4

MATRICES

Number, place, and combination - the three intersecting but distinct spheres of thought to which all mathematical ideas admit of being referred - Sylvester

- Introduction
- Formation of Matrices
- Types of Matrices
- Addition, Subtraction and Multiplication of matrices
- Matrix equations

James Joseph Sylvester

(1814-1897)

England

James Joseph Sylvester made fundamental contributions to matrix theory, invariant theory, number theory and combinatorics. He determined all matrices that commute with a given matrix. He introduced many mathematical terms including "discriminant".

In 1880, the Royal Society of London awarded Sylvester the Copley Medal, a highest award for scientific achievement. In 1901, Royal Society of London instituted the Sylvester medal in his memory, to encourage mathematical research.

4.1 Introduction

In this chapter we are going to discuss an important mathematical object called "MATRIX". Here, we shall introduce matrices and study the basics of matrix algebra.

Matrices were formulated and developed as a concept during 18th and 19th centuries. In the beginning, their development was due to transformation of geometric objects and solution of linear equations. However matrices are now one of the most powerful tools in mathematics. Matrices are useful because they enable us to consider an array of many numbers as a single object and perform calculations with these symbols in a very compact form. The " mathematical shorthand" thus obtained is very elegant and powerful and is suitable for various practical problems.

The term "Matrix" for arrangement of numbers, was introduced in 1850 by James Joseph Sylvester. "Matrix" is the Latin word for womb, and it retains that sense in English. It can also mean more generally any place in which something is formed or produced.

Now let us consider the following system of linear equations in x and y :

$$3x - 2y = 4 \qquad (1)$$

$$2x + 5y = 9 \qquad (2)$$

We already know how to get the solution (2, 1) of this system by the method of elimination (also known as Gaussian Elimination method), where only the coefficients are used and not the variables. The same method can easily be executed and the solution can thus be obtained using matrix algebra.

4.2 Formation of matrices

Let us consider some examples of the ways that matrices can arise.

Kumar has 10 pens. We may express it as (10), with the understanding that the number inside ( ) is the number of pens that Kumar has.

Now, if Kumar has 10 pens and 7 pencils, we may express it as (10 7) with the understanding that the first number inside ( ) is the number of pens while the other one is the number of pencils.

Look at the following information :

Pens and Pencils owned by Kumar and his friends Raju and Gopu are as given below.

Kumar has 10 pens and 7 pencils
Raju has 8 pens and 4 pencils
Gopu has 6 pens and 5 pencils

This can be arranged in tabular form as follows:

	Pens	Pencils
Kumar	10	7
Raju	8	4
Gopu	6	5

This can be expressed in a rectangular array where the entries denote the number of respective items.

(i) $\begin{pmatrix} 10 & 7 \\ 8 & 4 \\ 6 & 5 \end{pmatrix}$ ← first row, ← second row, ← third row

↑ first column, ↑ second column

The same information can also be arranged in tabular form as :

	Kumar	Raju	Gopu
Pens	10	8	6
Pencils	7	4	5

This can be expressed in a rectangular array.

(ii) $\begin{pmatrix} 10 & 8 & 6 \\ 7 & 4 & 5 \end{pmatrix}$ ← first row, ← second row

↑ first column, ↑ second column, ↑ third column

In arrangement (i), the entries in the first column represent the number of pens of Kumar, Raju and Gopu respectively and the second column represents the number of pencils owned by Kumar, Raju and Gopu respectively.

Similarly, in arrangement (ii), the entries in the first row represent the number of pens of Kumar, Raju and Gopu respectively. The entries in the second row represent the number of pencils owned by Kumar, Raju and Gopu respectively.

An arrangement or display of numbers of the above kind is called a MATRIX.

Definition

A matrix is a rectangular array of numbers in rows and columns enclosed within square brackets or parenthesis.

A matrix is usually denoted by a single capital letter like A, B, X, Y, $\cdots$. The numbers that make up a matrix are called entries or elements of the matrix. Each horizontal arrangement in a matrix is called a row of that matrix. Each vertical arrangement in a matrix is called a column of that matrix.

Some examples of matrices are

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -8 & 9 \\ 1 & 5 & -1 \end{bmatrix} \text{ and } C = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

4.2.1 General form of a matrix

A matrix A with m rows and n columns, is of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix}$$

where $a_{11}, a_{12}, a_{13}, \dots$ are the elements of the matrix. The above matrix can also be written as $A = [a_{ij}]_{m\times n}$ or $A = (a_{ij})_{m\times n}$, where $i = 1, 2, 3, \dots, m$. and $j = 1, 2, 3, \dots, n$.

Here, a_{ij} is the element of the matrix lying on the intersection of the i^{th} row and j^{th} column of A.

For example, if $A = \begin{pmatrix} 4 & 5 & 3 \\ 6 & 2 & 1 \\ 7 & 8 & 9 \end{pmatrix}$, then $a_{23} = 1$, the element which occurs in the second row and third column.

Similarly, $a_{11} = 4$, $a_{12} = 5$, $a_{13} = 3$, $a_{21} = 6$, $a_{22} = 2$, $a_{31} = 7$, $a_{32} = 8$ and $a_{33} = 9$.

4.2.2 Order or dimension of a matrix

If a matrix A has m rows and n columns, then we say that the order of A is $m \times n$ (Read as m by n).

The matrix

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ has 2 rows and 3 columns. So, the order of A is 2×3.

> *Note*
>
> In a $m \times n$ matrix, the first letter m always denotes the number of rows and the second letter n always denotes the number of columns.

4.3 Types of matrices

Let us learn certain types of matrices.

(i) Row matrix

A matrix is said to be a **row matrix** if it has only one row. A row matrix is also called as a row vector.

For example, $A = (5 \ \ 3 \ \ 4 \ \ 1)$ and $B = (-3 \ \ 0 \ \ 5)$ are row matrices of orders 1×4 and 1×3 respectively.

