5

COORDINATE GEOMETRY

No human investigation can be called real science if it cannot be demonstrated mathematically - Leonardo de Vinci

- Introduction
- Section Formula
- Area of Triangle and Quadrilateral
- Straight Lines

Pierre de Fermat
(1601-1665)
France

Together with Rene Descartes, Fermat was one of the two leading mathematicians of the first half of the 17th century. He discovered the fundamental principles of analytical geometry. He discovered an original method of finding the greatest and the smallest ordinates of curved lines.

He made notable contributions to coordinate geometry. Fermat's pioneering work in analytic geometry was circulated in manuscript form in 1636, predating the publication of Descarte's famous "La geometrie".

5.1 Introduction

Coordinate geometry, also known as analytical geometry is the study of geometry using a coordinate system and the principles of algebra and analysis. It helps us to interpret algebraic results geometrically and serves as a bridge between algebra and geometry. A systematic study of geometry using algebra was carried out by a French philosopher and a mathematician Rene Descartes. The use of coordinates was Descartes's great contribution to mathematics, which revolutionized the study of geometry. He published his book "La Geometry" in 1637. In this book, he converted a geometric problem into an algebraic equation, simplified and then solved the equation geometrically. French mathematician Pierre De Fermat also formulated the coordinate geometry at the same period and made great contribution to this field. In 1692, a German mathematician Gottfried Wilhelm Von Leibnitz introduced the modern terms like abscissa and ordinate in coordinate geometry . According to Nicholas Murray Butler, "The analytical geometry of Descartes and the calculus of Newton and Leibntiz have expanded into the marvelous mathematical method".

In class IX, we have studied the basic concepts of the coordinate geometry namely, the coordinate axes, plane, plotting of points in a plane and the distance between two points. In this chapter, we shall study about section formula, area of a triangle, slope and equation of a straight line.

5.2 Section formula

Let us look at the following problem.

Let A and B be two towns. Assume that one can reach town B from A by moving 60km towards east and then 30km towards north . A telephone company wants to raise a relay tower at

P which divides the line joining A and B in the ratio $1:2$ internally. Now, it wants to find the position of P where the relay tower is to be set up.

Choose the point A as the origin. Let $P(x,y)$ be the point. Draw the perpendiculars from P and B to the x-axis, meeting it in C and D respectively. Also draw a perpendicular from P to BD, intersecting at E.

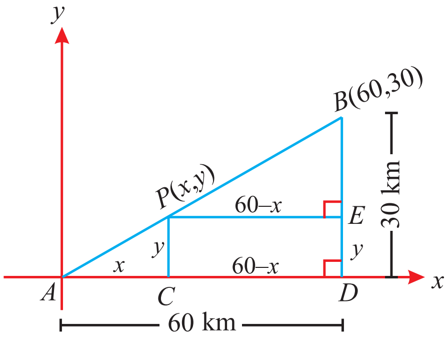


Fig. 5.1

Since ΔPAC and ΔBPE are similar, we have

$$\frac{AC}{PE} = \frac{PC}{BE} = \frac{AP}{PB} = \frac{1}{2}$$

Now $\frac{AC}{PE} = \frac{1}{2}$	
$\Longrightarrow \frac{x}{60-x} = \frac{1}{2}$	Also, $\frac{PC}{BE} = \frac{1}{2}$
$2x = 60 - x$	$\Longrightarrow \frac{y}{30-y} = \frac{1}{2}$
Thus, $x = 20.$	Thus, $2y = 30 - y \Longrightarrow y = 10.$

$\therefore$ The position of the relay tower is at $P(20,10)$.

Taking the above problem as a model, we shall derive the general section formula.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two distinct points such that a point $P(x, y)$ divides AB internally in the ratio $l:m$. That is, $\frac{AP}{PB} = \frac{l}{m}$

From the Fig. 5.2, we get

$$AF = CD = OD - OC = x - x_1$$

$$PG = DE = OE - OD = x_2 - x$$

Also, $$PF = PD - FD = y - y_1$$

$$BG = BE - GE = y_2 - y$$

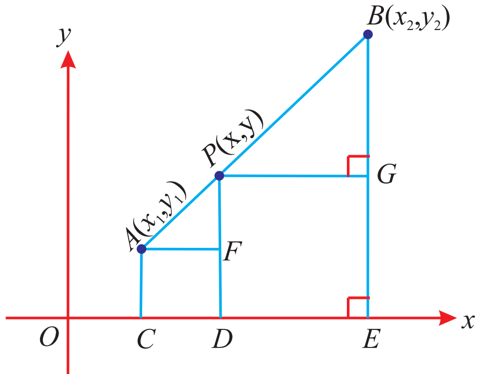


Fig. 5.2

Now, ΔAFP and ΔPGB are similar.
(Refer chapter 6, section 6.3)

Thus, $$\frac{AF}{PG} = \frac{PF}{BG} = \frac{AP}{PB} = \frac{l}{m}$$

$\therefore \frac{AF}{PG} = \frac{l}{m}$ and	$\frac{PF}{BG} = \frac{l}{m}$
$\Longrightarrow \frac{x-x_1}{x_2-x} = \frac{l}{m}$	$\Longrightarrow \frac{y-y_1}{y_2-y} = \frac{l}{m}$
$\Longrightarrow mx - mx_1 = lx_2 - lx$	$\Longrightarrow my - my_1 = ly_2 - ly$
$lx + mx = lx_2 + mx_1$	$ly + my = ly_2 + my_1$
$\Longrightarrow x = \frac{lx_2 + mx_1}{l+m}$	$\Longrightarrow y = \frac{ly_2 + my_1}{l+m}$

Thus, the point P which divides the line segment joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $l : m$ is

$$P\left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}\right)$$

This formula is known as **section formula.**

It is clear that the section formula can be used only when the related three points are collinear.

Results

(i) If P divides a line segment AB joining the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $l : m$, then the point P is $\left(\frac{lx_2 - mx_1}{l-m}, \frac{ly_2 - my_1}{l-m}\right)$. In this case $\frac{l}{m}$ is negative.

(ii) **Midpoint of AB**

If M is the midpoint of AB, then M divides the line segment AB internally in the ratio 1:1. By substituting $l = 1$ and $m = 1$ in the section formula, we obtain

the midpoint of AB as $M\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$.

The midpoint of the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

(iii) **Centroid of a triangle**

Consider a ΔABC whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let AD, BE and CF be the medians of the ΔABC.

We know that the medians of a triangle are concurrent and the point of concurrency is the centroid.

Let $G(x, y)$ be the centroid of ΔABC.

Now the midpoint of BC is $D\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$

By the property of triangle, the centroid G divides the median AD internally in the ratio 2 : 1

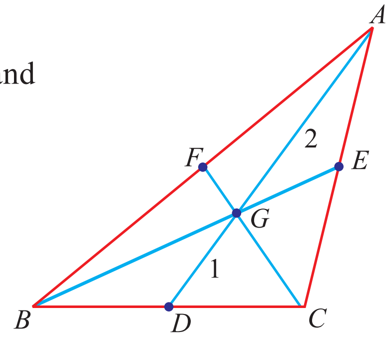


Fig. 5.3

$\therefore$ By section formula, the centroid

$$G(x, y) = G\left(\frac{2\frac{(x_2 + x_3)}{2} + 1(x_1)}{2+1}, \frac{2\frac{(y_2 + y_3)}{2} + 1(y_1)}{2+1}\right)$$

$$= G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

The centroid of the triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3), is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

Example 5.1

Find the midpoint of the line segment joining the points $(3, 0)$ and $(-1, 4)$.

Solution Midpoint $M(x, y)$ of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

$$M(x, y) = M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

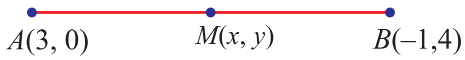


Fig. 5.4

$\therefore$ Midpoint of the line segment joining the points $(3, 0)$ and $(-1, 4)$ is

$$M(x, y) = \left(\frac{3 - 1}{2}, \frac{0 + 4}{2}\right) = M(1, 2).$$

Example 5.2

Find the point which divides the line segment joining the points (3 , 5) and (8 , 10) internally in the ratio 2 : 3.

Solution Let $A(3, 5)$ and $B(8, 10)$ be the given points.

Let the point $P(x, y)$ divide the line AB internally in the ratio 2 :3.

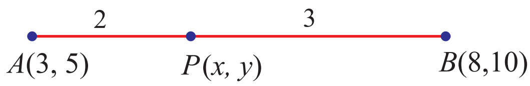


Fig. 5.5

By section formula, $P(x, y) = P\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right)$

Here $x_1 = 3, y_1 = 5, x_2 = 8, y_2 = 10$ and $l = 2, m = 3$

$$\therefore \quad P(x, y) = P\left(\frac{2(8) + 3(3)}{2 + 3}, \frac{2(10) + 3(5)}{2 + 3}\right) = P(5, 7)$$

Example 5.3

In what ratio does the point $P(-2, 3)$ divide the line segment joining the points $A(-3, 5)$ and $B(4, -9)$ internally?

Solution Given points are $A(-3, 5)$ and $B(4, -9)$.

Let $P(-2, 3)$ divide AB internally in the ratio $l : m$

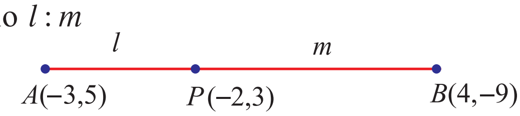


Fig. 5.6

By the section formula,

$$P\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right) = P(-2, 3) \quad (1)$$

Here $x_1 = -3, y_1 = 5, x_2 = 4, y_2 = -9$.

