3)$$

and

$$\frac{9}{x} + \frac{3}{y} = 11 \qquad (4)$$

Let $a = \frac{1}{x}$ and $b = \frac{1}{y}$.

Equations (3) and (4) become

$$3a + 6b = 7 \qquad (5)$$

$$9a + 3b = 11 \qquad (6)$$

which is a linear system in a and b.

To eliminate b, we have (6) $\times 2 \Longrightarrow \quad 18a + 6b = 22$ (7)

Subtracting (7) from (5) we get, $\; -15a = -15$. That is, $a = 1$.

Substituting $a = 1$ in (5) we get, $\quad b = \frac{2}{3}$. Thus, $a = 1$ and $b = \frac{2}{3}$.

When $a = 1$, we have $\frac{1}{x} = 1$. $\quad$ Thus, $x = 1$.

When $b = \frac{2}{3}$, we have $\frac{1}{y} = \frac{2}{3}$. $\quad$ Thus, $y = \frac{3}{2}$.

Thus, the system has two solutions $(1, \frac{3}{2})$ and $(0, 0)$.

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The given system of equations can also be solved in the following way.

Now, $\quad 3(2x + y) = 7xy \quad$ (1)

$3(x + 3y) = 11xy \quad$ (2)

Now, (2) × 2 – (1) $\Longrightarrow \quad 15y = 15xy$

$\Longrightarrow \quad 15y(1-x) = 0$. Thus, $x = 1$ and $y = 0$

When $x = 1$, we have $y = \frac{3}{2}$ and when $y = 0$, we have $x = 0$

Hence, the two solutions are $(1, \frac{3}{2})$ and $(0, 0)$.

Note : In $15y = 15xy$, y is not to be cancelled out as $y = 0$ gives another solution.

Exercise 3.1

Solve each of the following system of equations by elimination method.

1. $x + 2y = 7, \quad x - 2y = 1$
2. $3x + y = 8, \quad 5x + y = 10$
3. $x + \frac{y}{2} = 4, \quad \frac{x}{3} + 2y = 5$
4. $11x - 7y = xy, \quad 9x - 4y = 6xy$
5. $\frac{3}{x} + \frac{5}{y} = \frac{20}{xy}, \frac{2}{x} + \frac{5}{y} = \frac{15}{xy}, x \neq 0, y \neq 0$
6. $8x - 3y = 5xy, \quad 6x - 5y = -2xy$
7. $13x + 11y = 70, \quad 11x + 13y = 74$
8. $65x - 33y = 97, \quad 33x - 65y = 1$
9. $\frac{15}{x} + \frac{2}{y} = 17, \frac{1}{x} + \frac{1}{y} = \frac{36}{5}, x \neq 0, y \neq 0$
10. $\frac{2}{x} + \frac{2}{3y} = \frac{1}{6}, \frac{3}{x} + \frac{2}{y} = 0, x \neq 0, y \neq 0$

Cardinality of the set of solutions of the system of linear equations

Let us consider the system of two equations

$$a_1x + b_1y + c_1 = 0 \quad (1)$$

$$a_2x + b_2y + c_2 = 0 \quad (2)$$

where the coefficients are real numbers such that $a_1^2 + b_1^2 \neq 0$, $a_2^2 + b_2^2 \neq 0$.

Let us apply the elimination method for equating the coefficients of y.

Now, multiply equation (1) by b_2 and equation (2) by b_1, we get,

$$b_2 a_1 x + b_2 b_1 y + b_2 c_1 = 0 \qquad (3)$$

$$b_1 a_2 x + b_1 b_2 y + b_1 c_2 = 0 \qquad (4)$$

Subtracting equation (4) from (3), we get

$$(b_2 a_1 - b_1 a_2)x = b_1 c_2 - b_2 c_1 \implies x = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} \quad \text{provided } a_1 b_2 - a_2 b_1 \neq 0$$

Substituting the value of x in either (1) or (2) and solving for y, we get

$$y = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}, \quad \text{provided } a_1 b_2 - a_2 b_1 \neq 0.$$

Thus, we have

$$x = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1} \text{ and } y = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1},\ a_1 b_2 - a_2 b_1 \neq 0. \qquad (5)$$

Here, we have to consider two cases.

***Case* (i)** $a_1 b_2 - a_2 b_1 \neq 0$. That is, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

In this case, the pair of linear equations has a unique solution.

***Case* (ii)** $a_1 b_2 - a_2 b_1 = 0$. That is, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ if $a_2 \neq 0$ and $b_2 \neq 0$.

In this case, let $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \lambda$. Then $a_1 = \lambda a_2$, $b_1 = \lambda b_2$

Now, substituting the values of a_1 and b_1 in equation (1) we get,

$$\lambda(a_2 x + b_2 y) + c_1 = 0 \qquad (6)$$

It is easily observed that both the equations (6) and (2) can be satisfied only if

$$c_1 = \lambda c_2 \implies \frac{c_1}{c_2} = \lambda$$

If $c_1 = \lambda c_2$, any solution of equation (2) will also satisfy the equation (1) and vice versa.

So, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \lambda$; then there are infinitely many solutions to the pair of linear equations given by (1) and (2).

If $c_1 \neq \lambda c_2$, then any solution of equation (1) will not satisfy equation (2) and vice versa.

Hence, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the pair of linear equations given by (1) and (2) has no solution.

Note

Now, we summarise the above discussion.

For the system of equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0, \text{ where } a_1^2 + b_1^2 \neq 0 \text{ , } a_2^2 + b_2^2 \neq 0.$$

(i) If $a_1b_2 - b_1a_2 \neq 0$ or $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then the system of equations has a unique solution.

(ii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then the system of equations has infinitely many solutions.

(iii) If $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then the system of equations has no solution.

3.2.2 Cross multiplication method

While solving a pair of linear equations in two unknowns x and y using elimination method, we utilised the coefficients effectively to get the solution. There is another method called the cross multiplication method, which simplifies the procedure. Now, let us describe this method and see how it works.

Let us consider the system

$$a_1x + b_1y + c_1 = 0 \qquad (1)$$

$$a_2x + b_2y + c_2 = 0 \text{ with } a_1b_2 - b_1a_2 \neq 0 \qquad (2)$$

We have already established that the system has the solution

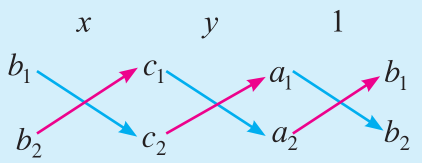
$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Thus, we can write $\frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$, $\frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$

Let us write the above in the following form

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

The following arrow diagram may be very useful in remembering the above relation.



The arrows between the two numbers indicate that they are multiplied, the second product (upward arrow) is to be subtracted from the first product (downward arrow).

Method of solving a linear system of equations by the above form is called the cross multiplication method.

Note that in the representation $\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$,

$b_1c_2 - b_2c_1$ or $c_1a_2 - c_2a_1$ may be equal to 0 but $a_1b_2 - a_2b_1 \neq 0$.

> Hence, for the system of equations $a_1x + b_1y + c_1 = 0$
> $a_2x + b_2y + c_2 = 0$
>
> (i) if $b_1c_2 - b_2c_1 = 0$ and $a_1b_2 - a_2b_1 \neq 0$, then $x = 0$
>
> (ii) if $c_1a_2 - c_2a_1 = 0$ and $a_1b_2 - a_2b_1 \neq 0$, then $y = 0$

Hereafter, we shall mostly restrict ourselves to the system of linear equations having unique solution and find the solution by the method of cross multiplication.

Example 3.6

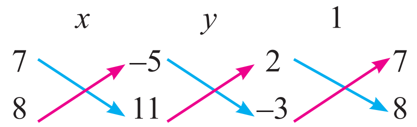
Solve

$$2x + 7y - 5 = 0$$
$$-3x + 8y = -11$$

Solution The given system of equations is

$$2x + 7y - 5 = 0$$
$$-3x + 8y + 11 = 0$$

For the cross multiplication method, we write the coefficients as



Hence, we get $\frac{x}{(7)(11)-(8)(-5)} = \frac{y}{(-5)(-3)-(2)(11)} = \frac{1}{(2)(8)-(-3)(7)}$.

That is, $\frac{x}{117} = \frac{y}{-7} = \frac{1}{37}$. i.e., $x = \frac{117}{37}$, $y = -\frac{7}{37}$.

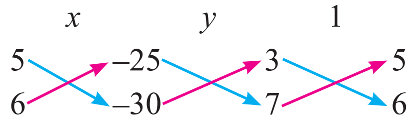
Hence, the solution is $\left(\frac{117}{37}, -\frac{7}{37}\right)$.

Example 3.7

Using cross multiplication method, solve $3x + 5y = 25$
$7x + 6y = 30$

Solution The given system of equations is $3x + 5y - 25 = 0$
$7x + 6y - 30 = 0$

Now, writing the coefficients for cross multiplication, we get



$$\Longrightarrow \quad \frac{x}{-150+150} = \frac{y}{-175+90} = \frac{1}{18-35}. \text{ i.e., } \frac{x}{0} = \frac{y}{-85} = \frac{1}{-17}.$$

Thus, we have $x = 0$, $y = 5$. Hence, the solution is (0, 5).

Note

Here, $\frac{x}{0} = -\frac{1}{17}$ is to mean $x = \frac{0}{-17} = 0$. Thus $\frac{x}{0}$ is only a notation and it is not division by zero. It is always true that division by zero is not defined.

Example 3.8

In a two digit number, the digit in the unit place is twice of the digit in the tenth place. If the digits are reversed, the new number is 27 more than the given number. Find the number.

Solution Let x denote the digit in the tenth place and y denote the digit in unit place.. So, the number may be written as $10x + y$ in the expanded form. (just like 35= 10(3) +5)

When the digits are reversed, x becomes the digit in unit place and y becomes the digit in the tenth place. The changed number, in the expanded form is $10y + x$.

According to the first condition, we have $y = 2x$ which is written as

$$2x - y = 0 \qquad (1)$$

Also, by second condition, we have

$$(10y + x) - (10x + y) = 27$$

That is, $-9x + 9y = 27 \Longrightarrow -x + y = 3 \qquad (2)$

Adding equations (1) and (2), we get $x = 3$.

Substituting $x = 3$ in the equation (2), we get $y = 6$.

Thus, the given number is $(3 \times 10) + 6 = 36$.

Example 3.9

A fraction is such that if the numerator is multiplied by 3 and the denominator is reduced by 3, we get $\frac{18}{11}$, but if the numerator is increased by 8 and the denominator is doubled, we get $\frac{2}{5}$. Find the fraction.

Solution Let the fraction be $\frac{x}{y}$. According to the given conditions, we have

$$\frac{3x}{y-3} = \frac{18}{11} \quad \text{and} \quad \frac{x+8}{2y} = \frac{2}{5}$$

$$\Longrightarrow \quad 11x = 6y - 18 \quad \text{and} \quad 5x + 40 = 4y$$

So, we have $\quad 11x - 6y + 18 = 0 \quad (1)$

$5x - 4y + 40 = 0 \quad (2)$

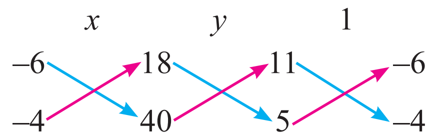
On comparing the coefficients of (1) and (2) with $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$,

we have $\quad a_1 = 11,\ b_1 = -6,\ c_1 = 18\ ;\quad a_2 = 5,\quad b_2 = -4,\quad c_2 = 40.$

Thus, $\quad a_1b_2 - a_2b_1 = (11)(-4) - (5)(-6) = -14 \neq 0.$

Hence, the system has a unique solution.

Now, writing the coefficients for the cross multiplication, we have



$$\Longrightarrow \quad \frac{x}{-240+72} = \frac{y}{90-440} = \frac{1}{-44+30}$$

$$\Longrightarrow \quad \frac{x}{-168} = \frac{y}{-350} = \frac{1}{-14}$$

Thus, $\quad x = \frac{168}{14} = 12\ ;\quad y = \frac{350}{14} = 25.$ Hence, the fraction is $\frac{12}{25}$.

Example 3.10

Eight men and twelve boys can finish a piece of work in 10 days while six men and eight boys can finish the same work in 14 days. Find the number of days taken by one man alone to complete the work and also one boy alone to complete the work.

Solution Let x denote the number of days needed for one man to finish the work and y denote the number of days needed for one boy to finish the work. Clearly, $x \neq 0$ and $y \neq 0$.

So, one man can complete $\frac{1}{x}$ part of the work in one day and one boy can complete $\frac{1}{y}$ part of the work in one day.

The amount of work done by 8 men and 12 boys in one day is $\frac{1}{10}$.

Thus, we have $\quad \frac{8}{x} + \frac{12}{y} = \frac{1}{10} \quad (1)$

The amount of work done by 6 men and 8 boys in one day is $\frac{1}{14}$.

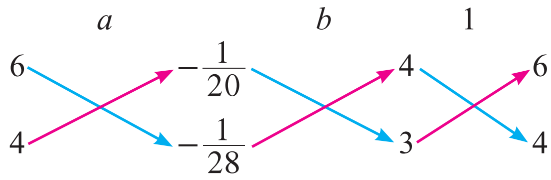
Thus, we have $\quad \frac{6}{x} + \frac{8}{y} = \frac{1}{14} \quad (2)$

Let $\quad a = \frac{1}{x}$ and $b = \frac{1}{y}$. Then (1) and (2) give, respectively,

$$8a + 12b = \frac{1}{10} \quad \Longrightarrow \quad 4a + 6b - \frac{1}{20} = 0. \quad (3)$$

$$6a + 8b = \frac{1}{14} \quad \Longrightarrow \quad 3a + 4b - \frac{1}{28} = 0. \quad (4)$$

Writing the coefficients of (3) and (4) for the cross multiplication, we have



Thus, we have $\dfrac{a}{-\frac{3}{14}+\frac{1}{5}} = \dfrac{b}{-\frac{3}{20}+\frac{1}{7}} = \dfrac{1}{16-18}$. i.e., $\dfrac{a}{-\frac{1}{70}} = \dfrac{b}{-\frac{1}{140}} = \dfrac{1}{-2}$.

That is, $a = \dfrac{1}{140}$, $b = \dfrac{1}{280}$

Thus, we have $x = \dfrac{1}{a} = 140$, $y = \dfrac{1}{b} = 280$.

Hence, one man can finish the work individually in 140 days and one boy can finish the work individually in 280 days.