In general, $A = (a_{ij})_{1 \times n}$ is a row matrix of order $1 \times n$.

(ii) Column matrix

A matrix is said to be a column matrix if it has only one column. It is also called as a column vector.

For example, $A = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$ are column matrices of orders 2×1 and 3×1 respectively.

In general, $A = [a_{ij}]_{m \times 1}$ is a column matrix of order $m \times 1$.

(iii) Square matrix

A matrix in which the number of rows and the number of columns are equal is said to be a square matrix**.** For example,

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 2 \\ 1 & 5 & -7 \\ 7 & 6 & 1 \end{pmatrix}$ are square matrices of orders 2 and 3 respectively.

In general, $A = [a_{ij}]_{m \times m}$ is a square matrix of order m.

The elements $a_{11}, a_{22}, a_{33}, \cdots, a_{mm}$ are called principal or leading diagonal elements of the square matrix A.

(iv) Diagonal matrix

A square matrix in which all the elements above and below the leading diagonal are equal to zero, is called a diagonal matrix. For example,

$A = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are diagonal matrices of orders 2 and 3 respectively. In general, $A = [a_{ij}]_{m \times m}$ is said to be a diagonal matrix if $a_{ij} = 0$ for all $i \neq j$.

Note Some of the leading diagonal elements of a diagonal matrix may be zero.

(v) Scalar matrix

A diagonal matrix in which all the elements along the leading diagonal are equal to a non-zero constant is called a scalar matrix. For example,

$$A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix} \text{ are scalar matrices of orders 2 and 3 respectively.}$$

In general, $A = [a_{ij}]_{m \times m}$ is said to be a scalar matrix if $a_{ij} = \begin{cases} 0, \text{when } & i \neq j \\ k, \text{when } & i = j \end{cases}$ where k is a constant.

(vi) Unit matrix

A diagonal matrix in which all the leading diagonal entries are 1 is called a unit matrix. A unit matrix of order n is denoted by I_n. For example,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ are unit matrices of orders 2 and 3 respectively.}$$

In general, a square matrix $A = (a_{ij})_{n \times n}$ is a unit matrix if $a_{ij} = \begin{cases} 1 \text{ if } & i = j \\ 0 \text{ if } & i \neq j \end{cases}$

Note A unit matrix is also called an identity matrix with respect to multiplication. Every unit matrix is clearly a scalar matrix. However a scalar matrix need not be a unit matrix. A unit matrix plays the role of the number 1 in numbers.

(vii) Null matrix or Zero-matrix

A matrix is said to be a null matrix or zero-matrix if each of its elements is zero. It is denoted by O. For example,

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ are null matrices of order } 2 \times 3 \text{ and } 2 \times 2.$$

Note (i) A zero-matrix need not be a square matrix. (ii) Zero-matrix plays the role of the number zero in numbers. (iii) A matrix does not change if the zero-matrix of same order is added to it or subtracted from it.

(viii) Transpose of a matrix

Definition The transpose of a matrix A is obtained by interchanging rows and columns of the matrix A and it is denoted by A^T (read as A transpose). For example,

$$\text{if } A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{pmatrix}, \text{ then } A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix}$$

In general, if $A = [a_{ij}]_{m \times n}$ then

$$A^T = [b_{ij}]_{n \times m}, \text{ where } b_{ij} = a_{ji}, \text{ for } i = 1, 2, \cdots, n \text{ and } j = 1, 2, \cdots, m.$$

Example 4.1

The table shows a five-day forecast indicating high (H) and low (L) temperatures in Fahrenheit. Organise the temperatures in a matrix where the first and second rows represent the High and Low temperatures respectively and identify which day will be the warmest?

Mon	Tue	Wed	Thu	Fri
H 88	H 90	H 86	H 84	H 85
L 54	L 56	L 53	L 52	L 52

Solution The above information can be represented in matrix form as

$$A = \begin{matrix} \\ H \\ L \end{matrix}\begin{matrix} \text{Mon} & \text{Tue} & \text{Wed} & \text{Thu} & \text{Fri} \\ \left(\begin{matrix} 88 \\ 54 \end{matrix}\right. & \begin{matrix} 90 \\ 56 \end{matrix} & \begin{matrix} 86 \\ 53 \end{matrix} & \begin{matrix} 84 \\ 52 \end{matrix} & \left.\begin{matrix} 85 \\ 52 \end{matrix}\right) \end{matrix}. \quad \text{That is , } A = \begin{pmatrix} 88 & 90 & 86 & 84 & 85 \\ 54 & 56 & 53 & 52 & 52 \end{pmatrix}$$

By reading through the first row (High), the warmest day is Tuesday.

Example 4.2

The amount of fat, carbohydrate and protein in grams present in each food item respectively are as follows:

	Item 1	Item 2	Item 3	Item 4
Fat	5	0	1	10
Carbohydrate	0	15	6	9
Protein	7	1	2	8

Use the information to write 3×4 and 4×3 matrices.

Solution The above information can be represented in the form of 3×4 matrix as

$$A = \begin{pmatrix} 5 & 0 & 1 & 10 \\ 0 & 15 & 6 & 9 \\ 7 & 1 & 2 & 8 \end{pmatrix}$$ where the columns correspond to food items. We write

a 4×3 matrix as $B = \begin{pmatrix} 5 & 0 & 7 \\ 0 & 15 & 1 \\ 1 & 6 & 2 \\ 10 & 9 & 8 \end{pmatrix}$ where the rows correspond to food items.

Example 4.3

Let $A = [a_{ij}] = \begin{pmatrix} 1 & 4 & 8 \\ 6 & 2 & 5 \\ 3 & 7 & 0 \\ 9 & -2 & -1 \end{pmatrix}$. Find

(i) the order of the matrix (ii) the elements a_{13} and a_{42} (iii) the position of the element 2.

Solution (i) Since the matrix A has 4 rows and 3 columns, A is of order 4×3.