(1) $\Longrightarrow \left(\frac{l(4)+m(-3)}{l+m}, \frac{l(-9)+m(5)}{l+m}\right) = (-2, 3)$

Equating the x-coordinates, we get

$$\frac{4l-3m}{l+m} = -2$$

$$\Longrightarrow \quad 6l = m$$

$$\frac{l}{m} = \frac{1}{6}$$

i.e., $\quad l : m = 1 : 6$

Hence P divides AB internally in the ratio 1 : 6

Note

(i) In the above example, one may get the ratio by equating y-coordinates also.

(ii) The ratios obtained by equating x-coordinates and by equating y-coordinates are same only when the three points are collinear.

(ii) If a point divides the line segment internally in the ratio $l : m$, then $\frac{l}{m}$ is positive.

(iii) If a point divides the line segment externally in the ratio $l : m$, then $\frac{l}{m}$ is negative.

Example 5.4

Find the points of trisection of the line segment joining $(4, -1)$ and $(-2, -3)$.

Solution Let $A(4,-1)$ and $B(-2,-3)$ be the given points.

Let $P(x,y)$ and Q(a,b) be the points of trisection of AB so that $AP = PQ = QB$

Fig. 5.7

Hence P divides AB internally in the ratio 1 : 2 and Q divides AB internally in the ratio 2 : 1

Fig. 5.8

$\therefore$ By the section formula, the required points are

$P\left(\frac{1(-2)+2(4)}{1+2}, \frac{1(-3)+2(-1)}{1+2}\right)$ and

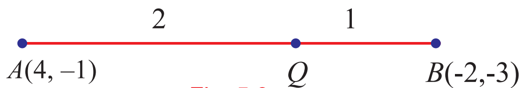


Fig. 5.9

$Q\left(\frac{2(-2)+1(4)}{2+1}, \frac{2(-3)+1(-1)}{2+1}\right)$

$\Longrightarrow \quad P(x,y) = P\left(\frac{-2+8}{3}, \frac{-3-2}{3}\right)$ and $\quad Q(a,b) = Q\left(\frac{-4+4}{3}, \frac{-6-1}{3}\right)$

$= P\left(2, -\frac{5}{3}\right) \qquad\qquad = Q\left(0, -\frac{7}{3}\right)$.

Note that Q is the midpoint of PB and P is the midpoint of AQ.

Example 5.5

Find the centroid of the triangle whose vertices are $A(4, -6)$, $B(3,-2)$ and $C(5, 2)$.

Solution The centroid $G(x , y)$ of a triangle whose vertices are

(x_1,y_1) , (x_2,y_2) and (x_3,y_3) is given by

$$G(x , y) = G\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).$$

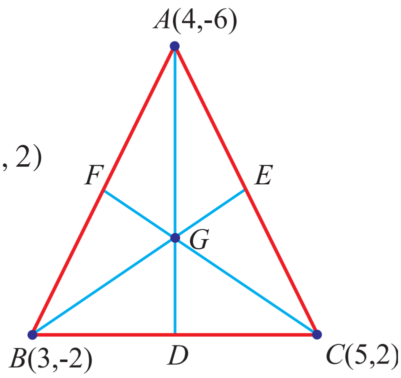


Fig. 5.10

We have $(x_1 , y_1) = (4, -6)$, $(x_2 , y_2) = (3, -2)$, $(x_3,y_3) = (5 , 2)$

$\therefore$ The centroid of the triangle whose vertices are $(4, -6), (3 , -2)$ and $(5, 2)$ is

$$G(x , y) = G\left(\frac{4 + 3 + 5}{3}, \frac{-6 - 2 + 2}{3}\right)$$

$$= G(4, -2).$$

Example 5.6

If $(7,3),(6,1),(8,2)$ and $(p,4)$ are the vertices of a parallelogram taken in order, then find the value of p.

Solution Let the vertices of the parallelogram be $A(7,3), B(6,1), C(8,2)$ and $D(p,4)$.

We know that the diagonals of a parallelogram bisect each other.

$\therefore$ The midpoints of the diagonal AC and the diagonal BD coincide.

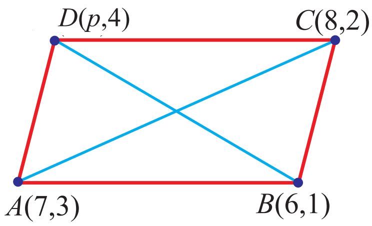


Fig. 5.11

Hence $$\left(\frac{7 + 8}{2}, \frac{3 + 2}{2}\right) = \left(\frac{6 + p}{2}, \frac{1 + 4}{2}\right)$$

$$\Longrightarrow \left(\frac{6 + p}{2}, \frac{5}{2}\right) = \left(\frac{15}{2}, \frac{5}{2}\right)$$

Equating the x-coordinates, we get,

$$\frac{6 + p}{2} = \frac{15}{2}$$

$$\therefore p = 9$$

Example 5.7

If C is the midpoint of the line segment joining $A(4 , 0)$ and $B(0 , 6)$ and if O is the origin, then show that C is equidistant from all the vertices of $\triangle OAB$.

Solution The midpoint of AB is $C\left(\frac{4 + 0}{2}, \frac{0 + 6}{2}\right) = C(2, 3)$

We know that the distance between $P(x_1,y_1)$ and $Q(x_2,y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Distance between $O(0,0)$ and $C(2,3)$ is

$$OC = \sqrt{(2 - 0)^2 + (3 - 0)^2} = \sqrt{13}.$$

Distance between $A(4,0)$ and $C(2,3)$,

$$AC = \sqrt{(2-4)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}$$

Distance between $B\ (0,6)$ and $C(2,3)$,

$$BC = \sqrt{(2-0)^2 + (3-6)^2} = \sqrt{4+9} = \sqrt{13}$$

$\therefore \quad OC = AC = BC$

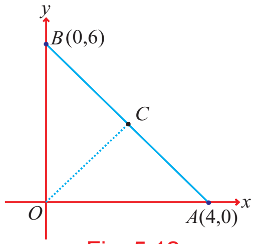


Fig. 5.12

$\therefore$ The point C is equidistant from all the vertices of the $\triangle OAB$.

Note The midpoint C of the hypotenuse, is the circumcentre of the right angled $\triangle OAB$.

Exercise 5.1

1. Find the midpoint of the line segment joining the points
   (i) $(1,-1)$ and $(-5,3)$ (ii) $(0,0)$ and $(0,4)$
2. Find the centroid of the triangle whose vertices are
   (i) $(1,3),(2,7)$ and $(12,-16)$ (ii) $(3,-5),(-7,4)$ and $(10,-2)$
3. The centre of a circle is at $(-6, 4)$. If one end of a diameter of the circle is at the origin, then find the other end.
4. If the centroid of a triangle is at $(1, 3)$ and two of its vertices are $(-7, 6)$ and $(8, 5)$ then find the third vertex of the triangle .
5. Using the section formula, show that the points $A(1,0)$, $B(5,3)$, $C(2,7)$ and $D(-2, 4)$ are the vertices of a parallelogram taken in order.
6. Find the coordinates of the point which divides the line segment joining $(3, 4)$ and $(-6, 2)$ in the ratio 3 : 2 externally.
7. Find the coordinates of the point which divides the line segment joining $(-3, 5)$ and $(4, -9)$ in the ratio 1 : 6 internally.
8. Let $A\ (-6,-5)$ and $B\ (-6, 4)$ be two points such that a point P on the line AB satisfies $AP = \frac{2}{9} AB$. Find the point P.
9. Find the points of trisection of the line segment joining the points $A(2,-2)$ and $B(-7, 4)$.
10. Find the points which divide the line segment joining $A(-4,0)$ and $B\ (0,6)$ into four equal parts.
11. Find the ratio in which the x-axis divides the line segment joining the points $(6, 4)$ and $(1,-7)$.
12. In what ratio is the line joining the points $(-5, 1)$ and $(2, 3)$ divided by the y-axis? Also, find the point of intersection .
13. Find the length of the medians of the triangle whose vertices are $(1, -1)$, $(0, 4)$ and $(-5, 3)$.

5.3 Area of a triangle

We have already learnt how to calculate the area of a triangle, when some measurements of the triangle are given. Now, if the coordinates of the vertices of a triangle are given, can we find its area ?

Let ABC be a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.

Draw the lines AD, BE and CF perpendicular to x-axis.

From the figure, $ED = x_1 - x_2$, $DF = x_3 - x_1$ and

$EF = x_3 - x_2$.

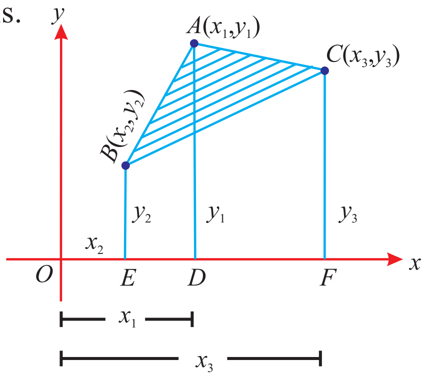


Fig. 5.13

Area of the triangle ABC

= Area of the trapezium $ABED$

+ Area of the trapezium $ADFC$

− Area of the trapezium $BEFC$

$$= \frac{1}{2}(BE + AD)ED + \frac{1}{2}(AD + CF)DF - \frac{1}{2}(BE + CF)EF$$

$$= \frac{1}{2}(y_2 + y_1)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}\{x_1y_2 - x_2y_2 + x_1y_1 - x_2y_1 + x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3 - x_3y_2 + x_2y_2 - x_3y_3 + x_2y_3\}$$

∴ Area of the ΔABC is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$.sq.units.