Exercise 3.2

1. Solve the following systems of equations using cross multiplication method.

   (i) $3x + 4y = 24$, $20x - 11y = 47$ (ii) $0.5x + 0.8y = 0.44$, $0.8x + 0.6y = 0.5$

   (iii) $\dfrac{3x}{2} - \dfrac{5y}{3} = -2$, $\dfrac{x}{3} + \dfrac{y}{2} = \dfrac{13}{6}$ (iv) $\dfrac{5}{x} - \dfrac{4}{y} = -2$, $\dfrac{2}{x} + \dfrac{3}{y} = 13$

2. Formulate the following problems as a pair of equations, and hence find their solutions:

   (i) One number is greater than thrice the other number by 2. If 4 times the smaller number exceeds the greater by 5, find the numbers.

   (ii) The ratio of income of two persons is 9 : 7 and the ratio of their expenditure is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly income.

   (iii) A two digit number is seven times the sum of its digits. The number formed by reversing the digits is 18 less than the given number. Find the given number.

   (iv) Three chairs and two tables cost ₹ 700 and five chairs and three tables cost ₹ 1100. What is the total cost of 2 chairs and 3 tables?

   (v) In a rectangle, if the length is increased and the breadth is reduced each by 2 cm then the area is reduced by 28 cm^2. If the length is reduced by 1 cm and the breadth increased by 2cm , then the area increases by 33 cm^2. Find the area of the rectangle.

   (vi) A train travelled a certain distance at a uniform speed. If the train had been 6 km/hr faster, it would have taken 4 hours less than the scheduled time. If the train were slower by 6 km/hr, then it would have taken 6 hours more than the scheduled time. Find the distance covered by the train.

3.3 Quadratic polynomials

A polynomial of degree n in the variable x is $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$ where $a_0 \neq 0$ and $a_1, a_2, a_3, ..., a_n$ are real constants.

A polynomial of degree two is called a quadratic polynomial and is normally written as $p(x) = ax^2 + bx + c$, where $a \neq 0$, b and c are real constants. Real constants are polynomials of degree zero.

For example, $x^2 + x + 1$, $3x^2 - 1$, $-\frac{3}{2}x^2 + 2x - \frac{7}{3}$ are quadratic polynomials.

The value of a quadratic polynomial $p(x) = ax^2 + bx + c$ at $x = k$ is obtained by replacing x by k in $p(x)$. Thus, the value of $p(x)$ at $x = k$ is $p(k) = ak^2 + bk + c$.

3.3.1 Zeros of a polynomial

Consider a polynomial $p(x)$. If k is a real number such that $p(k) = 0$, then k is called a zero of the polynomial $p(x)$.

For example,

the zeros of the polynomial $q(x) = x^2 - 5x + 6$ are 2 and 3 because $q(2) = 0$ and $q(3) = 0$.

Remarks

A polynomial may not have any zero in real numbers at all. For example, $p(x) = x^2 + 1$ has no zeros in real numbers. That is, there is no real k such that $p(k) = 0$. Geometrically a zero of any polynomial is nothing but the x-coordinate of the point of intersection of the graph of the polynomial and the x-axis if they intersect. (see Fig. 3.1 and Fig. 3.2)

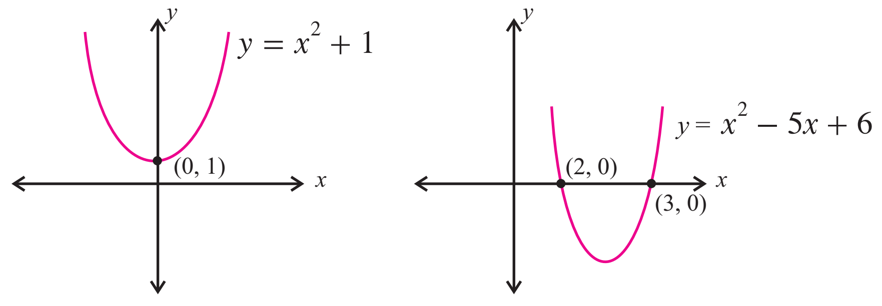


Fig. 3.1 Fig. 3.2

3.3.2 Relationship between zeros and coefficients of a quadratic polynomial

In general, if α and β are the zeros of the quadratic polynomial $p(x) = ax^2 + bx + c$, $a \neq 0$, then by factor theorem we get, $x - \alpha$ and $x - \beta$ are the factors of $p(x)$.

Therefore, $ax^2 + bx + c = k(x - \alpha)(x - \beta)$, where k is a non zero constant.

$$= k[x^2 - (\alpha + \beta)x + \alpha\beta]$$

Comparing the coefficients of x^2, x and the constant term on both sides, we obtain

$$a = k, \quad b = -k(\alpha + \beta) \quad \text{and} \quad c = k\alpha\beta$$

> The basic relationships between the zeros and the coefficients of $p(x) = ax^2 + bx + c$ are
>
> sum of zeros : $\alpha + \beta = -\dfrac{b}{a} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$.
>
> product of zeros : $\alpha\beta = \dfrac{c}{a} = \dfrac{\text{constant term}}{\text{coefficient of } x^2}$.

Example 3.11

Find the zeros of the quadratic polynomial $x^2 + 9x + 20$, and verify the basic relationships between the zeros and the coefficients.

Solution Let $p(x) = x^2 + 9x + 20 = (x+4)(x+5)$

So, $p(x) = 0 \implies (x+4)(x+5) = 0 \qquad \therefore\ x = -4$ or $x = -5$

Thus, $p(-4) = (-4+4)(-4+5) = 0$ and $p(-5) = (-5+4)(-5+5) = 0$

Hence, the zeros of $p(x)$ are -4 and -5

Thus, sum of zeros $= -9$ and the product of zeros $= 20$ (1)

From the basic relationships, we get

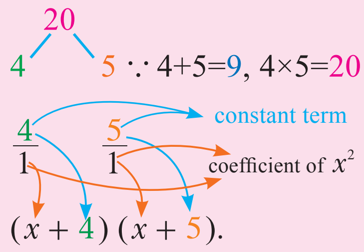
the sum of the zeros $= -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2} = -\dfrac{9}{1} = -9$ (2)

product of the zeros $= \dfrac{\text{constant term}}{\text{coefficient of } x^2} = \dfrac{20}{1} = 20$ (3)

Thus, the basic relationships are verified.

Remarks

To factorize $x^2 + 9x + 20$, one can proceed as follows



Note

A quadratic polynomial $p(x) = ax^2 + bx + c$ may have atmost two zeros.

Now, for any $a \neq 0$, $a(x^2 - (\alpha + \beta)x + \alpha\beta)$ is a polynomial with zeros α and β. Since we can choose any non zero a, there are infinitely many quadratic polynomials with zeros α and β.

Example 3.12

Find a quadratic polynomial if the sum and product of zeros of it are -4 and 3 respectively.

Solution Let α and β be the zeros of a quadratic polynomial.

Given that $\alpha + \beta = -4$ and $\alpha\beta = 3$.

One of the such polynomials is $p(x) = x^2 - (\alpha + \beta)x + \alpha\beta$

$$= x^2 - (-4)x + 3 = x^2 + 4x + 3$$

Example 3.13

Find a quadratic polynomial with zeros at $x = \frac{1}{4}$ and $x = -1$.

Solution

Let α and β be the zeros of $p(x)$

Using the relationship between zeros and coefficients, we have

$$p(x) = x^2 - (\alpha + \beta)x + \alpha\beta$$
$$= x^2 - \left(\frac{1}{4} - 1\right)x + \left(\frac{1}{4}\right)(-1)$$
$$= x^2 + \frac{3}{4}x - \frac{1}{4}$$

It is a polynomial with zeros $\frac{1}{4}$ and -1.

Aliter The required polynomial is obtained directly as follows:

$$p(x) = \left(x - \frac{1}{4}\right)(x + 1)$$
$$= x^2 + \frac{3}{4}x - \frac{1}{4}.$$

Any other polynomial with the desired property is obtained by multiplying $p(x)$ by any non-zero real number.

Note

$4x^2 + 3x - 1$ is also a polynomial with zeros $\frac{1}{4}$ and -1.

Exercise 3.3

1. Find the zeros of the following quadratic polynomials and verify the basic relationships between the zeros and the coefficients.

(i) $x^2 - 2x - 8$ (ii) $4x^2 - 4x + 1$ (iii) $6x^2 - 3 - 7x$ (iv) $4x^2 + 8x$

(v) $x^2 - 15$ (vi) $3x^2 - 5x + 2$ (vii) $2x^2 - 2\sqrt{2}x + 1$ (viii) $x^2 + 2x - 143$

2. Find a quadratic polynomial each with the given numbers as the sum and product of its zeros respectively.

(i) 3, 1 (ii) 2, 4 (iii) 0, 4 (iv) $\sqrt{2}, \frac{1}{5}$

(v) $\frac{1}{3}, 1$ (vi) $\frac{1}{2}, -4$ (vii) $\frac{1}{3}, -\frac{1}{3}$ (viii) $\sqrt{3}, 2$

3.4 Synthetic division

We know that when 29 is divided by 7 we get, 4 as the quotient and 1 as the remainder. Thus, $29 = 4(7) + 1$. Similarly one can divide a polynomial $p(x)$ by another polynomial $q(x)$ which results in getting the quotient and remainder such that

$$p(x) = (\text{quotient})q(x) + \text{remainder}$$

That is, $p(x) = s(x)q(x) + r(x)$, where $\deg r(x) < \deg q(x)$.

This is called the Division Algorithm.

If $q(x) = x + a$, then $\deg r(x) = 0$. Thus, $r(x)$ is a constant.

Hence, $p(x) = s(x)(x + a) + r$, where r is a constant.

Now if we put $x = -a$ in the above, we have $p(-a) = s(-a)(-a + a) + r \implies r = p(-a)$.

Thus, if $q(x) = x + a$, then the remainder can be calculated by simply evaluating $p(x)$ at $x = -a$.

Division algorithm :

If $p(x)$ is the dividend and $q(x)$ is the divisor, then by division algorithm we write, $p(x) = s(x)q(x) + r(x)$.

Now, we have the following results.

(i) If $q(x)$ is linear , then $r(x) = r$ is a constant.

(ii) If $\deg q(x) = 1$ (i.e., $q(x)$ is linear), then $\deg p(x) = 1 + \deg s(x)$

(iii) If $p(x)$ is divided by $x + a$, then the remainder is $p(-a)$.

(iv) If $r = 0$, we say $q(x)$ divides $p(x)$ or equivalently $q(x)$ is a factor of $p(x)$.

Remarks

An elegant way of dividing a polynomial by a linear polynomial was introduced by Paolo Ruffin in 1809. His method is known as **synthetic division.** It facilitates the division of a polynomial by a linear polynomial with the help of the coefficients involved.

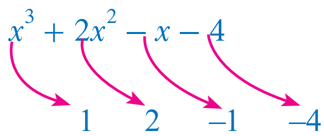
Paolo Ruffin

(1765-1822, Italy)

Let us explain the method of synthetic division with an example.

Let $p(x) = x^3 + 2x^2 - x - 4$ be the dividend and $q(x) = x + 2$ be the divisor. We shall find the quotient $s(x)$ and the remainder r, by proceeding as follows.

***Step* 1** Arrange the dividend and the divisor according to the descending powers of x and then write the coefficients of dividend in the first row (see figure). Insert 0 for missing terms.



***Step* 2** Find out the zero of the divisor.

Step 3 Put 0 for the first entry in the 2nd row.

Complete the entries of the 2nd row and 3rd row as shown below.

–2	1	2	–1	–4	
	0	1×(–2) → –2	0×(–2) → 0	–1×(–2) → 2	
	1+0 = 1	2+(–2) = 0	–1+0 = –1	–4+2 = –2	⟵ remainder

Step 4 Write down the quotient and the remainder accordingly. All the entries except the last one in the third row constitute the coefficients of the quotient.

Thus, the quotient is $x^2 - 1$ and the remainder is –2.

Example 3.14

Find the quotient and remainder when $x^3 + x^2 - 7x - 3$ is divided by $x - 3$.

Solution Let $p(x) = x^3 + x^2 - 7x - 3$. The zero of the divisor is 3. So we consider,

3	1	1	–7	–3
	0	3	12	15
	1	4	5	12 ⟶ Remainder.

∴ When $p(x)$ is divided by $x - 3$, the quotient is $x^2 + 4x + 5$ and the remainder is 12.

Example 3.15

If the quotient on dividing $2x^4 + x^3 - 14x^2 - 19x + 6$ by $2x + 1$ is $x^3 + ax^2 - bx - 6$. Find the values of a and b, also the remainder.

Solution Let $p(x) = 2x^4 + x^3 - 14x^2 - 19x + 6$.

Given that the divisor is $2x + 1$. Write $2x + 1 = 0$. Then $x = -\frac{1}{2}$

∴ The zero of the divisor is $-\frac{1}{2}$.

$-\frac{1}{2}$	2	1	–14	–19	6
	0	–1	0	7	6
	2	0	–14	–12	12 ⟶ Remainder

So, $2x^4 + x^3 - 14x^2 - 19x + 6 = \left(x + \frac{1}{2}\right)\{2x^3 - 14x - 12\} + 12$

$$= (2x + 1)\frac{1}{2}(2x^3 - 14x - 12) + 12$$

Thus, the quotient is $\frac{1}{2}(2x^3 - 14x - 12) = x^3 - 7x - 6$ and the remainder is 12.

But, given quotient is $x^3 + ax^2 - bx - 6$. Comparing this with the quotient obtained we get, $a = 0$ and $b = 7$. Thus, $a = 0$, $b = 7$ and the remainder is 12.

Exercise 3.4

1. Find the quotient and remainder using synthetic division.
   (i) $(x^3 + x^2 - 3x + 5) \div (x - 1)$ (ii) $(3x^3 - 2x^2 + 7x - 5) \div (x + 3)$
   (iii) $(3x^3 + 4x^2 - 10x + 6) \div (3x - 2)$ (iv) $(3x^3 - 4x^2 - 5) \div (3x + 1)$
   (v) $(8x^4 - 2x^2 + 6x - 5) \div (4x + 1)$ (vi) $(2x^4 - 7x^3 - 13x^2 + 63x - 48) \div (2x - 1)$
2. If the quotient on dividing $x^4 + 10x^3 + 35x^2 + 50x + 29$ by $x + 4$ is $x^3 - ax^2 + bx + 6$, then find a, b and also the remainder.
3. If the quotient on dividing, $8x^4 - 2x^2 + 6x - 7$ by $2x + 1$ is $4x^3 + px^2 - qx + 3$, then find p, q and also the remainder.