(ii) The element a_{13} is in the first row and third column. $\therefore\ a_{13} = 8$.

Similarly, $a_{42} = -2$, the element in 4th row and 2nd column.

(iii) The element 2 occurs in 2nd row and 2nd column $\therefore\ a_{22} = 2$.

Example 4.4

Construct a 2×3 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = |2i - 3j|$

Solution In general a 2×3 matrix is given by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Now, $a_{ij} = |2i - 3j|$ where $i = 1, 2$ and $j = 1, 2, 3$

$a_{11} = |2(1) - 3(1)| = |-1| = 1$, $a_{12} = |2(1) - 3(2)| = 4$, $a_{13} = |2(1) - 3(3)| = 7$

$a_{21} = |2(2) - 3| = 1$, $a_{22} = |2(2) - 3(2)| = 2$, $a_{23} = |2(2) - 9| = 5$

Hence the required matrix $A = \begin{pmatrix} 1 & 4 & 7 \\ 1 & 2 & 5 \end{pmatrix}$

Example 4.5

If $A = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$, then find A^T and $(A^T)^T$

Solution

$$A = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$$

The transpose A^T of a matrix A, is obtained by interchanging rows and columns of the matrix A.

Thus, $A^T = \begin{pmatrix} 8 & 1 \\ 5 & -3 \\ 2 & 4 \end{pmatrix}$

Similarly $(A^T)^T$ is obtained by interchanging rows and columns of the matrix A^T.

Hence, $(A^T)^T = \begin{pmatrix} 8 & 5 & 2 \\ 1 & -3 & 4 \end{pmatrix}$

Note

From the above example, we see that $(A^T)^T = A$. In fact, it is true that $(B^T)^T = B$ for any matrix B. Also, $(kA)^T = kA^T$ for any scalar k.

Exercise 4.1

1. The rates for the entrance tickets at a water theme park are listed below:

	Week Days rates(₹)	Week End rates(₹)
Adult	400	500
Children	200	250
Senior Citizen	300	400

Write down the matrices for the rates of entrance tickets for adults, children and senior citizens. Also find the dimensions of the matrices.

2. There are 6 Higher Secondary Schools, 8 High Schools and 13 Primary Schools in a town. Represent these data in the form of 3×1 and 1×3 matrices.

3. Find the order of the following matrices.

(i) $\begin{pmatrix} 1 & -1 & 5 \\ -2 & 3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 & -2 & 6 \\ 6 & -1 & 1 \\ 2 & 4 & 5 \end{pmatrix}$ (iv) $\begin{pmatrix} 3 & 4 & 5 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 2 \\ -2 & 3 \\ 9 & 7 \\ 6 & 4 \end{pmatrix}$

4. A matrix has 8 elements. What are the possible orders it can have?

5. A matrix consists of 30 elements. What are the possible orders it can have?.

6. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by

(i) $a_{ij} = ij$ (ii) $a_{ij} = 2i - j$ (iii) $a_{ij} = \dfrac{i-j}{i+j}$

7. Construct a 3×2 matrix $A = [a_{ij}]$ whose elements are given by

(i) $a_{ij} = \dfrac{i}{j}$ (ii) $a_{ij} = \dfrac{(i-2j)^2}{2}$ (iii) $a_{ij} = \dfrac{|2i - 3j|}{2}$

8. If $A = \begin{pmatrix} 1 & -1 & 3 & 2 \\ 5 & -4 & 7 & 4 \\ 6 & 0 & 9 & 8 \end{pmatrix}$, (i) find the order of the matrix (ii) write down the elements a_{24} and a_{32} (iii) in which row and column does the element 7 occur?

9. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \\ 5 & 0 \end{pmatrix}$, then find the transpose of A.

10. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{pmatrix}$, then verify that $(A^T)^T = A$.

4.4 Operation on matrices

In this section, we shall discuss the equality of matrices, multiplication of a matrix by a scalar, addition, subtraction and multiplication of matrices.

(i) Equality of matrices

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are said to be equal if

(i) they are of the same order and

(ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, the matrices $\begin{pmatrix} 6 & 3 \\ 0 & 9 \\ 1 & 5 \end{pmatrix}$ and $\begin{pmatrix} 6 & 0 & 1 \\ 3 & 9 & 5 \end{pmatrix}$ are not equal as the orders of the matrices are different.

Also $\begin{pmatrix} 1 & 2 \\ 8 & 5 \end{pmatrix} \neq \begin{pmatrix} 1 & 8 \\ 2 & 5 \end{pmatrix}$, since some of the corresponding elements are not equal.

Example 4.6

Find the values of x, y and z if $\begin{pmatrix} x & 5 & 4 \\ 5 & 9 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 & z \\ 5 & y & 1 \end{pmatrix}$

Solution As the given matrices are equal, their corresponding elements must be equal. Comparing the corresponding elements, we get $x = 3$, $y = 9$ and $z = 4$.

Example 4.7

Solve : $\begin{pmatrix} y \\ 3x \end{pmatrix} = \begin{pmatrix} 6 - 2x \\ 31 + 4y \end{pmatrix}$

Solution Since the matrices are equal, the corresponding elements are equal.

Comparing the corresponding elements, we get $y = 6 - 2x$ and $3x = 31 + 4y$.

Using $y = 6 - 2x$ in the other equation, we get $3x = 31 + 4(6 - 2x)$

$$3x = 31 + 24 - 8x$$

$\therefore$ $x = 5$ and hence $y = 6 - 2(5) = -4$.

Thus, $x = 5$ and $y = -4$.

(ii) Multiplication of a matrix by a scalar

Definition

For a given matrix $A = [a_{ij}]_{m\times n}$ and a scalar (real number) k, we define a new matrix $B = [b_{ij}]_{m\times n}$, where $b_{ij} = ka_{ij}$ for all i and j.