> If $A(x_1,y_1)$, $B(x_2, y_2)$, and $C(x_3,y_3)$ are the vertices of a ΔABC,
> then the area of the ΔABC is $\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$.sq.units.

Note

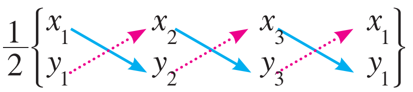
The area of the triangle can also be written as

$$\frac{1}{2}\{x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1 + x_3y_1 - x_3y_2\} \text{ sq.units.}$$

(or) $$\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}\text{sq.units}$$

The following pictorial representation helps us to write the above formula very easily.

Take the vertices $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ of ΔABC in counter clockwise direction and write them column-wise as shown below.



Add the diagonal products x_1y_2, x_2y_3 and x_3y_1 as shown in the dark arrows.

Also add the products x_2y_1, x_3y_2 and x_1y_3 as shown in the dotted arrows and then subtract the latter from the former to get the expression $\frac{1}{2}\left\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\}$

Note

To find the area of a triangle, the following steps may be useful.

(i) Plot the points in a rough diagram.

(ii) Take the vertices in counter clock-wise direction. Otherwise the formula gives a negative value.

(iii) Use the formula, area of the $\Delta ABC = \frac{1}{2}\left\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\right\}$

5.4 Collinearity of three points

Three or more points in a plane are said to be collinear, if they lie on the same straight line.

In other words, three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear if any one of these points lies on the straight line joining the other two points.

Suppose that the three points $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are collinear. Then they cannot form a triangle. Hence the area of the $\triangle ABC$ is zero.

i.e., $\frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\} = 0$

$$\Longrightarrow \quad x_1y_2 + x_2y_3 + x_3y_1 = x_2y_1 + x_3y_2 + x_1y_3$$

One can prove that the converse is also true.

Hence the area of $\triangle ABC$ is zero if and only if the points *A, B* and *C* are collinear.

5.5 Area of the Quadrilateral

Let $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ and $D(x_4,y_4)$ be the vertices of a quadrilateral *ABCD*.

Now the area of the quadrilateral *ABCD* = area of the ΔABD+area of the ΔBCD

$$= \frac{1}{2}\{(x_1y_2 + x_2y_4 + x_4y_1) - (x_2y_1 + x_4y_2 + x_1y_4)\}$$
$$+ \frac{1}{2}\{(x_2y_3 + x_3y_4 + x_4y_2) - (x_3y_2 + x_4y_3 + x_2y_4)\}$$

$\therefore$ Area of the quadrilateral *ABCD*

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\}$$

or

$$\frac{1}{2}\{(x_1 - x_3)(y_2 - y_4) - (x_2 - x_4)(y_1 - y_3)\} \text{ sq.units}$$

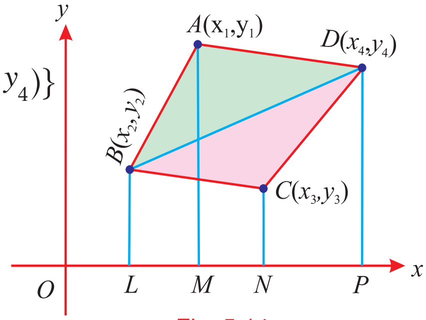


Fig. 5.14

The following pictorial representation helps us to write the above formula very easily.

Take the vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ in counter clockwise direction and write them column-wise as shown below. Follow the same technique as we did in the case of finding the area of a triangle.

$$\frac{1}{2}\begin{Bmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{Bmatrix}.$$

This helps us to get the required expression.

Thus, the area of the quadrilateral ABCD

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\} \text{ sq. units.}$$

Example 5.8

Find the area of the triangle whose vertices are (1, 2), (– 3 , 4), and (– 5 ,– 6).

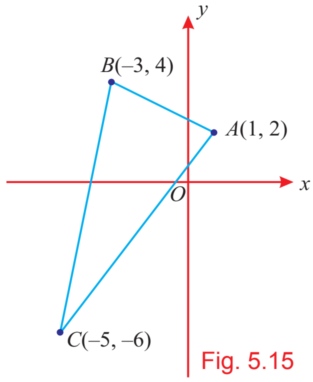


Fig. 5.15

Solution Plot the points in a rough diagram and take them in order.

Let the vertices be $A(1, 2)$, $B(-3, 4)$ and $C(-5, -6)$.

Now the area of $\triangle ABC$ is

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}$$

$$= \frac{1}{2}\{(4 + 18 - 10) - (-6 - 20 - 6)\} \qquad \text{use : } \frac{1}{2}\begin{Bmatrix} 1 & -3 & -5 & 1 \\ 2 & 4 & -6 & 2 \end{Bmatrix}$$

$$= \frac{1}{2}\{12 + 32\} = 22. \text{ sq. units}$$

Example 5.9

If the area of the ΔABC is 68 sq.units and the vertices are $A(6, 7)$, $B(-4, 1)$ and $C(a, -9)$ taken in order, then find the value of a.

Solution Area of $\triangle ABC$ is

$$\frac{1}{2}\{(6 + 36 + 7a) - (-28 + a - 54)\} = 68 \qquad \text{use : } \frac{1}{2}\begin{Bmatrix} 6 & -4 & a & 6 \\ 7 & 1 & -9 & 7 \end{Bmatrix}$$

$$\Longrightarrow (42 + 7a) - (a - 82) = 136$$

$$\Longrightarrow 6a = 12 \qquad \therefore \quad a = 2$$

Example 5.10

Show that the points $A(2, 3)$, $B(4, 0)$ and $C(6, -3)$ are collinear.

Solution Area of the ΔABC is

$$= \frac{1}{2}\{(0 - 12 + 18) - (12 + 0 - 6)\} \qquad \text{use : } \frac{1}{2}\begin{Bmatrix} 2 & 4 & 6 & 2 \\ 3 & 0 & -3 & 3 \end{Bmatrix}$$

$$= \frac{1}{2}\{6 - 6\} = 0.$$

$\therefore$ The given points are collinear.

Example 5.11

If $P(x\,,y)$ is any point on the line segment joining the points $(a\,,0)$ and $(0, b)$, then , prove that $\frac{x}{a}+\frac{y}{b}=1$, where $a, b \neq 0$.

Solution Now the points $(x\,,y)$, $(a\,,0)$ and $(0, b)$ are collinear.

$\therefore$ The area of the triangle formed by them is zero.

$\Longrightarrow \qquad ab - bx - ay = 0$ $\qquad$ use: $\frac{1}{2}\begin{Bmatrix} a & 0 & x & a \\ 0 & b & y & 0 \end{Bmatrix}$

$\therefore \qquad bx + ay = ab$

Dividing by ab on both sides, we get,

$$\frac{x}{a}+\frac{y}{b}=1, \qquad \text{where} \qquad a, b \neq 0$$

Example 5.12

Find the area of the quadrilateral formed by the points $(-4, -2)$, $(-3, -5)$, $(3, -2)$ and $(2\,, 3)$.

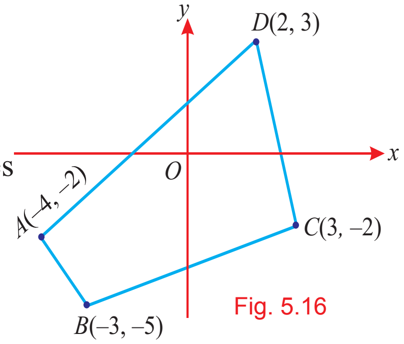


Fig. 5.16

Solution Let us plot the points roughly and take the vertices in counter clock-wise direction.

Let the vertices be

$A(-4, -2)$, $B(-3, -5)$, $C(3, -2)$ and $D(2, 3)$.

Area of the quadrilateral $ABCD$

$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1) - (x_2y_1 + x_3y_2 + x_4y_3 + x_1y_4)\}.$$

$$= \frac{1}{2}\{(20 + 6 + 9 - 4) - (6 - 15 - 4 - 12)\}$$

$$= \frac{1}{2}\{31 + 25\} = 28 \text{ sq.units.}$$

$$\frac{1}{2}\begin{Bmatrix} -4 & -3 & 3 & 2 & -4 \\ -2 & -5 & -2 & 3 & -2 \end{Bmatrix}$$

Exercise 5.2

1. Find the area of the triangle formed by the points
   (i) $(0, 0)$, $(3, 0)$ and $(0, 2)$ $\qquad$ (ii) $(5, 2)$, $(3, -5)$ and $(-5, -1)$
   (iii) $(-4, -5)$, $(4, 5)$ and $(-1, -6)$

2. Vertices of the triangles taken in order and their areas are given below. In each of the following find the value of a.