3.4.1 Factorization using synthetic division

We have already learnt in class IX, how to factorize quadratic polynomials. In this section, let us learn, how to factorize the cubic polynomial using synthetic division.

If we identify one linear factor of cubic polynomial $p(x)$, then using synthetic division we get the quadratic factor of $p(x)$. Further if possible one can factorize the quadratic factor into two linear factors. Hence the method of synthetic division helps us to factorize a cubic polynomial into linear factors if it can be factorized.

Note

(i) For any polynomial $p(x)$, $x = a$ is zero if and only if $p(a) = 0$.

(ii) $x - a$ is a factor for $p(x)$ if and only if $p(a) = 0$. (Factor theorem)

(iii) $x - 1$ is a factor of $p(x)$ if and only if the sum of coefficients of $p(x)$ is 0.

(iv) $x + 1$ is a factor of $p(x)$ if and only if sum of the coefficients of even powers of x, including constant is equal to sum of the coefficients of odd powers of x.

Example 3.16

(i) Prove that $x - 1$ is a factor of $x^3 - 6x^2 + 11x - 6$.

(ii) Prove that $x + 1$ is a factor of $x^3 + 6x^2 + 11x + 6$.

Solution

(i) Let $p(x) = x^3 - 6x^2 + 11x - 6$.

$p(1) = 1 - 6 + 11 - 6 = 0$. (note that sum of the coefficients is 0)

Thus, $(x - 1)$ is a factor of $p(x)$.

(ii) Let $q(x) = x^3 + 6x^2 + 11x + 6$.

$q(-1) = -1 + 6 - 11 + 6 = 0$. Hence, $x + 1$ is a factor of $q(x)$

Example 3.17

Factorize $2x^3 - 3x^2 - 3x + 2$ into linear factors.

Solution Let $p(x) = 2x^3 - 3x^2 - 3x + 2$

Now, $p(1) = -2 \neq 0$ (note that sum of the coefficients is not zero)

$\therefore$ $(x - 1)$ is not a factor of $p(x)$.

However, $p(-1) = 2(-1)^3 - 3(-1)^2 - 3(-1) + 2 = 0$.

So, $x + 1$ is a factor of $p(x)$.

We shall use synthetic division to find the other factors.

−1	2	−3	−3	2
	0	−2	5	−2
	2	−5	2	0 ⟶ Remainder

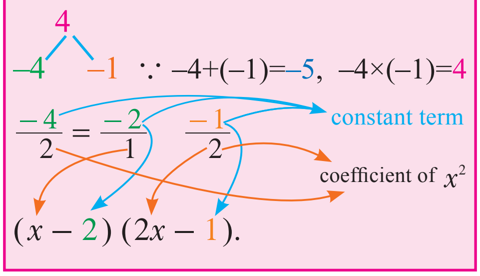
Thus, $p(x) = (x + 1)(2x^2 - 5x + 2)$

Now, $2x^2 - 5x + 2 = 2x^2 - 4x - x + 2 = (x - 2)(2x - 1)$.

Hence, $2x^3 - 3x^2 - 3x + 2 = (x + 1)(x - 2)(2x - 1)$.

Remarks

To factorize $2x^2 - 5x + 2$, one can proceed as follows



Example 3.18

Factorize $x^3 - 3x^2 - 10x + 24$

Solution Let $p(x) = x^3 - 3x^2 - 10x + 24$.

Since $p(1) \neq 0$ and $p(-1) \neq 0$, neither $x + 1$ nor $x - 1$ is a factor of $p(x)$. Therefore, we have to search for different values of x by trial and error method.

When $x = 2$, $p(2) = 0$. Thus, $x - 2$ is a factor of $p(x)$.

To find the other factors, let us use the synthetic division.

2	1	–3	–10	24	
	0	2	–2	–24	
	1	–1	–12	0	⟶ Remainder.

$\therefore$ The other factor is $x^2 - x - 12$.

Now, $x^2 - x - 12 = x^2 - 4x + 3x - 12 = (x - 4)(x + 3)$

Hence, $x^3 - 3x^2 - 10x + 24 = (x - 2)(x + 3)(x - 4)$

Exercise 3.5

1. Factorize each of the following polynomials.

(i) $x^3 - 2x^2 - 5x + 6$	(ii) $4x^3 - 7x + 3$	(iii) $x^3 - 23x^2 + 142x - 120$
(iv) $4x^3 - 5x^2 + 7x - 6$	(v) $x^3 - 7x + 6$	(vi) $x^3 + 13x^2 + 32x + 20$
(vii) $2x^3 - 9x^2 + 7x + 6$	(viii) $x^3 - 5x + 4$	(ix) $x^3 - 10x^2 - x + 10$
(x) $2x^3 + 11x^2 - 7x - 6$	(xi) $x^3 + x^2 + x - 14$	(xii) $x^3 - 5x^2 - 2x + 24$

3.5 Greatest Common Divisor (GCD) and Least Common Multiple (LCM)

3.5.1 Greatest Common Divisor (GCD)

The Highest Common Factor (HCF) or Greatest Common Divisor (GCD) of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

Consider the simple expressions

(i) a^4, a^3, a^5, a^6 (ii) a^3b^4, ab^5c^2, a^2b^7c

In (i), note that a, a^2, a^3 are the divisors of all these expressions. Out of them, a^3 is the divisor with highest power. Therefore a^3 is the GCD of the expressions a^4, a^3, a^5, a^6.

In (ii), similarly, one can easily see that ab^4 is the GCD of a^3b^4, ab^5c^2, a^2b^7c.

If the expressions have numerical coefficients, find their greatest common divisor, and prefix it as a coefficient to the greatest common divisor of the algebraic expressions.

Let us consider a few more examples to understand the greatest common divisor.

Examples 3.19

Find the GCD of the following : (i) 90, 150, 225 (ii) $15x^4y^3z^5$, $12x^2y^7z^2$

(iii) $6(2x^2 - 3x - 2)$, $8(4x^2 + 4x + 1)$, $12(2x^2 + 7x + 3)$

Solution

(i) Let us write the numbers 90, 150 and 225 in the product of their prime factors as

$$90 = 2\times3\times3\times5 \text{ , } 150 = 2\times3\times5\times5 \text{ and } 225 = 3\times3\times5\times5$$

From the above 3 and 5 are common prime factors of all the given numbers.

Hence the GCD $= 3\times5 = 15$

(ii) We shall use similar technique to find the GCD of algebraic expressions.

Now let us take the given expressions $15x^4y^3z^5$ and $12x^2y^7z^2$.

Here the common divisors of the given expressions are 3, x^2, y^3 and z^2.

Therefore, GCD $= 3\times x^2\times y^3\times z^2 = 3x^2y^3z^2$

(iii) Given expressions are $6(2x^2 - 3x - 2)$, $8(4x^2 + 4x + 1)$, $12(2x^2 + 7x + 3)$

Now, GCD of 6, 8, 12 is 2

Next let us find the factors of quadratic expressions.

$$2x^2 - 3x - 2 = (2x+1)(x-2)$$

$$4x^2 + 4x + 1 = (2x+1)(2x+1)$$

$$2x^2 + 7x + 3 = (2x+1)(x+3)$$

Common factor of the above quadratic expressions is $(2x + 1)$.

Therefore, GCD $= 2(2x + 1)$.

3.5.2 Greatest common divisor of polynomials using division algorithm

First let us consider the simple case of finding GCD of 924 and 105.

$924 = 8 \times 105 + 84$

$105 = 1 \times 84 + 21,$ (or)

$84 = 4 \times 21 + 0,$

21 is the GCD of 924 and 105

$$\begin{array}{r|r} & 8 \\ \hline 105 & 924 \\ & 840 \\ \hline & 84 \end{array} \quad \begin{array}{r|r} & 1 \\ \hline 84 & 105 \\ & 84 \\ \hline & 21 \end{array} \quad \begin{array}{r|r} & 4 \\ \hline 21 & 84 \\ & 84 \\ \hline & 0 \end{array}$$

Similar technique works with polynomials when they have GCD.

Let $f(x)$ and $g(x)$ be two non constant polynomials with $\deg(f(x)) \geq \deg(g(x))$. We want to find GCD of $f(x)$ and $g(x)$. If $f(x)$ and $g(x)$ can be factored into linear irreducible quadratic polynomials, then we can easily find the GCD by the method which we have learnt above. If the polynomials $f(x)$ and $g(x)$ are not easily factorable, then it will be a difficult problem.

However, the following method gives a systematic way of finding GCD.

Step 1 First, divide $f(x)$ by $g(x)$ to obtain $f(x) = g(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is remainder, so $\deg(g(x)) > \deg(r(x))$

If the remainder $r(x)$ is 0, then $g(x)$ is the GCD of $f(x)$ and $g(x)$.

Step 2 If the remainder $r(x)$ is non-zero, divide $g(x)$ by $r(x)$ to obtain $g(x) = r(x)q(x) + r_1(x)$ where $r_1(x)$ is the remainder. So $\deg r(x) > \deg r_1(x)$.

If the remainder $r_1(x)$ is 0, then $r(x)$ is the required GCD.

Step 3 If $r_1(x)$ is non-zero, then continue the process until we get zero as remainder. The remainder in the last but one step is the GCD of $f(x)$ and $g(x)$.
We write GCD($f(x)$, $g(x)$) to denote the GCD of the polynomials $f(x)$ and $g(x)$

Remarks

Euclid's division algorithm is based on the principle that GCD of two numbers does not change if the small number is subtracted from the larger number. Thus, GCD (252,105) = GCD (147,105) = GCD(42,105) = GCD(63,42) = GCD(21,42) = 21.

Example 3.20

Find the GCD of the polynomials $x^4 + 3x^3 - x - 3$ and $x^3 + x^2 - 5x + 3$.

Solution Let $f(x) = x^4 + 3x^3 - x - 3$ and $g(x) = x^3 + x^2 - 5x + 3$

Here degree of $f(x) >$ degree of $g(x)$. $\therefore$ Divisor is $x^3 + x^2 - 5x + 3$

$$\begin{array}{r|l} & x + 2 \\ \hline x^3 + x^2 - 5x + 3 & x^4 + 3x^3 + 0x^2 - x - 3 \\ & x^4 + x^3 - 5x^2 + 3x \\ \hline & 2x^3 + 5x^2 - 4x - 3 \\ & 2x^3 + 2x^2 - 10x + 6 \\ \hline & 3x^2 + 6x - 9 \\ & \Longrightarrow \quad x^2 + 2x - 3 \longrightarrow \text{remainder}(\neq 0) \end{array}$$

$$\begin{array}{r|l} & x-1 \\ \hline x^2 + 2x - 3 & x^3 + x^2 - 5x + 3 \\ & x^3 + 2x^2 - 3x \\ \hline & -x^2 - 2x + 3 \\ & -x^2 - 2x + 3 \\ \hline & 0 \longrightarrow \text{remainder} \end{array}$$

Therefore, GCD ($f(x)$, $g(x)$) $= x^2 + 2x - 3$.

Remarks

The two original expressions have no simple factors (constants). Thus their GCD can have none. Hence, in the above example we removed the simple factor 3 from $3x^2 + 6x - 9$ and took $x^2 + 2x - 3$ as the new divisor.

Example 3.21

Find the GCD of the following polynomials

$3x^4 + 6x^3 - 12x^2 - 24x$ and $4x^4 + 14x^3 + 8x^2 - 8x$.

Solution Let $f(x) = 3x^4 + 6x^3 - 12x^2 - 24x = 3x(x^3 + 2x^2 - 4x - 8)$.

Let $g(x) = 4x^4 + 14x^3 + 8x^2 - 8x = 2x(2x^3 + 7x^2 + 4x - 4)$

Let us find the GCD for the polynomials $x^3 + 2x^2 - 4x - 8$ and $2x^3 + 7x^2 + 4x - 4$

We choose the divisor to be $x^3 + 2x^2 - 4x - 8$.

$$\begin{array}{r|l} & \quad\quad 2 \\ \hline x^3 + 2x^2 - 4x - 8 & 2x^3 + 7x^2 + 4x - 4 \\ & 2x^3 + 4x^2 - 8x - 16 \\ \hline & \quad 3x^2 + 12x + 12 \\ & \quad (x^2 + 4x + 4) \\ & \quad \hookrightarrow \text{remainder}(\neq 0) \end{array} \qquad \begin{array}{r|l} & \quad x - 2 \\ \hline x^2 + 4x + 4 & x^3 + 2x^2 - 4x - 8 \\ & x^3 + 4x^2 + 4x \\ \hline & \quad -2x^2 - 8x - 8 \\ & \quad -2x^2 - 8x - 8 \\ \hline & \quad\quad 0 \longrightarrow \text{remainder} \end{array}$$

Common factor of $x^3 + 2x^2 - 4x - 8$ and $2x^3 + 7x^2 + 4x - 4$ is $x^2 + 4x + 4$

Also common factor of $3x$ and $2x$ is x.

Thus, GCD $(f(x), g(x)) = x(x^2 + 4x + 4)$.

Exercise 3.6

1. Find the greatest common divisor of
   (i) $7x^2yz^4,\ 21x^2y^5z^3$ (ii) $x^2y,\ x^3y,\ x^2y^2$
   (iii) $25bc^4d^3,\ 35b^2c^5,\ 45c^3d$ (iv) $35x^5y^3z^4,\ 49x^2yz^3,\ 14xy^2z^2$

2. Find the GCD of the following
   (i) $c^2 - d^2,\ c(c - d)$ (ii) $x^4 - 27a^3x,\ (x - 3a)^2$
   (iii) $m^2 - 3m - 18,\ m^2 + 5m + 6$ (iv) $x^2 + 14x + 33,\ x^3 + 10x^2 - 11x$
   (v) $x^2 + 3xy + 2y^2,\ x^2 + 5xy + 6y^2$ (vi) $2x^2 - x - 1,\ 4x^2 + 8x + 3$
   (vii) $x^2 - x - 2,\ x^2 + x - 6,\ 3x^2 - 13x + 14$ (viii) $x^3 - x^2 + x - 1,\ x^4 - 1$
   (ix) $24(6x^4 - x^3 - 2x^2),\ 20(2x^6 + 3x^5 + x^4)$
   (x) $(a - 1)^5(a + 3)^2,\ (a - 2)^2(a - 1)^3(a + 3)^4$

3. Find the GCD of the following pairs of polynomials using division algorithm.
   (i) $x^3 - 9x^2 + 23x - 15,\ 4x^2 - 16x + 12$
   (ii) $3x^3 + 18x^2 + 33x + 18,\ 3x^2 + 13x + 10$
   (iii) $2x^3 + 2x^2 + 2x + 2,\ 6x^3 + 12x^2 + 6x + 12$
   (iv) $x^3 - 3x^2 + 4x - 12,\ x^4 + x^3 + 4x^2 + 4x$

3.5.3 Least Common Multiple (LCM)

The least common multiple of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without remainder. For example, consider the simple expressions a^4, a^3, a^6.