Thus, the matrix B is obtained by multiplying each entry of A by the scalar k and written as $B = kA$. This multiplication is called scalar multiplication.

For example, if $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ then $kA = k\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} ka & kb & kc \\ kd & ke & kf \end{pmatrix}$

Example 4.8

If $A = \begin{pmatrix} -1 & 2 & 4 \\ 3 & 6 & -5 \end{pmatrix}$ then find $3A$

Solution The matrix $3A$ is obtained by multiplying every element of A by 3.

$$3A = 3\begin{pmatrix} -1 & 2 & 4 \\ 3 & 6 & -5 \end{pmatrix} = \begin{pmatrix} 3(-1) & 3(2) & 3(4) \\ 3(3) & 3(6) & 3(-5) \end{pmatrix} = \begin{pmatrix} -3 & 6 & 12 \\ 9 & 18 & -15 \end{pmatrix}$$

(iii) Addition of matrices

Matrices A and B given below show the marks obtained by 3 boys and 3 girls in the subjects Mathematics and Science respectively.

Mathematics: $A = \begin{pmatrix} 45 & 72 & 81 \\ 30 & 90 & 65 \end{pmatrix}\begin{matrix}\text{Boys} \\ \text{Girls}\end{matrix}$ Science: $B = \begin{pmatrix} 51 & 80 & 90 \\ 42 & 85 & 70 \end{pmatrix}\begin{matrix}\text{Boys} \\ \text{Girls}\end{matrix}$

To find the total marks obtained by each student, we shall add the corresponding entries of A and B. We write

$$A+B=\begin{pmatrix}45 & 72 & 81\\ 30 & 90 & 65\end{pmatrix}+\begin{pmatrix}51 & 80 & 90\\ 42 & 85 & 70\end{pmatrix}$$

$$=\begin{pmatrix}45+51 & 72+80 & 81+90\\ 30+42 & 90+85 & 65+70\end{pmatrix}=\begin{pmatrix}96 & 152 & 171\\ 72 & 175 & 135\end{pmatrix}$$

The final matrix shows that the first boy scores a total of 96 marks in Mathematics and Science. Similarly, the last girl scores a total of 135 marks in Mathematics and Science.

Hence, we observe that the sum of two matrices of same order is a matrix obtained by adding the corresponding entries of the given matrices.

Definition

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order, then the addition of A and B is a matrix $C=[c_{ij}]_{mxn}$, where $c_{ij}=a_{ij}+b_{ij}$ for all i and j.

Note that the operation of addition on matrices is defined as for numbers. The addition of two matrices A and B is denoted by $A+B$. Addition is not defined for matrices of different orders.

Example 4.9

Let $A=\begin{pmatrix}8 & 3 & 2\\ 5 & 9 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & -1\\ 3 & 0\end{pmatrix}$. Find $A+B$ if it exists.

Solution Since A is order of 2×3 and B is of order 2×2, addition of matrices A and B is not possible.

Example 4.10

If $A=\begin{pmatrix}5 & 6 & -2 & 3\\ 1 & 0 & 4 & 2\end{pmatrix}$ and $B=\begin{pmatrix}3 & -1 & 4 & 7\\ 2 & 8 & 2 & 3\end{pmatrix}$, then find $A+B$.

Solution Since A and B are of the same order 2×4, addition of A and B is defined.

So, $$A+B=\begin{pmatrix}5 & 6 & -2 & 3\\ 1 & 0 & 4 & 2\end{pmatrix}+\begin{pmatrix}3 & -1 & 4 & 7\\ 2 & 8 & 2 & 3\end{pmatrix}$$

$$=\begin{pmatrix}5+3 & 6-1 & -2+4 & 3+7\\ 1+2 & 0+8 & 4+2 & 2+3\end{pmatrix}$$

Thus, $$A+B=\begin{pmatrix}8 & 5 & 2 & 10\\ 3 & 8 & 6 & 5\end{pmatrix}$$

(iv) Negative of a matrix

The negative of a matrix $A=[a_{ij}]_{mxn}$ is denoted by $-A$ and is defined as $-A=(-1)A$. That is, $-A=[b_{ij}]_{mxn}$, where $b_{ij}=-a_{ij}$ for all i and j.

(v) Subtraction of matrices

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices of the same order, then the subtraction $A-B$ is defined as $A-B=A+(-1)B$. That is, $A-B=[c_{ij}]$ where $c_{ij}=a_{ij}-b_{ij}$ for all i and j.

Example 4.11

Matrix A shows the weight of four boys and four girls in kg at the beginning of a diet programme to lose weight. Matrix B shows the corresponding weights after the diet programme.

$$A = \begin{pmatrix} 35 & 40 & 28 & 45 \\ 42 & 38 & 41 & 30 \end{pmatrix}\begin{matrix}\text{Boys}\\ \text{Girls}\end{matrix}, \quad B = \begin{pmatrix} 32 & 35 & 27 & 41 \\ 40 & 30 & 34 & 27 \end{pmatrix}\begin{matrix}\text{Boys}\\ \text{Girls}\end{matrix}$$

Find the weight loss of the Boys and Girls.

Solution Weight loss matrix $A - B = \begin{pmatrix} 35 & 40 & 28 & 45 \\ 42 & 38 & 41 & 30 \end{pmatrix} - \begin{pmatrix} 32 & 35 & 27 & 41 \\ 40 & 30 & 34 & 27 \end{pmatrix}$

$$= \begin{pmatrix} 3 & 5 & 1 & 4 \\ 2 & 8 & 7 & 3 \end{pmatrix}.$$

4.5 Properties of matrix addition

(i) Matrix addition is commutative

If A and B are any two matrices of same order, then $A+B = B+A$

(ii) Matrix addition is associative

If A, B and C are any three matrices of same order, then $A + (B + C) = (A + B) + C$

(iii) Existence of additive identity

Null or zero matrix is the additive identity for matrix addition. If A is a matrix of order $m \times n$, then $A + O = O + A = A$, where O is the null matrix of order $m \times n$,

(iv) Existence of additive inverse

For a matrix A, B is called the additive inverse of A if $B + A = A + B = O$.