	Vertices	Area (in sq. units)
(i)	$(0\,,0)$, $(4, a)$, $(6, 4)$	17
(ii)	(a, a), $(4, 5)$, $(6, -1)$	9
(iii)	$(a, -3)$, $(3, a)$, $(-1, 5)$	12

3. Determine if the following set of points are collinear or not.

(i) $(4, 3), (1, 2)$ and $(-2, 1)$ (ii) $(-2, -2), (-6, -2)$ and $(-2, 2)$

(iii) $\left(-\frac{3}{2}, 3\right), (6, -2)$ and $(-3, 4)$

4. In each of the following, find the value of k for which the given points are collinear.

(i) $(k, -1), (2, 1)$ and $(4, 5)$ (ii) $(2, -5), (3, -4)$ and $(9, k)$

(iii) $(k, k), (2, 3)$ and $(4, -1)$

5. Find the area of the quadrilateral whose vertices are

(i) $(6, 9), (7, 4), (4, 2)$ and $(3, 7)$ (ii) $(-3, 4), (-5, -6), (4, -1)$ and $(1, 2)$

(iii) $(-4, 5), (0, 7), (5, -5)$ and $(-4, -2)$

6. If the three points $(h, 0), (a, b)$ and $(0, k)$ lie on a straight line, then using the area of the triangle formula, show that $\frac{a}{h} + \frac{b}{k} = 1$, where $h, k \neq 0$.

7. Find the area of the triangle formed by joining the midpoints of the sides of a triangle whose vertices are $(0, -1), (2, 1)$ and $(0, 3)$. Find the ratio of this area to the area of the given triangle.

5.6 Straight Lines

5.6.1 Angle of Inclination

Let a straight line l intersect the x-axis at A. The angle between the positive x-axis and the line l, measured in counter clockwise direction is called the angle of inclination of the straight line l.

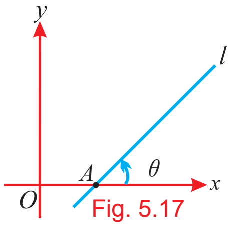


Fig. 5.17

Remarks

If θ is the angle of inclination of a straight line l, then

(i) $0^\circ \leq \theta \leq 180^\circ$

(ii) For horizontal lines, $\theta = 0^\circ$ or 180° and for vertical lines, $\theta = 90^\circ$

(iii) If a straight line initially lies along the x-axis and starts rotating about a fixed point A on the x-axis in the counter clockwise direction and finally coincides with the x-axis, then the angle of inclination of the straight line in the initial position is 0° and that of the line in the final position is 180°.

(iv) Lines which are perpendicular to x-axis are called as vertical lines. Other lines which are not perpendicular to x-axis are called as non vertical lines.

5.6.2 Slope of a straight line

Definition

If θ is the angle of inclination of a non-vertical straight line l, then $\tan\theta$ is called the Slope or Gradient of the line and is denoted by m.

$\therefore$ The slope of the straight line, $m = \tan\theta$ for $0^\circ \leq \theta \leq 180^\circ$, $\theta \neq 90^\circ$

Remarks

(i) Thus, the slope of x-axis or straight lines parallel to x-axis is zero.

(ii) The slope of y-axis or a straight line parallel to y-axis is not defined because $\tan 90^0$ is not defined. Therefore, whenever we talk about the slope of a straight line, we mean that of a non-vertical straight line.

(iii) If θ is acute, then the slope is positive, whereas if θ is obtuse then the slope is negative.

5.6.3 Slope of a straight line when any two points on the line are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on the straight line l whose angle of inclination is θ. Here, $0^\circ \leq \theta \leq 180^\circ$, $\theta \neq 90^\circ$

Let the straight line AB intersect the x-axis at C.

Now, the slope of the line l is $m = \tan\theta$ (1)

Draw AD and BE perpendicular to x-axis and draw the perpendicular AF line from A to BE.

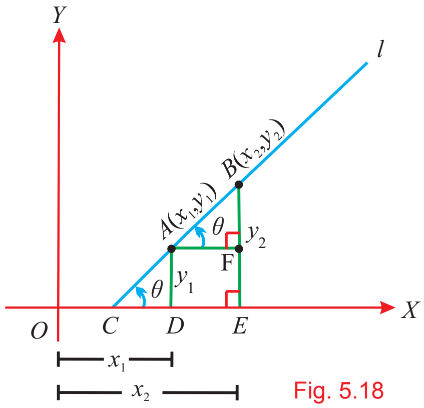


Fig. 5.18

From the figure, we have

$$AF = DE = OE - OD = x_2 - x_1$$

and $\quad BF = BE - EF = BE - AD = y_2 - y_1$

Also, we observe that $\angle DCA = \angle FAB = \theta$

In the right angled ΔABF, we have

$$\tan\theta = \frac{BF}{AF} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{if } x_1 \neq x_2 \qquad (2)$$

From (1) and (2), we get the slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

The slope of the straight line joining the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \text{ where } x_1 \neq x_2 \text{ as } \theta \neq 90^\circ.$$

Note

The slope of the straight line joining the points (x_1, y_1) and (x_2, y_2) is also interpreted as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y \text{ coordinates}}{\text{change in } x \text{ coordinates}}.$$

5.6.4 Condition for parallel lines in terms of their slopes

Consider parallel lines l_1 and l_2 whose angles of inclination are θ_1 and θ_2 and slopes are m_1 and m_2 respectively.

Since l_1 and l_2 are parallel, the angles of inclinations θ_1 and θ_2 are equal.

$$\therefore \quad \tan\theta_1 = \tan\theta_2 \quad \Longrightarrow \quad m_1 = m_2$$

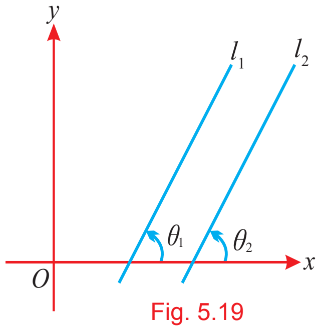


Fig. 5.19

$\therefore$ If two non-vertical straight lines are parallel, then their slopes are equal.

The converse is also true. i.e., if the slopes of two lines are equal, then the straight lines are parallel.

5.6.5 Condition for perpendicular lines in terms of their slopes

Let l_1 and l_2 be two perpendicular straight lines passing through the points $A(x_1,y_1)$ and $B(x_2,y_2)$ respectively.

Let m_1 and m_2 be their slopes.

Let $C(x_3,y_3)$ be their point of intersection.

The slope of the straight line l_1 is $m_1 = \dfrac{y_3 - y_1}{x_3 - x_1}$

The slope of the straight line l_2 is $m_2 = \dfrac{y_3 - y_2}{x_3 - x_2}$

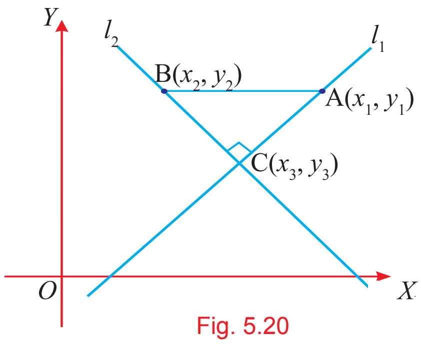


Fig. 5.20

In the right angled $\triangle ABC$, we have

$$AB^2 = AC^2 + BC^2$$

$$\Longrightarrow (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\Longrightarrow (x_2 - x_3 + x_3 - x_1)^2 + (y_2 - y_3 + y_3 - y_1)^2$$
$$= (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\Longrightarrow \left(x_2 - x_3\right)^2 + (x_3 - x_1\big)^2 + 2(x_2 - x_3)(x_3 - x_1) + (y_2 - y_3)^2 + (y_3 - y_1)^2 + 2(y_2 - y_3)(y_3 - y_1)$$
$$= (x_3 - x_1)^2 + (y_3 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$\Longrightarrow 2(x_2 - x_3)(x_3 - x_1) + 2(y_2 - y_3)(y_3 - y_1) = 0$$

$$\Longrightarrow (y_2 - y_3)(y_3 - y_1) = -(x_2 - x_3)(x_3 - x_1)$$

$$\left(\frac{y_3 - y_1}{x_3 - x_1}\right)\left(\frac{y_3 - y_2}{x_3 - x_2}\right) = -1.$$

$$\Longrightarrow \quad m_1 m_2 = -1 \text{ or } m_1 = -\frac{1}{m_2}$$

If two non-vertical straight lines with slopes m_1 and m_2, are perpendicular, then

$$m_1 m_2 = -1.$$

On the other hand, if $m_1 m_2 = -1$, then the two straight lines are perpendicular.

Note

The straight lines x-axis and y-axis are perpendicular to each other. But, the condition $m_1 m_2 = -1$ is not true because the slope of the x-axis is zero and the slope of the y-axis is not defined.

Example 5.13

Find the angle of inclination of the straight line whose slope is $\frac{1}{\sqrt{3}}$.

Solution If θ is the angle of inclination of the line, then the slope of the line is

$$m = \tan\theta \quad \text{where } 0^\circ \leq \theta \leq 180^\circ,\ \theta \neq 90^\circ.$$

$$\therefore \tan\theta = \frac{1}{\sqrt{3}} \implies \theta = 30^\circ$$

Example 5.14

Find the slope of the straight line whose angle of inclination is 45°.

Solution If θ is the angle of inclination of the line, then the slope of the line is $m = \tan\theta$

Given that $m = \tan 45^\circ \implies m = 1$.

Example 5.15

Find the slope of the straight line passing through the points $(3, -2)$ and $(-1, 4)$.

