Vol 1 Chapter 9 – Centre of Mass, Linear Momentum, Collision

1. \( m_1 = 1\text{kg}, \ m_2 = 2\text{kg}, \ m_3 = 3\text{kg}, \)
\( x_1 = 0, \ x_2 = 1, \ x_3 = \frac{1}{2} \)
\( y_1 = 0, \ y_2 = 0, \ y_3 = \frac{\sqrt{3}}{2} \)

The position of centre of mass is
\[
\text{C.M} = \left( \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}, \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \right)
\]
\[
= \left( \frac{(1 \times 0) + (2 \times 1) + (3 \times \frac{1}{2})}{1 + 2 + 3}, \frac{(1 \times 0) + (2 \times 0) + (3 \times (\sqrt{3}/2))}{1 + 2 + 3} \right)
\]
\[
= \left( \frac{7}{12}, \frac{3\sqrt{3}}{12} \right) \text{ from the point B.}
\]

2. Let \( B \) be the origin of the system

In the above figure
\( m_1 = 1\text{gm}, \ x_1 = -(0.96 \times 10^{-10})\sin 52^\circ, y_1 = 0 \)
\( m_2 = 1\text{gm}, \ x_2 = -(0.96 \times 10^{-10})\sin 52^\circ, y_2 = 0 \)
\( x_3 = 0, \ y_3 = (0.96 \times 10^{-10})\cos 52^\circ \)

The position of centre of mass
\[
\left( \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}, \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} \right)
\]
\[
= \left( -\frac{(0.96 \times 10^{-10}) \times \sin 52^\circ + (0.96 \times 10^{-10}) \sin 52^\circ + 16 \times 0}{1 + 1 + 16}, \frac{0 + 0 + 16 \times y_3}{18} \right)
\]
\[
= \left( 0, \frac{8 \times 0.96 \times 10^{-10} \cos 52^\circ}{18} \right)
\]

3. Let \( O (0,0) \) be the origin of the system.

Each brick is mass \( M \) & length \( L \).
Each brick is displaced w.r.t one in contact by \( L/10 \).

The \( X \) coordinate of the centre of mass
\[
\bar{x}_{cm} = \frac{m_1 \left( \frac{L}{2} + \frac{L}{10} \right) + m_2 \left( \frac{L}{2} + 2 \frac{L}{10} \right) + m_3 \left( \frac{L}{2} + 3 \frac{L}{10} \right) + m_4 \left( \frac{L}{2} + 4 \frac{L}{10} \right) + m_5 \left( \frac{L}{2} + 5 \frac{L}{10} \right) + m_6 \left( \frac{L}{2} + 6 \frac{L}{10} \right)}{7m}
\]
\[
= \frac{L}{2} + \frac{L}{10} + \frac{L}{5} + \frac{L}{10} + \frac{3L}{10} + \frac{L}{2} + \frac{L}{10} + \frac{L}{2} + \frac{L}{10} + \frac{L}{2} + \frac{L}{10} + \frac{L}{2} + \frac{L}{10} \]
\[
= \frac{7L}{2} + \frac{5L}{10} + \frac{2L}{10} + \frac{3L}{10} + \frac{L}{2} + \frac{L}{10} + \frac{L}{2} + \frac{L}{10} \]
\[
= \frac{7L}{2} + \frac{5L}{10} + \frac{2L}{10} + \frac{3L}{10} + \frac{L}{2} + \frac{L}{10} + \frac{L}{2} + \frac{L}{10} \]
\[
= \frac{35L + 5L + 4L}{10} \]
\[
= \frac{44L}{10} = \frac{11L}{3}
\]

4. Let the centre of the bigger disc be the origin.
\( 2R = \text{Radius of bigger disc} \)
\( R = \text{Radius of smaller disc} \)
\( m_1 = \pi R^2 \times T \times \rho \)
\( m_2 = \pi (2R)^2 \times T \times \rho \)
where \( T = \text{Thickness of the two discs} \)
\( \rho = \text{Density of the two discs} \)

\[
\bullet \text{ The position of the centre of mass}
\]
\[
\begin{aligned}
\left(\frac{m_1y_1 + m_2y_2}{m_1 + m_2}, \frac{m_1x_1 + m_2x_2}{m_1 + m_2}\right) \\
&= \left(\frac{0}{m_1 + m_2}, \frac{R}{m_1 + m_2}\right) \\
&= \left(\frac{\pi R^2 T \rho + 0}{\pi R^2 T \rho + \pi (2R)^2 T \rho}, \frac{0}{m_1 + m_2}\right) \\
&= \left(\frac{R}{5}, 0\right)
\end{aligned}
\]

At R/5 from the centre of bigger disc towards the centre of smaller disc.