Now, a^6, a^7, a^8, $\cdots$ are common multiples of a^3, a^4 and a^6.

Of all the common multiples, the least common multiple is a^6

Hence LCM of a^4, a^3, a^6 is a^6. Similarly, a^3b^7 is the LCM of a^3b^4, ab^5, a^2b^7.

We shall consider some more examples of finding LCM.

Example 3.22

Find the LCM of the following.

(i) 90, 150, 225 (ii) $35a^2c^3b$, $42a^3cb^2$, $30ac^2b^3$

(iii) $(a-1)^5(a+3)^2$, $(a-2)^2(a-1)^3(a+3)^4$

(iv) x^3+y^3, x^3-y^3, $x^4+x^2y^2+y^4$

Solution

(i) Now,

$$90 = 2\times3\times3\times5 = 2^1\times3^2\times5^1$$

$$150 = 2\times3\times5\times5 = 2^1\times3^1\times5^2$$

$$225 = 3\times3\times5\times5 = 3^2\times5^2$$

The product $2^1\times3^2\times5^2 = 450$ is the required LCM.

(ii) Now, LCM of 35, 42 and 30 is $5\times7\times6 = 210$

Hence, the required LCM $= 210\times a^3\times c^3\times b^3 = 210a^3c^3b^3$.

(iii) Now, LCM of $(a-1)^5(a+3)^2$, $(a-2)^2(a-1)^3(a+3)^4$ is $(a-1)^5(a+3)^4(a-2)^2$.

(iv) Let us first find the factors for each of the given expressions.

$$x^3+y^3 = (x+y)(x^2-xy+y^2)$$

$$x^3-y^3 = (x-y)(x^2+xy+y^2)$$

$$x^4+x^2y^2+y^4 = (x^2+y^2)^2-x^2y^2 = (x^2+xy+y^2)(x^2-xy+y^2)$$

Thus,

$$\text{LCM} = (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$$

$$= (x^3+y^3)(x^3-y^3) = x^6-y^6.$$

Exercise 3.7

Find the LCM of the following.

1. x^3y^2, xyz
2. $3x^2yz$, $4x^3y^3$
3. a^2bc, b^2ca, c^2ab
4. $66a^4b^2c^3$, $44a^3b^4c^2$, $24a^2b^3c^4$
5. a^{m+1}, a^{m+2}, a^{m+3}
6. x^2y+xy^2, x^2+xy
7. $3(a-1)$, $2(a-1)^2$, (a^2-1)
8. $2x^2-18y^2$, $5x^2y+15xy^2$, x^3+27y^3
9. $(x+4)^2(x-3)^3$, $(x-1)(x+4)(x-3)^2$
10. $10(9x^2+6xy+y^2)$, $12(3x^2-5xy-2y^2)$, $14(6x^4+2x^3)$.

3.5.4 Relation between LCM and GCD

We know that the product of two positive integers is equal to the product of their LCM and GCD. For example, $21 \times 35 = 105 \times 7$, where LCM (21,35) =105 and GCD (21,35) = 7.

In the same way, we have the following result:

The product of any two polynomials is equal to the product of their LCM and GCD.

That is, $f(x) \times g(x) = \text{LCM}\,(f(x), g(x)) \times \text{GCD}\,(f(x), g(x))$.

Let us justify this result with an example.

Let $f(x) = 12(x^4 - x^3)$ and $g(x) = 8(x^4 - 3x^3 + 2x^2)$ be two polynomials.

Now, $f(x) = 12(x^4 - x^3) = 2^2 \times 3 \times x^3 \times (x-1)$ (1)

Also, $g(x) = 8(x^4 - 3x^3 + 2x^2) = 2^3 \times x^2 \times (x-1) \times (x-2)$ (2)

From (1) and (2) we get,

$\text{LCM}\,(f(x), g(x)) = 2^3 \times 3^1 \times x^3 \times (x-1) \times (x-2) = 24x^3(x-1)(x-2)$

$\text{GCD}\,(f(x), g(x)) = 4x^2(x-1)$

Therefore, $\text{LCM} \times \text{GCD} = 24x^3(x-1)(x-2) \times 4x^2(x-1)$

$= 96x^5(x-1)^2(x-2)$ (3)

Also, $f(x) \times g(x) = 12x^3(x-1) \times 8x^2(x-1)(x-2)$

$= 96x^5(x-1)^2(x-2)$ (4)

From (3) and (4) we obtain, $\text{LCM} \times \text{GCD} = f(x) \times g(x)$.

Thus, the product of LCM and GCD of two polynomials is equal to the product of the two polynomials. Further, if $f(x)$, $g(x)$ and one of LCM and GCD are given, then the other can be found without ambiguity because LCM and GCD are unique, except for a factor of –1.

Example 3.23

The GCD of $x^4 + 3x^3 + 5x^2 + 26x + 56$ and $x^4 + 2x^3 - 4x^2 - x + 28$ is $x^2 + 5x + 7$. Find their LCM.

Solution Let $f(x) = x^4 + 3x^3 + 5x^2 + 26x + 56$ and $g(x) = x^4 + 2x^3 - 4x^2 - x + 28$

Given that $\text{GCD} = x^2 + 5x + 7$. Also, we have $\text{GCD} \times \text{LCM} = f(x) \times g(x)$.

Thus, $$\text{LCM} = \frac{f(x) \times g(x)}{\text{GCD}}$$ (1)

Now, GCD divides both $f(x)$ and $g(x)$.

Let us divide $f(x)$ by the GCD.

	1	–2	8		
1 5 7	1	3	5	26	56
	1	5	7		
		–2	–2	26	
		–2	–10	–14	
			8	40	56
			8	40	56
				0	

When $f(x)$ is divided by GCD, we get the quotient as $x^2 - 2x + 8$.

Now, (1) $\Longrightarrow$ LCM $= (x^2 - 2x + 8) \times g(x)$

Thus, LCM $= (x^2 - 2x + 8)(x^4 + 2x^3 - 4x^2 - x + 28)$.

> *Note*
>
> In the above problem, we can also divide $g(x)$ by GCD and multiply the quotient by $f(x)$ to get the required LCM.

Example 3.24

The GCD and LCM of two polynomials are $x + 1$ and $x^6 - 1$ respectively. If one of the polynomials is $x^3 + 1$, find the other.

Solution Given GCD = $x + 1$ and LCM = $x^6 - 1$

Let $f(x) = x^3 + 1$.

We know that LCM $\times$ GCD = $f(x) \times g(x)$

$$\Longrightarrow \quad g(x) = \frac{\text{LCM} \times \text{GCD}}{f(x)} = \frac{(x^6 - 1)(x + 1)}{x^3 + 1}$$

$$= \frac{(x^3 + 1)(x^3 - 1)(x + 1)}{x^3 + 1} = (x^3 - 1)(x + 1)$$

Hence, $g(x) = (x^3 - 1)(x + 1)$.

Exercise 3.8

1. Find the LCM of each pair of the following polynomials.
   (i) $x^2 - 5x + 6,\ x^2 + 4x - 12$ whose GCD is $x - 2$.
   (ii) $x^4 + 3x^3 + 6x^2 + 5x + 3,\ x^4 + 2x^2 + x + 2$ whose GCD is $x^2 + x + 1$.
   (iii) $2x^3 + 15x^2 + 2x - 35,\ x^3 + 8x^2 + 4x - 21$ whose GCD is $x + 7$.
   (iv) $2x^3 - 3x^2 - 9x + 5,\ 2x^4 - x^3 - 10x^2 - 11x + 8$ whose GCD is $2x - 1$.

2. Find the other polynomial $q(x)$ of each of the following, given that LCM and GCD and one polynomial $p(x)$ respectively.
   (i) $(x + 1)^2(x + 2)^2,\ (x + 1)(x + 2),\ (x + 1)^2(x + 2)$.
   (ii) $(4x + 5)^3(3x - 7)^3,\ (4x + 5)(3x - 7)^2,\ (4x + 5)^3(3x - 7)^2$.
   (iii) $(x^4 - y^4)(x^4 + x^2y^2 + y^4),\ x^2 - y^2,\ x^4 - y^4$.
   (iv) $(x^3 - 4x)(5x + 1),\ (5x^2 + x),\ (5x^3 - 9x^2 - 2x)$.
   (v) $(x - 1)(x - 2)(x^2 - 3x + 3),\ (x - 1),\ (x^3 - 4x^2 + 6x - 3)$.
   (vi) $2(x + 1)(x^2 - 4),\ (x + 1),\ (x + 1)(x - 2)$.

3.6 Rational expressions

A rational number is defined as a quotient $\frac{m}{n}$, of two integers m and $n \neq 0$. Similarly a rational expression is a quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$ is a non zero polynomial.

Every polynomial $p(x)$ is a rational expression, since $p(x)$ can be written as $\frac{p(x)}{1}$ where 1 is the constant polynomial.

However, a rational expression need not be a polynomial, for example $\frac{x}{x^2+1}$ is a rational expression but not a polynomial. Some examples of rational expressions are $2x+7$, $\frac{3x+2}{x^2+x+1}$, $\frac{x^3+\sqrt{2}x+5}{x^2+x-\sqrt{3}}$.

3.6.1 Rational expressions in lowest form

If the two polynomials $p(x)$ and $q(x)$ have the integer coefficients such that GCD of $p(x)$ and $q(x)$ is 1, then we say that $\frac{p(x)}{q(x)}$ is a rational expression in its lowest terms.

If a rational expression is not in its lowest terms, then it can be reduced to its lowest terms by dividing both numerator $p(x)$ and denominator $q(x)$ by the GCD of $p(x)$ and $q(x)$. Let us consider some examples.

Example 3.25

Simplify the rational expressions into lowest forms.

(i) $\frac{5x+20}{7x+28}$ (ii) $\frac{x^3-5x^2}{3x^3+2x^4}$

(iii) $\frac{6x^2-5x+1}{9x^2+12x-5}$ (iv) $\frac{(x-3)(x^2-5x+4)}{(x-1)(x^2-2x-3)}$

Solution

(i) Now, $\frac{5x+20}{7x+28} = \frac{5(x+4)}{7(x+4)} = \frac{5}{7}$

(ii) Now, $\frac{x^3-5x^2}{3x^3+2x^4} = \frac{x^2(x-5)}{x^3(2x+3)} = \frac{x-5}{x(2x+3)}$

(iii) Let $p(x) = 6x^2-5x+1 = (2x-1)(3x-1)$ and

$q(x) = 9x^2+12x-5 = (3x+5)(3x-1)$

Therefore, $\frac{p(x)}{q(x)} = \frac{(2x-1)(3x-1)}{(3x+5)(3x-1)} = \frac{2x-1}{3x+5}$

(iv) Let $f(x) = (x-3)(x^2-5x+4) = (x-3)(x-1)(x-4)$ and

$g(x) = (x-1)(x^2-2x-3) = (x-1)(x-3)(x+1)$

Therefore, $\frac{f(x)}{g(x)} = \frac{(x-3)(x-1)(x-4)}{(x-1)(x-3)(x+1)} = \frac{x-4}{x+1}$

Exercise 3.9

Simplify the following into their lowest forms.

(i) $\dfrac{6x^2+9x}{3x^2-12x}$ (ii) $\dfrac{x^2+1}{x^4-1}$ (iii) $\dfrac{x^3-1}{x^2+x+1}$

(iv) $\dfrac{x^3-27}{x^2-9}$ (v) $\dfrac{x^4+x^2+1}{x^2+x+1}$ (Hint: $x^4+x^2+1=(x^2+1)^2-x^2$)

(vi) $\dfrac{x^3+8}{x^4+4x^2+16}$ (vii) $\dfrac{2x^2+x-3}{2x^2+5x+3}$ (viii) $\dfrac{2x^4-162}{(x^2+9)(2x-6)}$

(ix) $\dfrac{(x-3)(x^2-5x+4)}{(x-4)(x^2-2x-3)}$ (x) $\dfrac{(x-8)(x^2+5x-50)}{(x+10)(x^2-13x+40)}$ (xi) $\dfrac{4x^2+9x+5}{8x^2+6x-5}$

(xii) $\dfrac{(x-1)(x-2)(x^2-9x+14)}{(x-7)(x^2-3x+2)}$

3.6.2 Multiplication and division of rational expressions

If $\dfrac{p(x)}{q(x)}$; $q(x)\neq 0$ and $\dfrac{g(x)}{h(x)}$; $h(x)\neq 0$ are two rational expressions, then

(i) their product $\dfrac{p(x)}{q(x)}\times\dfrac{g(x)}{h(x)}$ is defined as $\dfrac{p(x)\times g(x)}{q(x)\times h(x)}$

(ii) their division $\dfrac{p(x)}{q(x)}\div\dfrac{g(x)}{h(x)}$ is defined as $\dfrac{p(x)}{q(x)}\times\dfrac{h(x)}{g(x)}$.

Thus, $\dfrac{p(x)}{q(x)}\div\dfrac{g(x)}{h(x)} = \dfrac{p(x)\times h(x)}{q(x)\times g(x)}$

Example 3.26

Multiply (i) $\dfrac{x^3y^2}{9z^4}$ by $\dfrac{27z^5}{x^4y^2}$ (ii) $\dfrac{a^3+b^3}{a^2+2ab+b^2}$ by $\dfrac{a^2-b^2}{a-b}$ (iii) $\dfrac{x^3-8}{x^2-4}$ by $\dfrac{x^2+6x+8}{x^2+2x+4}$

Solution

(i) Now, $\dfrac{x^3y^2}{9z^4}\times\dfrac{27z^5}{x^4y^2}=\dfrac{(x^3y^2)(27z^5)}{(9z^4)(x^4y^2)}=\dfrac{3z}{x}$.

(ii) $\dfrac{a^3+b^3}{a^2+2ab+b^2}\times\dfrac{a^2-b^2}{a-b}=\dfrac{(a+b)(a^2-ab+b^2)}{(a+b)(a+b)}\times\dfrac{(a+b)(a-b)}{(a-b)}=a^2-ab+b^2$.

