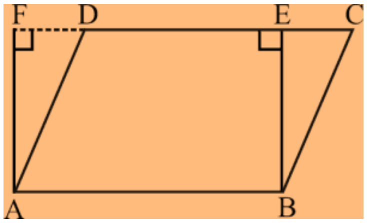
Exercise 9.4(Optional)* Page: 164

1. Parallelogram ABCD and rectangle ABEF are on the same base AB and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.

Solution:



Given,

|| gm ABCD and a rectangle ABEF have the same base AB and equal areas.

To prove,

Perimeter of || gm ABCD is greater than the perimeter of rectangle ABEF.

Proof,

We know that, the opposite sides of a|| gm and rectangle are equal.

∴, AB = DC [As ABCD is a || gm]

and, AB = EF [As ABEF is a rectangle]

∴, DC = EF ... (i)

Adding AB on both sides, we get,

⇒ AB + DC = AB + EF ... (ii)

We know that, the perpendicular segment is the shortest of all the segments that can be drawn to a given line from a point not lying on it.

∴, BE < BC and AF < AD

⇒ BC > BE and AD > AF

⇒ BC + AD > BE + AF ... (iii)

Adding (ii) and (iii), we get

AB + DC + BC + AD > AB + EF + BE + AF

⇒ AB + BC + CD + DA > AB + BE + EF + FA

⇒ perimeter of || gm ABCD > perimeter of rectangle ABEF.

∴, the perimeter of the parallelogram is greater than that of the rectangle.

Hence Proved.

2. In Fig. 9.30, D and E are two points on BC such that BD = DE = EC.
Show that ar (ABD) = ar (ADE) = ar (AEC).
Can you now answer the question that you have left in the 'Introduction' of this chapter, whether the field of Budhia has been actually divided into three parts of equal area?

[<u>Remark:</u> Note that by taking BD = DE = EC, the triangle ABC is divided into three triangles ABD, ADE and AEC of equal areas. In the same way, by dividing BC into *n* equal parts and joining the points of division so obtained to the opposite vertex of BC, you can divide DABC into *n* triangles of equal areas.]

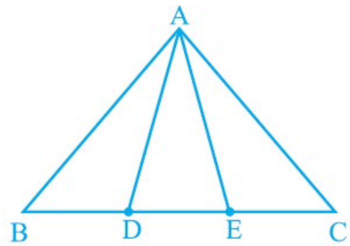


Fig. 9.30

Solution:

Given,

BD = DE = EC

To prove,

ar (△ABD) = ar (△ADE) = ar (△AEC)

Proof,

In (△ABE), AD is median [since, BD = DE, given]

We know that, the median of a triangle divides it into two parts of equal areas

∴, ar(△ABD) = ar(△AED) ---(i)

Similarly,

In (△ADC), AE is median [since, DE= EC, given]

∴, ar(ADE) = ar(AEC) ---(ii)

From the equation (i) and (ii), we get

ar(ABD) = ar(ADE) = ar(AEC)

3. In Fig. 9.31, ABCD, DCFE and ABFE are parallelograms. Show that ar (ADE) = ar (BCF).

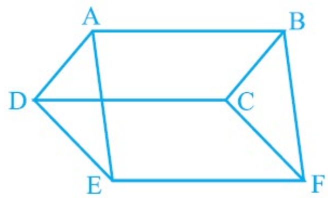


Fig. 9.31

Solution:

Given,

ABCD, DCFE and ABFE are parallelograms

To prove,

ar (△ADE) = ar (△BCF)

Proof,

In △ADE and △BCF,

AD=BC [Since, they are the opposite sides of the parallelogram ABCD]

DE= CF [Since, they are the opposite sides of the parallelogram DCFE]

AE= BF [Since, they are the opposite sides of the parallelogram ABFE]

∴, △ADE = △BCF [Using SSS Congruence theorem]

∴, ar(△ADE) = ar(△BCF) [By CPCT]

4. In Fig. 9.32, ABCD is a parallelogram and BC is produced to a point Q such that AD = CQ. If AQ intersect DC at P, show that ar (BPC) = ar (DPQ).
[Hint : Join AC.]