5. Let ‘O’ be the origin of the system.
\[R = \text{radius of the smaller disc}\]
\[2R = \text{radius of the bigger disc}\]
The smaller disc is cut out from the bigger disc
\[
\begin{aligned}
m_1 &= \pi R^2 T \rho \\
x_1 &= R \\
y_1 &= 0 \\
m_2 &= \pi (2R)^2 T \rho \\
x_2 &= 0 \\
y_2 &= 0
\end{aligned}
\]
The position of C.M. = \[
\left(\frac{-\pi R^2 T \rho R + 0}{\pi R^2 T \rho + \pi (2R)^2 T \rho}, \frac{0 + 0}{m_1 + m_2}\right)
\]
= \[
\left(\frac{-R}{3}, 0\right)
\]
C.M. is at R/3 from the centre of bigger disc away from centre of the hole.

6. Let \( m \) be the mass per unit area.
\[\text{Mass of the square plate} = M_1 = 3^2 m\]
Mass of the circular disc = \[M_2 = \frac{\pi d^2}{4} m\]
Let the centre of the circular disc be the origin of the system.
\[\text{Position of centre of mass}\]
\[= \left(\frac{d^2 m + \pi (d^2/4)m \times 0 + 0}{d^2 m + \pi (d^2/4)m \times M_1 + M_2}\right) = \left(\frac{d^2 m}{\pi (d^2/4)m \times M_1 + M_2}\right) = \left(\frac{4d}{\pi + 4}\right)\]
The new centre of mass is \(\left(\frac{4d}{\pi + 4}\right)\) right of the centre of circular disc.

7. \[m_1 = 1\text{kg.} \quad \vec{v}_1 = -1.5 \cos 37^\circ \hat{i} - 1.55 \sin 37^\circ \hat{j} = -1.2 \hat{i} - 0.9 \hat{j}\]
\[m_2 = 1.2\text{kg.} \quad \vec{v}_2 = 0.4 \hat{j}\]
\[m_3 = 1.5\text{kg.} \quad \vec{v}_3 = -0.8 \hat{i} + 0.6 \hat{j}\]
\[m_4 = 0.5\text{kg} \quad \vec{v}_4 = 3 \hat{i}\]
\[m_5 = 1\text{kg} \quad \vec{v}_5 = 1.6 \hat{i} - 1.2 \hat{j}\]
So, \[
\vec{v}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + m_4 \vec{v}_4 + m_5 \vec{v}_5}{m_1 + m_2 + m_3 + m_4 + m_5}
\]
\[= \frac{(-1.2 \hat{i} - 0.9 \hat{j}) + 1.2(0.4 \hat{j}) + 1.5(-0.8 \hat{i} + 0.6 \hat{j}) + 0.5(3 \hat{i}) + 1(1.6 \hat{i} - 1.2 \hat{j})}{5.2}
\]
\[= \frac{-1.2 \hat{i} - 0.9 \hat{j} + 0.48 \hat{j} - 1.2 \hat{i} + 0.3 \hat{j} + 1.5 \hat{i} + 1.6 \hat{i} - 1.2 \hat{j}}{5.2}
\]
\[= \frac{0.7 \hat{i} - 0.72 \hat{j}}{5.2}
\]
3. Two masses \( m_1 \) & \( m_2 \) are placed on the X-axis
\[ m_1 = 10 \text{ kg}, \quad m_2 = 20 \text{ kg}. \]
The first mass is displaced by a distance of 2 cm
\[ \therefore \overline{x}_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{10 \times 2 + 20 x_2}{30} \]
\[ \Rightarrow 0 = \frac{20 + 20 x_2}{30} \Rightarrow 20 + 20 x_2 = 0 \]
\[ \Rightarrow 20 = -20 x_2 \Rightarrow x_2 = -1. \]
\[ \therefore \text{ The 2nd mass should be displaced by a distance 1 cm} \] towards left so as to keep the position of centre of mass unchanged.

