Chapter 7

Matrices and Determinants

"Mathematics is the music of reason"

- Sylvester

7.1 Introduction

The beginnings of matrices and determinants go back to the second century BC although traces can be seen back to the fourth century BC. However, it was not until near the end of the seventeenth century that the ideas reappeared and development really got underway. It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. The Babylonians studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survive till now.

The evolution of the theory of **'matrices'** is the result of attempts to obtain compact and simple methods for solving systems of linear equations. It also began with the study of transformations of geometric objects. In 1850, it was **James Joseph Sylvester** an English Mathematician and lawyer, coined the word '**Matrix**' (originally from Latin: $M\bar{a}$ ter means Mother – Collin's Dictionary). Matrices are now one of the most powerful tools in mathematics.

Generally, a matrix is nothing but a rectangular array of objects. These matrices can be visualised in day-to-day applications where we use matrices to represent a military parade or a school assembly or vegetation.

The term **'determinant'** was first coined by **Carl F Gauss** in **Disquisitiones arithmeticae** (1801) while studying quadratic forms. But the concept is not the same as that of modern day determinant. In the same work Gauss laid out the coefficients of his quadratic forms in rectangular arrays where he described matrix multiplication.

It was Cauchy (in 1812) who used *determinant* in its modern sense and studied it in detail. He reproved the earlier results and gave new results of his own on minors and adjoints. It was **Arthur Cayley** whose major contribution was in developing the algebra of matrices and also published the theory of determinants in 1841. In that paper he used two vertical lines on either side of the array to denote the determinant, a notation which has now become standard. In 1858, he published

Sylvester
(1814 - 1897)

Memoir on the theory of matrices which was remarkable for containing the first abstract definition of a matrix. He showed that the coefficient arrays studied earlier for quadratic forms and for linear transformations were special cases of his general concept. They simplify our work to a great extent when compared with other straight forward methods which would involve tedious computation. The mathematicians **James Joseph Sylvester** (1814 – 1897), **William Rowan Hamilton** (1805 – 1865), and **Arthur Cayley** (1821 – 1895) played important roles in the development of matrix theory. English mathematician **Cullis** was the first to use modern bracket notation for matrices in 1913. The knowledge of matrices is absolutely necessary not only within the branches of mathematics but also in other areas of science, genetics, economics, sociology, modern psychology and industrial management.

Matrices are also useful for representing coefficients in systems of linear equations. Matrix notations and operations are used in electronic spreadsheet programs on computers, which in turn are used in different areas of business like budgeting, sales projection, cost estimation, and in science, for analyzing the results of an experiment etc.

Interestingly, many geometric operations such as magnification, rotation and reflection through a plane can also be represented mathematically by matrices. Economists use matrices for social accounting, input-output tables and in the study of inter-industry economics. Matrices are also used in communication theory and network analysis in electrical engineering. They are also used in Cryptography.

In this chapter, we now first discuss matrices and their various properties. Then we continue to study determinants, basic properties, minors and their cofactors. Here we now restrict the discussion up to determinants of order 3 only.

Learning Objectives

On completion of this chapter, the students are expected to
- visualise difficult problems in a simple manner in terms of matrices.
- understand different types of matrices and algebra of matrices.
- compute determinant values through expansion and using properties of determinants.
- apply the concepts of matrices and determinants to find the area of a triangle and collinearity of three points.

7.2 Matrices

A **matrix** is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket [ ].

In general, the entries of a matrix may be real or complex numbers or functions of one variable (such as polynomials, trigonometric functions or a combination of them) or more variables or any other object. Usually, matrices are denoted by capital letters $A, B, C, ...$ etc. In this chapter the entries of matrices are restricted to either real numbers or real valued functions on real variables.

General form of a matrix

If a matrix A has m rows and n columns, then it is written as

$A = [a_{ij}]_{m \times n}, 1 \leq i \leq m, 1 \leq j \leq n.$

That is,

$$n \text{ columns}$$

$$
A = [a_{ij}]_{m \times n} =
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}
\begin{matrix}
\leftarrow \text{Row 1} \\
\leftarrow \text{Row 2} \\
\\
\leftarrow \text{Row } i \\
\\
\leftarrow \text{Row } m
\end{matrix}
\quad m \text{ rows}
$$

(Column 1, Column 2, Column j, Column n)

Note that m and n are positive integers.

The following are some examples of matrices :

$$
A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 9 & -8 & 6 \end{bmatrix}, \quad
B = \begin{bmatrix} 7 & -9 & 1.2 & 0 \\ \sin\dfrac{x}{4} & 2 & x^2 & 4 \\ \cos\dfrac{x}{2} & 1 & 3 & -6 \end{bmatrix}, \text{ and } \quad
C = \begin{bmatrix} 1 & 5 & -7 \\ 3.4 & \dfrac{1}{2} & \sqrt{2} \\ e^2 & -3 & 4 \\ \sqrt{5} & 2 & a \end{bmatrix}.
$$

In a matrix, the horizontal lines of elements are known as **rows** and the vertical lines of elements are known as **columns**. Thus A has 3 rows and 3 columns, B has 3 rows and 4 columns, and C has 4 rows and 3 columns.

> **Definition 7.1**
> If a matrix A has m rows and n columns then the **order or size** of the matrix A is defined to be $m \times n$ (read as m by n).

The objects $a_{11}, a_{12}, \ldots, a_{mn}$ are called **elements or entries** of the matrix $A = [a_{ij}]_{m \times n}$. The element a_{ij} is common to i^{th} row and j^{th} column and is called $(i, j)^{th}$ element of A. Observe that the ith row and j^{th} column of A are $1 \times n$ and $m \times 1$ matrices respectively and are given by $[a_{i1} \ a_{i2} \ \ldots \ a_{in}]$ and $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

We shall now visualize the representation and construction of matrices for simplifying day-to-day problems.

Illustration 7.1

Consider the marks scored by a student in different subjects and in different terminal examinations. They are exhibited in a tabular form as given below :

	Tamil	English	Mathematics	Science	Social Science
Exam 1	48	71	80	62	55
Exam 2	70	68	91	73	60
Exam 3	77	84	95	82	62

This tabulation represents the above information in the form of matrix. What does the entry in the third row and second column represent?

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The above information may be represented in the form of a 3×5 matrix A as

$$A = \begin{bmatrix} 48 & 71 & 80 & 62 & 55 \\ 70 & 68 & 91 & 73 & 60 \\ 77 & \mathbf{84} & 95 & 82 & 62 \end{bmatrix}.$$

The entry 84 common to the third row and the second column in the matrix represents the mark scored by the student in English Exam 3.

Example 7.1

Suppose that a matrix has 12 elements. What are the possible orders it can have? What if it has 7 elements?

Solution

The number of elements is the product of number of rows and number of columns. Therefore, we will find all ordered pairs of natural numbers whose product is 12. Thus, all the possible orders of the matrix are 1×12, 12×1, 2×6, 6×2, 3×4 and 4×3.

Since 7 is prime, the only possible orders of the matrix are 1×7 and 7×1.

Example 7.2

Construct a 2×3 matrix whose $(i, j)^{\text{th}}$ element is given by

$$a_{ij} = \frac{\sqrt{3}}{2} |2i - 3j| \quad (1 \le i \le 2, \ 1 \le j \le 3).$$

Solution

In general, a 2×3 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

By definition of a_{ij}, we easily have $a_{11} = \frac{\sqrt{3}}{2} |2 - 3| = \frac{\sqrt{3}}{2}$ and other entries of the matrix

A may be computed similarly. Thus, the required matrix A is $\begin{bmatrix} \frac{\sqrt{3}}{2} & 2\sqrt{3} & \frac{7\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \sqrt{3} & \frac{5\sqrt{3}}{2} \end{bmatrix}.$

7.2.1 Types of Matrices

Row, Column, Zero matrices

Definition 7.2

A matrix having only one row is called a **row matrix**.

For instance, $A = [A]_{1 \times 4} = [1 \ \ 0 \ \ -1.1 \ \ \sqrt{2}]$ is a row matrix. More generally, $A = [a_{ij}]_{1 \times n} = [a_{1j}]_{1 \times n}$ is a row matrix of order $1 \times n$.

Definition 7.3

A matrix having only one column is called a **column matrix**.

For instance, $[A]_{4\times 1} = \begin{bmatrix} x+1 \\ x^2 \\ 3x \\ 4 \end{bmatrix}$ is a column matrix whose entries are real valued functions of real

variable x. More generally, $A = [a_{ij}]_{m\times 1} = [a_{i1}]_{m\times 1}$ is a column matrix of order $m\times 1$.

Definition 7.4

A matrix $A = [a_{ij}]_{m\times n}$ is said to be a **zero matrix** or **null matrix** or **void matrix** denoted by O if $a_{ij} = 0$ for all values of $1 \le i \le m$ and $1 \le j \le n$.

For instance, $[0]$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are zero matrices of order 1×1, 3×3 and 2×4 respectively.

A matrix A is said to be a **non-zero matrix** if at least one of the entries of A is non-zero.

Square, Diagonal, Unit, Triangular matrices

Definition 7.5

A matrix in which number of rows is equal to the number of columns, is called a **square matrix**. That is, a matrix of order $n \times n$ is often referred to as a square matrix of order n.

For instance, $A = \begin{bmatrix} a & b & c \\ d & c & f \\ g & h & l \end{bmatrix}$ is a square matrix of order 3.