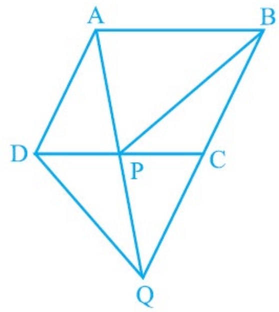


Fig. 9.32

Solution:

Given:

ABCD is a parallelogram

AD = CQ

To prove:

ar (△BPC) = ar (△DPQ)

Proof:

In △ADP and △QCP,

∠APD = ∠QPC [Vertically Opposite Angles]

∠ADP = ∠QCP [Alternate Angles]

AD=CQ [given]

∴, △ABO = △ACD [AAS congruency]

∴, DP= CP [CPCT]

In △CDQ, QP is median. [Since, DP= CP]

Since, median of a triangle divides it into two parts of equal areas.

∴, ar(△DPQ) = ar(△QPC) ---(i)

In △PBQ, PC is median. [Since, AD= CQ and AD= BC ⇒ BC= QC]

Since, median of a triangle divides it into two parts of equal areas.

∴, ar(△QPC) = ar(△BPC) ---(ii)

From the equation (i) and (ii), we get

ar(△BPC) = ar(△DPQ)

5. In Fig.9.33, ABC and BDE are two equilateral triangles such that D is the mid-point of BC. If AE intersects BC at F, show that:

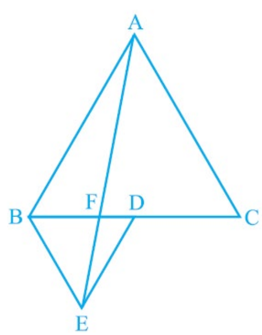
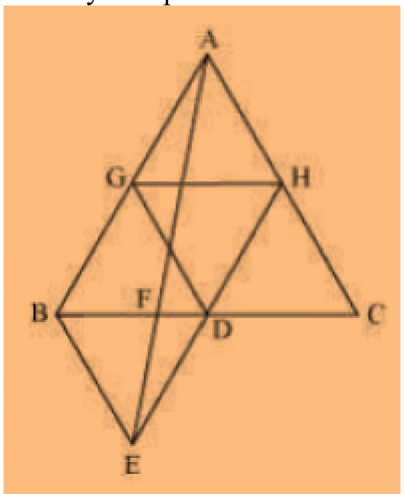


Fig. 9.33

(i) ar (BDE) = $\frac{1}{4}$ ar (ABC)
(ii) ar (BDE) = $\frac{1}{2}$ ar (BAE)
(iii) ar (ABC) = 2 ar (BEC)
(iv) ar (BFE) = ar (AFD)
(v) ar (BFE) = 2 ar (FED)
(vi) ar (FED) = $\frac{1}{8}$ ar (AFC)

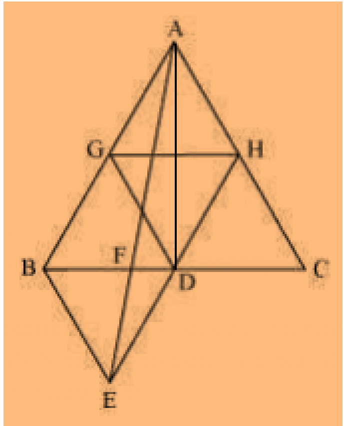
Solution:

(i) Assume that G and H are the mid-points of the sides AB and AC respectively.
Join the mid-points with line-segment GH. Here, GH is parallel to third side.
∴, BC will be half of the length of BC by mid-point theorem.



∴ GH = $\frac{1}{2}$ BC and GH || BD
∴ GH = BD = DC and GH || BD (Since, D is the mid-point of BC)
Similarly,
GD = HC = HA
HD = AG = BG
∴, ΔABC is divided into 4 equal equilateral triangles ΔBGD, ΔAGH, ΔDHC and ΔGHD
We can say that,
ΔBGD = $\frac{1}{4}$ ΔABC
Considering, ΔBDG and ΔBDE
BD = BD (Common base)
Since both triangles are equilateral triangle, we can say that,
BG = BE
DG = DE
∴, ΔBDG ≅ ΔBDE [By SSS congruency]
∴, area (ΔBDG) = area (ΔBDE)
ar (ΔBDE) = $\frac{1}{4}$ ar (ΔABC)
Hence proved