Solution Slope of the straight line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of the straight line passing through the points $(3, -2)$ and $(-1, 4)$ is

$$m = \frac{4+2}{-1-3} = -\frac{3}{2}.$$

Example 5.16

Using the concept of slope, show that the points $A(5, -2)$, $B(4, -1)$ and $C(1, 2)$ are collinear.

Solution Slope of the line joining the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

Slope of the line AB joining the points $A(5, -2)$ and $B(4 - 1)$ is $m_1 = \frac{-1+2}{4-5} = -1$

Slope of the line BC joining the points $B(4, -1)$ and $C(1, 2)$ is $m_2 = \frac{2+1}{1-4} = -1$

Thus, slope of AB = slope of BC.

Also, B is the common point.

Hence, the points A, B and C are collinear.

Example 5.17

Using the concept of slope, show that the points $(-2, -1)$, $(4, 0)$, $(3, 3)$ and $(-3, 2)$ taken in order form a parallelogram.

Solution Let $A(-2, -1)$, $B(4, 0)$, $C(3, 3)$ and $D(-3, 2)$ be the given points taken in order.

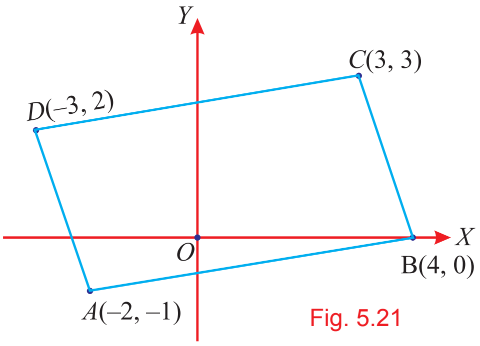


Fig. 5.21

Now the slope of $AB = \frac{0+1}{4+2} = \frac{1}{6}$

Slope of $CD = \frac{2-3}{-3-3} = \frac{1}{6}$

$\therefore$ Slope of AB = slope of CD

Hence, AB is parallel to CD. (1)

Now the slope of $BC = \frac{3-0}{3-4} = -3$

Slope of $AD = \frac{2+1}{-3+2} = -3$

$\therefore$ Slope of BC = slope of AD

Hence, BC is parallel to AD. (2)

From (1) and (2), we see that opposite sides of quadrilateral $ABCD$ are parallel

$\therefore$ $ABCD$ is a parallelogram.

Example 5.18

The vertices of a $\triangle ABC$ are $A(1, 2)$, $B(-4, 5)$ and $C(0, 1)$. Find the slopes of the altitudes of the triangle.

Solution Let AD, BE and CF be the altitudes of a $\triangle ABC$.

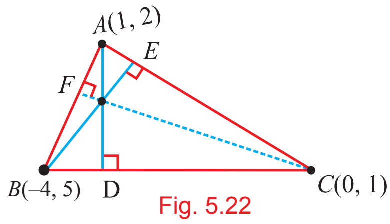


Fig. 5.22

slope of $BC = \frac{1-5}{0+4} = -1$

Since the altitude AD is perpendicular to BC,

slope of $AD = 1$ $\quad \because m_1 m_2 = -1$

slope of $AC = \frac{1-2}{0-1} = 1$

Thus, slope of $BE = -1$ $\quad \because BE \perp AC$

Also, slope of $AB = \frac{5-2}{-4-1} = -\frac{3}{5}$

$\therefore$ slope of $CF = \frac{5}{3}$ $\quad \because CF \perp AB$

Exercise 5.3

1. Find the angle of inclination of the straight line whose slope is
   (i) 1 (ii) $\sqrt{3}$ (iii) 0
2. Find the slope of the straight line whose angle of inclination is
   (i) 30° (ii) 60° (iii) 90°
3. Find the slope of the straight line passing through the points
   (i) $(3, -2)$ and $(7, 2)$ (ii) $(2, -4)$ and origin
   (iii) $(1+\sqrt{3}, 2)$ and $(3+\sqrt{3}, 4)$
4. Find the angle of inclination of the line passing through the points
   (i) $(1, 2)$ and $(2, 3)$ (ii) $(3, \sqrt{3})$ and $(0, 0)$
   (iii) (a, b) and $(-a, -b)$
5. Find the slope of the line which passes through the origin and the midpoint of the line segment joining the points $(0, -4)$ and $(8, 0)$.
6. The side AB of a square $ABCD$ is parallel to x-axis . Find the
   (i) slope of AB (ii) slope of BC (iii) slope of the diagonal AC
7. The side BC of an equilateral $\triangle ABC$ is parallel to x-axis. Find the slope of AB and the slope of BC.
8. Using the concept of slope, show that each of the following set of points are collinear.
   (i) $(2, 3)$, $(3, -1)$ and $(4, -5)$
   (ii) $(4, 1)$, $(-2, -3)$ and $(-5, -5)$ (iii) $(4, 4)$, $(-2, 6)$ and $(1, 5)$
9. If the points $(a, 1)$, $(1, 2)$ and $(0, b+1)$ are collinear, then show that $\frac{1}{a}+\frac{1}{b}=1$.
10. The line joining the points $A(-2, 3)$ and $B(a, 5)$ is parallel to the line joining the points $C(0, 5)$ and $D(-2, 1)$. Find the value of a.
11. The line joining the points $A(0, 5)$ and $B(4, 2)$ is perpendicular to the line joining the points $C(-1, -2)$ and $D(5, b)$. Find the value of b.
12. The vertices of $\triangle ABC$ are $A(1, 8)$, $B(-2, 4)$, $C(8, -5)$. If M and N are the midpoints of AB and AC respectively, find the slope of MN and hence verify that MN is parallel to BC.
13. A triangle has vertices at $(6, 7)$, $(2, -9)$ and $(-4, 1)$. Find the slopes of its medians.
14. The vertices of a $\triangle ABC$ are $A(-5, 7)$, $B(-4, -5)$ and $C(4, 5)$. Find the slopes of the altitudes of the triangle.

15. Using the concept of slope, show that the vertices $(1, 2)$, $(-2, 2)$, $(-4, -3)$ and $(-1, -3)$ taken in order form a parallelogram.

16. Show that the opposite sides of a quadrilateral with vertices $A(-2, -4)$, $B(5, -1)$, $C(6, 4)$ and $D(-1, 1)$ taken in order are parallel.

5.6.6 Equation of a straight line

Let L be a straight line in the plane. A first degree equation $px + qy + r = 0$ in the variables x and y is satisfied by the x-coordinate and y-coordinate of any point on the line L and any values of x and y that satisfy this equation will be the coordinates of a point on the line L. Hence this equation is called the equation of the straight line L. We want to describe this line L algebraically. That is, we want to describe L by an algebraic equation. Now L is in any one of the following forms:

(i) horizontal line (ii) vertical line (iii) neither vertical nor horizontal

(i) Horizontal line: Let L be a horizontal line.

Then either L is x-axis or L is a horizontal line other than x-axis.

Case (a) If L is x – axis, then a point (x, y) lies on L if and only if $y = 0$ and x can be any real number.

Thus, $y = 0$ describes x – axis.

$\therefore$ The equation of x-axis is $y = 0$

Case (b) L is a horizontal line other than x-axis.

That is, L is parallel to x-axis.

Now, a point (x, y) lies on L if and only if the y-coordinate must remain a constant and x can be any real number.

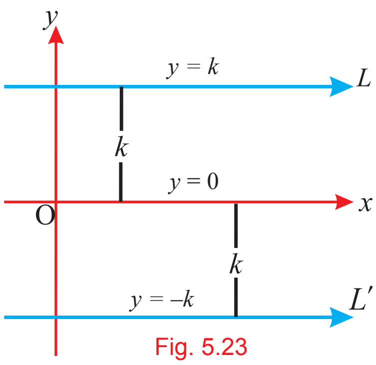


Fig. 5.23

$\therefore$ The equation of a straight line parallel to x-axis is $y = k$, where k is a constant.

Note that if $k > 0$, then L lies above x-axis and if $k < 0$, then L lies below x- axis. If $k = 0$, then L is nothing but the x-axis.

(ii) Vertical line: Let L be a vertical line.

Then either L is y-axis or L is a vertical line other than y-axis.

Case (a) If L is y-axis, then a point (x, y) in the plane lies on L if and only if $x = 0$ and y can be any real number.

Thus $x = 0$ describes y – axis.

$\therefore$ The equation of y-axis is $x = 0$

Case (b) If L is a vertical line other than y-axis, then it is parallel to y-axis.

Now a point (x, y) lies on L if and only if x-coordinate must remain constant and y can be any real number.

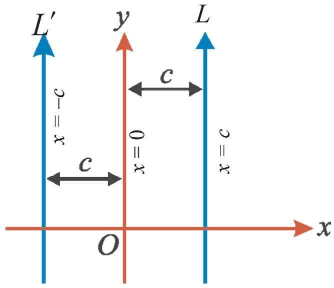


Fig. 5.24

$\therefore$ The equation of a straight line parallel to y-axis is $x = c$, where c is a constant.

Note that if $c > 0$, then L lies to the right y-axis and

if $c < 0$, then L lies to the left of y-axis.

If $c = 0$, then L is nothing but the y-axis.

(iii) Neither vertical nor horizontal: Let L be neither vertical nor horizontal.

In this case how do we describe L by an equation? Let θ denote the angle of inclination. Observe that if we know this θ and a point on L, then we can easily describe L.