(iii) Now, $\dfrac{x^3-8}{x^2-4}\times\dfrac{x^2+6x+8}{x^2+2x+4}=\dfrac{x^3-2^3}{x^2-2^2}\times\dfrac{(x+4)(x+2)}{x^2+2x+4}$

$$=\dfrac{(x-2)(x^2+2x+4)}{(x+2)(x-2)}\times\dfrac{(x+4)(x+2)}{x^2+2x+4}=x+4.$$

Example 3.27

Divide (i) $\dfrac{4x-4}{x^2-1}$ by $\dfrac{x-1}{x+1}$ (ii) $\dfrac{x^3-1}{x+3}$ by $\dfrac{x^2+x+1}{3x+9}$ (iii) $\dfrac{x^2-1}{x^2-25}$ by $\dfrac{x^2-4x-5}{x^2+4x-5}$

Solution

(i) $\dfrac{4x-4}{x^2-1} \div \dfrac{x-1}{x+1} = \dfrac{4(x-1)}{(x+1)(x-1)} \times \dfrac{(x+1)}{(x-1)} = \dfrac{4}{x-1}$.

(ii) $\dfrac{x^3-1}{x+3} \div \dfrac{x^2+x+1}{3x+9} = \dfrac{(x-1)(x^2+x+1)}{x+3} \times \dfrac{3(x+3)}{x^2+x+1} = 3(x-1)$.

(iii) $\dfrac{x^2-1}{x^2-25} \div \dfrac{x^2-4x-5}{x^2+4x-5} = \dfrac{(x+1)(x-1)}{(x+5)(x-5)} \times \dfrac{(x+5)(x-1)}{(x-5)(x+1)}$

$= \dfrac{(x-1)(x-1)}{(x-5)(x-5)} = \dfrac{x^2-2x+1}{x^2-10x+25}$.

Exercise 3.10

1. Multiply the following and write your answer in lowest terms.

(i) $\dfrac{x^2-2x}{x+2} \times \dfrac{3x+6}{x-2}$ (ii) $\dfrac{x^2-81}{x^2-4} \times \dfrac{x^2+6x+8}{x^2-5x-36}$

(iii) $\dfrac{x^2-3x-10}{x^2-x-20} \times \dfrac{x^2-2x+4}{x^3+8}$ (iv) $\dfrac{x^2-16}{x^2-3x+2} \times \dfrac{x^2-4}{x^3+64} \times \dfrac{x^2-4x+16}{x^2-2x-8}$

(v) $\dfrac{3x^2+2x-1}{x^2-x-2} \times \dfrac{2x^2-3x-2}{3x^2+5x-2}$ (vi) $\dfrac{2x-1}{x^2+2x+4} \times \dfrac{x^4-8x}{2x^2+5x-3} \times \dfrac{x+3}{x^2-2x}$

2. Divide the following and write your answer in lowest terms.

(i) $\dfrac{x}{x+1} \div \dfrac{x^2}{x^2-1}$ (ii) $\dfrac{x^2-36}{x^2-49} \div \dfrac{x+6}{x+7}$

(iii) $\dfrac{x^2-4x-5}{x^2-25} \div \dfrac{x^2-3x-10}{x^2+7x+10}$ (iv) $\dfrac{x^2+11x+28}{x^2-4x-77} \div \dfrac{x^2+7x+12}{x^2-2x-15}$

(v) $\dfrac{2x^2+13x+15}{x^2+3x-10} \div \dfrac{2x^2-x-6}{x^2-4x+4}$ (vi) $\dfrac{3x^2-x-4}{9x^2-16} \div \dfrac{4x^2-4}{3x^2-2x-1}$

(vii) $\dfrac{2x^2+5x-3}{2x^2+9x+9} \div \dfrac{2x^2+x-1}{2x^2+x-3}$

3.6.3 Addition and subtraction of rational expressions

If $\dfrac{p(x)}{q(x)}$ and $\dfrac{r(x)}{s(x)}$ are any two rational expressions with $q(x) \neq 0$ and $s(x) \neq 0$, then we define the sum and the difference (subtraction) as

$$\frac{p(x)}{q(x)} \pm \frac{r(x)}{s(x)} = \frac{p(x).s(x) \pm q(x)r(x)}{q(x).s(x)}$$

Example 3.28

Simplify (i) $\dfrac{x+2}{x+3} + \dfrac{x-1}{x-2}$ (ii) $\dfrac{x+1}{(x-1)^2} + \dfrac{1}{x+1}$ (iii) $\dfrac{x^2-x-6}{x^2-9} + \dfrac{x^2+2x-24}{x^2-x-12}$

Solution

(i) $$\frac{x+2}{x+3}+\frac{x-1}{x-2} = \frac{(x+2)(x-2)+(x-1)(x+3)}{(x+3)(x-2)} = \frac{2x^2+2x-7}{x^2+x-6}$$

(ii) $$\frac{x+1}{(x-1)^2}+\frac{1}{x+1} = \frac{(x+1)^2+(x-1)^2}{(x-1)^2(x+1)} = \frac{2x^2+2}{(x-1)^2(x+1)}$$

$$= \frac{2x^2+2}{x^3-x^2-x+1}$$

(iii) $$\frac{x^2-x-6}{x^2-9}+\frac{x^2+2x-24}{x^2-x-12} = \frac{(x-3)(x+2)}{(x+3)(x-3)}+\frac{(x+6)(x-4)}{(x+3)(x-4)}$$

$$= \frac{x+2}{x+3}+\frac{x+6}{x+3} = \frac{x+2+x+6}{x+3} = \frac{2x+8}{x+3}$$

Example 3.29

What rational expression should be added to $\frac{x^3-1}{x^2+2}$ to get $\frac{2x^3-x^2+3}{x^2+2}$?

Solution Let $p(x)$ be the required rational expression.

Given that $$\frac{x^3-1}{x^2+2} + p(x) = \frac{2x^3-x^2+3}{x^2+2}$$

$$p(x) = \frac{2x^3-x^2+3}{x^2+2} - \frac{x^3-1}{x^2+2}$$

$$= \frac{2x^3-x^2+3-x^3+1}{x^2+2} = \frac{x^3-x^2+4}{x^2+2}$$

Example 3.30

Simplify $\left(\frac{2x-1}{x-1}-\frac{x+1}{2x+1}\right)+\frac{x+2}{x+1}$ as a quotient of two polynomials in the simplest form.

Solution Now, $\left(\frac{2x-1}{x-1}-\frac{x+1}{2x+1}\right)+\frac{x+2}{x+1}$

$$= \left[\frac{(2x-1)(2x+1)-(x+1)(x-1)}{(x-1)(2x+1)}\right]+\frac{x+2}{x+1}$$

$$= \frac{(4x^2-1)-(x^2-1)}{(x-1)(2x+1)}+\frac{x+2}{x+1} = \frac{3x^2}{(x-1)(2x+1)}+\frac{x+2}{x+1}$$

$$= \frac{3x^2(x+1)+(x+2)(x-1)(2x+1)}{(x^2-1)(2x+1)} = \frac{5x^3+6x^2-3x-2}{2x^3+x^2-2x-1}$$

Exercise 3.11

1. Simplify the following as a quotient of two polynomials in the simplest form.

(i) $\frac{x^3}{x-2}+\frac{8}{2-x}$

(ii) $\frac{x+2}{x^2+3x+2}+\frac{x-3}{x^2-2x-3}$

(iii) $\frac{x^2-x-6}{x^2-9}+\frac{x^2+2x-24}{x^2-x-12}$

(iv) $\frac{x-2}{x^2-7x+10}+\frac{x+3}{x^2-2x-15}$

(v) $\frac{2x^2-5x+3}{x^2-3x+2} - \frac{2x^2-7x-4}{2x^2-3x-2}$ (vi) $\frac{x^2-4}{x^2+6x+8} - \frac{x^2-11x+30}{x^2-x-20}$

(vii) $\left[\frac{2x+5}{x+1} + \frac{x^2+1}{x^2-1}\right] - \left(\frac{3x-2}{x-1}\right)$ (viii) $\frac{1}{x^2+3x+2} + \frac{1}{x^2+5x+6} - \frac{2}{x^2+4x+3}$.

2. Which rational expression should be added to $\frac{x^3-1}{x^2+2}$ to get $\frac{3x^3+2x^2+4}{x^2+2}$?

3. Which rational expression should be subtracted from

$\frac{4x^3-7x^2+5}{2x-1}$ to get $2x^2-5x+1$?

4. If $P = \frac{x}{x+y}$, $Q = \frac{y}{x+y}$, then find $\frac{1}{P-Q} - \frac{2Q}{P^2-Q^2}$.

3.7 Square root

Let $a \in \mathbb{R}$ be a non negative real number. A square root of a, is a real number b such that $b^2 = a$. The positive square root of a is denoted by $\sqrt[2]{a}$ or $\sqrt{a}$. Even though both $(-3)^2 = 9$ and $(+3)^2 = 9$ are true, the radical sign $\sqrt{\ }$ is used to indicate the positive square root of the number under it. Hence $\sqrt{9} = 3$. Similarly, we have $\sqrt{121} = 11$, $\sqrt{10000} = 100$.

In the same way, the square root of any expression or a polynomial is an expression whose square is equal to the given expression. In the case of polynomials, we take

$$\sqrt{(p(x))^2} = |p(x)|, \text{ where } |p(x)| = \begin{cases} p(x) & \text{if } p(x) \geq 0 \\ -p(x) & \text{if } p(x) < 0 \end{cases}. \quad \text{For example,}$$

$$\sqrt{(x-a)^2} = |(x-a)| \text{ and } \sqrt{(a-b)^2} = |(a-b)|.$$

In general, the following two methods are very familiar to find the square root of a given polynomial (i) factorization method (ii) division method.

In this section, let us learn the factorization method through some examples for both the expressions and polynomials when they are factorable.

3.7.1 Square root by factorization method

Example 3.31

Find the square root of

(i) $121(x-a)^4(x-b)^6(x-c)^{12}$ (ii) $\frac{81x^4y^6z^8}{64w^{12}s^{14}}$ (iii) $(2x+3y)^2 - 24xy$

Solution

(i) $\sqrt{121(x-a)^4(x-b)^6(x-c)^{12}} = 11|(x-a)^2(x-b)^3(x-c)^6|$

(ii) $\sqrt{\frac{81x^4y^6z^8}{64w^{12}s^{14}}} = \frac{9}{8}\left|\frac{x^2y^3z^4}{w^6s^7}\right|$

(iii) $\sqrt{(2x+3y)^2 - 24xy} = \sqrt{4x^2+12xy+9y^2-24xy} = \sqrt{(2x-3y)^2}$

$= |(2x-3y)|$

Example 3.32

Find the square root of (i) $4x^2 + 20xy + 25y^2$ (ii) $x^6 + \frac{1}{x^6} - 2$

(iii) $(6x^2 - x - 2)(3x^2 - 5x + 2)(2x^2 - x - 1)$

Solution

(i) $\sqrt{4x^2 + 20xy + 25y^2} = \sqrt{(2x+5y)^2} = |(2x+5y)|$

(ii) $\sqrt{x^6 + \frac{1}{x^6} - 2} = \sqrt{\left(x^3 - \frac{1}{x^3}\right)^2} = \left|\left(x^3 - \frac{1}{x^3}\right)\right|$

(iii) First, let us factorize the polynomials

$6x^2 - x - 2 = (2x+1)(3x-2)$; $3x^2 - 5x + 2 = (3x-2)(x-1)$ and

$2x^2 - x - 1 = (x-1)(2x+1)$

Now, $\sqrt{(6x^2 - x - 2)(3x^2 - 5x + 2)(2x^2 - x - 1)}$

$= \sqrt{(2x+1)(3x-2)\times(3x-2)(x-1)\times(x-1)(2x+1)}$

$= \sqrt{(2x+1)^2(3x-2)^2(x-1)^2} = |(2x+1)(3x-2)(x-1)|$

Exercise 3.12

1. Find the square root of the following

(i) $196a^6b^8c^{10}$ (ii) $289(a-b)^4(b-c)^6$ (iii) $(x+11)^2 - 44x$

(iv) $(x-y)^2 + 4xy$ (v) $121x^8y^6 \div 81x^4y^8$ (vi) $\frac{64(a+b)^4(x-y)^8(b-c)^6}{25(x+y)^4(a-b)^6(b+c)^{10}}$

2. Find the square root of the following:

(i) $16x^2 - 24x + 9$

(ii) $(x^2 - 25)(x^2 + 8x + 15)(x^2 - 2x - 15)$

(iii) $4x^2 + 9y^2 + 25z^2 - 12xy + 30yz - 20zx$

(iv) $x^4 + \frac{1}{x^4} + 2$

(v) $(6x^2 + 5x - 6)(6x^2 - x - 2)(4x^2 + 8x + 3)$

(vi) $(2x^2 - 5x + 2)(3x^2 - 5x - 2)(6x^2 - x - 1)$

3.7.2 Finding the square root of a polynomial by division method

In this method, we find the square root of a polynomial which cannot easily be reduced into product of factors. Also division method is a convenient one when the polynomials are of higher degrees.

One can find the square root of a polynomial the same way of finding the square root of a positive integer. Let us explain this method with the following examples.

To find (i) $\sqrt{66564}$

$$\begin{array}{r|l} & 2\ \ 5\ \ 8 \\ \hline 2 & 6\ \ 65\ \ 64 \\ & 4 \\ \hline 45 & 2\ \ 65 \\ & 2\ \ 25 \\ \hline 508 & 40\ \ 64 \\ & 40\ \ 64 \\ \hline & 0 \end{array}$$

(ii) $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1}$

Let $p(x) = 9x^4 + 12x^3 + 10x^2 + 4x + 1$

$$\begin{array}{r|l} & 3x^2 + 2x + 1 \\ \hline 3x^2 & 9x^4 + 12x^3 + 10x^2 + 4x + 1 \\ & 9x^4 \\ \hline 6x^2 + 2x & 12x^3 + 10x^2 \\ & 12x^3 + 4x^2 \\ \hline 6x^2 + 4x + 1 & 6x^2 + 4x + 1 \\ & 6x^2 + 4x + 1 \\ \hline & 0 \end{array}$$

Therefore, $\sqrt{66564} = 258$ and $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = |3x^2 + 2x + 1|$

Remarks

(i) While writing the polynomial in **ascending or descending** powers of x, insert zeros for missing terms.

(ii) The above method can be compared with the following procedure.

$$\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = \sqrt{(a+b+c)^2}$$

Therefore, it is a matter of finding the suitable a, b and c.