(ii)



ar(ΔBDE) = ar(ΔAED) (Common base DE and DE||AB)
ar(ΔBDE) − ar(ΔFED) = ar(ΔAED) − ar (ΔFED)
ar(ΔBEF) = ar(ΔAFD) …(i)
Now,
ar(ΔABD) = ar(ΔABF) + ar(ΔAFD)
ar(ΔABD) = ar(ΔABF) + ar(ΔBEF) [From equation (i)]
ar(ΔABD) = ar(ΔABE) …(ii)
AD is the median of ΔABC.

ar(ΔABD) = $\frac{1}{2}$ ar (ΔABC)
= $\frac{4}{2}$ ar (ΔBDE)
= 2 ar (ΔBDE) …(iii)

From (ii) and (iii), we obtain
2 ar (ΔBDE) = ar (ΔABE)
ar (BDE) = $\frac{1}{2}$ ar (BAE)
Hence proved

(iii) ar(ΔABE) = ar(ΔBEC) [Common base BE and BE||AC]
ar(ΔABF) + ar(ΔBEF) = ar(ΔBEC)
From eq^n (i), we get,
ar(ΔABF) + ar(ΔAFD) = ar(ΔBEC)
ar(ΔABD) = ar(ΔBEC)
$\frac{1}{2}$ar(ΔABC) = ar(ΔBEC)
ar(ΔABC) = 2 ar(ΔBEC)
Hence proved

(iv) ΔBDE and ΔAED lie on the same base (DE) and are in-between the parallel lines DE and AB.
∴ ar (ΔBDE) = ar (ΔAED)
Subtracting ar(ΔFED) from L.H.S and R.H.S,
We get,
∴ ar (ΔBDE) − ar (ΔFED) = ar (ΔAED) − ar (ΔFED)
∴ ar (ΔBFE) = ar (ΔAFD)
Hence proved

(v) Assume that h is the height of vertex E, corresponding to the side BD in ΔBDE.
Also assume that H is the height of vertex A, corresponding to the side BC in ΔABC.
While solving Question (i),
We saw that,

$$\text{ar}(\Delta BDE) = \frac{1}{4}\text{ar}(\Delta ABC)$$

While solving Question (iv),
We saw that,

$$\text{ar}(\Delta BFE) = \text{ar}(\Delta AFD).$$

$$\therefore \text{ar}(\Delta BFE) = \text{ar}(\Delta AFD)$$

$$= 2\,\text{ar}(\Delta FED)$$

Hence, ar (ΔBFE) = 2 ar (ΔFED)

Hence proved

(vi) $\text{ar}(\Delta AFC) = \text{ar}(\Delta AFD) + \text{ar}(\Delta ADC)$

$$= 2\,\text{ar}(\Delta FED) + \frac{1}{2}\text{ar}(\Delta ABC) \quad \text{[using (v)]}$$

$$= 2\,\text{ar}(\Delta FED) + \frac{1}{2}[4\text{ar}(\Delta BDE)] \quad \text{[Using result of Question (i)]}$$

$$= 2\,\text{ar}(\Delta FED) + 2\,\text{ar}(\Delta BDE)$$

Since, ΔBDE and ΔAED are on the same base and between same parallels

$$= 2\,\text{ar}(\Delta FED) + 2\,\text{ar}(\Delta AED)$$

$$= 2\,\text{ar}(\Delta FED) + 2\,[\text{ar}(\Delta AFD) + \text{ar}(\Delta FED)]$$

$$= 2\,\text{ar}(\Delta FED) + 2\,\text{ar}(\Delta AFD) + 2\,\text{ar}(\Delta FED) \quad \text{[From question (viii)]}$$

$$= 4\,\text{ar}(\Delta FED) + 4\,\text{ar}(\Delta FED)$$

$$\Rightarrow \text{ar}(\Delta AFC) = 8\,\text{ar}(\Delta FED)$$

$$\Rightarrow \text{ar}(\Delta FED) = \frac{1}{8}\text{ar}(\Delta AFC)$$

Hence proved

6. Diagonals AC and BD of a quadrilateral ABCD intersect each other at P. Show that ar (APB) × ar (CPD) = ar (APD) × ar (BPC).
[Hint : From A and C, draw perpendiculars to BD.]