9. Two masses \( m_1 \) & \( m_2 \) are kept in a vertical line
\[ m_1 = 10 \text{ kg}, \quad m_2 = 30 \text{ kg} \]
The first block is raised through a height of 7 cm.
The centre of mass is raised by 1 cm:
\[ \therefore 1 = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{10 \times 7 + 30 y_2}{40} \]
\[ \Rightarrow 1 = \frac{70 + 30 y_2}{40} \Rightarrow 70 + 30 y_2 = 40 \Rightarrow 30 y_2 = -30 \Rightarrow y_2 = -1. \]
The 30 kg body should be displaced 1 cm downward inorder to raise the centre of mass through 1 cm.

10. As the hall is gravity free, after the ice melts, it would tend to acquire a spherical shape. But there is no external force acting on the system. So, the centre of mass of the system would not move.

11. The centre of mass of the plate will be on the symmetrical axis.
\[ \overline{y}_{cm} = \frac{\frac{\pi R_2^2}{2} + \frac{4 \pi R_2^3}{3 \pi}}{\frac{\pi R_1^2}{2} - \frac{\pi R_1^3}{3 \pi}} \]
\[ = \frac{(2/3) R_2^3 - (2/3) R_1^3}{\pi/2 (R_2^2 - R_1^2)} = \frac{4 (R_2 R_1^2 + R_1 R_2^2)}{3 \pi} (R_2 - R_1) (R_2 + R_1) \]
\[ = \frac{4 (R_2^2 + R_1^2) R_2 R_1}{3 \pi (R_1 + R_2)} \] above the centre.

12. \( m_1 = 60 \text{ kg}, \quad m_2 = 40 \text{ kg}, \quad m_3 = 50 \text{ kg} \)
Let \( A \) be the origin of the system.
Initially Mr. Verma & Mr. Mathur are at extreme position of the boat.
\[ \therefore \text{ The centre of mass will be at a distance} \]
\[ = \frac{60 \times 0 + 40 \times 2 + 50 \times 4}{150} = \frac{280}{150} = 1.87 \text{ m from 'A'} \]
When they come to the mid point of the boat the CM lies at 2 m from 'A'.
\[ \therefore \text{ The shift in CM} = 2 - 1.87 = 0.13 \text{ m towards right.} \]
But as there is no external force in longitudinal direction their CM would not shift.
So, the boat moves 0.13 m or 13 cm towards right.

13. Let the bob fall at \( A \). The mass of bob = \( m \).
The mass of cart = \( M \).
Initially their centre of mass will be at
\[ \frac{m \times L + M \times 0}{M + m} - \frac{m}{M + m} \]
Distance from \( P \)
When, the bob falls in the slot the CM is at a distance ‘O’ from \( P \).
Shift in CM = 0 - \frac{mL}{M+m} = - \frac{mL}{M+m} \text{ towards left.}

But there is no external force in horizontal direction.

So the cart displaces a distance \frac{mL}{M+m} \text{ towards right.}

14. Initially the monkey & balloon are at rest.
So the CM is at ‘P’

When the monkey descends through a distance ‘L’
The CM will shift
t = \frac{m \times L}{M+m} = \frac{mL}{M+m} \text{ from P}

So, the balloon descends through a distance \frac{mL}{M+m}

15. Let the mass of the two particles be \(m_1\) & \(m_2\) respectively

\(m_1 = 1\) kg, \(m_2 = 4\) kg

\(\Rightarrow \frac{m_1}{m_2} = \frac{1}{4}\)

\(\Rightarrow \frac{\sqrt{m_1}}{\sqrt{m_2}} = \frac{1}{2}\)

\(\Rightarrow \frac{\sqrt{m_1}v_1}{\sqrt{m_2}v_2} = 1:2\)

16. As uranium 238 nucleus emits a \(\alpha\)-particle with a speed of \(1.4 \times 10^7\) m/sec. Let \(v_2\) be the speed of the residual nucleus thorium 234.