Now, $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
$= a^2 + b^2 + 2ab + 2ac + 2bc + c^2$
$= a^2 + (2a+b)b + (2a+2b+c)c$
$= (3x^2)^2 + (6x^2 + 2x)(2x) + (6x^2 + 4x + 1)(1)$

Thus, $\sqrt{9x^4 + 12x^3 + 10x^2 + 4x + 1} = |3x^2 + 2x + 1|$, where $a = 3x^2$, $b = 2x$ and $c = 1$

Aliter : To find the square root, first write $9x^4 + 12x^3 + 10x^2 + 4x + 1$
$= (mx^2 + nx + l)^2 = m^2x^4 + 2mnx^3 + (n^2 + 2lm)x^2 + 2nlx + l^2$

Compare the coefficients and then find the suitable constants m, n, l.

(iii) It is also quite interesting to note the following :

$25x^4 - 30x^3 + 29x^2 - 12x + 4 = 25x^4 - 30x^3 + 9x^2 + 20x^2 - 12x + 4$
$= (5x^2)^2 + [10x^2 + (-3x)](-3x) + (10x^2 - 6x + 2)2$
$= (5x^2)^2 + [2(5x^2) + (-3x)](-3x) + [2(5x^2) + 2(-3x) + 2]2$
$= a^2 + [2a + (-b)](-b) + [2a + 2(-b) + c]c$
$= a^2 + (-b)^2 + c^2 + 2a(-b) + 2(-b)c + 2ac$
$= (a - b + c)^2$, where $a = 5x^2$, $b = 3x$, $c = 2$

$\therefore \sqrt{25x^4 - 30x^3 + 29x^2 - 12x + 4} = |5x^2 - 3x + 2|$.

Example 3.33

Find the square root of $x^4 - 10x^3 + 37x^2 - 60x + 36$.

Solution Given polynomial is already in descending powers of x.

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\
\hline
x^2 & x^4 - 10x^3 + 37x^2 - 60x + 36 \\
 & x^4 \\
\hline
2x^2 - 5x & \quad -10x^3 + 37x^2 \\
 & \quad -10x^3 + 25x^2 \\
\hline
2x^2 - 10x + 6 & \qquad\qquad 12x^2 - 60x + 36 \\
 & \qquad\qquad 12x^2 - 60x + 36 \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

Thus, $\sqrt{x^4 - 10x^3 + 37x^2 - 60x + 36} = |(x^2 - 5x + 6)|$

Example 3.34

Find the square root of $x^4 - 6x^3 + 19x^2 - 30x + 25$

Solution Let us write the polynomial in ascending powers of x and find the square root.

$$
\begin{array}{r|l}
 & 5 - 3x + x^2 \\
\hline
5 & 25 - 30x + 19x^2 - 6x^3 + x^4 \\
 & 25 \\
\hline
10 - 3x & \quad -30x + 19x^2 \\
 & \quad -30x + 9x^2 \\
\hline
10 - 6x + x^2 & \qquad\qquad 10x^2 - 6x^3 + x^4 \\
 & \qquad\qquad 10x^2 - 6x^3 + x^4 \\
\hline
 & \qquad\qquad\quad 0
\end{array}
$$

Hence, the square root of the given polynomial is $|x^2 - 3x + 5|$

Example 3.35

If $m - nx + 28x^2 + 12x^3 + 9x^4$ is a perfect square, then find the values of m and n.

Solution Arrange the polynomial in descending power of x.

$$9x^4 + 12x^3 + 28x^2 - nx + m.$$

Now,

	$3x^2 + 2x + 4$
$3x^2$	$9x^4 + 12x^3 + 28x^2 - nx + m$
	$9x^4$
$6x^2 + 2x$	$12x^3 + 28x^2$
	$12x^3 + 4x^2$
$6x^2 + 4x + 4$	$24x^2 - nx + m$
	$24x^2 + 16x + 16$
	0

Since the given polynomial is a perfect square, we must have $n = -16$ and $m = 16$.

Exercise 3.13

1. Find the square root of the following polynomials by division method.
   (i) $x^4 - 4x^3 + 10x^2 - 12x + 9$ (ii) $4x^4 + 8x^3 + 8x^2 + 4x + 1$
   (iii) $9x^4 - 6x^3 + 7x^2 - 2x + 1$ (iv) $4 + 25x^2 - 12x - 24x^3 + 16x^4$
2. Find the values of a and b if the following polynomials are perfect squares.
   (i) $4x^4 - 12x^3 + 37x^2 + ax + b$ (ii) $x^4 - 4x^3 + 10x^2 - ax + b$
   (iii) $ax^4 + bx^3 + 109x^2 - 60x + 36$ (iv) $ax^4 - bx^3 + 40x^2 + 24x + 36$

3.8 Quadratic equations

Greek mathematician Euclid developed a geometrical approach for finding out lengths which in our present day terminology, are solutions of quadratic equations. Solving quadratic equations in general form is often credited to ancient Indian Mathematicians. In fact, Brahma Gupta (A.D 598 - 665) gave an explicit formula to solve a quadratic equation of the form $ax^2 + bx = c$. Later *Sridhar Acharya* (1025 A.D) derived a formula, now known as the quadratic formula, (as quoted by Bhaskara II) for solving a quadratic equation by the method of completing the square.

In this section, we will learn solving quadratic equations, by various methods. We shall also see some applications of quadratic equations.

Definition

A quadratic equation in the variable x is an equation of the form $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is a quadratic equation, whose standard form is $ax^2 + bx + c = 0$, $a \neq 0$.

For example, $2x^2 - 3x + 4 = 0$, $1 - x + x^2 = 0$ are some quadratic equations.

3.8.1 Solution of a quadratic equation by factorization method

Factorization method can be used when the quadratic equation can be factorized into linear factors. Given a product, if any factor is zero, then the whole product is zero. Conversely, if a product is equal to zero, then some factor of that product must be zero, and any factor which contains an unknown may be equal to zero. Thus, in solving a quadratic equation, we find the values of x which make each of the factors zero. That is, we may equate each factor to zero and solve for the unknown.

Example 3.36

Solve $6x^2 - 5x - 25 = 0$

Solution Given $6x^2 - 5x - 25 = 0.$

First, let us find α and β such that $\alpha + \beta = -5$ and $\alpha\beta = 6 \times (-25) = -150$, where -5 is the coefficient of x. Thus, we get $\alpha = -15$ and $\beta = 10$.

Next, $6x^2 - 5x - 25 = 6x^2 - 15x + 10x - 25 = 3x(2x-5) + 5(2x-5)$

$$= (2x-5)(3x+5).$$

Therefore, the solution set is obtained from $2x - 5 = 0$ and $3x + 5 = 0$

Thus, $x = \frac{5}{2}$, $x = -\frac{5}{3}$.

Hence, solution set is $\left\{-\frac{5}{3}, \frac{5}{2}\right\}$.

Example 3.37

Solve $\frac{6}{7x-21} - \frac{1}{x^2-6x+9} + \frac{1}{x^2-9} = 0$

Solution Given equation appears to be a non-quadratic equation. But when we simplify the equation, it will reduce to a quadratic equation.

Now, $\frac{6}{7(x-3)} - \frac{1}{(x-3)^2} + \frac{1}{(x+3)(x-3)} = 0$

$$\implies \frac{6(x^2-9) - 7(x+3) + 7(x-3)}{7(x-3)^2(x+3)} = 0$$

$$\implies 6x^2 - 54 - 42 = 0 \implies x^2 - 16 = 0$$

The equation $x^2 = 16$ is quadratic and hence we have two values $x = 4$ and $x = -4$.

$\therefore$ Solution set is $\{-4, 4\}$

Example 3.38

Solve $\sqrt{24 - 10x} = 3 - 4x$, $3 - 4x > 0$

Solution Given $\sqrt{24 - 10x} = 3 - 4x$

Squaring on both sides, we get, $24 - 10x = (3-4x)^2$

$$\implies 16x^2 - 14x - 15 = 0 \qquad \implies 16x^2 - 24x + 10x - 15 = 0$$

$$\Longrightarrow \quad (8x+5)(2x-3)=0 \text{ which gives } x=\frac{3}{2} \text{ or } -\frac{5}{8}$$

When $x=\frac{3}{2}$, $3-4x=3-4\left(\frac{3}{2}\right)<0$ and hence, $x=\frac{3}{2}$ is not a solution of the equation.

When $x=-\frac{5}{8}$, $3-4x>0$ and hence, the solution set is $\left\{-\frac{5}{8}\right\}$.

Remarks

To solve radical equation like the above, we rely on the **squaring property** :

$a=b \Longrightarrow a^2=b^2$. Unfortunately, this squaring property does not guarantee that all solutions of the new equation are solutions of the original equation. For example, on squaring the equation $x=5$ we get $x^2=25$, which in turn gives $x=5$ and $x=-5$. But $x=-5$ is not a solution of the original equation. Such a solution is called an **extraneous** solution.

Thus, the above example shows that when squaring on both sides of a radical equation, the solution of the final equation must be checked to determine whether they are solutions of the original equation or not. This is necessary because no solution of the original equation will be lost by squaring but certain values may be introduced which are roots of the new equation but not of the original equation.

Exercise 3.14

Solve the following quadratic equations by factorization method.

(i) $(2x+3)^2-81=0$ (ii) $3x^2-5x-12=0$ (iii) $\sqrt{5}x^2+2x-3\sqrt{5}=0$

(iv) $3(x^2-6)=x(x+7)-3$ (v) $3x-\frac{8}{x}=2$ (vi) $x+\frac{1}{x}=\frac{26}{5}$

(vii) $\frac{x}{x+1}+\frac{x+1}{x}=\frac{34}{15}$ (viii) $a^2b^2x^2-(a^2+b^2)x+1=0$

(ix) $2(x+1)^2-5(x+1)=12$ (x) $3(x-4)^2-5(x-4)=12$

3.8.2 Solution of a quadratic equation by completing square

From $\left(x+\frac{b}{2}\right)^2=x^2+bx+\left(\frac{b}{2}\right)^2$, note that the last term $\left(\frac{b}{2}\right)^2$ is the square of half the coefficient of x. Hence, the x^2+bx lacks only the term $\left(\frac{b}{2}\right)^2$ of being the square of $x+\frac{b}{2}$. Thus, if the square of half the coefficient of x be added to an expression of the form x^2+bx, the result is the square of a binomial. Such an addition is usually known as completing the square. In this section, we shall find the solution of a quadratic equation by the method of completing the square through the following steps.

***Step* 1** If the coefficient of x^2 is 1, go to step 2. If not, divide both sides of the equation by the coefficient of x^2. Get all the terms with variable on one side of equation.

***Step* 2** Find half the coefficient of x and square it. Add this number to both sides of the equation. To solve the equation, use the square root property:

$$x^2=t \Longrightarrow x=\sqrt{t} \text{ or } x=-\sqrt{t} \text{ where } t \text{ is non-negative.}$$

Example 3.39

Solve the quadratic equation $5x^2 - 6x - 2 = 0$ by completing the square.

Solution Given quadratic equation is $5x^2 - 6x - 2 = 0$

$$\Longrightarrow \quad x^2 - \frac{6}{5}x - \frac{2}{5} = 0 \qquad \text{(Divide on both sides by 5)}$$

$$\Longrightarrow \quad x^2 - 2\left(\frac{3}{5}\right)x = \frac{2}{5} \qquad \left(\ \frac{3}{5} \text{ is the half of the coefficient of } x\ \right)$$

$$\Longrightarrow \quad x^2 - 2\left(\frac{3}{5}\right)x + \frac{9}{25} = \frac{9}{25} + \frac{2}{5} \qquad \left(\text{ add } \left(\frac{3}{5}\right)^2 = \frac{9}{25} \text{ on both sides }\right)$$

$$\Longrightarrow \quad \left(x - \frac{3}{5}\right)^2 = \frac{19}{25}$$

$$\Longrightarrow \quad x - \frac{3}{5} = \pm\sqrt{\frac{19}{25}} \qquad \text{(take square root on both sides)}$$

Thus, we have $x = \frac{3}{5} \pm \frac{\sqrt{19}}{5} = \frac{3 \pm \sqrt{19}}{5}$.

Hence, the solution set is $\left\{\frac{3+\sqrt{19}}{5}, \frac{3-\sqrt{19}}{5}\right\}$.

Example 3.40

Solve the equation $a^2x^2 - 3abx + 2b^2 = 0$ by completing the square

Solution There is nothing to prove if $a = 0$. For $a \neq 0$, we have

$$a^2x^2 - 3abx + 2b^2 = 0$$

$$\Longrightarrow \quad x^2 - \frac{3b}{a}x + \frac{2b^2}{a^2} = 0 \qquad \Longrightarrow \quad x^2 - 2\left(\frac{3b}{2a}\right)x = \frac{-2b^2}{a^2}$$

$$\Longrightarrow \quad x^2 - 2\left(\frac{3b}{2a}\right)x + \frac{9b^2}{4a^2} = \frac{9b^2}{4a^2} - \frac{2b^2}{a^2}$$

$$\Longrightarrow \quad \left(x - \frac{3b}{2a}\right)^2 = \frac{9b^2 - 8b^2}{4a^2} \qquad \Longrightarrow \quad \left(x - \frac{3b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$\Longrightarrow \quad x - \frac{3b}{2a} = \pm\frac{b}{2a} \qquad \Longrightarrow \quad x = \frac{3b \pm b}{2a}$$

Therefore, the solution set is $\left\{\frac{b}{a}, \frac{2b}{a}\right\}$.

3.8.3 Solution of quadratic equation by formula method

In this section, we shall derive the quadratic formula, which is useful for finding the roots of a quadratic equation. Consider a quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$. We rewrite the given equation as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\Longrightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \frac{c}{a} = 0 \qquad \Longrightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x = -\frac{c}{a}$$

Adding $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ both sides we get, $x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

That is, $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

$$\Longrightarrow \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

So, we have $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (1)

The solution set is $\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\}$.

The formula given in equation (1) is known as quadratic formula.

Now, let us solve some quadratic equations using quadratic formula.

Example 3.41

Solve the equation $\frac{1}{x+1} + \frac{2}{x+2} = \frac{4}{x+4}$, where $x + 1 \neq 0$, $x + 2 \neq 0$ and $x + 4 \neq 0$ using quadratic formula.

Solution Note that the given equation is not in the standard form of a quadratic equation.

Consider $\frac{1}{x+1} + \frac{2}{x+2} = \frac{4}{x+4}$

That is, $\frac{1}{x+1} = 2\left[\frac{2}{x+4} - \frac{1}{x+2}\right] = 2\left[\frac{2x + 4 - x - 4}{(x+4)(x+2)}\right]$

$$\frac{1}{x+1} = 2\left[\frac{x}{(x+2)(x+4)}\right]$$

$$x^2 + 6x + 8 = 2x^2 + 2x$$

Thus, we have $x^2 - 4x - 8 = 0$, which is a quadratic equation.