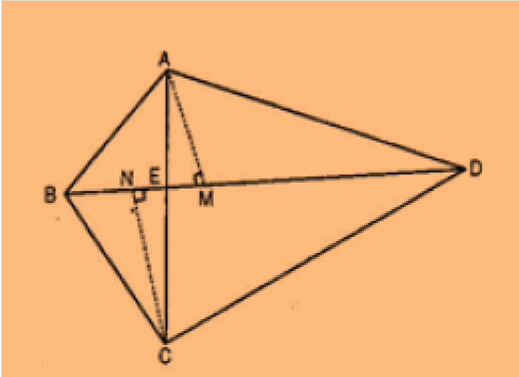
Solution:

Given:

The diagonal AC and BD of the quadrilateral ABCD, intersect each other at point E.

Construction:

From A, draw AM perpendicular to BD
From C, draw CN perpendicular to BD



To Prove,

$ar(\Delta AED) \times ar(\Delta BEC) = ar(\Delta ABE) \times ar(\Delta CDE)$

Proof,

$ar(\Delta ABE) = \frac{1}{2}\times BE\times AM$.............. i

$ar(\Delta AED) = \frac{1}{2}\times DE\times AM$.............. ii

Dividing eq. ii by i , we get,

$$\frac{ar(\Delta AED)}{ar(\Delta ABE)} = \frac{\frac{1}{2}\times DE\times AM}{\frac{1}{2}\times BE\times AM}$$

$$\frac{ar(\Delta AED)}{ar(\Delta ABE)} = \frac{DE}{BE} \ldots\ldots\ldots\ldots iii$$

Similarly,

$$\frac{ar(\Delta CDE)}{ar(\Delta BEC)} = \frac{DE}{BE} \ldots\ldots\ldots\ldots iv$$

From eq. iii and iv , we get

$$\frac{ar(\Delta AED)}{ar(\Delta ABE)} = \frac{ar(\Delta CDE)}{ar(\Delta BEC)}$$

$\therefore$, $ar(\Delta AED) \times ar(\Delta BEC) = ar(\Delta ABE) \times ar(\Delta CDE)$

Hence proved.

7. P and Q are respectively the mid-points of sides AB and BC of a triangle ABC and R is the mid-point of AP, show that:

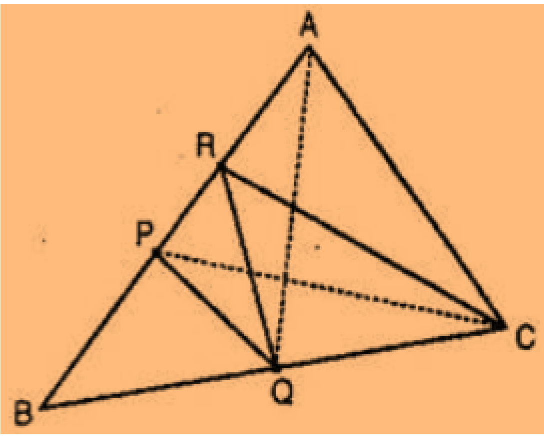
(i) ar (PRQ) = $\frac{1}{2}$ ar (ARC)

(ii) ar (RQC) = $\frac{3}{8}$ ar (ABC)

(iii) ar (PBQ) = ar (ARC)

Solution:

(i)



We know that, median divides the triangle into two triangles of equal area,

PC is the median of ABC.

ar (ΔBPC) = ar (ΔAPC) ……….(i)

RC is the median of APC.

ar (ΔARC) = $\frac{1}{2}$ ar (ΔAPC) ……….(ii)

PQ is the median of BPC.

ar (ΔPQC) = $\frac{1}{2}$ ar (ΔBPC) ……….(iii)

From eq. (i) and (iii), we get,

ar (ΔPQC) = $\frac{1}{2}$ ar (ΔAPC) ……….(iv)

From eq. (ii) and (iv), we get,

ar (ΔPQC) = ar (ΔARC) ……….(v)

P and Q are the mid-points of AB and BC respectively [given]

∴PQ||AC

and, PA = $\frac{1}{2}$ AC

Since, triangles between same parallel are equal in area, we get,

ar (ΔAPQ) = ar (ΔPQC) ……….(vi)

From eq. (v) and (vi), we obtain,

ar (ΔAPQ) = ar (ΔARC) ……….(vii)

R is the mid-point of AP.

∴, RQ is the median of APQ.

ar (ΔPRQ) = $\frac{1}{2}$ ar (ΔAPQ) ……….(viii)

From (vii) and (viii), we get,

ar (ΔPRQ) = $\frac{1}{2}$ ar (ΔARC)

Hence Proved.