Definition 7.6

In a square matrix $A = [a_{ij}]_{n\times n}$ of order n, the elements $a_{11}, a_{22}, a_{33}, ..., a_{nn}$ are called the **principal diagonal** or simply the **diagonal** or **main diagonal** or **leading diagonal elements.**

Definition 7.7

A square matrix $A = [a_{ij}]_{n\times n}$ is called a **diagonal matrix** if $a_{ij} = 0$ whenever $i \ne j$.

Thus, in a diagonal matrix all the entries except the entries along the main diagonal are zero. For instance,

$A = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$, $B = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$, $C = [6]$, and $D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$

are diagonal matrices of order 3, 2, 1, and n respectively.

Is a square zero matrix, a diagonal matrix?

Definition 7.8

A diagonal matrix whose entries along the principal diagonal are equal is called a **Scalar matrix**.

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That is, a square matrix $A = [a_{ij}]_{n \times n}$ is said to be a scalar matrix if $a_{ij} = \begin{cases} c & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

where c is a fixed number. For instance,

$$A = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, B = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}, C = \begin{bmatrix} \sqrt{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \\ R_n \end{matrix}$$

are scalar matrices of order 3, 2, 1, and n respectively.

Observe that any square zero matrix can be considered as a scalar matrix with scalar 0.

> **Definition 7.9**
> A square matrix in which all the diagonal entries are 1 and the rest are all zero is called a
>
> **unit matrix**. Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is said to be a unit matrix if $a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

We represent the unit matrix of order n as I_n. For instance,

$$I_1 = [1], I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ \\ R_n \end{matrix}$$

are unit matrices of order 1, 2, 3 and n respectively.

Note 7.1

Unit matrix is an example of a scalar matrix.

There are two kinds of triangular matrices namely upper triangular and lower triangular matrices.

> **Definition 7.10**
> A square matrix is said to be an **upper triangular matrix** if all the elements below the main diagonal are zero.

Thus, the square matrix $A = [a_{ij}]_{n \times n}$ is said to be an upper triangular matrix if $a_{ij} = 0$ for all $i > j$. For instance,

$$\begin{bmatrix} 4 & 3 & 0 \\ 0 & 7 & 8 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} \text{ are all upper triangular matrices.}$$

> **Definition 7.11**
> A square matrix is said to be a **lower triangular matrix** if all the elements above the main diagonal are zero.

More precisely, a square matrix $A = [a_{ij}]_{n \times n}$ is said to be a lower triangular matrix if $a_{ij} = 0$ for all $i < j$. For instance,

$$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & -5 & 7 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 9 & -3 \end{bmatrix}, \text{ and } \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \text{ are all lower triangular}$$

matrices.

> **Definition 7.12**
>
> A square matrix which is either upper triangular or lower triangular is called a **triangular matrix**.

Observe that a square matrix that is both upper and lower triangular simultaneously will turn out to be a diagonal matrix.

7.2.2 Equality of Matrices

> **Definition 7.13**
>
> Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** (written as $A = B$) if and only if
> (i) both A and B are of the same order
> (ii) the corresponding entries of A and B are equal. That is, $a_{ij} = b_{ij}$ for all i and j.

For instance, if

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 2.5 & -1 \\ \dfrac{1}{\sqrt{2}} & \dfrac{3}{5} \end{bmatrix}, \text{ then we must have } x = 2.5, y = -1, u = \dfrac{1}{\sqrt{2}} \text{ and } v = \dfrac{3}{5}.$$

> **Definition 7.14**
>
> Two matrices A and B are called **unequal** if either of condition (i) or (ii) of Definition 7.13 does not hold.

For instance, $\begin{bmatrix} 4 & -3 \\ 0 & 8 \end{bmatrix} \neq \begin{bmatrix} 8 & -5 \\ 0 & 4 \end{bmatrix}$ as the corresponding entries are not equal. Also

$\begin{bmatrix} 4 & -3 \\ 0 & 8 \end{bmatrix} \neq \begin{bmatrix} 5 & -8 \\ 3 & 4 \\ 6 & 7 \end{bmatrix}$ as the orders are not the same.

Example 7.3

Find $x, y, a,$ and b if $\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}.$

Solution

As the orders of the two matrices are same, they are equal if and only if the corresponding entries are equal. Thus, by comparing the corresponding elements, we get

$3x+4y = 2,\ x-2y = 4,\ a+b = 5,$ and $2a-b = -5.$

Solving these equations, we get $x = 2,\ y = -1,\ a = 0,$ and $b = 5.$

7.2.3 Algebraic Operations on Matrices

Basic operations on matrices are

(1) multiplication of a matrix by a scalar,

(2) addition/subtraction of two matrices, and

(3) multiplication of two matrices.

There is no concept of dividing a matrix by another matrix and thus, the operation $\dfrac{A}{B}$, where A and B are matrices, is not defined.

(1) Multiplication of a matrix by a scalar

For a given matrix $A = [a_{ij}]_{m \times n}$ and a scalar k, we define a new matrix $kA = [b_{ij}]_{m \times n}$, where $b_{ij} = ka_{ij}$ for all i and j.

For instance, if $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, then $kA = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$

In particular if $k = -1$, we obtain $-A = [-a_{ij}]_{m \times n}$. This $-A$ is called **negative** of the matrix A. Don't say $-A$ as a negative matrix.

(2) Addition and Subtraction of two matrices

If A and B are two matrices of the same order, then their sum denoted by $A + B$, is again a matrix of same order, obtained by adding the corresponding entries of A and B.

More precisely, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices, then the sum $A + B$ of A and B is a matrix given by

$A + B = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ for all i and j.

Similarly subtraction $A - B$ is defined as $A - B = A + (-1)B$.

That is, $A - B = [d_{ij}]_{m \times n}$, where $d_{ij} = a_{ij} - b_{ij}$ $\forall$ i and j. (The symbol $\forall$ denotes for every or for all).

Note 7.2

(i) If A and B are not of the same order, then $A + B$ and $A - B$ are not defined.

(ii) The addition and subtraction can be extended to any finite number of matrices.

Example 7.4
Compute $A + B$ and $A - B$ if

$$A = \begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} \sqrt{3} & \sqrt{5} & 7.3 \\ 1 & \dfrac{1}{3} & \dfrac{1}{4} \end{bmatrix}.$$

Solution
By the definitions of addition and subtraction of matrices, we have

$$A + B = \begin{bmatrix} 4 + \sqrt{3} & 2\sqrt{5} & 14.3 \\ 0 & \dfrac{1}{3} & \dfrac{3}{4} \end{bmatrix} \text{ and } A - B = \begin{bmatrix} 4 - \sqrt{3} & 0 & -0.3 \\ -2 & -\dfrac{1}{3} & \dfrac{1}{4} \end{bmatrix}.$$

Example 7.5

Find the sum $A + B + C$ if A, B, C are given by

$$A = \begin{bmatrix} \sin^2\theta & 1 \\ \cot^2\theta & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \cos^2\theta & 0 \\ -\csc^2\theta & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Solution

By the definition of sum of matrices, we have

$$A + B + C = \begin{bmatrix} \sin^2\theta + \cos^2\theta + 0 & 1 + 0 - 1 \\ \cot^2\theta - \csc^2\theta - 1 & 0 + 1 + 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Example 7.6

Determine $3B + 4C - D$ if $B, C,$ and D are given by

$$B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -2 & 3 \\ -1 & 0 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 4 & -1 \\ 5 & 6 & -5 \end{bmatrix}.$$

Solution

$$3B + 4C - D = \begin{bmatrix} 6 & 9 & 0 \\ 3 & -3 & 15 \end{bmatrix} + \begin{bmatrix} -4 & -8 & 12 \\ -4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 0 & -4 & 1 \\ -5 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 13 \\ -6 & -9 & 28 \end{bmatrix}.$$

Example 7.7

Simplify :

$$\sec\theta \begin{bmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{bmatrix} - \tan\theta \begin{bmatrix} \tan\theta & \sec\theta \\ \sec\theta & \tan\theta \end{bmatrix}.$$

Solution

If we denote the given expression by A, then using the scalar multiplication rule, we get

$$A = \begin{bmatrix} \sec^2\theta & \sec\theta\tan\theta \\ \sec\theta\tan\theta & \sec^2\theta \end{bmatrix} - \begin{bmatrix} \tan^2\theta & \tan\theta\sec\theta \\ \sec\theta\tan\theta & \tan^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(3) Multiplication of matrices

Definition 7.15

A matrix A is said to be **conformable for multiplication** with a matrix B if the number of columns of A is equal to the number of rows of B.

That is, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are given two matrices, then the product of matrices A and B is denoted by AB and its order is $m \times p$.

The order of AB is $m \times p = ($number of rows of $A) \times ($number of columns of $B)$.

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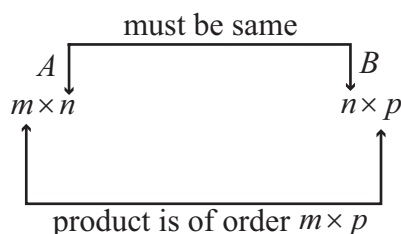
If $A = [a_1 \; a_2 \; ... \; a_n]_{1 \times n}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1}$,then AB is a matrix of order 1×1, that gives a single

element which is defined by $AB = [a_1 \; a_2 \; ... \; a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [a_1 b_1 + a_2 b_2 + ... + a_n b_n] = \left[\sum_{k=1}^{n} a_k b_k \right].$

For instance,

$$[1 \; 2 \; 3] \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = [1(-2) + 2(3) + 3(5)] = [-2 + 6 + 15] = [19].$$

In general,

if $A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B = [b_{ij}]_{n \times p} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$ then

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

and the product $AB = [c_{ij}]_{m \times p} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$,

where $c_{ij} = [a_{i1} \; a_{i2} \; ... \; a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj}$, since c_{ij} is an element.