\(\Rightarrow \frac{m_1}{m_2} = \frac{1}{4}\)

\(\Rightarrow 4 \times 1.4 \times 10^7 = 234 \times v_2\)

\(\Rightarrow v_2 = \frac{4 \times 1.4 \times 10^7}{234} = 2.4 \times 10^5 \text{ m/sec.}\)

17. \(m_1v_1 = m_2v_2\)

\(\Rightarrow 50 \times 1.8 = 6 \times 10^{24} \times v_2\)

\(\Rightarrow v_2 = \frac{50 \times 1.8}{6 \times 10^{24}} = 1.5 \times 10^{-23} \text{ m/sec.}\)

So, the earth will recoil at a speed of \(1.5 \times 10^{-23}\) m/sec.

18. Mass of proton = \(1.67 \times 10^{-27}\) kg

Let \(\gamma_p\) be the velocity of proton

Given momentum of electron = \(1.4 \times 10^{-26}\) kg m/sec

Given momentum of antineutrino = \(6.4 \times 10^{-27}\) kg m/sec

a) The electron & the antineutrino are ejected in the same direction. As the total momentum is conserved the proton should be ejected in the opposite direction.

\(1.67 \times 10^{-27} \times \gamma_p = 1.4 \times 10^{-26} + 6.4 \times 10^{-27} = 20.4 \times 10^{-27}\)

\(\Rightarrow \gamma_p = \frac{20.4}{1.67} = 12.2 \text{ m/sec in the opposite direction.}\)

b) The electron & antineutrino are ejected \(\perp\) to each other.

Total momentum of electron and antineutrino,

\[= \sqrt{(14)^2 + (6.4)^2} \times 10^{-27} \text{ kg m/s} = 15.4 \times 10^{-27} \text{ kg m/s}\]

Since, \(1.67 \times 10^{-27} \gamma_p = 15.4 \times 10^{-27} \text{ kg m/s}\)

So \(\gamma_p = 9.2 \text{ m/sec}\)
19. Mass of man = M, Initial velocity = 0
Mass of bad = m
Let the throw the bag towards left with a velocity v towards left. So, there is no external force in the horizontal direction.
The momentum will be conserved. Let he goes right with a velocity
\[ mv = MV \Rightarrow v = \frac{mv}{M} \]
\[ \Rightarrow v = \frac{m}{M} \]

Let the total time he will take to reach ground = \( \sqrt{\frac{2H}{g}} \)
Let the total time he will take to reach the height h = \( \sqrt{\frac{2H-h}{g}} \)
Then the time of his flying = \( t_1 = \frac{\sqrt{2H/h}}{g} - \frac{\sqrt{2H-h}}{g} = \sqrt{\frac{g}{H-H-h}} \)
Within this time he reaches the ground in the pond covering a horizontal distance x
\[ \Rightarrow x = V \times t \Rightarrow V = \frac{x}{t} \]
\[ \Rightarrow v = \frac{M}{m} \times \frac{\sqrt{g}}{\sqrt{H-H-h}} \]
As there is no external force in horizontal direction, the x-coordinate of CM will remain at that position.
\[ \Rightarrow 0 = \frac{M}{m} \times \frac{x}{m} \Rightarrow x_t = -\frac{M}{m} \times \frac{x}{m} \]
:. The bag will reach the bottom at a distance (M/m) x towards left of the line it falls.

20. Mass = 50g, M = 0.05kg
\[ v = 2 \cos 45^\circ \hat{i} - 2 \sin 45^\circ \hat{j} \]
\[ v_t = v - 2 \cos 45^\circ \hat{i} - 2 \sin 45^\circ \hat{j} \]
\[ \text{a) change in momentum} = m \cdot (v_t) - m \cdot v, \]
\[ = 0.05 \times (2 \cos 45^\circ \hat{i} - 2 \sin 45^\circ \hat{j}) - 0.05 \times (-2 \cos 45^\circ \hat{i} - 2 \sin 45^\circ \hat{j}) \]
\[ = 0.1 \cos 45^\circ \hat{i} + 0.1 \sin 45^\circ \hat{j} + 0.1 \cos 45^\circ \hat{i} + 0.1 \sin 45^\circ \hat{j} \]
\[ = 0.2 \cos 45^\circ \hat{i} \]
\[ \Rightarrow \text{magnitude} = \sqrt{\left( \frac{0.2}{\sqrt{2}} \right)^2} = 0.14 \text{ kg m/s} \]
\[ \text{c) The change in magnitude of the momentum of the ball} \]
\[ \Rightarrow |P_t| - |P| = 2 \times 0.5 - 2 \times 0.5 = 0. \]

