DATE: **MATHS**

CLASS: **JEE ADVANCE PAPER 2**

ALL CENTRE

SECTION - 1

1. Let f: R → R be given by f(x) = (x – 1) (x – 2) (x – 5). Define F(x) = $\int_0^x f(t)\,dt, x > 0.$ Then which of the following options is/are correct?

(a) F has a local minimum at x = 1 (b) F has a local maximum at x = 2

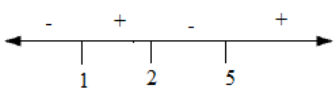
(c) $F(x) \neq 0$ for all x ϵ (0, 5) (d) F has two local maxima and one local minimum in $(0, \infty)$

Solution:

f(x) = (x – 1) (x – 2) (x – 5)

Given F(x) = $\int_0^x f(t).dt$

$$F'(x) = (x-1)(x-2)(x-5)$$



At x = 1 and x = 5, $F'(x)$ changes from – to +

∴ F(x) has two local minima points at x = 1 and x = 5

F(x) has one local maxima point at x = 2.

2. For a ϵ R, |a| > 1, let $\lim_{n\to\infty}\left(\dfrac{1+\sqrt[3]{2}+.....\sqrt[3]{n}}{n^{7/3}\left(\dfrac{1}{(an+1)^2}+\dfrac{1}{(an+2)^2}+....+\dfrac{1}{(an+n)^2}\right)}\right) = 54.$ Then the possible value(s) of a is/are:

(a) 8 (b) – 9 (c) – 6 (d) 7

Solution:

$$\lim_{n\to\infty}\frac{\sqrt[3]{1}+\sqrt[3]{2}+......+\sqrt[3]{n}}{n^{7/3}\left[\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+.....+\frac{1}{(an+n)^2}\right]}=54$$

$$\Rightarrow \lim_{n\to\infty}\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\frac{1}{n}\left[\frac{n^2}{(an+1)^2}+\frac{n^2}{(an+2)^2}+.....+\frac{n^2}{(an+n)^2}\right]}=54$$

$$\Rightarrow \frac{\int_0^1 x^{1/3}dx}{\int_0^1 \frac{dx}{(a+x)^2}}=54 \qquad \Rightarrow \frac{\left[\frac{3}{4}x^{4/3}\right]_0^1}{\left[\frac{-1}{a+x}\right]_0^1}=\frac{3/4}{\frac{1}{a}-\frac{1}{a+1}}=54$$

$$\Rightarrow \frac{(a+1)-a}{a(a+1)}=\frac{3}{4}\times\frac{1}{54} \qquad \Rightarrow \frac{1}{a(a+1)}=\frac{1}{72} \qquad \Rightarrow a(a+1)=72$$

$$\Rightarrow a=8\, or\, a=-9$$

3. Three lines

$$L_1:\vec{r}=\lambda\hat{i},\lambda\in R,$$
$$L_2:\vec{r}=\vec{k}+\mu\hat{j},\mu\in R\, and$$
$$L_3:\vec{r}=\hat{i}+\hat{j}+v\hat{k},v\in R$$

are given. For which point(s) Q and L_2 can we find a point P on L_1 and a point R on L_3 so that P, Q and R are collinear?

(a) $\hat{k}+\hat{j}$ (b) $\hat{k}$ (c) $\hat{k}+\frac{1}{2}\hat{j}$ (d) $\hat{k}-\frac{1}{2}\hat{j}$

Solution:

P (λ, 0, 0), Q (0, μ, 1), R (1, 1, r)

Given $\overline{PQ}=k.\overline{PR}\Rightarrow\frac{\lambda}{\lambda-1}=\frac{-\mu}{-1}=\frac{-1}{-r}$

∴ μ cannot take the values 0 and 1

4. Let F: R → R be a function. We say that f has

PROPERTY 1 $if \lim_{h\to 0} \frac{f(\text{h})-\text{f}(0)}{\sqrt{|\text{h}|}}$ exists and is finite and

PROPERTY 2 $f \lim_{h\to 0} \frac{f(\text{h})-\text{f}(0)}{h^2}$ exists and is finite

Then which of the following options is/are correct?