(The above equation can also be obtained by taking LCM)

Using the quadratic formula we get,

$$x = \frac{4 \pm \sqrt{16 - 4(1)(-8)}}{2(1)} = \frac{4 \pm \sqrt{48}}{2}$$

Thus, $x = 2 + 2\sqrt{3}$ or $2 - 2\sqrt{3}$

Hence, the solution set is $\{2 - 2\sqrt{3}, 2 + 2\sqrt{3}\}$

Exercise 3.15

1 Solve the following quadratic equations by completing the square .

(i) $x^2 + 6x - 7 = 0$

(ii) $x^2 + 3x + 1 = 0$

(iii) $2x^2 + 5x - 3 = 0$

(iv) $4x^2 + 4bx - (a^2 - b^2) = 0$

(v) $x^2 - (\sqrt{3} + 1)x + \sqrt{3} = 0$

(vi) $\frac{5x + 7}{x - 1} = 3x + 2$

2. Solve the following quadratic equations using quadratic formula.

(i) $x^2 - 7x + 12 = 0$ (ii) $15x^2 - 11x + 2 = 0$

(iii) $x + \frac{1}{x} = 2\frac{1}{2}$ (iv) $3a^2x^2 - abx - 2b^2 = 0$

(v) $a(x^2 + 1) = x(a^2 + 1)$ (vi) $36x^2 - 12ax + (a^2 - b^2) = 0$

(vii) $\frac{x-1}{x+1} + \frac{x-3}{x-4} = \frac{10}{3}$ (viii) $a^2x^2 + (a^2 - b^2)x - b^2 = 0$

3.8.4 Solution of problems involving quadratic equations

In this section, we will solve some simple problems expressed in words and some problems describing day-to-day life situations involving quadratic equation. First we shall form an equation translating the given statement and then solve it. Finally, we choose the solution that is relevant to the given problem.

Example 3.42

The sum of a number and its reciprocal is $5\frac{1}{5}$. Find the number.

Solution Let x denote the required number. Then its reciprocal is $\frac{1}{x}$

By the given condition, $x + \frac{1}{x} = 5\frac{1}{5} \implies \frac{x^2+1}{x} = \frac{26}{5}$

So, $5x^2 - 26x + 5 = 0$

$$\implies 5x^2 - 25x - x + 5 = 0$$

That is, $(5x-1)(x-5) = 0 \implies x = 5$ or $\frac{1}{5}$

Thus, the required numbers are $5, \frac{1}{5}$.

Example 3.43

The base of a triangle is 4cm longer than its altitude. If the area of the triangle is 48 sq. cm, then find its base and altitude.

Solution Let the altitude of the triangle be x cm.

By the given condition, the base of the triangle is $(x+4)$ cm.

Now, the area of the triangle = $\frac{1}{2}(\text{base}) \times \text{height}$

By the given condition $\frac{1}{2}(x+4)(x) = 48$

$$\implies x^2 + 4x - 96 = 0 \implies (x+12)(x-8) = 0$$

$$\implies x = -12 \text{ or } 8$$

But $x = -12$ is not possible (since the length should be positive)

Therefore, $x = 8$ and hence, $x + 4 = 12$.

Thus, the altitude of the triangle is 8 cm and the base of the triangle is 12 cm.

Example 3.44

A car left 30 minutes later than the scheduled time. In order to reach its destination 150km away in time, it has to increase its speed by 25km/hr from its usual speed. Find its usual speed.

Solution Let the usual speed of the car be x km/hr.

Thus, the increased speed of the car is $(x+25)$ km/hr

Total distance $=150$ km; Time taken $=\dfrac{\text{Distance}}{\text{Speed}}$.

Let T_1 and T_2 be the time taken in hours by the car to cover the given distance in scheduled time and decreased time (as the speed is increased) respectively.

By the given information $T_1 - T_2 = \frac{1}{2}$hr (30 minutes $= \frac{1}{2}$hr)

$$\Longrightarrow \quad \frac{150}{x} - \frac{150}{x+25} = \frac{1}{2} \quad \Longrightarrow \quad 150\left[\frac{x+25-x}{x(x+25)}\right] = \frac{1}{2}$$

$$\Longrightarrow \quad x^2 + 25x - 7500 = 0 \quad \Longrightarrow \quad (x+100)(x-75) = 0$$

Thus, $x = 75$ or -100, but $x = -100$ is not an admissible value.

Therefore, the usual speed of the car is 75 km/hr.

Exercise 3.16

1. The sum of a number and its reciprocal is $\frac{65}{8}$. Find the number.
2. The difference of the squares of two positive numbers is 45. The square of the smaller number is four times the larger number. Find the numbers.
3. A farmer wishes to start a 100 sq.m rectangular vegetable garden. Since he has only 30 m barbed wire, he fences the sides of the rectangular garden letting his house compound wall act as the fourth side fence. Find the dimension of the garden.
4. A rectangular field is 20 m long and 14 m wide. There is a path of equal width all around it having an area of 111 sq. metres. Find the width of the path on the outside.
5. A train covers a distance of 90 km at a uniform speed. Had the speed been 15 km/hr more, it would have taken 30 minutes less for the journey. Find the original speed of the train.
6. The speed of a boat in still water is 15 km/hr. It goes 30 km upstream and return downstream to the original point in 4 hrs 30 minutes. Find the speed of the stream.
7. One year ago, a man was 8 times as old as his son. Now his age is equal to the square of his son's age. Find their present ages.
8. A chess board contains 64 equal squares and the area of each square is 6.25 cm^2. A border around the board is 2 cm wide. Find the length of the side of the chess board.

9. A takes 6 days less than the time taken by B to finish a piece of work. If both A and B together can finish it in 4 days, find the time that B would take to finish this work by himself.

10. Two trains leave a railway station at the same time. The first train travels due west and the second train due north. The first train travels 5 km/hr faster than the second train. If after two hours, they are 50 km apart, find the average speed of each train.

3.8.5 Nature of roots of a quadratic equation

The roots of the equation $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

If $b^2 - 4ac > 0$, we get two distinct real roots

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac = 0$, then the equation has two equal roots $x = \frac{-b}{2a}$.

If $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is not a real number. Therefore there is no real root for the given quadratic equation.

So, evidently the nature of roots depends on the values of $b^2 - 4ac$. The value of the expression $b^2 - 4ac$ discriminates the nature of the roots of $ax^2 + bx + c = 0$ and so it is called the discriminant of the quadratic equation and denoted by the symbol $\triangle$.

Discriminant $\triangle = b^2 - 4ac$	Nature of roots
$\triangle > 0$	Real and unequal
$\triangle = 0$	Real and equal.
$\triangle < 0$	No real roots. (It has imaginary roots)

Example 3.45

Determine the nature of roots of the following quadratic equations

(i) $x^2 - 11x - 10 = 0$ (ii) $4x^2 - 28x + 49 = 0$ (iii) $2x^2 + 5x + 5 = 0$

Solution For $ax^2 + bx + c = 0$, the discriminant, $\triangle = b^2 - 4ac$.

(i) Here, $a = 1$; $b = -11$ and $c = -10$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$$= (-11)^2 - 4(1)(-10) = 121 + 40 = 161$$

Thus, $\triangle > 0$. Therefore, the roots are real and unequal.

(ii) Here, $a = 4$, $b = -28$ and $c = 49$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$$= (-28)^2 - 4(4)(49) = 0$$

Since $\triangle = 0$, the roots of the given equation are real and equal.

(iii) Here, $a = 2$, $b = 5$ and $c = 5$.

Now, the discriminant $\triangle = b^2 - 4ac$

$= (5)^2 - 4(2)(5)$

$= 25 - 40 = -15$

Since $\triangle < 0$, the equation has no real roots.

Example 3.46

Prove that the roots of the equation $(a - b + c)x^2 + 2(a - b)x + (a - b - c) = 0$ are rational numbers for all real numbers a and b and for all rational c.

Solution Let the given equation be of the form $Ax^2 + Bx + C = 0$. Then,

$A = a - b + c$, $B = 2(a - b)$ and $C = a - b - c$.

Now, the discriminant of $Ax^2 + Bx + C = 0$ is

$$B^2 - 4AC = [2(a-b)]^2 - 4(a-b+c)(a-b-c)$$

$$= 4(a-b)^2 - 4[(a-b)+c][(a-b)-c]$$

$$= 4(a-b)^2 - 4[(a-b)^2 - c^2]$$

$\triangle = 4(a-b)^2 - 4(a-b)^2 + 4c^2 = 4c^2$, a perfect square.

Therefore, $\triangle > 0$ and it is a perfect square.

Hence, the roots of the given equation are rational numbers.

Example 3.47

Find the values of k so that the equation $x^2 - 2x(1 + 3k) + 7(3 + 2k) = 0$ has real and equal roots.

Solution The given equation is $x^2 - 2x(1 + 3k) + 7(3 + 2k) = 0$. (1)

Let the equation (1) be in the form $ax^2 + bx + c = 0$

Here, $a = 1$, $b = -2(3k + 1)$, $c = 7(3 + 2k)$.

Now, the discriminant is $\triangle = b^2 - 4ac$

$= (-2(3k+1))^2 - 4(1)(7)(3+2k)$

$= 4(9k^2 + 6k + 1) - 28(3 + 2k) = 4(9k^2 - 8k - 20)$

Given that the equation has equal roots. Thus, $\triangle = 0$

$\Longrightarrow 9k^2 - 8k - 20 = 0$

$\Longrightarrow (k-2)(9k+10) = 0$

Thus, $k = 2, -\frac{10}{9}$.

Exercise 3.17

1. Determine the nature of the roots of the equation.

   (i) $x^2 - 8x + 12 = 0$ (ii) $2x^2 - 3x + 4 = 0$

   (iii) $9x^2 + 12x + 4 = 0$ (iv) $3x^2 - 2\sqrt{6}x + 2 = 0$

   (v) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$ (vi) $(x - 2a)(x - 2b) = 4ab$

2. Find the values of k for which the roots are real and equal in each of the following equations.

   (i) $2x^2 - 10x + k = 0$ (ii) $12x^2 + 4kx + 3 = 0$

   (iii) $x^2 + 2k(x - 2) + 5 = 0$ (iv) $(k + 1)x^2 - 2(k - 1)x + 1 = 0$

3. Show that the roots of the equation $x^2 + 2(a + b)x + 2(a^2 + b^2) = 0$ are unreal.

4. Show that the roots of the equation $3p^2x^2 - 2pqx + q^2 = 0$ are not real.

5. If the roots of the equation $(a^2 + b^2)x^2 - 2(ac + bd)x + c^2 + d^2 = 0$, where a, b, c and $d \neq 0$, are equal, prove that $\frac{a}{b} = \frac{c}{d}$.

6. Show that the roots of the equation $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$ are always real and they cannot be equal unless $a = b = c$.

7. If the equation $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ has equal roots, then prove that $c^2 = a^2(1 + m^2)$.

3.8.6 Relations between roots and coefficients of a quadratic equation

Consider a quadratic equation $ax^2 + bx + c = 0$, where a , b , c are real numbers and $a \neq 0$. The roots of the given equation are α and β, where

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Then, the sum of the roots, $\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-b}{a} = \frac{-\text{coefficient of } x}{\text{coefficient of } x^2}$$

and the product of roots, $\alpha\beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \times \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2}$$

$$= \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

Therefore, if α, β are the roots of $ax^2 + bx + c = 0$, then

(i) the sum of the roots, $\alpha + \beta = -\frac{b}{a}$

(ii) the product of roots, $\alpha\beta = \frac{c}{a}$

Formation of quadratic equation when roots are given

Let α and β be the roots of a quadratic equation.

Then $(x - \alpha)$ and $(x - \beta)$ are factors.

$\therefore \quad (x - \alpha)(x - \beta) = 0$

$\Longrightarrow \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0$

That is, $x^2 - (\text{sum of roots})x + \text{product of roots} = 0$

> *Note*
>
> There are infinitely many quadratic equations with the same roots.

Example 3.48

If one of the roots of the equation $3x^2 - 10x + k = 0$ is $\frac{1}{3}$, then find the other root and also the value of k.

Solution The given equation is $3x^2 - 10x + k = 0$.

Let the two roots be α and β.

$$\therefore \quad \alpha + \beta = \frac{-(-10)}{3} = \frac{10}{3} \qquad (1)$$

Substituting $\alpha = \frac{1}{3}$ in (1) we get $\beta = 3$

Also, $\alpha\beta = \frac{k}{3}, \quad \Longrightarrow \quad k = 3$

Thus, the other root $\beta = 3$ and the value of $k = 3$.

Example 3.49

If the sum and product of the roots of the quadratic equation $ax^2 - 5x + c = 0$ are both equal to 10, then find the values of a and c.

Solution The given equation is $ax^2 - 5x + c = 0$.

Sum of the roots, $\frac{5}{a} = 10, \Longrightarrow a = \frac{1}{2}$

Product of the roots, $\frac{c}{a} = 10$

$\Longrightarrow \quad c = 10a = 10 \times \frac{1}{2} = 5$

Hence, $a = \frac{1}{2}$ and $c = 5$

Note

If α and β are the roots of $ax^2 + bx + c = 0$, then many expressions in α and β like $\alpha^2 + \beta^2$, $\alpha^2\beta^2$, $\alpha^2 - \beta^2$ etc., can be evaluated using the values of $\alpha + \beta$ and $\alpha\beta$.

Let us write some results involving α and β.

(i) $|\alpha - \beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

(ii) $\alpha^2 + \beta^2 = [(\alpha+\beta)^2 - 2\alpha\beta]$

(iii) $\alpha^2 - \beta^2 = (\alpha+\beta)(\alpha-\beta) = (\alpha+\beta)[\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}]$ only if $\alpha \geq \beta$

(iv) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

(v) $\alpha^3 - \beta^3 = (\alpha-\beta)^3 + 3\alpha\beta(\alpha-\beta)$

(vi) $\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2 = [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

(vii) $\alpha^4 - \beta^4 = (\alpha+\beta)(\alpha-\beta)(\alpha^2+\beta^2)$

Example 3.50

If α and β are the roots of the equation $2x^2 - 3x - 1 = 0$, find the values of

(i) $\alpha^2 + \beta^2$ (ii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(iii) $\alpha - \beta$ if $\alpha > \beta$ (iv) $\left(\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}\right)$

(v) $\left(\alpha + \dfrac{1}{\beta}\right)\left(\dfrac{1}{\alpha} + \beta\right)$ (vi) $\alpha^4 + \beta^4$ (vii) $\dfrac{\alpha^3}{\beta} + \dfrac{\beta^3}{\alpha}$

Solution Given equation is $2x^2 - 3x - 1 = 0$

Let the given equation be written as $ax^2 + bx + c = 0$

Then, $a = 2$, $b = -3$, $c = -1$. Given α and β are the roots of the equation.