21. \[ \text{P}_{\text{incident}} = (h/l) \cos \theta \hat{i} - (h/l) \sin \theta \hat{j} \]
\[ \text{P}_{\text{reflected}} = -(h/l) \cos \theta \hat{i} - (h/l) \sin \theta \hat{j} \]
The change in momentum will be only in the x-axis direction. i.e.
\[ \vec{AP} = -(h/l) \cos \theta - (h/l) \sin \theta = 2(h/l) \cos \theta \]

22. As the block is exploded only due to its internal energy. So net external force during this process is 0. So the centre mass will not change.
Let the body while exploded was at the origin of the co-ordinate system.
If the two bodies of equal mass are moving at a speed of 10 m/s in +x & +y axis direction respectively,
\[ \sqrt{10^2 + 10^2 + 210.10 \cos 90^\circ} = 10\sqrt{2} \text{ m/s 45^\circ w.r.t. x axis} \]
If the centre mass is at rest, then the third mass which have equal mass with other two, will move in the opposite direction (i.e. 135^\circ w.r.t. +x-axis) of the resultant at the same velocity.

23. Since the spaceship is removed from any material object & totally isolated from surrounding, the masses by astronauts couldn't slip away from the spaceship. So the total mass of the spaceship remain unchanged and also its velocity.
24. \( d = 1 \text{cm}, \quad v = 20 \text{ m/s}, \quad u = 0, \quad \rho = 900 \text{ kg/m}^3 = 0.9 \text{ gm/cm}^3 \)

\[ \text{volume} = \frac{(4/3)\pi r^3}{(4/3) \pi (0.5)^3} = 0.5236 \text{ cm}^3 \]

\[ \therefore \text{mass} = \rho v = 0.5236 \times 0.9 = 0.47142857 \text{ gm} \]

\[ \therefore \text{mass of 2000 hailstone} = 2000 \times 0.4714 = 947.857 \text{ gm} \]

\[ \therefore \text{Rate of change in momentum per unit area} = 947.857 \times 2000 = 18 \text{ Nm}^3 \]

\[ \therefore \text{Total force exerted} = 18 \times 180 = 1800 \text{ N.} \]

25. A ball of mass \( m \) is dropped onto a floor from a certain height let ‘\( h \)’.

\[ \therefore v_1 = \sqrt{2gh}, \quad v_1 = 0, \quad v_2 = -\sqrt{2gh} \quad \& \quad v = 0 \]

\[ \therefore \text{Rate of change of velocity} \]

\[ F = -\frac{m \times 2\sqrt{2gh}}{t} \]

\[ \therefore v = \sqrt{2gh}, \quad s = h, \quad v = 0 \]

\[ \Rightarrow v = u + at \]

\[ \Rightarrow \sqrt{2gh} = g t \quad \Rightarrow t = \frac{2h}{g} \]

\[ \therefore \text{Total time} \]

\[ t = \sqrt{\frac{2h}{g}} \]

\[ \therefore F = \frac{m \times 2\sqrt{2gh}}{\sqrt{g}} = mg \]

26. A railroad car of mass \( M \) is at rest on frictionless rails when a man of mass \( m \) starts moving on the car towards the engine. The car recoils with a speed \( v \) backward on the rails.

Let the mass is moving with a velocity \( x \) w.r.t. the engine.

\[ \therefore \text{The velocity of the mass w.r.t earth is} \ (x - v) \ \text{towards right} \]

\[ V_{cm} = 0 \] (Initially at rest)

\[ \therefore 0 = -Mv + m(x - v) \]

\[ \Rightarrow Mv = m(x - v) \Rightarrow mx = Mv + mv \Rightarrow x = \left( \frac{M + m}{m} \right)v \Rightarrow x = \left( 1 + \frac{M}{m} \right)v \]