> Slope m of a non-vertical line L can be calculated using
> (i) $m = \tan\theta$ if we know the angle of inclination θ.
> (ii) $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ if we know two distinct points (x_1, y_1), (x_2, y_2) on L.
> (iii) $m = 0$ if and only if L is horizontal.

Now consider the case where L is not a vertical line and derive the equation of a straight line in the following forms: (a) Slope-Point form (b) Two-Points form (c) Slope-Intercept form (d) Intercepts form

(a) Slope-Point form

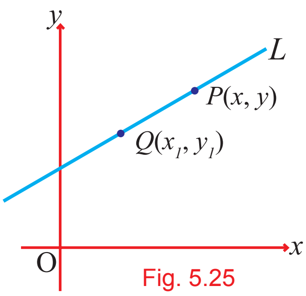


Fig. 5.25

Let m be the slope of L and $Q(x_1, y_1)$ be a point on L.

Let $P(x, y)$ be an arbitrary point on L other than Q. Then, we have

$$m = \frac{y - y_1}{x - x_1} \Leftrightarrow m(x - x_1) = y - y_1$$

Thus, the equation of a straight line with slope m and passing through (x_1, y_1) is

$$y - y_1 = m(x - x_1) \text{ for all points } (x, y) \text{ on } L. \qquad (1)$$

Remarks

(i) Now the first degree equation (1) in the variables x and y is satisfied by the x-coordinate and y-coordinate of any point on the line L. Any value of x and y that satisfies this equation will be the coordinates of a point on the line L. Hence the equation (1) is called the equation of the straight line L.

(ii) The equation (1) says that the change in y-coordinates of the points on L is directly proportional to the change in x-coordinates. The proportionality constant m is the slope.

(b) Two-Points form

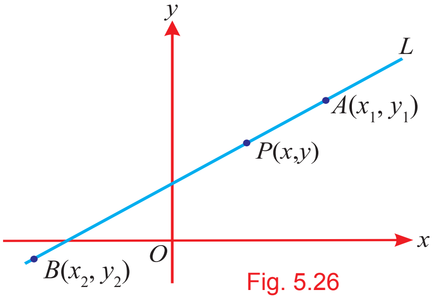


Fig. 5.26

Suppose that two distinct points (x_1, y_1), (x_2, y_2) are given on a non-vertical line L.

To find the equation of L, we find the slope of L first and then use (1) .

The slope of L is

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where } x_2 \neq x_1 \text{ as } L \text{ is non-vertical.}$$

Now, the formula (1) gives

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$\Longrightarrow \frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \quad \Longrightarrow \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \quad \text{for all points } (x, y) \text{ on } L \qquad (2)$$

Note

To get the equation of L, we can also use the point (x_2, y_2) instead of (x_1, y_1).

(c) Slope-Intercept form

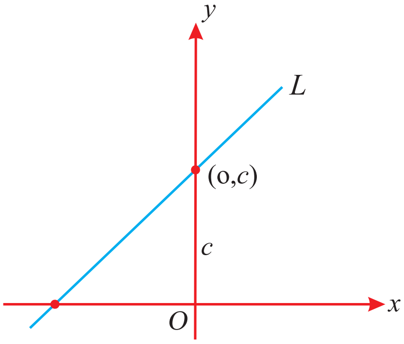


Fig. 5.27

Suppose that m is the slope of L and c is the y-intercept of L.

Since c is the y-intercept, the point $(0, c)$ lies on L. Now using (1) with $(x_1, y_1) = (0, c)$ we obtain, $y - c = m(x - 0)$

$$\Longrightarrow \quad y = mx + c \quad \text{for all points } (x, y) \text{ on } L. \qquad (3)$$

Thus, $y = mx + c$ is the equation of straight line in the **Slope-Intercept form.**

(d) Intercepts form

Suppose that the straight line L makes non-zero intercepts a and b on the x-axis and on the y-axis respectively.

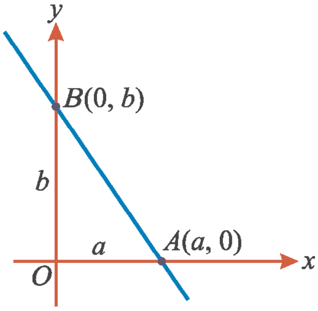


Fig. 5.28

$\therefore$ The straight line cuts the x-axis at $A(a, 0)$ and the y-axis at $B(0, b)$

The slope of AB is $m = -\frac{b}{a}$.

Now (1) gives, $\quad y - 0 = -\frac{b}{a}(x - a)$

$\Longrightarrow \quad ay = -bx + ab$

$bx + ay = ab$

Divide by ab to get $\frac{x}{a} + \frac{y}{b} = 1$

$\therefore$ Equation of a straight line having x-intercept a and y-intercept b is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{for all points } (x, y) \text{ on } L \qquad (4)$$

Note

(i) If the line L with slope m, makes x-intercept d, then the equation of the line is $y = m(x - d)$.

(ii) The straight line $y = mx$ passes through the origin.(both x and y-intercepts are zero for $m \neq 0$).

(iii) Equations (1), (2) and (4) can be simplified to slope-intercept form given by (3).

(iv) Each equation in (1), (2), (3) and (4) can be rewritten in the form $px + qy + r = 0$ for all points (x, y) on L, which is called the general form of equation of a straight line.

Example 5.19

Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(3, -4)$.

Solution Let L and L' be the straight lines passing through the point $(3, -4)$ and parallel to x-axis and y-axis respectively.

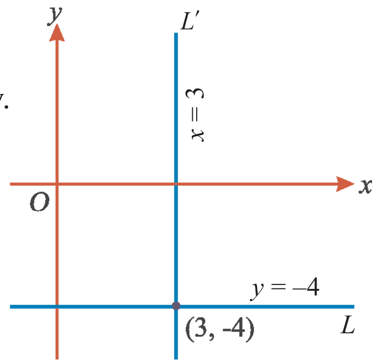


Fig. 5.29

The y-coordinate of every point on the line L is -4.

Hence, the equation of the line L is $\quad y = -4$

Similarly, the x-coordinate of every point on the straight line L' is 3

Hence, the equation of the line L' is $x = 3$.

Example 5.20

Find the equation of straight line whose angle of inclination is 45° and y-intercept is $\frac{2}{5}$.

Solution Slope of the line, $m = \tan\theta$

$$= \tan 45^\circ = 1$$

y-intercept is $c = \frac{2}{5}$

By the slope-intercept form, the equation of the straight line is

$$y = mx + c$$

$$y = x + \frac{2}{5} \implies y = \frac{5x+2}{5}$$

$\therefore$ The equation of the straight line is $5x - 5y + 2 = 0$

Example 5.21

Find the equation of the straight line passing through the point $(-2,3)$ with slope $\frac{1}{3}$.

Solution Given that the slope $m = \frac{1}{3}$ and a point $(x_1, y_1) = (-2, 3)$

By slope-point formula, the equation of the straight line is

$$y - y_1 = m(x - x_1)$$

$$\implies y - 3 = \frac{1}{3}(x+2)$$

Thus, $x - 3y + 11 = 0$ is the required equation.

Example 5.22

Find the equation of the straight line passing through the points $(-1,1)$ and $(2,-4)$.

Solution Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the given points.

Here $x_1 = -1$, $y_1 = 1$ and $x_2 = 2$, $y_2 = -4$.

Using two-points formula, the equation of the straight line is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\implies \frac{y-1}{-4-1} = \frac{x+1}{2+1}$$

$$\implies 3y - 3 = -5x - 5$$

Hence, $5x + 3y + 2 = 0$ is the required equation of the straight line.

Example 5.23

The vertices of a $\triangle ABC$ are $A(2, 1)$, $B(-2, 3)$ and $C(4, 5)$. Find the equation of the median through the vertex A.

Solution Median is a straight line joining a vertex and the midpoint of the opposite side.

Let D be the midpoint of BC.

$\therefore$ Midpoint of BC is $D\left(\frac{-2+4}{2}, \frac{3+5}{2}\right) = D(1, 4)$

Now the equation of the median AD is

$$\frac{y-1}{4-1} = \frac{x-2}{1-2} \qquad \because (x_1, y_1) = (2,1) \text{ and } (x_2, y_2) = (1,4)$$

$$\frac{y-1}{3} = \frac{x-2}{-1}$$

$\therefore$ $3x + y - 7 = 0$ is the required equation.

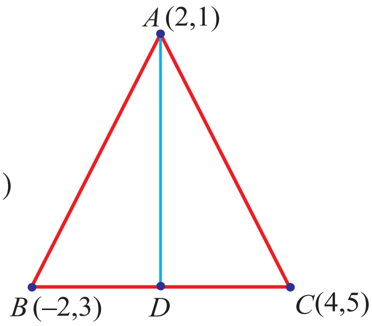


Fig. 5.30

Example 5.24

If the x-intercept and y-intercept of a straight line are $\frac{2}{3}$ and $\frac{3}{4}$ respectively, then find the equation of the straight line.

Solution Given that x-intercept of the straight line, $a = \frac{2}{3}$

and the y-intercept of the straight line, $b = \frac{3}{4}$

Using intercept form, the equation of the straight line is

$$\frac{x}{a} + \frac{y}{b} = 1 \implies \frac{x}{\frac{2}{3}} + \frac{y}{\frac{3}{4}} = 1$$

$$\implies \frac{3x}{2} + \frac{4y}{3} = 1$$

Hence, $9x + 8y - 6 = 0$ is the required equation.