Example 7.8

If $A = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$, compute A^2.

Solution

$$A^2 = AA = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix} \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

where $c_{11} = [0 \ c \ b]\begin{bmatrix} 0 \\ c \\ b \end{bmatrix} = 0 \cdot 0 + c \cdot c + b \cdot b = c^2 + b^2$ and other elements c_{ij} may be computed similarly.

Finally, we easily obtain that

$$A^2 = \begin{bmatrix} 0 + c^2 + b^2 & 0 + 0 + ab & 0 + ac + 0 \\ 0 + 0 + ab & c^2 + 0 + a^2 & bc + 0 + 0 \\ 0 + ac + 0 & bc + 0 + 0 & b^2 + a^2 + 0 \end{bmatrix} = \begin{bmatrix} b^2 + c^2 & ab & ac \\ ab & c^2 + a^2 & bc \\ ac & bc & a^2 + b^2 \end{bmatrix}.$$

Example 7.9

Solve for x if $[x \ \ 2 \ \ -1] \begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O.$

Solution

$$[x \ \ 2 \ \ -1] \begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$$

That is, $\quad [x - 2 + 1 \ \ \ x - 8 + 1 \ \ \ 2x + 2 + 2] \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$

$$[x - 1 \ \ \ x - 7 \ \ \ 2x + 4] \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$$

$$x(x - 1) + 2(x - 7) + 1(2x + 4) = 0$$

$$x^2 + 3x - 10 = 0 \implies x = -5, 2.$$

Note 7.3

We have the following important observations:

(1) If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$, and $m \ne p$, then the product AB is defined but not BA.

(2) The fundamental properties of real numbers namely,

$ab = ba \quad \forall a, b \in \mathbb{R}$

$ab = ac \implies b = c \ \forall \ a, b, c \in \mathbb{R}, \ a \ne 0$

$ab = 0 \implies a = 0 \text{ or } b = 0 \ \forall \ a, b \in \mathbb{R}.$

Can we discuss these in matrices also?

(i)   Even if AB and BA are defined, then $AB = BA$ is not necessarily true.

For instance, we consider

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

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and observe that $AB \neq BA$, since

$$AB = \begin{bmatrix} 5 & 0 \\ 4 & -2 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix}$$

In this case we say that A and B do not commute (with respect to multiplication)

Observe that $AB = BA$ is also possible. For instance,

$$\text{if } A = \begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \text{ then, } AB = BA = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

In this case we say that A and B commute with respect to multiplication.

(ii) Cancellation property does not hold for matrix multiplication. That is, $A \neq O$, B, and C are three square matrices of same order $n \times n$ with $n > 1$, then $AB = AC$ does not imply $B = C$ and $BA = CA$ does not imply $B = C$.

As a simple demonstration of these facts, we observe that for instance,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}.$$

(iii) It is possible that $AB = O$ with $A \neq O$ and $B \neq O$; Equivalently, $AB = O$ is not necessarily imply either $A = O$ or $B = O$. The following relation demonstrates this fact :

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(3) In general, for any two matrices A and B which are conformable for addition and multiplication, for the below operations, we have

- $(A \pm B)^2$ need not be equal to $A^2 \pm 2AB + B^2$
- $A^2 - B^2$ need not be equal to $(A+B)(A-B)$.

Example 7.10

If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 0 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$ find AB and BA if they exist.

Solution

The order of A is 3×3 and the order of B is 3×2. Therefore the order of AB is 3×2.

A and B are conformable for the product AB. Call $C = AB$. Then,

$c_{11} = $ (first row of A) (first column of B)

$$\Rightarrow c_{11} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1+1+2 = 4, \text{ since } c_{11} \text{ is an element.}$$

Similarly $c_{12} = 0, c_{21} = 0, c_{22} = 13, c_{31} = 7, c_{32} = 5$.

$$\text{Therefore, } AB = C = [c_{ij}] = \begin{bmatrix} 4 & 0 \\ 0 & 13 \\ 7 & 5 \end{bmatrix}$$

The product BA does not exist, because the number of columns in B is not equal to the number of rows in A.

Example 7.11

A fruit shop keeper prepares 3 different varieties of gift packages. Pack-I contains 6 apples, 3 oranges and 3 pomegranates. Pack-II contains 5 apples, 4 oranges and 4 pomegranates and Pack –III contains 6 apples, 6 oranges and 6 pomegranates. The cost of an apple, an orange and a pomegranate respectively are ₹ 30, ₹ 15 and ₹ 45. What is the cost of preparing each package of fruits?

Solution

$$\text{Cost matrix } A = [30 \ 15 \ 45], \qquad \text{Fruit matrix } B = \begin{matrix} & \text{P-I} & \text{P-II} & \text{P-III} & \\ & \begin{bmatrix} 6 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix} & \begin{matrix} \text{Apples} \\ \text{Oranges} \\ \text{Pomegranates} \end{matrix} \end{matrix}$$

Cost of packages are obtained by computing AB. That is, by multiplying cost of each item in A (cost matrix A) with number of items in B (Fruit matrix B).

$$AB = [30 \ 15 \ 45] \begin{bmatrix} 6 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 360 \\ 390 \\ 540 \end{bmatrix}$$

Pack-I cost ₹ 360, Pack-II cost ₹ 390, Pack-III costs ₹ 540.

7.2.4 Properties of Matrix Addition, Scalar Multiplication and Product of Matrices

Let $A, B,$ and C be three matrices of same order which are conformable for addition and a, b be two scalars. Then we have the following:

(1) $A + B$ yields a matrix of the same order

(2) $A + B = B + A$ (Matrix addition is commutative)

(3) $(A + B) + C = A + (B + C)$ (Matrix addition is associative)

(4) $A + O = O + A = A$ (O is additive identity)

(5) $A + (-A) = O = (-A) + A$ ($-A$ is the additive inverse of A)

(6) $(a + b)A = aA + bA$ and $a(A + B) = aA + aB$

(7) $a(bA) = (ab)A, \quad 1A = A$ and $0A = O$.

Properties of matrix multiplication

Using the algebraic properties of matrices we have,

- If $A, B,$ and C are three matrices of orders $m \times n, n \times p$ and $p \times q$ respectively, then $A(BC)$ and $(AB)C$ are matrices of same order $m \times q$ and

  $A(BC) = (AB)C$ (Matrix multiplication is associative).

- If A, B, and C are three matrices of orders $m \times n$, $n \times p$, and $n \times p$ respectively, then $A(B + C)$ and $AB + AC$ are matrices of the same order $m \times p$ and

  $A(B + C) = AB + AC$. (Matrix multiplication is left distributive over addition)
- If A, B, and C are three matrices of orders $m \times n$, $m \times n$, and $n \times p$ respectively, then $(A + B)C$ and $AC + BC$ are matrices of the same order $m \times p$ and

  $(A + B)C = AC + BC$. (Matrix multiplication is right distributive over addition).
- If A, B are two matrices of orders $m \times n$ and $n \times p$ respectively and α is scalar, then

  $\alpha(AB) = A(\alpha B) = (\alpha A)B$ is a matrix of order $m \times p$.
- If I is the unit matrix, then $AI = IA = A$ (I is called multiplicative identity).

7.2.5 Operation of Transpose of a Matrix and its Properties

Definition 7.16

The **transpose** of a matrix is obtained by interchanging rows and columns of A and is denoted by A^T.

More precisely, if $A = [a_{ij}]_{m \times n}$, then $A^T = [b_{ij}]_{n \times m}$, where $b_{ij} = a_{ji}$ so that the $(i, j)^{th}$ entry of A^T is a_{ji}.

For instance,

$$A = \begin{bmatrix} 1 & \sqrt{2} & 4 \\ -8 & 0 & 0.2 \end{bmatrix} \quad \text{implies} \quad A^T = \begin{bmatrix} 1 & -8 \\ \sqrt{2} & 0 \\ 4 & 0.2 \end{bmatrix}$$

We state a few basic results on transpose whose proofs are straight forward.

For any two matrices A and B of suitable orders, we have

(i) $(A^T)^T = A$ (ii) $(kA)^T = kA^T$ (where k is any scalar)

(iii) $(A + B)^T = A^T + B^T$ (iv) $(AB)^T = B^T A^T$ (reversal law on transpose)

Example 7.12

If $A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$,

verify (i) $(AB)^T = B^T A^T$ (ii) $(A + B)^T = A^T + B^T$ (iii) $(A - B)^T = A^T - B^T$ (iv) $(3A)^T = 3A^T$

Solution

(i) $AB = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 2 & 22 \\ -2 & 9 & 9 \\ 7 & 1 & 14 \end{bmatrix}$

$$(AB)^T = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \quad \quad \quad \quad \text{... (1)}$$

$$B^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \qquad \dots (2)$$

From (1) and (2), $\quad (AB)^T = B^T A^T.$

(ii) $\quad A + B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 3 \end{bmatrix}$

$$(A+B)^T = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \qquad \dots(3)$$

$$A^T + B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \qquad \dots (4)$$

From (3) and (4), $\quad (A+B)^T = A^T + B^T.$

(iii) $\quad A - B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 3 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$(A-B)^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \qquad \dots (5)$$

$$A^T - B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \qquad \dots (6)$$

From (5) and (6) $\quad (A-B)^T = A^T - B^T.$

(iv) $\quad 3A = \begin{bmatrix} 12 & 18 & 6 \\ 0 & 3 & 15 \\ 0 & 9 & 6 \end{bmatrix}$

$$(3A)^T = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix} = 3\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = 3(A^T).$$

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7.2.6 Symmetric and Skew-symmetric Matrices

Definition 7.17

A square matrix A is said to be **symmetric** if $A^T = A$.