$\therefore\ \alpha + \beta = \dfrac{-b}{a} = \dfrac{-(-3)}{2} = \dfrac{3}{2}$ and $\alpha\beta = -\dfrac{1}{2}$

(i) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = \left(\dfrac{3}{2}\right)^2 - 2\left(-\dfrac{1}{2}\right) = \dfrac{9}{4} + 1 = \dfrac{13}{4}$

(ii) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \dfrac{\left(\frac{3}{2}\right)^2 - 2\left(-\frac{1}{2}\right)}{-\frac{1}{2}} = \dfrac{13}{4} \times (-2) = -\dfrac{13}{2}$

(iii) $\alpha - \beta = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= \left[\left(\dfrac{3}{2}\right)^2 - 4 \times \left(-\dfrac{1}{2}\right)\right]^{\frac{1}{2}} = \left(\dfrac{9}{4} + 2\right)^{\frac{1}{2}} = \dfrac{\sqrt{17}}{2}$

(iv) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}=\dfrac{\alpha^3+\beta^3}{\alpha\beta}=\dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\dfrac{\frac{27}{8}+\frac{9}{4}}{\frac{-1}{2}}=-\dfrac{45}{4}$

(v) $\left(\alpha+\dfrac{1}{\beta}\right)\left(\dfrac{1}{\alpha}+\beta\right)=\dfrac{(\alpha\beta+1)(1+\alpha\beta)}{\alpha\beta}$

$$=\frac{(1+\alpha\beta)^2}{\alpha\beta}=\frac{\left(1-\frac{1}{2}\right)^2}{-\frac{1}{2}}=-\frac{1}{2}$$

(vi) $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$

$$=\left(\frac{13}{4}\right)^2-2\left(-\frac{1}{2}\right)^2=\left(\frac{169}{16}-\frac{1}{2}\right)=\frac{161}{16}.$$

(vii) $\dfrac{\alpha^3}{\beta}+\dfrac{\beta^3}{\alpha}=\dfrac{\alpha^4+\beta^4}{\alpha\beta}=\left(\dfrac{161}{16}\right)\left(-\dfrac{2}{1}\right)=-\dfrac{161}{8}.$

Example 3.51

Form the quadratic equation whose roots are $7+\sqrt{3}$ and $7-\sqrt{3}$.

Solution Given roots are $7+\sqrt{3}$ and $7-\sqrt{3}$.

$\therefore$ Sum of the roots $=7+\sqrt{3}+7-\sqrt{3}=14$.

Product of roots $=(7+\sqrt{3})(7-\sqrt{3})=(7)^2-(\sqrt{3})^2=49-3=46$.

The required equation is $x^2-(\text{sum of the roots})x+(\text{product of the roots})=0$

Thus, the required equation is $x^2-14x+46=0$

Example 3.52

If α and β are the roots of the equation

$3x^2-4x+1=0$, form a quadratic equation whose roots are $\dfrac{\alpha^2}{\beta}$ and $\dfrac{\beta^2}{\alpha}$.

Solution Since α,β are the roots of the equation $3x^2-4x+1=0$,

we have $\alpha+\beta=\dfrac{4}{3}$, $\alpha\beta=\dfrac{1}{3}$

Now, for the required equation, the sum of the roots $=\left(\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}\right)=\dfrac{\alpha^3+\beta^3}{\alpha\beta}$

$$=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}=\frac{\left(\frac{4}{3}\right)^3-3\times\frac{1}{3}\times\frac{4}{3}}{\frac{1}{3}}=\frac{28}{9}$$

Also, product of the roots $=\left(\dfrac{\alpha^2}{\beta}\right)\left(\dfrac{\beta^2}{\alpha}\right)=\alpha\beta=\dfrac{1}{3}$

$\therefore$ The required equation is $x^2-\dfrac{28}{9}x+\dfrac{1}{3}=0$ or $9x^2-28x+3=0$

Exercise 3.18

1. Find the sum and the product of the roots of the following equations.

   (i) $x^2 - 6x + 5 = 0$ (ii) $kx^2 + rx + pk = 0$

   (iii) $3x^2 - 5x = 0$ (iv) $8x^2 - 25 = 0$

2. Form a quadratic equation whose roots are

   (i) 3 , 4 (ii) $3+\sqrt{7}, 3-\sqrt{7}$ (iii) $\frac{4+\sqrt{7}}{2}, \frac{4-\sqrt{7}}{2}$

3. If α and β are the roots of the equation $3x^2 - 5x + 2 = 0$, then find the values of

   (i) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (ii) $\alpha - \beta$ (iii) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

4. If α and β are the roots of the equation $3x^2 - 6x + 4 = 0$, find the value of $\alpha^2 + \beta^2$.

5. If α, β are the roots of $2x^2 - 3x - 5 = 0$, form a equation whose roots are α^2 and β^2.

6. If α, β are the roots of $x^2 - 3x + 2 = 0$, form a quadratic equation whose roots are $-\alpha$ and $-\beta$.

7. If α and β are the roots of $x^2 - 3x - 1 = 0$, then form a quadratic equation whose roots are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.

8. If α and β are the roots of the equation $3x^2 - 6x + 1 = 0$, form an equation whose roots are (i) $\frac{1}{\alpha}, \frac{1}{\beta}$ (ii) $\alpha^2\beta, \beta^2\alpha$ (iii) $2\alpha + \beta, 2\beta + \alpha$

9. Find a quadratic equation whose roots are the reciprocal of the roots of the equation $4x^2 - 3x - 1 = 0$.

10. If one root of the equation $3x^2 + kx - 81 = 0$ is the square of the other, find k.

11. If one root of the equation $2x^2 - ax + 64 = 0$ is twice the other, then find the value of a

12. If α and β are the roots of $5x^2 - px + 1 = 0$ and $\alpha - \beta = 1$, then find p.

Exercise 3.19

Choose the correct answer.

1. If the system $6x - 2y = 3$, $kx - y = 2$ has a unique solution, then

   (A) $k = 3$ (B) $k \neq 3$ (C) $k = 4$ (D) $k \neq 4$

2. A system of two linear equations in two variables is inconsistent, if their graphs

   (A) coincide (B) intersect only at a point

   (C) do not intersect at any point (D) cut the x-axis

3. The system of equations $x - 4y = 8$, $3x - 12y = 24$

   (A) has infinitely many solutions (B) has no solution

   (C) has a unique solution (D) may or may not have a solution

4. If one zero of the polynomial $p(x) = (k+4)x^2+13x+3k$ is reciprocal of the other, then k is equal to

(A) 2 (B) 3 (C) 4 (D) 5

5. The sum of two zeros of the polynomial $f(x) = 2x^2 + (p+3)x + 5$ is zero, then the value of p is

(A) 3 (B) 4 (C) –3 (D) –4

6. The remainder when $x^2 - 2x + 7$ is divided by $x+4$ is

(A) 28 (B) 29 (C) 30 (D) 31

7. The quotient when $x^3 - 5x^2 + 7x - 4$ is divided by $x-1$ is

(A) $x^2 + 4x + 3$ (B) $x^2 - 4x + 3$ (C) $x^2 - 4x - 3$ (D) $x^2 + 4x - 3$

8. The GCD of $(x^3 + 1)$ and $x^4 - 1$ is

(A) $x^3 - 1$ (B) $x^3 + 1$ (C) $x + 1$ (D) $x - 1$

9. The GCD of $x^2 - 2xy + y^2$ and $x^4 - y^4$ is

(A) 1 (B) $x+y$ (C) $x-y$ (D) $x^2 - y^2$

10. The LCM of $x^3 - a^3$ and $(x - a)^2$ is

(A) $(x^3 - a^3)(x + a)$ (B) $(x^3 - a^3)(x - a)^2$

(C) $(x - a)^2(x^2 + ax + a^2)$ (D) $(x + a)^2(x^2 + ax + a^2)$

11. The LCM of a^k, a^{k+3}, a^{k+5} where $k \in \mathbb{N}$ is

(A) a^{k+9} (B) a^k (C) a^{k+6} (D) a^{k+5}

12. The lowest form of the rational expression $\dfrac{x^2 + 5x + 6}{x^2 - x - 6}$ is

(A) $\dfrac{x-3}{x+3}$ (B) $\dfrac{x+3}{x-3}$ (C) $\dfrac{x+2}{x-3}$ (D) $\dfrac{x-3}{x+2}$

13. If $\dfrac{a+b}{a-b}$ and $\dfrac{a^3 - b^3}{a^3 + b^3}$ are the two rational expressions, then their product is

(A) $\dfrac{a^2 + ab + b^2}{a^2 - ab + b^2}$ (B) $\dfrac{a^2 - ab + b^2}{a^2 + ab + b^2}$ (C) $\dfrac{a^2 - ab - b^2}{a^2 + ab + b^2}$ (D) $\dfrac{a^2 + ab + b^2}{a^2 - ab - b^2}$

14. On dividing $\dfrac{x^2 - 25}{x+3}$ by $\dfrac{x+5}{x^2 - 9}$ is equal to

(A) $(x-5)(x-3)$ (B) $(x-5)(x+3)$ (C) $(x+5)(x-3)$ (D) $(x+5)(x+3)$

15. If $\dfrac{a^3}{a-b}$ is added with $\dfrac{b^3}{b-a}$, then the new expression is

(A) $a^2 + ab + b^2$ (B) $a^2 - ab + b^2$ (C) $a^3 + b^3$ (D) $a^3 - b^3$

16. The square root of $49(x^2 - 2xy + y^2)^2$ is

(A) $7|x - y|$ (B) $7(x + y)(x - y)$ (C) $7(x + y)^2$ (D) $7(x - y)^2$

17. The square root of $x^2 + y^2 + z^2 - 2xy + 2yz - 2zx$

(A) $|x + y - z|$ (B) $|x - y + z|$ (C) $|x + y + z|$ (D) $|x - y - z|$

18. The square root of $121x^4y^8z^6(l-m)^2$ is

(A) $11x^2y^4z^4|l-m|$ (B) $11x^4y^4|z^3(l-m)|$

(C) $11x^2y^4z^6|l-m|$ (D) $11x^2y^4|z^3(l-m)|$

19. If $ax^2+bx+c=0$ has equal roots, then c is equal

(A) $\frac{b^2}{2a}$ (B) $\frac{b^2}{4a}$ (C) $-\frac{b^2}{2a}$ (D) $-\frac{b^2}{4a}$

20. If $x^2+5kx+16=0$ has no real roots, then

(A) $k>\frac{8}{5}$ (B) $k>-\frac{8}{5}$ (C) $-\frac{8}{5}<k<\frac{8}{5}$ (D) $0<k<\frac{8}{5}$

21. A quadratic equation whose one root is 3 is

(A) $x^2-6x-5=0$ (B) $x^2+6x-5=0$

(C) $x^2-5x-6=0$ (D) $x^2-5x+6=0$

22. The common root of the equations $x^2-bx+c=0$ and $x^2+bx-a=0$ is

(A) $\frac{c+a}{2b}$ (B) $\frac{c-a}{2b}$ (C) $\frac{c+b}{2a}$ (D) $\frac{a+b}{2c}$

23. If α,β are the roots of $ax^2+bx+c=0$ $a\neq 0$, then the wrong statement is

(A) $\alpha^2+\beta^2=\frac{b^2-2ac}{a^2}$ (B) $\alpha\beta=\frac{c}{a}$

(C) $\alpha+\beta=\frac{b}{a}$ (D) $\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{b}{c}$

24. If α and β are the roots of $ax^2+bx+c=0$, then one of the quadratic equations whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$, is

(A) $ax^2+bx+c=0$ (B) $bx^2+ax+c=0$

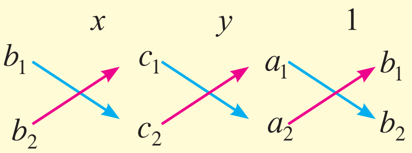
(C) $cx^2+bx+a=0$ (D) $cx^2+ax+b=0$

25. Let $b=a+c$. Then the equation $ax^2+bx+c=0$ has equl roots, if

(A) a = c (B) a = – c (C) a = 2c (D) a = –2c

Points to Remember

- A set of finite number of linear equations in two variables x and y is called a system of linear equations in x and y. Such a system is also called simultaneous equations.
- Eliminating one of the variables first and then solving a system is called method of elimination.
- The following arrow diagram helps us very much to apply the method of cross multiplication in solving $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$.



- A real number k is said to be a zero of a polynomial $p(x)$, if $p(k)=0$.

- The basic relationships between zeros and coefficients of a quadratic polynomial $p(x) = ax^2 + bx + c$ are

  Sum of zeros $= -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

  Product of zeros $= \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

- (i) For any polynomial $p(x)$, $x = a$ is zero if and only if $p(a) = 0$.

  (ii) $x - a$ is a factor for $p(x)$ if and only if $p(a) = 0$.

- GCD of two or more algebraic expressions is the expression of highest degree which divides each of them without remainder.

- LCM of two or more algebraic expressions is the expression of lowest degree which is divisible by each of them without remainder.

- The product of LCM and GCD of any two polynomials is equal to the product of the two polynomials.

- Let $a \in \mathbb{R}$ be a non negative real number. A square root of a, is a real number b such that $b^2 = a$. The square root of a is denoted by $\sqrt[2]{a}$ or $\sqrt{a}$.

- A quadratic equation in the variable x is of the form $ax^2 + bx + c = 0$, where a,b,c are real numbers and $a \neq 0$.

- A quadratic equation can be solved by (i) the method of factorization (ii) the method of completing square (iii) using a quadratic formula.

- The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided $b^2 - 4ac \geq 0$.

- A quadratic equation $ax^2 + bx + c = 0$ has

  (i) two distinct real roots if $b^2 - 4ac > 0$

  (ii) two equal roots if $b^2 - 4ac = 0$, and

  (iii) no real roots if $b^2 - 4ac < 0$

Do you know?

Fermat's last theorem: The equation $x^n + y^n = z^n$ has no integer solution when $n > 2$. Fermat wrote, " I have discovered a truely remarkable proof which this margin is too small to contain ". No one was able to solve this for over 300 years until British mathematician Andrew Wiles solved it in 1994. Interestingly he came to know about this problem in his city library when he was a high school student.`