(a) f(x) = x|x| has PROPERTY 2

(b) F(x) = $x^{2/3}$ has PROPERTY 1

(c) f(x) = sin x has PROPERTY 2

(d) f(x) = |x| has PROPERTY 1

Solution:

(a) f (x) = x|x|

$$\underset{h\to 0}{Lt} \frac{f(\text{h})-\text{f}(0)}{h^2} = \underset{h\to 0}{Lim} \frac{h|\text{h}|-0}{h^2}$$ which does not exist.

(b) $\underset{h\to 0}{Lim} \frac{h^{2/3}-0}{\sqrt{|\text{h}|}} = 0$

(c) $\underset{h\to 0}{Lim} \frac{\sinh-0}{h^2}\, does\, not\, exist$

(d) $\underset{h\to 0}{Lim} \frac{|\text{h}|-0}{\sqrt{|\text{h}|}} = 0$

5. For non-negative integers n, let

$$\text{f(n)} = \frac{\sum_{k=0}^{n} \sin\left(\frac{k+1}{x+2}\pi\right)\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n} \sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

Assuming cos $^{-1}$ x takes value in [0, π], which of the following options is/are correct?

(a) sin (7 $\cos^{-1}$ f(5)) = 0

(b) f(4) = $\frac{\sqrt{3}}{2}$

(c) $\lim_{n\to\infty} f(\text{n}) = \frac{1}{2}$

(d) If α = tan ($\cos^{-1}$ f(6)), then $\alpha^2 + 2\alpha - 1 = 0$

Solution:

$$f(\mathrm{n})=\frac{\sum_{k=0}^{n}\sin\left(\frac{k+1}{n+2}\pi\right)\cdot\sin\left(\frac{k+2}{n+2}\pi\right)}{\sum_{k=0}^{n}2\sin^2\left(\frac{k+1}{n+2}\pi\right)}$$

$$=\frac{\sum_{k=0}^{n}\cos\frac{\pi}{n+2}-\cos\left(\frac{2k+3}{n+2}\right)\pi}{\sum_{k=0}^{n}2\sin^2\left(\frac{k+1}{n+2}\right)\pi}$$

$$=\frac{(\mathrm{n}+1)\cos\frac{\pi}{n+2}-\frac{\cos\left(\frac{n+3}{n+2}\right)\pi\cdot\sin\left(\frac{n+1}{n+2}\right)\pi}{\sin\frac{\pi}{n-2}}}{(\mathrm{n}+1)-\frac{\cos\pi\cdot\sin\left(\frac{n+1}{n+2}\right)\pi}{\sin\left(\frac{\pi}{n+2}\right)}}$$

$$=\frac{(\mathrm{n}+1)\cos\left(\frac{\pi}{n+2}\right)+\cos\left(\frac{n+3}{n+2}\right)\pi}{(\mathrm{n}+1)+1}$$

$$=\cos\left(\frac{\pi}{n+2}\right)$$

$$(\mathrm{A})\ \alpha=Tan\left(\cos^{-1}f(6)\right)=Tan\cos^{-1}\left(\cos{\pi/8}\right)=Tan\,{\pi/8}$$

$$\alpha^2+2\alpha-1=Tan^2\,{\pi/8}+2Tan\,{\pi/8}-1$$

$$Tan2\left(\frac{\pi}{8}\right)=\frac{2Tan\,{\pi/8}}{1-Tan^2\,{\pi/8}}$$

$$\Rightarrow 1=\frac{2\alpha}{1-\alpha^2}\qquad\Rightarrow \alpha^2+2\alpha-1=0$$

$\therefore$ option (A) is correct.