That is, $A = [a_{ij}]_{n \times n}$ is a symmetric matrix, then $a_{ij} = a_{ji}$ for all i and j.

For instance, $A = \begin{bmatrix} 3 & -6 & 9 \\ -6 & 8 & 5 \\ 9 & 5 & 2 \end{bmatrix}$ is a symmetric matrix since $A^T = A$.

Observe that transpose of A^T is the matrix A itself. That is $\left(A^T \right)^T = A$.

Definition 7.18

A square matrix A is said to be **skew-symmetric** if $A^T = -A$.

If $A = [a_{ij}]_{n \times n}$ is a skew-symmetric matrix, then $a_{ij} = -a_{ji}$ for all i and j.

Now, if we put $i = j$, then $2a_{ii} = 0$ or $a_{ii} = 0$ for all i. This means that all the diagonal elements of a skew-symmetric matrix are zero.

For instance, $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is a skew-symmetric matrix since $A^T = -A$.

It is interesting to note that any square matrix can be written as the sum of symmetric and skew-symmetric matrices.

Theorem 7.1

For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew-symmetric matrix.

Proof

Let $B = A + A^T$.

$$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B .$$

This implies $A + A^T$ is a symmetric matrix.

Next, we let $C = A - A^T$. Then we see that

$$C^T = (A + (-A^T))^T = A^T + (-A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$$

This implies $A - A^T$ is a skew-symmetric matrix.

Theorem 7.2

Any square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof

Let A be a square matrix. Then, we can write

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

From Theorem 7.1, it follows that $(A + A^T)$ and $(A - A^T)$ are symmetric and skew-symmetric matrices respectively. Since $(kA)^T = kA^T$, it follows that $\frac{1}{2}(A + A^T)$ and $\frac{1}{2}(A - A^T)$ are symmetric and skew-symmetric matrices, respectively. Now, the desired result follows.

Note 7.4

A matrix which is both symmetric and skew-symmetric is a zero matrix.

Example 7.13

Express the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrices.

Solution

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$$

Let $P = \dfrac{1}{2}(A + A^T) = \dfrac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}$

Now $P^T = \dfrac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} = P$

Thus, $P = \dfrac{1}{2}(A + A^T)$ is a symmetric matrix.

Let $Q = \dfrac{1}{2}(A - A^T)$

$= \dfrac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$

Then $Q^T = \dfrac{1}{2}\begin{bmatrix} 0 & -9 & -9 \\ 9 & 0 & 3 \\ 9 & -3 & 0 \end{bmatrix} = -Q$

Thus $Q = \dfrac{1}{2}(A - A^T)$ is a skew-symmetric matrix.

$$A = P + Q = \dfrac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} + \dfrac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$$

Thus A is expressed as the sum of symmetric and skew-symmetric matrices.

EXERCISE 7.1

(1) Construct an $m \times n$ matrix $A = [a_{ij}]$, where a_{ij} is given by

(i) $a_{ij} = \dfrac{(i - 2j)^2}{2}$ with $m = 2, n = 3$ (ii) $a_{ij} = \dfrac{|3i - 4j|}{4}$ with $m = 3, n = 4$

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(2) Find the values of $p, q, r,$ and s if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \dfrac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}.$$

(3) Determine the value of $x + y$ if $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}.$

(4) Determine the matrices A and B if they satisfy

$$2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0 \text{ and } A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

(5) If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then compute A^4.

(6) Consider the matrix $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

   (i)   Show that $A_\alpha A_\beta = A_{(\alpha+\beta)}$.

   (ii) Find all possible real values of α satisfying the condition $A_\alpha + A_\alpha^T = I$.

(7) If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and such that $(A-2I)(A-3I) = O$, find the value of x.

(8) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, show that A^2 is a unit matrix.

(9) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI = O$, find the value of k.

(10) Give your own examples of matrices satisfying the following conditions in each case:

   (i) A and B such that $AB \neq BA$.

   (ii) A and B such that $AB = O = BA$, $A \neq O$ and $B \neq O$.

   (iii) A and B such that $AB = O$ and $BA \neq O$.

(11) Show that $f(x)f(y) = f(x+y)$, where $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

(12) If A is a square matrix such that $A^2 = A$, find the value of $7A - (I + A)^3$.

(13) Verify the property $A(B + C) = AB + AC$, when the matrices $A, B,$ and C are given by

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

(14) Find the matrix A which satisfies the matrix relation $A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$.

(15) If $A^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$, verify the following

(i) $(A+B)^T = A^T + B^T = B^T + A^T$ (ii) $(A-B)^T = A^T - B^T$ (iii) $(B^T)^T = B$.

(16) If A is a 3×4 matrix and B is a matrix such that both $A^T B$ and BA^T are defined, what is the order of the matrix B?

(17) Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

(i) $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ and   (ii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$.

(18) Find the matrix A such that $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A^T = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$.

(19) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ is a matrix such that $AA^T = 9I$, find the values of x and y.

(20) (i) For what value of x, the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix}$ is skew-symmetric.

(ii) If $\begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of p, q, and r.

(21) Construct the matrix $A = [a_{ij}]_{3\times 3}$, where $a_{ij} = i - j$. State whether A is symmetric or skew-symmetric.

(22) Let A and B be two symmetric matrices. Prove that $AB = BA$ if and only if AB is a symmetric matrix.

(23) If A and B are symmetric matrices of same order, prove that

(i) $AB + BA$ is a symmetric matrix.

(ii) $AB - BA$ is a skew-symmetric matrix.

(24) A shopkeeper in a Nuts and Spices shop makes gift packs of cashew nuts, raisins and almonds.

Pack I contains 100 gm of cashew nuts, 100 gm of raisins and 50 gm of almonds.

Pack-II contains 200 gm of cashew nuts, 100 gm of raisins and 100 gm of almonds.

Pack-III contains 250 gm of cashew nuts, 250 gm of raisins and 150 gm of almonds.

The cost of 50 gm of cashew nuts is ₹ 50, 50 gm of raisins is ₹10, and 50 gm of almonds is ₹ 60. What is the cost of each gift pack?

7.3 Determinants

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number called **determinant** of the matrix A.

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Note 7.5

(i) Determinants can be defined only for square matrices.

(ii) For a square matrix A, $|A|$ is read as **determinant of A.**

(iii) Matrix is only a representation whereas determinant is a value of a matrix.

7.3.1 Determinants of Matrices of different order

Determinant of a matrix of order 1

Let $A = [a]$ be the matrix of order 1, then the determinant of A is defined as 'a'.

Determinant of a matrix of order 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \, a_{22} - a_{21} \, a_{12}.$$

Example 7.14

Evaluate : (i) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$.

Solution

(i) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = (2 \times 2) - (-1 \times 4) = 4 + 4 = 8.$

(ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = (\cos\theta\cos\theta) - (-\sin\theta\sin\theta) = \cos^2\theta + \sin^2\theta = 1.$

Determinant of a Matrix of order 3

We consider the determinant of a 3×3 matrix with entries as real numbers or real valued functions defined on $\mathbb{R}$ and study its properties and discuss various methods of evaluation of certain determinants.

Definition 7.19

Let $A = [a_{ij}]_{3\times3}$ be a given square matrix of order 3. The **minor** of an arbitrary element a_{ij} is the determinant obtained by deleting the ith row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is usually denoted by M_{ij}.

> **Definition 7.20**
>
> The **cofactor** is a signed minor. The cofactor of a_{ij} is usually denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

For instance, consider the 3×3 matrix defined by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then the minors and cofactors of the elements a_{11}, a_{12}, a_{13} are given as follows :

(i)   Minor of a_{11} is M$_{11}$ $= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

Cofactor of a_{11} is A_{11} $= (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

(ii)   Minor of a_{12} is M_{12} $= \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

Cofactor a_{12} is A_{12} $= (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$

(iii)   Minor of a_{13} is M_{13} $= \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

Cofactor of a_{13} is A_{13} $= (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$.

Result 7.1 (Laplace Expansion)

For a given matrix $A = [a_{ij}]_{3\times3}$, the sum of the product of elements of the first row with their corresponding cofactors is the **determinant** of A.

That is, $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

This can also be written using minors. That is, $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$.

The determinant can be computed by expanding along any row or column and it is important to note that the value in all cases remains the same. For example,

expansion along R_1 is $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

along R_2 is $|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$.

along C_1 is $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

> **Example 7.15**
>
> Compute all minors, cofactors of A and hence compute $|A|$ if $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ -3 & 5 & 2 \end{bmatrix}$. Also check

that $|A|$ remains unaltered by expanding along any row or any column.

Solution

Minors :

$$M_{11} = \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = -10 - 30 = -40$$

$$M_{12} = \begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} = 8 + 18 = 26$$

$$M_{13} = \begin{vmatrix} 4 & -5 \\ -3 & 5 \end{vmatrix} = 20 - 15 = 5$$

$$M_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 6 + 10 = 16$$

$$M_{22} = \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = 2 - 6 = -4$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} = 5 + 9 = 14$$

$$M_{31} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 18 - 10 = 8$$

$$M_{32} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 6 + 8 = 14$$

$$M_{33} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -5 - 12 = -17$$

Cofactors :

$$A_{11} = (-1)^{1+1}(-40) = -40$$

$$A_{12} = (-1)^{1+2}(+26) = -26$$

$$A_{13} = (-1)^{1+3}(5) = 5$$

$$A_{21} = (-1)^{2+1}(16) = -16$$

$$A_{22} = (-1)^{2+2}(-4) = -4$$

$$A_{23} = (-1)^{2+3}(14) = -14$$

$$A_{31} = (-1)^{3+1}(8) = 8$$

$$A_{32} = (-1)^{3+2}(14) = -14$$

$$A_{33} = (-1)^{3+3}(-17) = -17$$

Expanding along R_1 yields

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \ .$$

$$|A| = 1(-40) + (3)(-26) + (-2)(5) = -128 \ . \qquad\qquad ... (3)$$

Expanding along C_1 yields

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \ .$$
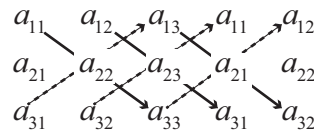
$$= 1(-40) + 4(-16) + -3(8) = -128 \qquad \text{... (4)}$$

From (3) and (4), we have

|A| obtained by expanding along R_1 is equal to expanding along C_1.