Since $A + (-A) = (-A) + A = O$, $-A$ is the additive inverse of A.

Note

The additive inverse of a matrix is its negative matrix and it is unique (only one).

Exercise 4.2

1. Find the values of x, y and z from the matrix equation
$$\begin{pmatrix} 5x + 2 & y - 4 \\ 0 & 4z + 6 \end{pmatrix} = \begin{pmatrix} 12 & -8 \\ 0 & 2 \end{pmatrix}$$

2. Solve for x and y if $\begin{pmatrix} 2x + y \\ x - 3y \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix}$

3. If $A = \begin{pmatrix} 2 & 3 \\ -9 & 5 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 7 & -1 \end{pmatrix}$, then find the additive inverse of A.

4. Let $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & -1 \\ 4 & 3 \end{pmatrix}$. Find the matrix C if $C = 2A + B$.

5. If $A = \begin{pmatrix} 4 & -2 \\ 5 & -9 \end{pmatrix}$ and $B = \begin{pmatrix} 8 & 2 \\ -1 & -3 \end{pmatrix}$ find $6A - 3B$.

6. Find a and b if $a\begin{pmatrix} 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$.

7. Find X and Y if $2X + 3Y = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$ and $3X + 2Y = \begin{pmatrix} 2 & -2 \\ -1 & 5 \end{pmatrix}$.

8. Solve for x and y if $\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} + 3\begin{pmatrix} 2x \\ -y \end{pmatrix} = \begin{pmatrix} -9 \\ 4 \end{pmatrix}$.

9. If $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then verify: (i) $A + B = B + A$ (ii) $A + (-A) = O = (-A) + A$.

10. If $A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & -2 & 3 \\ 0 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 8 \\ 2 & 4 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}$, then verify that $A + (B + C) = (A + B) + C$.

11. An electronic company records each type of entertainment device sold at three of their branch stores so that they can monitor their purchases of supplies. The sales in two weeks are shown in the following spreadsheets.

		T.V.	**DVD**	**Videogames**	**CD Players**
Week I	Store I	30	15	12	10
	Store II	40	20	15	15
	Store III	25	18	10	12
Week II	Store I	25	12	8	6
	Store II	32	10	10	12
	Store III	22	15	8	10

Find the sum of the items sold out in two weeks using matrix addition.

12. The fees structure for one-day admission to a swimming pool is as follows:

Daily Admission Fees in ₹		
Member	Children	Adult
Before 2.00 p.m.	20	30
After 2.00 p.m.	30	40
Non-Member		
Before 2.00 p.m.	25	35
After 2.00 p.m.	40	50

Write the matrix that represents the additional cost for non-membership.

4.6 Multiplication of matrices

Suppose that Selvi wants to buy 3 pens and 2 pencils, while Meena needs 4 pens and 5 pencils. Each pen and pencil cost ₹10 and ₹5 respectively. How much money does each need to spend?

Clearly, Since $3 \times 10 + 2 \times 5 = 40$, Selvi needs ₹ 40.

Since $4 \times 10 + 5 \times 5 = 65$, Meena needs ₹ 65.

We can also do this using matrix multiplication.

Let us write the above information as follows:

Requirements	Price (in ₹)	Money Needed (in ₹)
Selvi, Meena $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$	$\begin{pmatrix} 10 \\ 5 \end{pmatrix}$	$\begin{pmatrix} 3 \times 10 + 2 \times 5 \\ 4 \times 10 + 5 \times 5 \end{pmatrix} = \begin{pmatrix} 40 \\ 65 \end{pmatrix}$

Suppose the cost of each pen and pencil in another shop are ₹8 and ₹4 respectively. The money required by Selvi and Meena will be $3 \times 8 + 2 \times 4 =$ ₹32 and $4 \times 8 + 5 \times 4 =$ ₹ 52 . The above information can be represented as

Requirements	Price (in ₹)	Money Needed (in ₹)
Selvi, Meena $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$	$\begin{pmatrix} 8 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 3 \times 8 + 2 \times 4 \\ 4 \times 8 + 5 \times 4 \end{pmatrix} = \begin{pmatrix} 32 \\ 52 \end{pmatrix}$

Now, the above information in both the cases can be combined in matrix form as shown below.

Requirements	Price (in ₹)	Money needed (in ₹)
Selvi, Meena $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$	$\begin{pmatrix} 10 & 8 \\ 5 & 4 \end{pmatrix}$	$\begin{pmatrix} 3 \times 10 + 2 \times 5 & 3 \times 8 + 2 \times 4 \\ 4 \times 10 + 5 \times 5 & 4 \times 8 + 5 \times 4 \end{pmatrix} = \begin{pmatrix} 40 & 32 \\ 65 & 52 \end{pmatrix}$

From the above example, we observe that multiplication of two matrices is possible if the number of columns in the first matrix is equal to the number of rows in the second matrix. Further, for getting the elements of the product matrix, we take rows of the first matrix and columns of the second matrix, multiply them element-wise and sum it.

The following simple example illustrates how to get the elements of the product matrix when the product is defined.

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix}$. Then the product of AB is defined and is given by

$$AB = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix}$$

Step 1 : Multiply the numbers in the first row of A by the numbers in the first column of B, add the products, and put the result in the first row and first column of AB.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3) + (-1)5 & \\ & \end{pmatrix}$$

Step 2: Follow the same procedure as in step 1, using the first row of A and second column of B. Write the result in the first row and second column of AB.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ & \end{pmatrix}$$

Step 3: Follow the same procedure with the second row of A and first column of B. Write the result in the second row and first column of AB.

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & \end{pmatrix}$$

Step 4: The procedure is the same for the numbers in the second row of A and second column of B.

