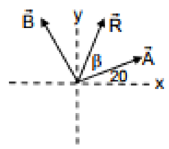
Vol 1 Chapter 2 - Physics and Mathematics

CHAPTER – 2

1. As shown in the figure,



The angle between $\vec{A}$ and $\vec{B}$ = 110° – 20° = 90°

$|\vec{A}|$ = 3 and $|\vec{B}|$ = 4m

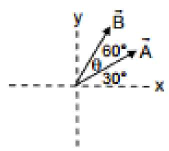
Resultant R = $\sqrt{A^2 + B^2 + 2AB\cos\theta}$ = 5 m

Let β be the angle between $\vec{R}$ and $\vec{A}$

$\beta = \tan^{-1}\left(\dfrac{4\sin 90°}{3 + 4\cos 90°}\right) = \tan^{-1}(4/3) = 53°$

∴ Resultant vector makes angle (53° + 20°) = 73° with x-axis.

2. Angle between $\vec{A}$ and $\vec{B}$ is θ = 60° – 30° =30°



$|\vec{A}|$ and $|\vec{B}|$ = 10 unit

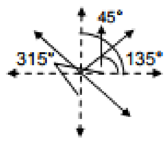
R = $\sqrt{10^2 + 10^2 + 2.10.10.\cos 30°}$ = 19.3

β be the angle between $\vec{R}$ and $\vec{A}$

$\beta = \tan^{-1}\left(\dfrac{10\sin 30°}{10 + 10\cos 30°}\right) = \tan^{-1}\left(\dfrac{1}{2+\sqrt{3}}\right) = \tan^{-1}(0.26795) = 15°$

∴ Resultant makes 15° + 30° = 45° angle with x-axis.

3. x component of $\vec{A}$ = 100 cos 45° = $100/\sqrt{2}$ unit



x component of $\vec{B}$ = 100 cos 135° = $100/\sqrt{2}$

x component of $\vec{C}$ = 100 cos 315° = $100/\sqrt{2}$

Resultant x component = $100/\sqrt{2} - 100/\sqrt{2} + 100/\sqrt{2} = 100/\sqrt{2}$

y component of $\vec{A}$ = 100 sin 45° = $100/\sqrt{2}$ unit

y component of $\vec{B}$ = 100 sin 135° = $100/\sqrt{2}$

y component of $\vec{C}$ = 100 sin 315° = $-100/\sqrt{2}$

Resultant y component = $100/\sqrt{2} + 100/\sqrt{2} - 100/\sqrt{2} = 100/\sqrt{2}$

Resultant = 100

$\tan\alpha = \dfrac{\text{y component}}{\text{x component}} = 1$

$\Rightarrow \alpha = \tan^{-1}(1) = 45°$

The resultant is 100 unit at 45° with x-axis.