Evaluation of determinant of order 3 by using Sarrus Rule (named after the French Mathematician Pierre Frédéic Sarrus)

Let $A = [a_{ij}]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Write the entries of Matrix A as follows :

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

Then $|A|$ is computed as follows :

$$|A| = \left[a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \right] - \left[a_{33}a_{21}a_{12} + a_{32}a_{23}a_{11} + a_{31}a_{22}a_{13} \right]$$

Example 7.16

Find $|A|$ if $A = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{bmatrix}$.

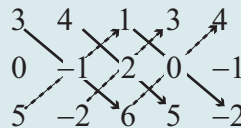**Solution**

$$\begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix} = 0 M_{11} - \sin\alpha M_{12} + \cos\alpha M_{13}$$

$$= 0 - \sin\alpha(0 - \cos\alpha \sin\beta) + \cos\alpha(-\sin\alpha \sin\beta - 0) = 0.$$

Example 7.17

Compute $|A|$ using Sarrus rule if $A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{bmatrix}$.

Solution

$$\begin{matrix} 3 & 4 & 1 & 3 & 4 \\ 0 & -1 & 2 & 0 & -1 \\ 5 & -2 & 6 & 5 & -2 \end{matrix}$$

$$|A| = \left[3(-1)(6) + 4(2)(5) + 1(0)(-2) \right] - \left[5(-1)(1) + (-2)(2)3 + 6(0)(4) \right]$$

$$= [-18 + 40 + 0] - [-5 - 12 + 0] = 22 + 17 = 39.$$

Note 7.6

For easier calculations, we expand the determinant along a row or column which contains maximum number of zeros.

Determinant of square matrix of order *n*, *n* ≥ 4

The concept of determinant can be extended to the case of square matrix of order $n, n \geq 4$. Let $A = [a_{ij}]_{n \times n}, n \geq 4$.

If we delete the ith row and j^{th} column from the matrix of $A = [a_{ij}]_{n \times m}$, we obtain a determinant of order $(n-1)$, which is called the **minor of the element** a_{ij}. We denote this minor by M_{ij}. The cofactor of the element a_{ij} is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

Result 7.2

For a given square matrix $A = [a_{ij}]_{n \times n}$ of order *n*, the sum of the products of elements of the first row with their corresponding cofactors is the determinant of *A*. That is,

$$|A| = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} = \sum_{j=1}^{n} a_{1j}A_{1j}$$ which, by the definition of cofactors and minors, is same as

$$|A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}M_{1j},$$

where A_{1j} denotes the cofactor of a_{1j} and M_{1j} denotes the minor of $a_{1j}, j = 1, 2, \ldots, n.$ ■

Note 7.7

(i) If $A = [a_{ij}]_{n \times n}$ then determinant of *A* can also be denoted as det(*A*) or *det A* or Δ.

(ii) It can be computed by using any row or column.

7.3.2 Properties of Determinants

We can use one or more of the following properties of the determinants to simplify the evaluation of determinants.

Property 1

The determinant of a matrix remains unaltered if its rows are changed into columns and columns into rows. That is, $|A| = |A^T|$.

Since the row-wise expansion is same as the column-wise expansion, the result holds good.

Property 2

If any two rows / columns of a determinant are interchanged, then the determinant changes in sign but its absolute value remains unaltered.

Verification

$$\text{Let } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$\text{Let } |A_1| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad (\text{since } R_2 \leftrightarrow R_3)$$

$$= a_1(b_3c_2 - b_2c_3) - b_1(a_3c_2 - a_2c_3) + c_1(a_3b_2 - a_2b_3)$$

$$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$= -|A|$$

Therefore, $|A_1| = -|A|$. Thus the property is verified. ■

Property 3

If there are n interchanges of rows (columns) of a matrix A then the determinant of the resulting matrix is $(-1)^n|A|$.

Property 4

If two rows (columns) of a matrix are identical, then its determinant is zero.

Verification

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}$, with 2nd and 3rd rows are identical.

Interchanging second and third rows, we get $-|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = |A|$.

$$\Rightarrow 2|A| = 0 \Rightarrow |A| = 0.$$ ■

Property 5

If a row (column) of a matrix A is a scalar multiple of another row (or column) of A, then its determinant is zero.

Note 7.8

(i) If all entries of a row or a column are zero, then the determinant is zero.

(ii) The determinant of a triangular matrix is obtained by the product of the principal diagonal elements.

Property 6

If each element in a row (or column) of a matrix is multiplied by a scalar k, then the determinant is multiplied by the same scalar k.

Verification

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Let $|A_1| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= ka_1(b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2) = k|A|$$

$$= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] = k|A|$$

$$\Rightarrow |A_1| = k|A|.$$ ■

Note 7.9

If A is a square matrix of order n, then

    (i) $|AB| = |A||B|$

    (ii) If $AB = O$ then either $|A| = 0$ or $|B| = 0$.

    (iii) $|A^n| = (|A|)^n$

Property 7

    If each element of a row (or column) of a determinant is expressed as sum of two or more terms then the whole determinant is expressed as sum of two or more determinants.

That is,
$$\begin{vmatrix} a_1 + m_1 & b_1 & c_1 \\ a_2 + m_2 & b_2 & c_2 \\ a_3 + m_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}.$$

Verification

    By taking first column expansion it can be verified easily.

$$\text{LHS} = (a_1 + m_1)(b_2c_3 - b_3c_2) - (a_2 + m_2)(b_1c_3 - b_3c_1) + (a_3 + m_3)(b_1c_2 - b_2c_1)$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) + m_1(b_2c_3 - b_3c_2) - m_2(b_1c_3 - b_3c_1)$$

$$+ m_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix} = \text{RHS.}$$

∎

Property 8

    If, to each element of any row (column) of a determinant the equi-multiples of the corresponding entries of one or more rows (columns) are added or subtracted, then the value of the determinant remains unchanged.

Verification

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and $|A_1| = \begin{vmatrix} a_1 + pa_2 + qa_3 & b_1 + pb_2 + qb_3 & c_1 + pc_2 + qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pa_2 & pb_2 & pc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qa_3 & qb_3 & qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{cases} \text{using} \\ \text{Property 7} \end{cases}$$

$$|A_1| = |A| + p\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + q\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$|A_1| = |A| + p(0) + q(0) = |A|$ (using Property 4)

Therefore $|A_1| = |A|$

This property is independent of any fixed row or column.

Example 7.18

If a, b, c and x are positive real numbers, then show that $\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^x + b^{-x})^2 & (b^x - b^{-x})^2 & 1 \\ (c^x + c^{-x})^2 & (c^x - c^{-x})^2 & 1 \end{vmatrix}$ is zero.

Solution

Applying $C_1 \to C_1 - C_2$, we get $\begin{vmatrix} 4 & (a^x - a^{-x})^2 & 1 \\ 4 & (b^x - b^{-x})^2 & 1 \\ 4 & (c^x - c^{-x})^2 & 1 \end{vmatrix} = 0$, since C_1 and C_3 are proportional.

Example 7.19

Without expanding the determinants, show that $|B| = 2|A|$.

Where $B = \begin{bmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Solution

We have $|B| = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$ $(R_1 \to R_1 + R_2 + R_3)$

$= 2 \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$

$= 2 \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$ $(R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$

$= 2 \begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix}$ $(R_1 \to R_1 + R_2 + R_3)$

$= 2(-1)^2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= 2|A|$.

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Example 7.20

Evaluate $\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix}$.

Solution

$\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 2017-2014 & 0 \\ 2020 & 2023-2020 & 1 \\ 2023 & 2026-2023 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 3 & 0 \\ 2020 & 3 & 1 \\ 2023 & 3 & 0 \end{vmatrix} = 3\begin{vmatrix} 2014 & 1 & 0 \\ 2020 & 1 & 1 \\ 2023 & 1 & 0 \end{vmatrix}$

$= -3(2014 - 2023) = -3(-9) = 27.$

Example 7.21

Find the value of x if $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$.

Solution

Since all the entries below the principal diagonal are zero, the value of the determinant is $(x-1)(x-2)(x-3) = 0$ which gives $x = 1, 2, 3$.

Example 7.22

Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x).$

Solution

Applying $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$, we get

$\text{LHS} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$

$= (y-x)(z-x)[(z+x)-(y+x)] .$

$= (y-x)(z-x)(z-y) .$

$= (x-y)(y-z)(z-x) = \text{RHS}.$

EXERCISE 7.2

(1) Without expanding the determinant, prove that $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0.$

(2) Show that $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0.$

(3) Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

(4) Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)$.

(5) Prove that $\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0.$

(6) Show that $\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = 0.$

(7) Write the general form of a 3×3 skew-symmetric matrix and prove that its determinant is 0.