$$(\mathrm{B})\lim_{n\to\infty}f(\mathrm{x})=\lim_{n\to\infty}\cos\left(\frac{\pi}{n+2}\right)=\lim_{\frac{1}{n}\to 0}\cos\left(\frac{{\pi/n}}{1+{2/n}}\right)=1$$

Option (B) correct.

$$(C)\, f(4) = \cos\left(\frac{\pi}{4+2}\right) = \cos \pi/6 = \sqrt{3}/2$$

Option (C) wrong

$$(D) \sin\left[7\cos^{-1} f(5)\right] = \sin\left[7\cos^{-1}\left(\cos \pi/7\right)\right] = \sin\left[7\times\frac{\pi}{7}\right] = 0$$

6. Let $P_1 = I = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, P_2 = \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, P_3 = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, P_4 = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}, P_5 = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix},$

$P_6 = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ and $X = \sum_{k=1}^{6} P_K \begin{bmatrix}2&1&3\\1&0&2\\3&2&1\end{bmatrix} P_K^T$

Where P_K^T denotes the transpose of the matrix P_K. Then which of the following options is/are correct?

(a) X – 30I is an invertible matrix

(b) The sum of diagonal entries of X is 18

(c) If $X\begin{bmatrix}1\\1\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$, then $\alpha = 30$

(d) X is a symmetric matrix

Solution:

From the given data it is clear that

$P_1 = P_1^T = P_1^{-1}$

$P_2 = P_2^T = P_2^{-1}$

$P_6 = P_6^T = P_6^{-1}$

And Let $A = \begin{bmatrix}2&1&3\\1&0&2\\3&2&1\end{bmatrix}$

Here $A^T = A$ → A is symmetric matrix

$X^T = \left(P_1AP_1^T + \dots\dots + P_6AP_6^T\right)^T$

$$= P_1 A^T P_1^T + \ldots\ldots + P_6 A^T P_6^T$$

$$= X$$

$$\therefore X \text{ is symmetric}$$

$$Let\ B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$XB = P_1 A P_1^T G + P_2 A P_2^T B + \ldots\ldots + P_6 A P_6^T B$$

$$= P_1 AB + P_2 AB + \ldots. + P_6 AB$$

$$= \left(P_1 + P_2 + P_3 + \ldots\ldots + P_6\right)\begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix} = 30B \qquad \Rightarrow \propto = 30$$

$$\text{Since } X\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 30\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left(X - 30I\right) \text{B = 0 has a nontrivial solution } B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left(X - 30I\right) = 0$$

$$X = P_1 A P_1^T + \ldots\ldots + P_6 A P_6^T$$

$$\text{Trace}\left(X\right)=tr\left(P_1AP_1^T\right)+....+Tr\left(P_6AP_6^T\right)$$

= (2 + 0 + 1) + …. + (2 + 0 + 1) = 3 + 3 + …. (6 times) = 18

7. Let x ϵ R and let $P=\begin{bmatrix}1&1&1\\0&2&2\\0&0&3\end{bmatrix}, Q=\begin{bmatrix}2&x&x\\0&4&0\\x&x&6\end{bmatrix}$ $and\ R=PQP^{-1}$

Then which of the following options is/are correct?

(a) For x = 1, there exists a unit vector $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ for which R $\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(b) There exists a real number x such that PQ = QP

(c) det R = det $\begin{bmatrix}2&x&x\\0&4&0\\x&x&5\end{bmatrix}+8,\ for\ all\ x\,\varepsilon\,R$

(d) for x = 0, if $R\begin{bmatrix}1\\a\\b\end{bmatrix}=6\begin{bmatrix}1\\a\\b\end{bmatrix}, then\ a+b=5$

Solution:

R = PQP^{-1}

$|R|=|P||Q|.|P^{-1}|$

$\Rightarrow \det Q=2(24)-x(0)+x(-4x)=48-4x^2$

$$P=\begin{bmatrix}1&1&1\\0&2&2\\0&0&3\end{bmatrix}.Q\left(X=0\right)=\begin{bmatrix}2&0&0\\0&4&0\\0&0&6\end{bmatrix}$$

R = PQR^{-1}

$$=\begin{bmatrix}2&4&6\\0&8&12\\0&0&18\end{bmatrix}.\frac{1}{6}\begin{bmatrix}6&-3&0\\0&3&-2\\0&0&2\end{bmatrix}$$

$$=\frac{1}{6}\begin{bmatrix}12&6&4\\0&24&8\\0&0&36\end{bmatrix}=\begin{bmatrix}2&1&2/3\\0&4&4/3\\0&0&6\end{bmatrix}$$

$$(R-6I)\begin{pmatrix}1\\a\\b\end{pmatrix}=\begin{pmatrix}-4&1&2/3\\0&-2&4/3\\0&0&0\end{pmatrix}\begin{pmatrix}1\\a\\b\end{pmatrix}=\begin{bmatrix}-4&+a&+\frac{2b}{3}\\0&-2a&+4b/3\\0&0&0\end{bmatrix}$$

$$-4+a+\frac{2b}{3}=0 \ and \ -2a+\frac{4b}{3}=0\Rightarrow a=2\ \&\ b=3$$

$$\therefore \qquad a+b=5$$

$PQ = QP \Rightarrow x+4+x=2+2x+0 \Rightarrow$ No value exist

8. Let $f(\text{x})=\frac{\sin \pi x}{x^2}, x>0$

Let $x_1 < x_2 < x_3 < \ldots. < x_n < \ldots$ be all the points of local maximum of f
and $y_1 < y_2 < y_3 < \ldots. < y_n < \ldots..$ be all the points of local minimum of f.
Then which of the following options is/are correct?

(a) $|x_n - y_n| > 1$ for every n

(b) $x_1 < y_1$

(c) $x_n \in \left(2n, 2n+\frac{1}{2}\right)$ *for every n*

(d) $x_{n+1} - x_n > 2$ *for every n*

Solution:

$$f(\text{x})=\frac{\sin \pi x}{x^2} \Rightarrow f'(\text{x})=\frac{x^2\cdot(\cos \pi\,\text{x})\cdot(\pi)-\sin \pi\,\text{x}\cdot(2\,\text{x})}{x^4}$$

$$\Rightarrow f'(\text{x})=\frac{2x\cos \pi x\left(\frac{\pi x}{2}-\tan \pi x\right)}{x^4}$$

By using graph we can say that option (1) (3) (4) are correct.

SECTION – 2

1. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$ equals

Solution:

$$\sec^{-1}\pi\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$$

$$=\sec^{-1}\left(\frac{-1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\operatorname{cosec}\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)$$

$$=\sec^{-1}\left(\frac{-1}{4}\sum_{k=0}^{10}\frac{2}{\sin\left(\frac{7\pi}{6}+k\pi\right)}\right)$$

$$=\sec^{-1}\left(\frac{-1}{2}\sum_{k=0}^{10}\frac{1}{(-1)^{k+1}\sin\frac{\pi}{6}}\right)$$

$$=\sec^{-1}\left(-\sum_{k=0}^{10}\frac{1}{(-1)^{k+1}}\right)=\sec^{-1}(1)=0$$

2. Let |X| denote the number of elements in set X. Let S = {1,2,3,4,5,6} be a sample space, where each element is equally likely to occur. If A and B are independent events associated with S, then the number of ordered pairs (A,B) such that $1 \le |B| < |A|$, equals.