4. $\vec{a} = 4\vec{i} + 3\vec{j}$, $\vec{b} = 3\vec{i} + 4\vec{j}$

a) $|\vec{a}| = \sqrt{4^2 + 3^2} = 5$

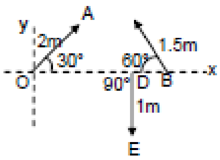
b) $|\vec{b}| = \sqrt{9 + 16} = 5$

c) $|\vec{a} + \vec{b}| = |7\vec{i} + 7\vec{j}| = 7\sqrt{2}$

d) $\vec{a} - \vec{b} = (-3+4)\hat{i} + (-4+3)\hat{j} = \hat{i} - \hat{j}$

$|\vec{a} - \vec{b}| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

5. x component of $\vec{OA}$ = 2cos30° = $\sqrt{3}$



x component of $\vec{BC}$ = 1.5 cos 120° = –0.75

x component of $\vec{DE}$ = 1 cos 270° = 0

y component of $\vec{OA}$ = 2 sin 30° = 1

y component of $\vec{BC}$ = 1.5 sin 120° = 1.3

y component of $\vec{DE}$ = 1 sin 270° = –1

R_x = x component of resultant = $\sqrt{3} - 0.75 + 0 = 0.98$ m

R_y = resultant y component = 1 + 1.3 – 1 = 1.3 m

So, R = Resultant = 1.6 m

If it makes and angle α with positive x-axis

Tan α = $\frac{\text{y component}}{\text{x component}}$ = 1.32

⇒ α = $\tan^{-1}$ 1.32

6. $|\vec{a}|$ = 3m $|\vec{b}|$ = 4

a) If R = 1 unit ⇒ $\sqrt{3^2 + 4^2 + 2.3.4.\cos\theta}$ = 1

θ = 180°

b) $\sqrt{3^2 + 4^2 + 2.3.4.\cos\theta} = 5$

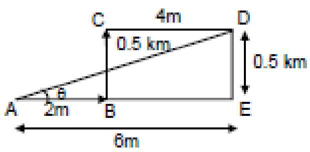
θ = 90°

c) $\sqrt{3^2 + 4^2 + 2.3.4.\cos\theta}$ = 7

θ = 0°

Angle between them is 0°.

7. $\vec{AD} = 2\hat{i} + 0.5\hat{J} + 4\hat{K} = 6\hat{i} + 0.5\hat{j}$



AD = $\sqrt{AE^2 + DE^2}$ = 6.02 KM

Tan θ = DE / AE = 1/12

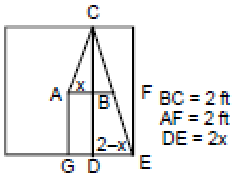
θ = $\tan^{-1}$ (1/12)

The displacement of the car is 6.02 km along the distance $\tan^{-1}$ (1/12) with positive x-axis.

8. In ΔABC, tanθ = x/2 and in ΔDCE, tanθ = (2 – x)/4 tan θ = (x/2) = (2 – x)/4 = 4x

⇒ 4 – 2x = 4x

⇒ 6x = 4 ⇒ x = 2/3 ft



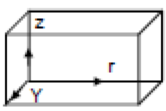
a) In ΔABC, AC = $\sqrt{AB^2 + BC^2} = \frac{2}{3}\sqrt{10}$ ft

b) In ΔCDE, DE = 1 – (2/3) = 4/3 ft

CD = 4 ft. So, CE = $\sqrt{CD^2 + DE^2}$ = $\frac{4}{3}\sqrt{10}$ ft

c) In ΔAGE, AE = $\sqrt{AG^2 + GE^2} = 2\sqrt{2}$ ft.

9. Here the displacement vector $\vec{r} = 7\hat{i} + 4\hat{j} + 3\hat{k}$



a) magnitude of displacement = $\sqrt{74}$ ft

b) the components of the displacement vector are 7 ft, 4 ft and 3 ft.

10. $\vec{a}$ is a vector of magnitude 4.5 unit due north.

a) $3|\vec{a}| = 3 \times 4.5 = 13.5$

$3\vec{a}$ is along north having magnitude 13.5 units.

b) $-4|\vec{a}| = -4 \times 1.5 = -6$ unit

$-4\vec{a}$ is a vector of magnitude 6 unit due south.

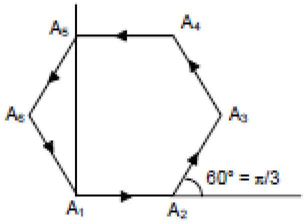
11. $|\vec{a}| = 2$ m, $|\vec{b}| = 3$ m

angle between them $\theta = 60°$

a) $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos 60° = 2 \times 3 \times 1/2 = 3\ m^2$

b) $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}| \sin 60° = 2 \times 3 \times \sqrt{3/2} = 3\sqrt{3}\ m^2$.

12. We know that according to polygon law of vector addition, the resultant of these six vectors is zero.



Here A = B = C = D = E = F (magnitude)

So, Rx = A cosθ + A cos π/3 + A cos 2π/3 + A cos 3π/3 + A cos 4π/4 + A cos 5π/5 = 0

[As resultant is zero. X component of resultant $R_x = 0$]

= cos θ + cos π/3 + cos 2π/3 + cos 3π/3 + cos 4π/3 + cos 5π/3 = 0

Note : Similarly it can be proved that,

sin θ + sin π/3 + sin 2π/3 + sin 3π/3 + sin 4π/3 + sin 5π/3 = 0

13. $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$; $\vec{b} = 3\vec{i} + 4\vec{j} + 5\vec{k}$

$\vec{a} \cdot \vec{b} = ab\cos\theta \Rightarrow \theta = \cos^{-1}\dfrac{\vec{a} \cdot \vec{b}}{ab}$

$\Rightarrow \cos^{-1}\dfrac{2 \times 3 + 3 \times 4 + 4 \times 5}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{3^2 + 4^2 + 5^2}} = \cos^{-1}\left(\dfrac{38}{\sqrt{1450}}\right)$

14. $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$ (claim)

As, $\vec{A} \times \vec{B} = AB\sin\theta\hat{n}$

AB sin θ $\hat{n}$ is a vector which is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, this implies that it is also perpendicular to $\vec{A}$. As dot product of two perpendicular vector is zero.