(8) If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$,

prove that a, b, c are in *G.P.* or α is a root of $ax^2 + 2bx + c = 0$.

(9) Prove that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$.

(10) If a, b, c are p^{th}, q^{th} and r^{th} terms of an *A.P*, find the value of $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$.

(11) Show that $\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by x^4.

(12) If a, b, c are all positive, and are p^{th}, q^{th} and r^{th} terms of a *G.P.*, show that $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$

(13) Find the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ if $x, y, z \neq 1$.

Matrices and Determinants

(14) If $A = \begin{bmatrix} \dfrac{1}{2} & \alpha \\ 0 & \dfrac{1}{2} \end{bmatrix}$, prove that $\displaystyle\sum_{k=1}^{n} \det(A^k) = \dfrac{1}{3}\left(1 - \dfrac{1}{4^n}\right)$.

(15) Without expanding, evaluate the following determinants :

(i) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ (ii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

(16) If A is a square matrix and $|A| = 2$, find the value of $|AA^T|$.

(17) If A and B are square matrices of order 3 such that $|A| = -1$ and $|B| = 3$, find the value of $|3AB|$.

(18) If $\lambda = -2$, determine the value of $\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$.

(19) Determine the roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$.

(20) Verify that $\det(AB) = (\det A)(\det B)$ for $A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$.

(21) Using cofactors of elements of second row, evaluate $|A|$, where $A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

7.3.3 Application of Factor Theorem to Determinants.

Theorem 7.3 (Factor Theorem)

If each element of a matrix A is a polynomial in x and if $|A|$ vanishes for $x = a$, then $(x - a)$ is a factor of $|A|$.

Note 7.10

(i) This theorem is very much useful when we have to obtain the value of the determinant in 'factors' form.

(ii) If we substitute b for a in the determinant $|A|$, any two of its rows or columns become identical, then $|A| = 0$, and hence by factor theorem $(a - b)$ is a factor of $|A|$.

(iii) If r rows (columns) are identical in a determinant of order n ($n \geq r$), when we put $x = a$, then $(x - a)^{r-1}$ is a factor of $|A|$.

(iv) A square matrix (or its determinant) is said to be in cyclic symmetric form if each row is obtained from the first row by changing the variables cyclically.

(v) If the determinant is in cyclic symmetric form and if m is the difference between the degree of the product of the factors (obtained by substitution) and the degree of the product of the leading diagonal elements and if

(1) m is zero, then the required factor is a constant k

(2) m is 1, then the required factor is $k(a + b + c)$ and

(3) m is 2, then the required factor is $k(a^2 + b^2 + c^2) + l\,(ab + bc + ca)$.

Example 7.23

Using Factor Theorem, prove that $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2(x+9)$.

Solution

$$\text{Let } |A| = \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix}.$$

$$\text{Putting } x = 1, \text{ we get } |A| = \begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 0$$

Since all the three rows are identical, $(x-1)^2$ is a factor of $|A|$

$$\text{Putting } x = -9 \text{ in } |A|, \text{we get} |A| = \begin{vmatrix} -8 & 3 & 5 \\ 2 & -7 & 5 \\ 2 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 5 \\ 0 & -7 & 5 \\ 0 & 3 & -5 \end{vmatrix} = 0$$

Therefore $(x+9)$ is a factor of $|A|$ [since $C_1 \to C_1 + C_2 + C_3$].

The product $(x-1)^2 (x+9)$ is a factor of $|A|$. Now the determinant is a cubic polynomial in x.

Therefore the remaining factor must be a constant 'k'.

$$\text{Therefore } \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = k(x-1)^2(x+9).$$

Equating x^3 term on both sides, we get $k = 1$. Thus $|A| = (x-1)^2 (x+9)$.

Example 7.24

Prove that $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)\,(y-z)\,(z-x)\,(xy+yz+zx)$.

Solution

$$\text{Let } |A| = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}.$$

$$\text{Putting } x = y \text{ gives } |A| = \begin{vmatrix} 1 & y^2 & y^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = 0 \quad (\text{since } R_1 \equiv R_2).$$

Therefore $(x - y)$ is a factor.

The given determinant is in cyclic symmetric form in x, y and z. Therefore $(y - z)$ and $(z - x)$ are also factors.

The degree of the product of the factors $(x - y)(y - z)(z - x)$ is 3 and the degree of the product of the leading diagonal elements $1 \times y^2 \times z^3$ is 5.

Therefore the other factor is $k(x^2 + y^2 + z^2) + \ell(xy + yz + zx)$.

Thus
$$\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = [k(x^2 + y^2 + z^2) + \ell(xy + yz + zx)] \times (x - y)(y - z)(z - x) .$$

Putting $x = 0$, $y = 1$ and $z = 2$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 8 \end{vmatrix} = [k(0 + 1 + 4) + \ell(0 + 2 + 0)](-1)(1 - 2)(2 - 0)$$

$\Rightarrow$
$$(8 - 4) = [(5k + 2\ell)](-1)(-1)(2)$$

$$4 = 10k + 4\ell \Rightarrow 5k + 2\ell = 2. \qquad \ldots (1)$$

Putting $x = 0$, $y = -1$ and $z = 1$, We get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = [k(2) + \ell(-1)](1)(-2)(1)$$

$\Rightarrow$
$$[(2k - \ell)(-2)] = 2$$

$$2k - \ell = -1. \qquad \ldots (2)$$

Solving (1) and (2), we get $k = 0$, $\ell = 1$.

Therefore
$$\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x - y)(y - z)(z - x)(xy + yz + zx).$$

Example 7.25

Prove that $|A| = \begin{vmatrix} (q + r)^2 & p^2 & p^2 \\ q^2 & (r + p)^2 & q^2 \\ r^2 & r^2 & (p + q)^2 \end{vmatrix} = 2pqr\,(p + q + r)^3.$

Solution :

Taking $p = 0$, we get $|A| = \begin{vmatrix} (q + r)^2 & 0 & 0 \\ q^2 & r^2 & q^2 \\ r^2 & r^2 & q^2 \end{vmatrix} = 0.$

Therefore, $(p - 0)$ is a factor. That is, p is a factor.

Since $|A|$ is in cyclic symmetric form in p, q, r and hence q and r also factors.

Putting $p + q + r = 0 \Rightarrow q + r = -p$; $r + p = -q$; and $p + q = -r$.

$$|A| = \begin{vmatrix} p^2 & p^2 & p^2 \\ q^2 & q^2 & q^2 \\ r^2 & r^2 & r^2 \end{vmatrix} = 0 \text{ since 3 columns are identical.}$$

Therefore, $(p + q + r)^2$ is a factor of $|A|$.

The degree of the obtained factor $pqr (p + q + r)^2$ is 5. The degree of $|A|$ is 6.

Therefore, required factor is $k (p + q + r)$.

$$\begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = k(p+q+r)\,(p+q+r)^2 \times pqr$$

Taking $p = 1$, $q = 1$, $c = 1$, we get

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k(1+1+1)^3 \,(1)\,(1)\,(1).$$

$4(16 - 1) - 1(4 - 1) + 1(1 - 4) = 27k$

$60 - 3 - 3 = 27\,k \Rightarrow k = 2.$

$|A| = 2pqr\,(p + q + r)^3.$

Example 7.26

In a triangle ABC, if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix} = 0,$

prove that $\triangle ABC$ is an isosceles triangle.

Solution :

By putting $\sin A = \sin B$, we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin A & 1+\sin C \\ \sin A(1+\sin A) & \sin A(1+\sin A) & \sin C(1+\sin C) \end{vmatrix} = 0$$

That is, by putting $\sin A = \sin B$ we see that, the given equation is satisfied.

Similarly by putting $\sin B = \sin C$ and $\sin C = \sin A$, the given equation is satisfied.

Thus, we have $A = B$ or $B = C$ or $C = A$.

In all cases atleast two angles are equal. Thus the triangle is isosceles.

EXERCISE 7.3

Solve the following problems by using Factor Theorem :

(1) Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$.

(2) Show that $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$.

(3) Solve $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0.$

(4) Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)\,(a-b)\,(b-c)\,(c-a).$

(5) Solve $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$.

(6) Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x).$

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7.3.4 Product of Determinants

While multiplying two matrices "row-by-column" rule alone can be followed. The process of interchanging the rows and columns will not affect the value of the determinant (by Property 1). Therefore we can also adopt the following procedures for multiplication of two determinants.

   (i) Row by column multiplication rule
  (ii) Row by row multiplication rule
 (iii) Column by column multiplication rule
 (iv) Column by row multiplication rule

Note 7.11

   (i) If A and B are square matrices of the same order n, then $|AB| = |A||B|$ holds.

  (ii) In matrices, although $AB \neq BA$ in general, we do have $|AB| = |BA|$ always.

Example 7.27

Verify that $|AB| = |A||B|$ if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$

Solution

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |AB| = 1. \qquad \qquad \dots (1)$$

$$|A| = \cos^2\theta + \sin^2\theta = 1.$$

$$|B| = \cos^2\theta + \sin^2\theta = 1.$$

$$|A|\,|B| = 1. \qquad \qquad \dots (2)$$

From (1) and (2), $|AB| = |A|\,|B|$.

Example 7.28

Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2 + c^2 & ab & ac \\ ab & c^2 + a^2 & bc \\ ab & bc & a^2 + b^2 \end{vmatrix}$.