Example 5.25

Find the equations of the straight lines each passing through the point $(6, -2)$ and whose sum of the intercepts is 5.

Solution Let a and b be the x-intercept and y-intercept of the required straight line respectively.

Given that sum of the intercepts, $a + b = 5$

$$\implies b = 5 - a$$

Now, the equation of the straight line in the intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1 \implies \frac{x}{a} + \frac{y}{5-a} = 1$$

$$\implies \frac{(5-a)x + ay}{a(5-a)} = 1$$

Thus, $(5-a)x + ay = a(5-a)$ (1)

Since the straight line given by (1) passes through $(6, -2)$, we get,

$$(5-a)6 + a(-2) = a(5-a)$$

$$\Longrightarrow \quad a^2 - 13a + 30 = 0.$$

That is, $\quad (a-3)(a-10) = 0$

$\therefore \quad a = 3$ or $a = 10$

When $a = 3$, (1) $\Longrightarrow (5-3)x + 3y = 3(5-3)$

$$\Longrightarrow \quad 2x + 3y = 6 \qquad (2)$$

When $a = 10$, (1) $\Longrightarrow (5-10)x + 10y = 10(5-10)$

$$\Longrightarrow \quad -5x + 10y = -50$$

That is, $\quad x - 2y - 10 = 0. \qquad (3)$

Hence, $2x + 3y = 6$ and $x - 2y - 10 = 0$ are the equations of required straight lines.

Exercise 5.4

1. Write the equations of the straight lines parallel to x- axis which are at a distance of 5 units from the x-axis.

2. Find the equations of the straight lines parallel to the coordinate axes and passing through the point $(-5,-2)$.

3. Find the equation of a straight line whose
   (i) slope is -3 and y-intercept is 4.
   (ii) angle of inclination is 60^0 and y-intercept is 3.

4. Find the equation of the line intersecting the y- axis at a distance of 3 units above the origin and $\tan\theta = \frac{1}{2}$, where θ is the angle of inclination.

5. Find the slope and y-intercept of the line whose equation is
   (i) $y = x + 1$ (ii) $5x = 3y$ (iii) $4x - 2y + 1 = 0$ (iv) $10x + 15y + 6 = 0$

6. Find the equation of the straight line whose
   (i) slope is -4 and passing through (1, 2)
   (ii) slope is $\frac{2}{3}$ and passing through $(5, -4)$

7. Find the equation of the straight line which passes through the midpoint of the line segment joining (4, 2) and (3, 1) whose angle of inclination is 30^0.

8. Find the equation of the straight line passing through the points
   (i) $(-2, 5)$ and $(3, 6)$ (ii) $(0, -6)$ and $(-8, 2)$

9. Find the equation of the median from the vertex R in a $\triangle PQR$ with vertices at $P(1, -3)$, $Q(-2, 5)$ and $R(-3, 4)$.

10. By using the concept of the equation of the straight line, prove that the given three points are collinear.

(i) (4, 2), (7, 5) and (9, 7) (ii) (1, 4), (3, −2) and (−3, 16)

11. Find the equation of the straight line whose x and y-intercepts on the axes are given by

(i) 2 and 3 (ii) $-\frac{1}{3}$ and $\frac{3}{2}$ (iii) $\frac{2}{5}$ and $-\frac{3}{4}$

12. Find the x and y intercepts of the straight line

(i) $5x + 3y - 15 = 0$ (ii) $2x - y + 16 = 0$ (iii) $3x + 10y + 4 = 0$

13. Find the equation of the straight line passing through the point (3, 4) and has intercepts which are in the ratio 3 : 2.

14. Find the equation of the straight lines passing through the point (2, 2) and the sum of the intercepts is 9.

15. Find the equation of the straight line passing through the point (5, −3) and whose intercepts on the axes are equal in magnitude but opposite in sign.

16. Find the equation of the line passing through the point (9, −1) and having its x-intercept thrice as its y-intercept.

17. A straight line cuts the coordinate axes at A and B. If the midpoint of AB is (3, 2), then find the equation of AB.

18. Find the equation of the line passing through (22, −6) and having intercept on x-axis exceeds the intercept on y-axis by 5.

19. If $A(3, 6)$ and $C(-1, 2)$ are two vertices of a rhombus $ABCD$, then find the equation of straight line that lies along the diagonal BD.

20. Find the equation of the line whose gradient is $\frac{3}{2}$ and which passes through P, where P divides the line segment joining $A(-2, 6)$ and B (3, −4) in the ratio 2 : 3 internally.

5.7 General Form of Equation of a straight line

We have already pointed out that different forms of the equation of a straight line may be converted into the standard form $ax + by + c = 0$, where a , b and c are real constants such that either $a \neq 0$ or $b \neq 0$.

Now let us find out

(i) the slope of $ax + by + c = 0$

(ii) the equation of a straight line parallel to $ax + by + c = 0$

(iii) the equation of a straight line perpendicular to $ax + by + c = 0$ and

(iv) the point of intersection of two intersecting straight lines.

(i) The general form of the equation of a straight line is $ax + by + c = 0$.

The above equation is rewritten as $y = -\frac{a}{b}x - \frac{c}{b},\ b \neq 0$ (1)

Comparing (1) with the slope-intercept form $y = mx + k$, we get,

$$\text{slope, } m = -\frac{a}{b} \text{ and the } y\text{-intercept } = -\frac{c}{b}$$

$\therefore$ For the equation $ax + by + c = 0$, we have

$$\text{slope } m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} \quad \text{and} \quad \text{the } y\text{-intercept is } -\frac{\text{constant term}}{\text{coefficient of } y}.$$

(ii) Equation of a line parallel to the line $ax + by + c = 0$.

We know that two straight lines are parallel if and only if their slopes are equal.

Hence the equations of all lines parallel to the line $ax + by + c = 0$ are of the form

$$ax + by + k = 0, \text{ for different values of } k.$$

(iii) Equation of a line perpendicular to the line $ax + by + c = 0$

We know that two non-vertical lines are perpendicular if and only if the product of their slopes is -1.

Hence the equations of all lines perpendicular to the line $ax + by + c = 0$ are

$$bx - ay + k = 0, \text{ for different values of } k.$$

Note

Two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where the coefficients are non-zero,

(i) are parallel if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

(ii) are perpendicular if and only if $a_1a_2 + b_1b_2 = 0$

(iv) The point of intersection of two straight lines

If two straight lines are not parallel, then they will intersect at a point. This point lies on both the straight lines. Hence, the point of intersection is obtained by solving the given two equations.

Example 5.26

Show that the straight lines $3x + 2y - 12 = 0$ and $6x + 4y + 8 = 0$ are parallel.

Solution Slope of the straight line $3x + 2y - 12 = 0$ is $m_1 = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{3}{2}$

Similarly, the slope of the line $6x + 4y + 8 = 0$ is $m_2 = -\frac{6}{4} = -\frac{3}{2}$

$\therefore$ $m_1 = m_2$. Hence, the two straight lines are parallel.

Example 5.27

Prove that the straight lines $x+2y+1=0$ and $2x-y+5=0$ are perpendicular to each other.

Solution Slope of the straight line $x+2y+1=0$ is $m_1 = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{1}{2}$

Slope of the straight line $2x-y+5=0$ is $m_2 = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = \frac{-2}{-1} = 2$

Product of the slopes $m_1 m_2 = -\frac{1}{2}\times 2 = -1$

$\therefore$ The two straight lines are perpendicular.

Example 5.28

Find the equation of the straight line parallel to the line $x-8y+13=0$ and passing through the point (2, 5).

Solution Equation of the straight line parallel to $x-8y+13=0$ is $x-8y+k=0$

Since it passes through the point (2, 5)

$$2-8(5)+k = 0 \implies k = 38$$

$\therefore$ Equation of the required straight line is $x-8y+38=0$

Example 5.29

The vertices of $\triangle ABC$ are $A(2, 1)$, $B(6, -1)$ and $C(4, 11)$. Find the equation of the straight line along the altitude from the vertex A.

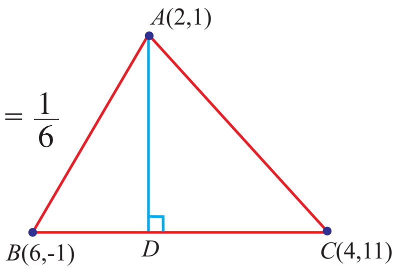


Fig. 5.31

Solution Slope of $BC = \frac{11+1}{4-6} = -6$

Since the line AD is perpendicular to the line BC, slope of $AD = \frac{1}{6}$

$\therefore$ Equation of AD is $y-y_1 = m(x-x_1)$

$$y-1=\frac{1}{6}(x-2) \implies 6y-6 = x-2$$

$\therefore$ Equation of the required straight line is $x-6y+4=0$

Exercise 5.5

1. Find the slope of the straight line

   (i) $3x+4y-6=0$ (ii) $y=7x+6$ (iii) $4x=5y+3$.

2. Show that the straight lines $x+2y+1=0$ and $3x+6y+2=0$ are parallel.

3. Show that the straight lines $3x-5y+7=0$ and $15x+9y+4=0$ are perpendicular.

4. If the straight lines $\frac{y}{2} = x-p$ and $ax+5=3y$ are parallel, then find a.

5. Find the value of a if the straight lines $5x-2y-9=0$ and $ay+2x-11=0$ are perpendicular to each other.

6. Find the values of p for which the straight lines $8px + (2 - 3p)y + 1 = 0$ and $px + 8y - 7 = 0$ are perpendicular to each other.