Solution:

The number of ordered pairs of (A, B) are

$6c_1\,(6c_2 + 6c_3 + \ldots + 6c_6) + 6c_2\,(6c_2(6c_3 + 6c_4 \ldots + 6c_6) + 6c_3(6c_4 + 6c_5 + 6c_6) + 6c_4(6c_5 + 6c_6) + 6c_5 \,.\, 6c_6$

$= (6c_1 .\, 6c_2 + 6c_1 \,.\, 6c_3 + \ldots + 6c_1 6c_6) + (6c_2.6c_3 + 6c_2.6c_4 + \ldots + 6c_2.6c_6) + (6c_3.6c_4 + 6c_3.6c_5 + 6c_3.6c_6)$

$+ {}^6c_4.{}^6c_5 + {}^6c_4.{}^6c_6 + {}^6c_5.{}^6c_6.$

$= ({}^{12}c_5 – {}^6c_1) + ({}^{12}c_4 – {}^6c_2) + ({}^{12}c_3 – {}^6c_3) + ({}^{12}c_2 – {}^6c_4) + ({}^{12}c_1 – {}^6c_5)$

$= ({}^{12}c_1 + {}^{12}c_2 + {}^{12}c_3 + {}^{12}c_4 + {}^{12}c_5) – ({}^6c_1 + {}^6c_2 + \ldots + {}^6c_5)$

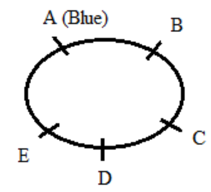
$= 1585 – 62 = 1523.$

3. Five person A, B, C, D and E are seated in a circular arrangement. If each of them is given a hat of one of the three colours red, blue and green, then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is

Solution:

Maximum number of hats used of same colour are 2.

They cannot be 3 otherwise atleast 2 hats of same colour are consecutive.



Now the hats used are consider as B B G G B

Which can be selected in 3 ways.

It can be R G G B B or R R G B B

The number of ways of distributing blue hat (single one) in 5 persons equal to 5

Now either position B and D are filled by green hats and C and E are filled by Red hats or B & D are filled by

Red hats and C & E are filled by Green hats.

➔2 ways are possible.

Hence number of ways = 3×5×2 = 30 ways.

4. Suppose

$$\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals

Solution:

$$\begin{vmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k \cdot k^2 \\ \sum_{k=0}^{n} {}^nC_k \cdot k & \sum_{k=0}^{n} {}^nC_k \cdot 3^k \end{vmatrix} = 0$$

$$\begin{vmatrix} \frac{n(n+1)}{2} & n \cdot 2^{n-1} + n(n-1) \cdot 2^{n-2} \\ n \cdot 2^{n-1} & 4^n \end{vmatrix} = 0$$

$$\Rightarrow \frac{n(n+1)}{2} \cdot 4^n - n \cdot 2^{2n-1}\left(n \cdot 2^{n-1} + n(n-1) \cdot 2^{n-2}\right) = 0$$

$$\Rightarrow \frac{n(n+1)}{2} \cdot 4^n - n^2 \cdot 2^{2n-2} \cdot - n^2(n-1) \cdot 2^{2n-3} \cdot = 0$$

$$\Rightarrow \frac{n(n+1)}{2} - \frac{n^2}{4} - \frac{n^2(n-1)}{8} = 0 \quad \Rightarrow \frac{n}{2}\left[n+1-\frac{n}{2}-\frac{n(n-1)}{4}\right] = 0$$

$$\Rightarrow n = 0 \; or \; 4(n+1) - 2n - 1(n-1) = 0 \qquad \Rightarrow n = 0 \; or \; n = 4$$

$$\sum_{\pi=0}^{4} \frac{4c\pi}{r+1} = \sum_{r=0}^{4} \frac{5cr+1}{5} = \frac{2^5-1}{5} = \frac{31}{5} = 6.20$$