Solution

$$\text{LHS} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}.$$

$$= \begin{vmatrix} 0 + c^2 + b^2 & 0 + 0 + ab & 0 + ac + 0 \\ 0 + 0 + ab & c^2 + 0 + a^2 & bc + 0 + 0 \\ 0 + ac + 0 & bc + 0 + 0 & b^2 + a^2 + 0 \end{vmatrix}.$$

$$= \begin{vmatrix} c^2 + b^2 & ab & ac \\ ab & c^2 + a^2 & bc \\ ac & bc & b^2 + a^2 \end{vmatrix} = \text{RHS}.$$

Example 7.29

Show that $\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$.

Solution

$$\text{RHS} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \quad \text{[In the 2}^{\text{nd}}\text{ determinant } R_2 \leftrightarrow R_3]$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Taking row by column method, we get

$$= \begin{vmatrix} -a^2+bc+cb & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \text{RHS}.$$

Example 7.30

Prove that $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}.$

Solution

$$\text{LHS} = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times (-1)\,(-1) \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-x^2-x^2 & x-x-x^2 & x-x^2-x \\ x-x-x^2 & x^2-1-x^2 & x^2-x-x \\ x-x^2-x & x^2-x-x & x^2-x^2-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}.$$

$= \text{R.H.S.}$

7.3.5 Relation between a Determinant and its Cofactor Determinant

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$

Let $A_1, B_1, C_1 \ldots$ be the cofactors of $a_1, b_1, c_1 \ldots$ in $|A|$.

Hence, the cofactor determinant is $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

$|A| = a_1 A_1 + b_1 B_1 + c_1 C_1$

Similarly, $|A| = a_2 A_2 + b_2 B_2 + c_2 C_2$ and $|A| = a_3 A_3 + b_3 B_3 + c_3 C_3$

Note that the sum of the product of elements of any row (or column) with their corresponding cofactors is the value of the determinant.

Now $a_1 A_2 + b_1 B_2 + c_1 C_2 = -a_1 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

$= -a_1 (b_1 c_3 - b_3 c_1) + b_1 (a_1 c_3 - a_3 c_1) - c_1 (a_1 b_3 - a_3 b_1)$

$= a_1 b_1 c_3 + a_1 b_3 c_1 + a_1 b_1 c_3 - a_3 b_1 c_1 - a_1 b_3 c_1 + a_3 b_1 c_1 = 0$

Similarly we get

$a_1 A_3 + b_1 B_3 + c_1 C_3 = 0$; $a_2 A_1 + b_2 B_1 + c_2 C_1 = 0$;

$a_2 A_3 + b_2 B_3 + c_2 C_3 = 0$; $a_3 A_1 + b_3 B_1 + c_3 C_1 = 0$ and $a_3 A_2 + b_3 B_2 + c_3 C_2 = 0$.

Note 7.12

If elements of a row (or column) are multiplied with corresponding cofactors of any other row (or column) then their sum is zero.

Example 7.31

If A_i, B_i, C_i are the cofactors of a_i, b_i, c_i, respectively, $i = 1$ to 3 in

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ show that } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2 .$$

Solution

Consider the product $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 A_1 + b_1 B_1 + c_1 C_1 & a_1 A_2 + b_1 B_2 + c_1 C_2 & a_1 A_3 + b_1 B_3 + c_1 C_3 \\ a_2 A_1 + b_2 B_1 + c_2 C_1 & a_2 A_2 + b_2 B_2 + c_2 C_2 & a_2 A_3 + b_2 B_3 + c_2 C_3 \\ a_3 A_1 + b_3 B_1 + c_3 C_1 & a_3 A_2 + b_3 B_2 + c_3 C_2 & a_3 A_3 + b_3 B_3 + c_3 C_3 \end{vmatrix}$$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3$$

That is, $|A| \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^3$.

$\Rightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$.

7.3.6 Area of a Triangle

We know that the **area of a triangle** whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is equal to the absolute value of

$$\frac{1}{2}(x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3).$$

This expression can be written in the form of a determinant as the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Example 7.32

If the area of the triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is 9 square units, find the values of k.

Solution

$$\text{Area of the triangle} = \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

$$9 = \left| \frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} \right| = \left| \frac{1}{2}(-k)(-3-3) \right|$$

$$\Rightarrow \quad 9 = 3|k| \text{ and hence, } k = \pm 3.$$

Note 7.13

The area of the triangle formed by three points is zero if and only if the three points are collinear. Also, we remind the reader that the determinant could be negative whereas area is always non-negative.

Example 7.33

Find the area of the triangle whose vertices are $(-2, -3)$, $(3, 2)$, and $(-1, -8)$.

Solution

$$\text{Area of the triangle} = \left| \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|.$$

$$\left| \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix} \right| = \left| \frac{1}{2}(-20+12-22) \right| = |-15| = 15$$

and therefore required area is 15 sq.units.

Example 7.34

Show that the points $(a, b+c)$, $(b, c+a)$, and $(c, a+b)$ are collinear.

Solution

To prove the given points are collinear, it suffices to prove $|A| = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} = 0$.

Applying $C_1 \to C_1 + C_2$, we deduce that

$$|A| = \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = (a+b+c) \times 0 = 0$$

which shows that the given points are collinear.

7.2.11 Singular and non-singular Matrices

Definition 7.21

A square matrix A is said to be **singular** if $|A| = 0$. A square matrix A is said to be **non-singular** if $|A| \neq 0$.

For instance, the matrix $A = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix}$ is a singular matrix, since

$$|A| = 3(1-1) - 8(-4+4) + 1(-4+4) = 0.$$

If $B = \begin{bmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{bmatrix}$ then $|B| = 2(0-20) - (-3)(-42-4) + 5(30-0) = -28 \neq 0.$

Thus B is a non-singular matrix.

Note 7.14

If A and B are non-singular matrices of the same order then AB and BA are also non-singular matrices because $|AB| = |A| \, |B| = |BA|$.

EXERCISE 7.4

(1) Find the area of the triangle whose vertices are $(0, 0)$, $(1, 2)$ and $(4, 3)$.

(2) If $(k, 2)$, $(2, 4)$ and $(3, 2)$ are vertices of the triangle of area 4 square units then determine the value of k.

(3) Identify the singular and non-singular matrices:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{bmatrix}$

(4) Determine the values of a and b so that the following matrices are singular:

(i) $A = \begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$

(ii) $B = \begin{bmatrix} b-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

(5) If $\cos 2\theta = 0$, determine $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$.

(6) Find the value of the product ; $\begin{vmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$.

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EXERCISE 7.5

Choose the correct or the most suitable answer from the given four alternatives.

(1) If $a_{ij} = \dfrac{1}{2}(3i - 2j)$ and $A = [a_{ij}]_{2 \times 2}$ is

(1) $\begin{bmatrix} \dfrac{1}{2} & 2 \\ -\dfrac{1}{2} & 1 \end{bmatrix}$
(2) $\begin{bmatrix} \dfrac{1}{2} & -\dfrac{1}{2} \\ 2 & 1 \end{bmatrix}$
(3) $\begin{bmatrix} 2 & 2 \\ 1 & -\dfrac{1}{2} \end{bmatrix}$
(4) $\begin{bmatrix} -\dfrac{1}{2} & \dfrac{1}{2} \\ 1 & 2 \end{bmatrix}$

(2) What must be the matrix X, if $2X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$?

(1) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$
(2) $\begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$
(3) $\begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix}$
(4) $\begin{bmatrix} 2 & -6 \\ 4 & -2 \end{bmatrix}$

(3) Which one of the following is not true about the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$?

(1) a scalar matrix   (2) a diagonal matrix
(3) an upper triangular matrix   (4) a lower triangular matrix

(4) If A and B are two matrices such that $A + B$ and AB are both defined, then
    (1) A and B are two matrices not necessarily of same order
    (2) A and B are square matrices of same order
    (3) Number of columns of A is equal to the number of rows of B
    (4) $A = B$.

(5) If $A = \begin{bmatrix} \lambda & 1 \\ -1 & -\lambda \end{bmatrix}$, then for what value of λ, $A^2 = O$?

(1) 0   (2) ± 1 (3) -1 (4) 1

(6) If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$, then the values of a and b are

(1) $a = 4, b = 1$ (2) $a = 1, b = 4$ (3) $a = 0, b = 4$ (4) $a = 2, b = 4$

(7) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity

matrix, then the ordered pair (a, b) is equal to

(1) $(2, -1)$ (2) $(-2, 1)$ (3) $(2, 1)$ (4) $(-2, -1)$

(8) If A is a square matrix, then which of the following is not symmetric?

(1) $A + A^T$ (2) AA^T (3) $A^T A$ (4) $A - A^T$

(9) If A and B are symmetric matrices of order n, where $(A \neq B)$, then

(1) $A + B$ is skew-symmetric   (2) $A + B$ is symmetric

(3) $A + B$ is a diagonal matrix   (4) $A + B$ is a zero matrix

(10) If $A = \begin{bmatrix} a & x \\ y & a \end{bmatrix}$ and if $xy = 1$, then det $(A A^T)$ is equal to

(1) $(a-1)^2$ (2) $(a^2 + 1)^2$ (3) $a^2 - 1$ (4) $(a^2 - 1)^2$

(11) The value of x, for which the matrix $A = \begin{bmatrix} e^{x-2} & e^{7+x} \\ e^{2+x} & e^{2x+3} \end{bmatrix}$ is singular is

(1) 9   (2) 8   (3) 7   (4) 6

(12) If the points $(x, -2)$, $(5, 2)$, $(8, 8)$ are collinear, then x is equal to

(1) -3 (2) $\dfrac{1}{3}$ (3) 1   (4) 3

(13) If $\begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix} = \dfrac{abc}{2} \neq 0$, then the area of the triangle whose vertices are

$\left(\dfrac{x_1}{a}, \dfrac{y_1}{a} \right), \left(\dfrac{x_2}{b}, \dfrac{y_2}{b} \right), \left(\dfrac{x_3}{c}, \dfrac{y_3}{c} \right)$ is

(1) $\dfrac{1}{4}$ (2) $\dfrac{1}{4} abc$ (3) $\dfrac{1}{8}$ (4) $\dfrac{1}{8} abc$

(14) If the square of the matrix $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is the unit matrix of order 2, then α, β and γ should satisfy the relation.