7. If the straight line passing through the points $(h, 3)$ and (4, 1) intersects the line $7x - 9y - 19 = 0$ at right angle, then find the value of h.

8. Find the equation of the straight line parallel to the line $3x - y + 7 = 0$ and passing through the point $(1, -2)$.

9. Find the equation of the straight line perpendicular to the straight line $x - 2y + 3 = 0$ and passing through the point $(1, -2)$.

10. Find the equation of the perpendicular bisector of the straight line segment joining the points (3, 4) and $(-1, 2)$.

11. Find the equation of the straight line passing through the point of intersection of the lines $2x + y - 3 = 0$ and $5x + y - 6 = 0$ and parallel to the line joining the points (1, 2) and (2, 1).

12. Find the equation of the straight line which passes through the point of intersection of the straight lines $5x - 6y = 1$ and $3x + 2y + 5 = 0$ and is perpendicular to the straight line $3x - 5y + 11 = 0$.

13. Find the equation of the straight line joining the point of intersection of the lines $3x - y + 9 = 0$ and $x + 2y = 4$ and the point of intersection of the lines $2x + y - 4 = 0$ and $x - 2y + 3 = 0$.

14. If the vertices of a $\triangle ABC$ are $A(2, -4)$, $B(3, 3)$ and $C(-1, 5)$. Find the equation of the straight line along the altitude from the vertex B.

15. If the vertices of a $\triangle ABC$ are $A(-4,4)$, $B(8,4)$ and $C(8,10)$. Find the equation of the straight line along the median from the vertex A.

16. Find the coordinates of the foot of the perpendicular from the origin on the straight line $3x + 2y = 13$.

17. If $x + 2y = 7$ and $2x + y = 8$ are the equations of the lines of two diameters of a circle, find the radius of the circle if the point $(0, -2)$ lies on the circle.

18. Find the equation of the straight line segment whose end points are the point of intersection of the straight lines $2x - 3y + 4 = 0$, $x - 2y + 3 = 0$ and the midpoint of the line joining the points $(3, -2)$ and $(-5, 8)$.

19. In an isosceles $\triangle PQR$, $PQ = PR$. The base QR lies on the x-axis, P lies on the y- axis and $2x - 3y + 9 = 0$ is the equation of PQ. Find the equation of the straight line along PR.

Exercise 5.6

Choose the correct answer

1. The midpoint of the line joining $(a, -b)$ and $(3a, 5b)$ is
   (A) $(-a, 2b)$ (B) $(2a, 4b)$ (C) $(2a, 2b)$ (D) $(-a, -3b)$

2. The point P which divides the line segment joining the points $A(1, -3)$ and $B(-3, 9)$ internally in the ratio 1:3 is
   (A) $(2, 1)$ (B) $(0, 0)$ (C) $\left(\frac{5}{3}, 2\right)$ (D) $(1, -2)$

3. If the line segment joining the points $A(3, 4)$ and $B(14, -3)$ meets the x-axis at P, then the ratio in which P divides the segment AB is
   (A) 4 : 3 (B) 3 : 4 (C) 2 : 3 (D) 4 : 1

4. The centroid of the triangle with vertices at $(-2, -5)$, $(-2, 12)$ and $(10, -1)$ is
   (A) $(6, 6)$ (B) $(4, 4)$ (C) $(3, 3)$ (D) $(2, 2)$

5. If $(1, 2)$, $(4, 6)$, $(x, 6)$ and $(3, 2)$ are the vertices of a parallelogram taken in order, then the value of x is
   (A) 6 (B) 2 (C) 1 (D) 3

6. Area of the triangle formed by the points (0,0), $(2, 0)$ and $(0, 2)$ is
   (A) 1 sq. units (B) 2 sq. units (C) 4 sq. units (D) 8 sq. units

7. Area of the quadrilateral formed by the points $(1, 1)$, $(0, 1)$, $(0, 0)$ and $(1, 0)$ is
   (A) 3 sq. units (B) 2 sq. units (C) 4 sq. units (D) 1 sq. units

8. The angle of inclination of a straight line parallel to x-axis is equal to
   (A) 0° (B) 60° (C) 45° (D) 90°

9. Slope of the line joining the points $(3, -2)$ and $(-1, a)$ is $-\frac{3}{2}$, then the value of a is equal to
   (A) 1 (B) 2 (C) 3 (D) 4

10. Slope of the straight line which is perpendicular to the straight line joining the points $(-2, 6)$ and $(4, 8)$ is equal to
    (A) $\frac{1}{3}$ (B) 3 (C) -3 (D) $-\frac{1}{3}$

11. The point of intersection of the straight lines $9x - y - 2 = 0$ and $2x + y - 9 = 0$ is
    (A) $(-1, 7)$ (B) $(7, 1)$ (C) $(1, 7)$ (D) $(-1, -7)$

12. The straight line $4x + 3y - 12 = 0$ intersects the y- axis at
    (A) $(3, 0)$ (B) $(0, 4)$ (C) $(3, 4)$ (D) $(0, -4)$

13. The slope of the straight line $7y - 2x = 11$ is equal to
    (A) $-\frac{7}{2}$ (B) $\frac{7}{2}$ (C) $\frac{2}{7}$ (D) $-\frac{2}{7}$

14. The equation of a straight line passing through the point $(2, -7)$ and parallel to x-axis is
    (A) $x = 2$ (B) $x = -7$ (C) $y = -7$ (D) $y = 2$

15. The x and y-intercepts of the line $2x - 3y + 6 = 0$, respectively are

(A) 2, 3 (B) 3, 2 (C) $-3, 2$ (D) $3, -2$

16. The centre of a circle is $(-6, 4)$. If one end of the diameter of the circle is at $(-12, 8)$, then the other end is at

(A) $(-18, 12)$ (B) $(-9, 6)$ (C) $(-3, 2)$ (D) $(0, 0)$

17. The equation of the straight line passing through the origin and perpendicular to the straight line $2x + 3y - 7 = 0$ is

(A) $2x + 3y = 0$ (B) $3x - 2y = 0$ (C) $y + 5 = 0$ (D) $y - 5 = 0$

18. The equation of a straight line parallel to y-axis and passing through the point $(-2, 5)$ is

(A) $x - 2 = 0$ (B) $x + 2 = 0$ (C) $y + 5 = 0$ (D) $y - 5 = 0$

19. If the points (2, 5), (4, 6) and (a, a) are collinear, then the value of a is equal to

(A) -8 (B) 4 (C) -4 (D) 8

20. If a straight line $y = 2x + k$ passes through the point (1, 2), then the value of k is equal to

(A) 0 (B) 4 (C) 5 (D) -3

21. The equation of a straight line having slope 3 and y-intercept -4 is

(A) $3x - y - 4 = 0$ (B) $3x + y - 4 = 0$

(C) $3x - y + 4 = 0$ (D) $3x + y + 4 = 0$

22. The point of intersection of the straight lines $y = 0$ and $x = -4$ is

(A) $(0, -4)$ (B) $(-4, 0)$ (C) $(0, 4)$ (D) $(4, 0)$

23. The value of k if the straight lines $3x + 6y + 7 = 0$ and $2x + ky = 5$ are perpendicular is

(A) 1 (B) -1 (C) 2 (D) $\frac{1}{2}$

Points to Remember

- The distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- The point P which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $l : m$ is $\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right)$.
- The point Q which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $l : m$ is $\left(\frac{lx_2 - mx_1}{l - m}, \frac{ly_2 - my_1}{l - m}\right)$.
- Midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

- The area of the triangle formed by the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\sum x_1(y_2 - y_3) = \frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$
$$= \frac{1}{2}\{(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)\}.$$

- Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if
  (i) $x_1y_2 + x_2y_3 + x_3y_1 = x_2y_1 + x_3y_2 + x_1y_3$ (or)
  (ii) Slope of AB = Slope of BC or slope of AC.
- If a line makes an angle θ with the positive direction of x- axis, then the slope $m = \tan\theta$.
- Slope of the non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is
  $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$
- Slope of the line $ax + by + c = 0$ is $m = -\frac{\text{coefficient of } x}{\text{coefficient of } y} = -\frac{a}{b}$, $b \neq 0$
- Slope of the horizontal line is 0 and slope of the vertical line is undefined.
- Two lines are parallel if and only if their slopes are equal.
- Two non-vertical lines are perpendicular if and only if the product of their slopes is -1. That is, $m_1m_2 = -1$.

Equation of straight lines

Sl.No	Straight line	Equation
1.	x-axis	$y = 0$
2.	y-axis	$x = 0$
3.	Parallel to x-axis	$y = k$
4.	Parallel to y-axis	$x = k$
5.	Parallel to $ax+by+c=0$	$ax+by+k = 0$
6.	Perpendicular to $ax+by+c=0$	$bx-ay+k = 0$
	Given	**Equation**
1.	Passing through the origin	$y = mx$
2.	Slope m, y-intercept c	$y = mx + c$
3.	Slope m, a point (x_1, y_1)	$y - y_1 = m(x-x_1)$
4.	Passing through two points (x_1, y_1), (x_2, y_2)	$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$
5.	x-intercept a , y-intercept b	$\frac{x}{a} + \frac{y}{b} = 1$