(ii) PQ is the median of BPC

ar (ΔPQC) = $\frac{1}{2}$ ar (ΔBPC)

= $\frac{1}{2} \times \frac{1}{2}$ ar (ΔABC)

$= \frac{1}{4}$ ar (ΔABC) ……….(ix)

Also,

ar (ΔPRC) = $\frac{1}{2}$ ar (ΔAPC) [From (iv)]

ar (ΔPRC) = $\frac{1}{2} \times \frac{1}{2}$ ar (ABC)

= $\frac{1}{4}$ ar (ABC) ……….(x)

Add eq. (ix) and (x), we get,

ar (ΔPQC) + ar (ΔPRC) = $\frac{1}{4} \times \frac{1}{4}$ ar (ΔABC)

ar (quad. PQCR) = $\frac{1}{4}$ ar (ΔABC) ……….(xi)

Subtracting ar (ΔPRQ) from L.H.S and R.H.S,

ar (quad. PQCR) – ar (ΔPRQ) = $\frac{1}{2}$ ar (ΔABC) – ar (ΔPRQ)

ar (ΔRQC) = $\frac{1}{2}$ ar (ΔABC) – $\frac{1}{2}$ ar (ΔARC) [From result (i)]

ar (ΔARC) = $\frac{1}{2}$ ar (ΔABC) – $\frac{1}{2} \times \frac{1}{2}$ ar (ΔAPC)

ar (ΔRQC) = $\frac{1}{2}$ ar (ΔABC) – $\frac{1}{4}$ ar (ΔAPC)

ar (ΔRQC) = $\frac{1}{2}$ ar (ΔABC) – $\frac{1}{4} \times \frac{1}{2}$ ar (ΔABC) [As, PC is median of ΔABC]

ar (ΔRQC) = $\frac{1}{2}$ ar (ΔABC) – $\frac{1}{8}$ ar (ΔABC)

ar (ΔRQC) = $(\frac{1}{2} - \frac{1}{8})$ ar (ΔABC)

ar (ΔRQC) = $\frac{3}{8}$ ar (ΔABC)

(iii) ar (ΔPRQ) = $\frac{1}{2}$ ar (ΔARC) [From result (i)]

2ar (ΔPRQ) = ar (ΔARC) ……………..(xii)

ar (ΔPRQ) = $\frac{1}{2}$ ar (ΔAPQ) [RQ is the median of APQ] ……….(xiii)

But, we know that,

ar (ΔAPQ) = ar (ΔPQC) [From the reason mentioned in eq. (vi)] ……….(xiv)

From eq. (xiii) and (xiv), we get,

ar (ΔPRQ) = $\frac{1}{2}$ ar (ΔPQC) ……….(xv)

At the same time,

ar (ΔBPQ) = ar (ΔPQC) [PQ is the median of BPC] ……….(xvi)

From eq. (xv) and (xvi), we get,

ar (ΔPRQ) = $\frac{1}{2}$ ar (ΔBPQ) ……….(xvii)

From eq. (xii) and (xvii), we get,

$2\times \frac{1}{2}$ ar (ΔBPQ) = ar (ΔARC)

$\Longrightarrow$ ar (ΔBPQ) = ar (ΔARC)

Hence Proved.

8. In Fig. 9.34, ABC is a right triangle right angled at A. BCED, ACFG and ABMN are squares on the sides BC, CA and AB respectively. Line segment AX ^ DE meets BC at Y. Show that:

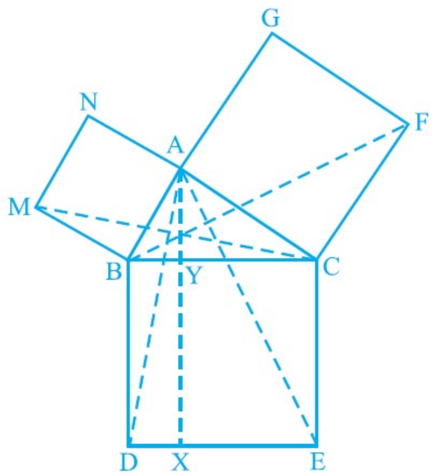


Fig. 9.34

(i) $\Delta MBC \cong \Delta ABD$
(ii) ar(BYXD) = 2ar(MBC)
(iii) ar(BYXD) = ar(ABMN)
(iv) $\Delta FCB \cong \Delta ACE$
(v) ar(CYXE) = 2ar(FCB)
(vi) ar(CYXE) = ar(ACFG)
(vii) ar(BCED) = ar(ABMN) + ar(ACFG)
Note : Result (vii) is the famous Theorem of Pythagoras. You shall learn a simpler proof of this theorem in Class X.