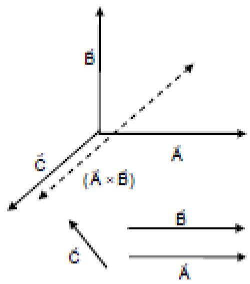
Thus $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$.

15. $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{B} = 4\hat{i} + 3\hat{j} + 2\hat{k}$

$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{vmatrix} \Rightarrow \hat{i}(6 - 12) - \hat{j}(4 - 16) + \hat{k}(6 - 12) = -6\hat{i} + 12\hat{j} - 6\hat{k}$.

16. Given that $\vec{A}$, $\vec{B}$ and $\vec{C}$ are mutually perpendicular

$\vec{A} \times \vec{B}$ is a vector which direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.



Also $\vec{C}$ is perpendicular to $\vec{A}$ and $\vec{B}$

∴ Angle between $\vec{C}$ and $\vec{A} \times \vec{B}$ is 0° or 180° (fig.1)

So, $\vec{C} \times (\vec{A} \times \vec{B}) = 0$

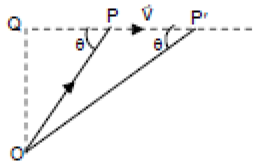
The converse is not true.

For example, if two of the vector are parallel, (fig.2), then also

$\vec{C} \times (\vec{A} \times \vec{B}) = 0$

So, they need not be mutually perpendicular.

17. The particle moves on the straight line PP' at speed v.



From the figure,

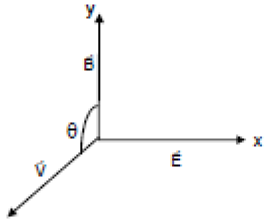
$\overrightarrow{OP} \times v = (OP) v \sin\theta \hat{n} = v(OP) \sin\theta\ \hat{n} = v(OQ)\hat{n}$

It can be seen from the figure, OQ = OP sin θ = OP' sin θ'

So, whatever may be the position of the particle, the magnitude and direction of $\overrightarrow{OP} \times \vec{v}$ remain constant.

$\therefore\ \overrightarrow{OP} \times \vec{v}$ is independent of the position P.

18. Give $\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}) = 0$



$\Rightarrow \vec{E} = -(\vec{v} \times \vec{B})$

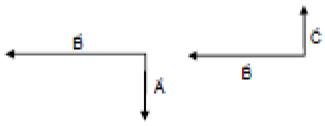
So, the direction of $\vec{v} \times \vec{B}$ should be opposite to the direction of $\vec{E}$. Hence, $\vec{v}$ should be in the positive yz-plane.

Again, E = vB sin θ $\Rightarrow v = \dfrac{E}{B \sin\theta}$

For v to be minimum, θ = 90° and so v_{min} = F/B

So, the particle must be projected at a minimum speed of E/B along +ve z-axis (θ = 90°) as shown in the figure, so that the force is zero.

19. For example, as shown in the figure,



$\vec{A} \perp \vec{B}$ $\quad$ $\vec{B}$ along west

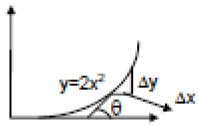
$\vec{B} \perp \vec{C}$ $\quad$ $\vec{A}$ along south

$\vec{C}$ along north

$\vec{A} \cdot \vec{B} = 0$ $\quad$ $\therefore\ \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{C}$

$\vec{B} \cdot \vec{C} = 0$ $\quad$ But $\vec{B} \neq \vec{C}$

20. The graph $y = 2x^2$ should be drawn by the student on a graph paper for exact results.



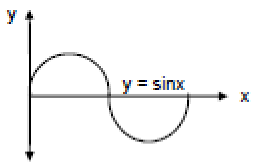
To find slope at any point, draw a tangent at the point and extend the line to meet x-axis. Then find tan θ as shown in the figure.