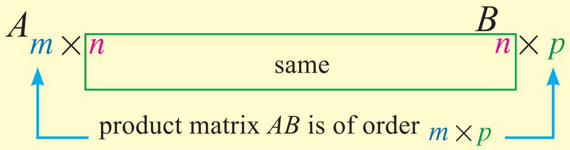
$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 3 & -9 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & 3(-9)+4(7) \end{pmatrix}$$

Step 5: Simplify to get the product matrix AB

$$\begin{pmatrix} 2(3)+(-1)5 & 2(-9)+(-1)7 \\ 3(3)+4(5) & 3(-9)+4(7) \end{pmatrix} = \begin{pmatrix} 1 & -25 \\ 29 & 1 \end{pmatrix}$$

Definition

If $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then the product matrix AB is defined and is of order $m\times p$. This fact is explained in the following diagram.



Example 4.12

Determine whether each matrix product is defined or not. If the product is defined, state the dimension of the product matrix.

(i) $A_{2\times 5}$ and $B_{5\times 4}$ (ii) $A_{1\times 3}$ and $B_{4\times 3}$

Solution

(i) Now, the number of columns in A and the number of rows in B are equal.

So, the product AB is defined.

Also, the product matrix AB is of order 2×4.

(ii) Given that A is of order 1×3 and B is of order 4×3

Now, the number of columns in A and the number of rows in B are not equal.

So, the matrix product AB is not defined.

Example 4.13

Solve $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$

Solution Given that $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$

$$\Longrightarrow \quad \begin{pmatrix} 3x + 2y \\ 4x + 5y \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

Equating the corresponding elements, we get

$$3x + 2y = 8 \quad \text{and} \quad 4x + 5y = 13$$

$$\Longrightarrow 3x + 2y - 8 = 0 \quad \text{and} \quad 4x + 5y - 13 = 0.$$

Solving the equations by the method of cross multiplication, we get

$$\begin{array}{cccccc} & x & & y & & 1 \\ 2 & & -8 & & 3 & & 2 \\ 5 & & -13 & & 4 & & 5 \end{array}$$

$$\Longrightarrow \frac{x}{-26 + 40} = \frac{y}{-32 + 39} = \frac{1}{15 - 8} \quad \Longrightarrow \quad \frac{x}{14} = \frac{y}{7} = \frac{1}{7}$$

Thus, $x = 2, \quad y = 1$

Example 4.14

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then show that $A^2 - (a + d)A = (bc - ad)I_2$.

Solution Consider $A^2 = A \times A$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} \qquad (1)$$

Now, $(a + d)A = (a + d)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix} \qquad (2)$$

From (1) and (2) we get,

$$A^2 - (a + d)A = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} - \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix}$$

$$= \begin{pmatrix} bc - ad & 0 \\ 0 & bc - ad \end{pmatrix} = (bc - ad)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, $A^2 - (a + d)A = (bc - ad)I_2$.

4.7 Properties of matrix multiplication

The matrix multiplication does not retain some important properties enjoyed by multiplication of numbers. Some of such properties are (i) $AB \neq BA$ (in general) (ii) $AB = 0$ does not imply that either A or B is a zero-matrix and (iii) $AB = AC$, A is a non-zero matrix, does not imply always that $B = C$.

For example, let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, C = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then,

(i) $AB \neq BA$ (ii) $AD = O$, however, A and D are not zero-matrices and (iii) $AB = AC$, but $B \neq C$. Let us see some properties of matrix multiplication through examples.

(i) Matrix multiplication is not commutative in general

If A and B are two matrices and if AB and BA both are defined, it is not necessary that $AB = BA$.

Example 4.15

If $A = \begin{pmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{pmatrix}$, then find AB and BA if they exist.

Solution The matrix A is of order 3×2 and B is of order 2×3. Thus, both the products AB and BA are defined.

$$\text{Now, } AB = \begin{pmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 72 - 42 & -24 + 7 & 16 + 35 \\ -18 + 24 & 6 - 4 & -4 - 20 \\ 0 + 18 & 0 - 3 & 0 - 15 \end{pmatrix} = \begin{pmatrix} 30 & -17 & 51 \\ 6 & 2 & -24 \\ 18 & -3 & -15 \end{pmatrix}$$

Similarly,

$$BA = \begin{pmatrix} 9 & -3 & 2 \\ 6 & -1 & -5 \end{pmatrix} \begin{pmatrix} 8 & -7 \\ -2 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 78 & -69 \\ 50 & -61 \end{pmatrix}. \qquad (\text{Note that } AB \neq BA)$$

Remarks

Multiplication of two diagonal matrices of same order is commutative. Also, under matrix multiplication unit matrix commutes with any square matrix of same order.

(ii) Matrix multiplication is always associative

For any three matrices A, B and C, we have $(AB)C = A(BC)$, whenever both sides of the equality are defined.

(iii) Matrix multiplication is distributive over addition

For any three matrices A, B and C, we have (i) $A(B + C) = AB + AC$

(ii) $(A + B)C = AC + BC$, whenever both sides of equality are defined.

Example 4.16

If $A = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 5 \\ 6 & 7 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix}$ verify that $A(B + C) = AB + AC$

Solution Now, $B + C = \begin{pmatrix} -2 & 5 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 1 & 10 \end{pmatrix}$

$$\text{Thus, } A(B + C) = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} -1 & 6 \\ 1 & 10 \end{pmatrix} = \begin{pmatrix} -1 & 38 \\ 5 & 34 \end{pmatrix} \qquad (1)$$

Now, $AB + AC = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & 7 \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -5 & 3 \end{pmatrix}$

$$= \begin{pmatrix} -6+12 & 15+14 \\ 2+24 & -5+28 \end{pmatrix} + \begin{pmatrix} 3-10 & 3+6 \\ -1-20 & -1+12 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 29 \\ 26 & 23 \end{pmatrix} + \begin{pmatrix} -7 & 9 \\ -21 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 38 \\ 5 & 34 \end{pmatrix} \qquad (2)$$

From (1) and (2), we have $A(B + C) = AB + AC$.