Solution:

(i) We know that each angle of a square is 90°. Hence, $\angle ABM = \angle DBC = 90^\circ$
$\therefore \angle ABM + \angle ABC = \angle DBC + \angle ABC$
$\therefore \angle MBC = \angle ABD$

In ΔMBC and ΔABD,
$\angle MBC = \angle ABD$ (Proved above)
MB = AB (Sides of square ABMN)
BC = BD (Sides of square BCED)
$\therefore \Delta MBC \cong \Delta ABD$ (SAS congruency)

(ii) We have
$\Delta MBC \cong \Delta ABD$
$\therefore$ ar (ΔMBC) = ar (ΔABD) ... (i)
It is given that $AX \perp DE$ and $BD \perp DE$ (Adjacent sides of square BDEC)
$\therefore BD \parallel AX$ (Two lines perpendicular to same line are parallel to each other)

ΔABD and parallelogram BYXD are on the same base BD and between the same parallels BD and AX.
Area (ΔYXD) = 2 Area (ΔMBC) [From equation (i)] ... (ii)

(iii) ΔMBC and parallelogram ABMN are lying on the same base MB and between same parallels MB and NC.

2 ar (ΔMBC) = ar (ABMN)
ar (ΔYXD) = ar (ABMN) [From equation (ii)] ... (iii)

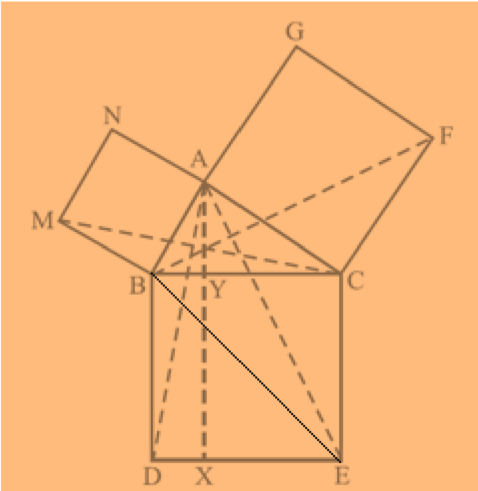
(iv) We know that each angle of a square is 90°.

$\therefore \angle FCA = \angle BCE = 90^\circ$
$\therefore \angle FCA + \angle ACB = \angle BCE + \angle ACB$
$\therefore \angle FCB = \angle ACE$

In ΔFCB and ΔACE,

$\angle FCB = \angle ACE$
FC = AC (Sides of square ACFG)
CB = CE (Sides of square BCED)
$\Delta FCB \cong \Delta ACE$ (SAS congruency)

(v) AX $\perp$ DE and CE $\perp$ DE (Adjacent sides of square BDEC) [given]
Hence,
CE || AX (Two lines perpendicular to the same line are parallel to each other)



Consider BACE and parallelogram CYXE
BACE and parallelogram CYXE are on the same base CE and between the same parallels CE and AX.

$\therefore$ ar (ΔYXE) = 2 ar (ΔACE) ... (iv)

We had proved that

$\therefore \Delta FCB \cong \Delta ACE$
ar (ΔFCB) $\cong$ ar (ΔACE) ... (v)

From equations (iv) and (v), we get

ar (CYXE) = 2 ar (ΔFCB) ... (vi)

(vi) Consider BFCB and parallelogram ACFG
BFCB and parallelogram ACFG are lying on the same base CF and between the same parallels CF and BG.
∴ ar (ACFG) = 2 ar (ΔFCB)
∴ ar (ACFG) = ar (CYXE) [From equation (vi)] ... (vii)

(vii) From the figure, we can observe that
ar (ΔCED) = ar (ΔYXD) + ar (CYXE)
∴ ar (ΔCED) = ar (ABMN) + ar (ACFG) [From equations (iii) and (vii)].