It can be checked that,

Slope = tan θ = $\dfrac{dy}{dx} = \dfrac{d}{dx}(2x^2) = 4x$

Where x = the x-coordinate of the point where the slope is to be measured.

21. y = sinx



So, y + Δy = sin (x + Δx)

Δy = sin (x + Δx) – sin x

$= \left(\dfrac{\pi}{3} + \dfrac{\pi}{100}\right) - \sin\dfrac{\pi}{3} = 0.0157.$

22. Given that, $i = i_0 e^{-t/RC}$

$\therefore$ Rate of change of current = $\dfrac{di}{dt} = \dfrac{d}{dt} i_0 e^{-t/RC} = i_0 \dfrac{d}{dt} e^{-t/RC} = \dfrac{-i_0}{RC} \times e^{-t/RC}$

When $\quad$ a) t = 0, $\dfrac{di}{dt} = \dfrac{-i}{RC}$

b) when t = RC, $\dfrac{di}{dt} = \dfrac{-i}{RCe}$

c) when t = 10 RC, $\dfrac{di}{dt} = \dfrac{-i_0}{RCe^{10}}$

23. Equation $i = i_0 e^{-t/RC}$

$i_0 = 2A$, $R = 6 \times 10^{-5}\ \Omega$, $C = 0.0500 \times 10^{-6}\ F = 5 \times 10^{-7}\ F$

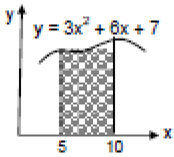
a) $i = 2 \times e^{\left(\frac{-0.3}{6\times 0^3 \times 5 \times 10^{-7}}\right)} = 2 \times e^{\left(\frac{-0.3}{0.3}\right)} = \frac{2}{e}\text{amp}.$

b) $\frac{di}{dt} = \frac{-i_0}{RC} e^{-t/RC}$ when t = 0.3 sec $\Rightarrow \frac{di}{dt} = -\frac{2}{0.30} e^{(-0.3/0.3)} = \frac{-20}{3e}\text{Amp/sec}$

c) At t = 0.31 sec, $i = 2e^{(-0.3/0.3)} = \frac{5.8}{3e}\text{Amp}.$

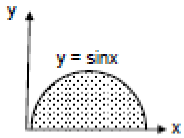
24. $y = 3x^2 + 6x + 7$

∴ Area bounded by the curve, x axis with coordinates with x = 5 and x = 10 is given by,



$$\text{Area} = \int_0^y dy = \int_5^{10} (3x^2 + 6x + 7)dx = 3\left.\frac{x^3}{3}\right]_5^{10} + 5\left.\frac{x^2}{3}\right]_5^{10} + 7x]_5^{10} = 1135 \text{ sq.units.}$$

25. $$\text{Area} = \int_0^y dy = \int_0^{\pi} \sin x dx = -[\cos x]_0^{\pi} = 2$$

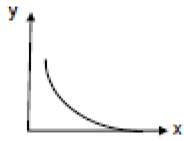


26. The given function is $y = e^{-x}$

When x = 0, $y = e^{-0} = 1$

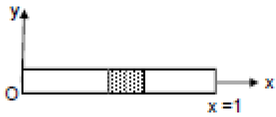
x increases, y value deceases and only at x = ∞, y = 0.

So, the required area can be found out by integrating the function from 0 to ∞.



So, Area $= \int_0^{\infty} e^{-x}dx = -[e^{-x}]_0^{\infty} = 1.$

27. $\rho = \frac{\text{mass}}{\text{length}} = a + bx$



a) S.I. unit of 'a' = kg/m and SI unit of 'b' = kg/m^2 (from principle of homogeneity of dimensions)

b) Let us consider a small element of length 'dx' at a distance x from the origin as shown in the figure.

∴ dm = mass of the element = ρ dx = (a + bx) dx

So, mass of the rod $= m = \int dm = \int_0^L (a + bx)dx = \left[ax + \frac{bx^2}{2}\right]_0^L = aL + \frac{bL^2}{2}$

28. $\frac{dp}{dt}$ = (10 N) + (2 N/S)t

momentum is zero at t = 0

∴ momentum at t = 10 sec will be

dp = [(10 N) + 2Ns t]dt

$$\int_0^p dp = \int_0^{10} 10dt + \int_0^{10} (2tdt) = 10t]_0^{10} + 2\left.\frac{t^2}{2}\right]_0^{10} = 200 \text{ kg m/s.}$$