(1) $1 + \alpha^2 + \beta\gamma = 0$ (2) $1 - \alpha^2 - \beta\gamma = 0$

(3) $1 - \alpha^2 + \beta\gamma = 0$ (4) $1 + \alpha^2 - \beta\gamma = 0$

     Matrices and Determinants

(15) If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, then $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix}$ is

    (1) Δ (2) $k\Delta$ (3) $3k\Delta$ (4) $k^3\Delta$

(16) A root of the equation $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$ is

    (1) 6   (2) 3   (3) 0   (4) – 6

(17) The value of the determinant of $A = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is

    (1) $-2abc$ (2) abc (3) 0   (4) $a^2 + b^2 + c^2$

(18) If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in geometric progression with the same common ratio, then the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are

    (1) vertices of an equilateral triangle

    (2) vertices of a right angled triangle

    (3) vertices of a right angled isosceles triangle

    (4) collinear

(19) If $\lfloor . \rfloor$ denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0$, $0 \le y < 1$, $1 \le z < 2$, then the value of the determinant $\begin{vmatrix} \lfloor x \rfloor + 1 & \lfloor y \rfloor & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor + 1 & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor & \lfloor z \rfloor + 1 \end{vmatrix}$ is

    (1) $\lfloor z \rfloor$ (2) $\lfloor y \rfloor$ (3) $\lfloor x \rfloor$ (4) $\lfloor x \rfloor + 1$

(20) If $a \ne b, b, c$ satisfy $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$, then $abc =$

    (1) $a+b+c$ (2) 0   (3) b^3 (4) $ab+bc$

(21) If $A = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} -2 & 4 & 2 \\ 6 & 2 & 0 \\ -2 & 4 & 8 \end{vmatrix}$, then B is given by

    (1) $B = 4A$ (2) $B = -4A$ (3) $B = -A$ (4) $B = 6A$

(22) If A is skew-symmetric of order n and C is a column matrix of order $n \times 1$, then $C^T A C$ is

    (1) an identity matrix of order n (2) an identity matrix of order 1

    (3) a zero matrix of order 1   (4) an identity matrix of order 2

(23) The matrix A satisfying the equation $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is

  (1) $\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -4 \\ 1 & 0 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$

(24) If $A + I = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, then $(A+I)(A-I)$ is equal to

  (1) $\begin{bmatrix} -5 & -4 \\ 8 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -5 & 4 \\ -8 & 9 \end{bmatrix}$ (3) $\begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$ (4) $\begin{bmatrix} -5 & -4 \\ -8 & -9 \end{bmatrix}$

(25) Let A and B be two symmetric matrices of same order. Then which one of the following statement is not true?

    (1) $A + B$ is a symmetric matrix   (2) AB is a symmetric matrix

    (3) $AB = (BA)^T$ (4) $A^T B = A B^T$

SUMMARY

In this chapter we have acquired the knowledge of

- A matrix is a rectangular array of real numbers or real functions on $\mathbb{R}$ or complex numbers.

- A matrix having m rows and n columns, then the order of the matrix is $m \times n$.

- A matrix $A = [a_{ij}]_{m \times n}$ is said to be a

  square matrix if $m = n$

  row matrix if $m = 1$

  column matrix if $n = 1$

  zero matrix if $a_{ij} = 0 \; \forall \, i$ and j

  diagonal matrix if $m = n$ and $a_{ij} = 0 \; \forall \, i \neq j$

  scalar matrix if $m = n$ and $a_{ij} = 0 \; \forall \, i \neq j$ and $a_{ii} = \lambda$ for all i

  unit matrix or identity matrix if $m = n$ and $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = 1 \; \forall \, i$

  upper triangular matrix if $m = n$ and $a_{ij} = 0 \; \forall \; i > j$

  lower triangular matrix if $m = n$ and $a_{ij} = 0 \; \forall \; i < j$.

- Matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, are said to be equal if $a_{ij} = b_{ij} \; \forall \, i$ and j

- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A + B = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$

- If $A = [a_{ij}]_{m \times n}$ and λ is a scalar, then $\lambda A = [\lambda a_{ij}]_{m \times n}$

- $-A = (-1)A$
- $A + B = B + A$
- $A - B = A + (-1)B$
- $(A + B) + C = A + (B + C)$ where A, B and C have the same order.
- $A(BC) = (AB)C$ (ii) $A(B+C) = AB+AC$ (iii) $(A+B)C = AC+BC$
- The transpose of A, denoted by A^T is obtained by interchanging rows and columns of A.

  (i) $(A^T)^T = A$,   (ii) $(kA)^T = kA^T$,   (iii) $(A + B)^T = A^T + B^T$,   (iv) $(AB)^T = B^T A^T$
- A square matrix A is called

  (i) symmetric if $A^T = A$ and   (ii) skew-symmetric if $A^T = -A$
- Any square matrix can be expressed as sum of a symmetric and skew-symmetric matrices.
- The diagonal entries of a skew-symmetric must be zero.
- For any square matrix A with real entries, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric and further $A = \dfrac{1}{2}(A + A^T) + \dfrac{1}{2}(A - A^T)$.
- Determinant is defined only for square matrices.
- $|A^T| = |A|$.
- $|AB| = |A| \, |B|$ where A and B are square matrices of same order.
- If $A = [a_{ij}]_{m \times n}$, then $|kA| = k^n |A|$, where k is a scalar.
- A determinant of a square matrix A is the sum of products of elements of any row (or column) with its corresponding cofactors; for instance, $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.
- If the elements of a row or column is multiplied by the cofactors of another row or column, then their sum is zero; for example, $a_{11}A_{13} + a_{12}A_{23} + a_{13}A_{33} = 0$.
- The determinant value remains unchanged if all its rows are interchanged by its columns.
- If all the elements of a row or a column are zero, then the determinant is zero.
- If any two rows or columns are interchanged, then the determinant changes its sign.
- If any two rows or columns are identical or proportional, then the determinant is zero.
- If each element of a row or a column is multiplied by constant k, then determinant gets multiplied by k.
- If each element in any row (column) is the sum of r terms, then the determinant can be expressed as the sum of r determinants.
- A determinant remains unaltered under a row (R_i) operation of the form $R_i + \alpha R_j + \beta R_k (j, k \neq i)$ or a Column (C_i) operation of the form $C_i + \alpha C_j + \beta C_k (j, k \neq i)$ where α, β are scalars.
- **Factor theorem** : If each element of $|A|$ is a polynomial in x and if $|A|$ vanishes for $x = a$, then $x - a$ is a factor of $|A|$.
- Area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the absolute value of $\dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

  If the area is zero, then the three points are collinear.
- A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

Matrices and Determinants
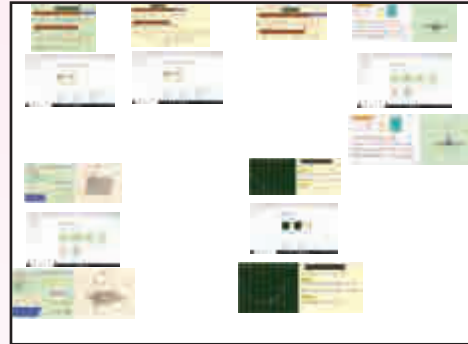
Expected Outcome

Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "Matrices and Determinants" will appear. In that there are several worksheets related to your lesson.
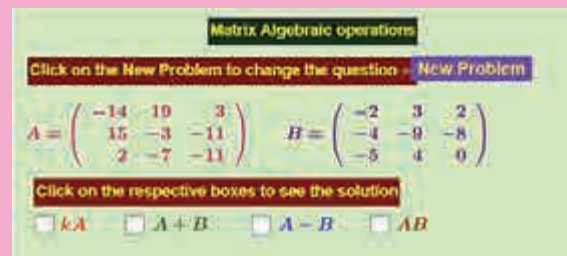
Step 2

Select the work sheet "Matrices-Algebraic operations"work out the operations given and Select the check boxes to verify corresponding answers.

Click on "New Problem" to get new question.
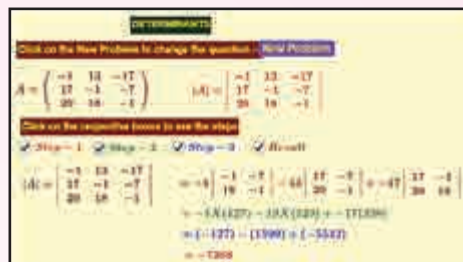

Step1

Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

Matrices and Determinants

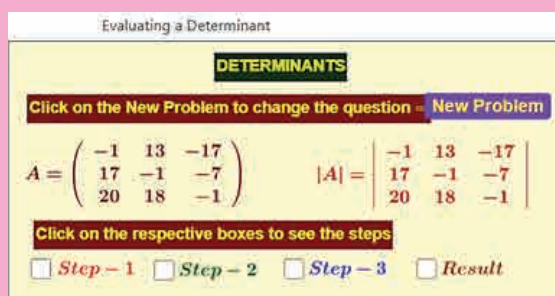**Expected Outcome**

Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "Matrices and Determinants" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Determinants" Evaluate the determinant for the matrix given and Select the check boxes to verify steps.

Click on "New Problem" to get new question.

Step1

Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

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