(iv) Existence of multiplicative identity

In ordinary algebra we have the number 1, which has the property that its product with any number is the number itself. We now introduce an analogous concept in matrix algebra.

For any square matrix A of order n, we have $AI = IA = A$, where I is the unit matrix of order n. Hence, I is known as the identity matrix under multiplication.

Example 4.17

If $A = \begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix}$, then verify $AI = IA = A$, where I is the unit matrix of order 2.

Solution

Now, $AI = \begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 0+3 \\ 9+0 & 0-6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix} = A$

Also, $IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix} = \begin{pmatrix} 1+0 & 3+0 \\ 0+9 & 0-6 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 9 & -6 \end{pmatrix} = A$

Hence $AI = IA = A$.

(v) Existence of multiplicative inverse

If A is a square matrix of order n, and if there exists a square matrix B of the same order n, such that $AB = BA = I$, where I is the unit matrix of order n, then B is called the multiplicative inverse matrix of A and it is denoted by A^{-1}.

Note

(i) Some of the square matrices like $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ do not have multiplicative inverses.

(ii) If B is the multiplicative inverse of A, then A is the multiplicative inverse of B.

(iii) If multiplicative inverse of a square matrix exists, then it is unique.

Example 4.18

Prove that $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ are multiplicative inverses to each other.

Solution Now, $\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6-5 & -15+15 \\ 2-2 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

Also, $\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\therefore$ The given matrices are inverses to each other under matrix multiplication.

(vi) Reversal law for transpose of matrices

If A and B are two matrices and if AB is defined, then $(AB)^T = B^T A^T$.

Example 4.19

If $A = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & -6 \end{pmatrix}$, then verify that $(AB)^T = B^T A^T$.

Solution Now, $AB = \begin{pmatrix} -2 \\ 4 \\ 5 \end{pmatrix}\begin{pmatrix} 1 & 3 & -6 \end{pmatrix} = \begin{pmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{pmatrix}$

Thus, $(AB)^T = \begin{pmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{pmatrix}$ (1)

Now, $B^T A^T = \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix}\begin{pmatrix} -2 & 4 & 5 \end{pmatrix}$

$= \begin{pmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{pmatrix}$ (2)

From (1) and (2), we get $(AB)^T = B^T A^T$.

Exercise 4.3

1. Determine whether the product of the matrices is defined in each case. If so, state the order of the product.

   (i) AB, where $A = [a_{ij}]_{4\text{x}3}$, $B = [b_{ij}]_{3\text{x}2}$ (ii) PQ, where $P = [p_{ij}]_{4\text{x}3}$, $Q = [q_{ij}]_{4\text{x}3}$

   (iii) MN, where $M = [m_{ij}]_{3\text{x}1}$, $N = [n_{ij}]_{1\text{x}5}$ (iv) RS, where $R = [r_{ij}]_{2\text{x}2}$, $S = [s_{ij}]_{2\text{x}2}$

2. Find the product of the matrices, if exists,

   (i) $\begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 7 \end{pmatrix}$

   (iii) $\begin{pmatrix} 2 & 9 & -3 \\ 4 & -1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -6 & 7 \\ -2 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 \\ -3 \end{pmatrix}\begin{pmatrix} 2 & -7 \end{pmatrix}$

3. A fruit vendor sells fruits from his shop. Selling prices of Apple, Mango and Orange are ₹ 20, ₹ 10 and ₹ 5 each respectively. The sales in three days are given below

Day	Apples	Mangoes	Oranges
1	50	60	30
2	40	70	20
3	60	40	10

Write the matrix indicating the total amount collected on each day and hence find the total amount collected from selling of all three fruits combined.

4. Find the values of x and y if $\begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 9 & 0 \end{pmatrix}$.

5. If $A = \begin{pmatrix} 5 & 3 \\ 7 & 5 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} -5 \\ -11 \end{pmatrix}$ and if $AX = C$, then find the values of x and y.

6. If $A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$ then show that $A^2 - 4A + 5I_2 = O$.

7. If $A = \begin{pmatrix} 3 & 2 \\ 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 3 & 2 \end{pmatrix}$ then find AB and BA. Are they equal?

8. If $A = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $C = (2 \quad 1)$ verify $(AB)C = A(BC)$.

9. If $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ verify that $(AB)^T = B^T A^T$.

10. Prove that $A = \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$ are inverses to each other under matrix multiplication.

11. Solve $(x \quad 1)\begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ 5 \end{pmatrix} = (\ 0\ )$.

12. If $A = \begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 6 \\ 3 & -2 \end{pmatrix}$, then prove that $(A+B)^2 \neq A^2 + 2AB + B^2$.

13. If $A = \begin{pmatrix} 3 & 3 \\ 7 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 7 \\ 0 & 9 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$, find $(A+B)C$ and $AC + BC$.

Is $(A+B)C = AC + BC$?

Exercise 4.4

Choose the correct answer.

1. Which one of the following statements is not true?

   (A) A scalar matrix is a square matrix

   (B) A diagonal matrix is a square matrix

   (C) A scalar matrix is a diagonal matrix

   (D) A diagonal matrix is a scalar matrix.

2. Matrix $A = [a_{ij}]_{m\times n}$ is a square matrix if

   (A) $m < n$ (B) $m > n$ (C) $m = 1$ (D) $m = n$

3. If $\begin{pmatrix} 3x+7 & 5 \\ y+1 & 2-3x \end{pmatrix} = \begin{pmatrix} 1 & y-2 \\ 8 & 8 \end{pmatrix}$ then the values of x and y respectively are

   (A) $-2, 7$ (B) $-\frac{1}{3}, 7$ (C) $-\frac{1}{3}, -\frac{2}{3}$ (D) $2, -7$

4. If $A = \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ then $A + B$

   (A) $(0\ \ 0\ \ 0)$ (B) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

   (C) (-14) (D) not defined

5. If a matrix is of order 2×3, then the number of elements in the matrix is

   (A) 5 (B) 6 (C) 2 (D) 3

6. If $\begin{pmatrix} 8 & 4 \\ x & 8 \end{pmatrix} = 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ then the value of x is

   (A) 1 (B) 2 (C) $\frac{1}{4}$ (D) 4

7. If A is of order 3×4 and B is of order 4×3, then the order of BA is

   (A) 3×3 (B) 4×4 (C) 4×3 (D) not defined

8. If $A\times\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix}$, then the order of A is

   (A) 2×1 (B) 2×2 (C) 1×2 (D) 3×2

9. If A and B are square matrices such that $AB = I$ and $BA = I$, then B is

   (A) Unit matrix (B) Null matrix

   (C) Multiplicative inverse matrix of A (D) $-A$

10. If $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, then the values of x and y respectively, are

    (A) $2, 0$ (B) $0, 2$ (C) $0, -2$ (D) $1, 1$

11. If $A = \begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ and $A + B = O$, then B is

(A) $\begin{pmatrix} 1 & -2 \\ -3 & 4 \end{pmatrix}$ (B) $\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}$ (C) $\begin{pmatrix} -1 & -2 \\ -3 & -4 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

12. If $A = \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$, then A^2 is

(A) $\begin{pmatrix} 16 & 4 \\ 36 & 9 \end{pmatrix}$ (B) $\begin{pmatrix} 8 & -4 \\ 12 & -6 \end{pmatrix}$ (C) $\begin{pmatrix} -4 & 2 \\ -6 & 3 \end{pmatrix}$ (D) $\begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix}$

13. A is of order $m \times n$ and B is of order $p \times q$, addition of A and B is possible only if

(A) $m = p$ (B) $n = q$ (C) $n = p$ (D) $m = p, n = q$

14. If $\begin{pmatrix} a & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, then the value of a is

(A) 8 (B) 4 (C) 2 (D) 11

15. If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$ is such that $A^2 = I$, then

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$

16. If $A = [a_{ij}]_{2\times 2}$ and $a_{ij} = i + j$, then $A =$

(A) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (B) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ (C) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ (D) $\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$

17. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the values of a, b, c and d respectively are

(A) $-1, 0, 0, -1$ (B) $1, 0, 0, 1$ (C) $-1, 0, 1, 0$ (D) $1, 0, 0, 0$

18. If $A = \begin{pmatrix} 7 & 2 \\ 1 & 3 \end{pmatrix}$ and $A + B = \begin{pmatrix} -1 & 0 \\ 2 & -4 \end{pmatrix}$, then the matrix $B =$

(A) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 6 & 2 \\ 3 & -1 \end{pmatrix}$ (C) $\begin{pmatrix} -8 & -2 \\ 1 & -7 \end{pmatrix}$ (D) $\begin{pmatrix} 8 & 2 \\ -1 & 7 \end{pmatrix}$

19. If $\begin{pmatrix} 5 & x & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 20 \end{pmatrix}$, then the value of x is

(A) 7 (B) -7 (C) $\frac{1}{7}$ (D) 0

20. Which one of the following is true for any two square matrices A and B of same order?.

(A) $(AB)^T = A^T B^T$ (B) $(A^T B)^T = A^T B^T$ (C) $(AB)^T = BA$ (D) $(AB)^T = B^T A^T$

Points to Remember

- A matrix is a rectangular array of numbers.
- A matrix having m rows and n columns, is of the order $m \times n$.
- $A = [a_{ij}]_{m\times n}$ is a row matrix if $m = 1$.
- $A = [a_{ij}]_{m\times n}$ is a column matrix if $n = 1$.
- $A = [a_{ij}]_{m\times n}$ is a square matrix if $m = n$.
- $A = [a_{ij}]_{n\times n}$ is diagonal matrix if $a_{ij} = 0$, when $i \neq j$.
- $A = [a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = 0$, when $i \neq j$ and $a_{ij} = k$, when $i = j$. (k is a non-zero constant).
- $A = [a_{ij}]$ is unit matrix if $a_{ij} = 1$, when $i = j$ and $a_{ij} = 0$, when $i \neq j$.
- A matrix is said to be a zero matrix if all its elements are zero.
- Two matrices A and B are equal if the matrices A and B are of same order and their corresponding entries are equal.
- Addition or subtraction of two matrices are possible only when they are of same order.
- Matrix addition is commutative.

  That is, $A + B = B + A$, if A and B are matrices of same order.
- Matrix addition is Associative.

  That is, $(A + B) + C = A + (B + C)$, if A, B and C are matrices of same order.
- If A is a matrix of order $m\times n$ and B is a matrix of order $n\times p$, then the product matrix AB is defined and is of order $m\times p$.
- Matrix multiplication is not commutative in general. i.e., $AB \neq BA$.
- Matrix multiplication is associative. i.e., $(AB)C = A(BC)$, if both sides are defined.
- $(A^T)^T = A$, $(A + B)^T = A^T + B^T$ and $(AB)^T = B^T A^T$.
- Matrices A and B are multiplicative inverses to each other if $AB = BA = I$.
- If $AB = O$, it is not necessary that $A = O$ or $B = O$.

  That is, product of two non-zero matrices may be a zero matrix.

Do you know?

The Abel Prize , which was awarded for the first time in 2003, amounts to One Million US dollar. It is an International Prize awarded by Norwegian Academy of Science and presented annually by the King of Norway to one or more outstanding Mathematicians.

S.R. Srinivasa Varadhan, an Indian-American Mathematician born in Chennai, was awarded the Abel Prize in 2007 for his fundamental contributions to Probability Theory and in particular for creating a unified theory of large deviations.