5. The value of the integral $\int_{0}^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ equals

Solution:

$$I = \int_{0}^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\sin\theta}+\sqrt{\cos\theta}\right)^5} \cdot d\theta$$

$$I = 3\int_{0}^{\pi/2} \frac{\sqrt{\cos\theta}}{\left(\sqrt{\sin\theta}+\sqrt{\cos\theta}\right)^5} \qquad \rightarrow 1$$

$$I=3\int_0^{\pi/2}\frac{\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}\qquad\rightarrow 2\qquad\left[\because\int_0^a f(\mathrm{x})\,\mathrm{dx}=\int_0^a f(\mathrm{a-x})\cdot\mathrm{dx}\right]$$

$$2I=3\int_0^{\pi/2}\frac{\sqrt{\cos\theta}\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}\cdot d\theta=3\int_0^{\pi/2}\frac{d\theta}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^4}$$

$$\frac{2I}{3}=\int_0^{\pi/2}\frac{\sec 2\theta\cdot d\theta}{\left(\sqrt{\tan\theta}+1\right)^4}$$

Let $Tan\theta=t^2 \quad\Rightarrow\quad \sec 2\theta\cdot d\theta=2t\,dt$

$$\frac{2I}{3}=\int_0^{\infty}\frac{2tdt}{(\mathrm{t}+1)^4}$$

$$\frac{I}{3}=\int_0^{\infty}\left[\frac{1}{(\mathrm{t}+1)^3}-\frac{1}{(\mathrm{t}+1)^4}\right]dt$$

$$I=\left[\frac{-3}{2(\mathrm{t}+1)^2}+\frac{1}{(\mathrm{t}+1)^3}\right]_0^{\infty}$$

$$=\frac{3}{2}-1=\frac{1}{2}$$

6. Let $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$ be two vectors. Consider a vector $\vec{c}=\alpha\vec{a}+\beta\vec{b}+\alpha,\beta\,\varepsilon\,\mathbb{R}$. If the projection of $\vec{c}$ on the vector $\left(\vec{a}+\vec{b}\right) is\, 3\sqrt{2}$, then the minimum value of $\left(\vec{c}-\left(\vec{a}\times\vec{b}\right)\right).\vec{c}$ equals

Solution:

$\vec{a}=2i+j-k$

$\vec{b}=i+2j+k$

$\vec{c}=\alpha\vec{a}+\beta\vec{b}=\alpha(2\text{i}+\text{j}-\text{k})+\beta(\text{i}+2\,\text{j}+\text{k})$

$=(2\alpha+\beta)\text{i}+(\alpha+2\beta)\,\text{j}+(\beta-\alpha)\,\text{k}$

Given $\dfrac{\vec{c}\cdot(\text{a}+\text{b})}{|\vec{\text{a}}+\vec{\text{b}}|}=3\sqrt{2}$

$\Rightarrow 9(\alpha+\beta)=18 \;\; \Rightarrow \alpha+\beta=2$

$(\vec{\text{c}}-\text{a}\times\text{b})\,\text{c}=\left(\alpha\vec{a}+\beta\vec{\text{b}}+\vec{a}\times\vec{b}\right)\cdot\left(\alpha\vec{a}+\vec{b}\beta\right)$

$=6\alpha^2+6\alpha\beta+6\beta^2=6\left[\alpha^2+\alpha(2-\alpha)+(2-\alpha)^2\right]$

$=6\left(\alpha^2-2\alpha+4\right)$

Minimum value = 18

SECTION – 3

1. Answer the following by appropriately matching the lists based on the information given in the paragraph
Let f(x) = sin(π cosx) and g(x) = cos(2π sinx) be two functions defined for x > 0. Define the following sets whose element are written in the increasing order:

$X=\{x: f(\text{x})=0\}$, $Y=\{x: f'(\text{x})=0\}$

$Z=\{x: g(\text{x})=0\}$, $W=\{x: g'(\text{x})=0\}$

List –I contains the sets X,Y,Z and W. List – II contains some information regarding these sets.

List I	**List – II**
(I) X	(P) $\supseteq\left\{\dfrac{\pi}{2},\dfrac{3\pi}{2},4\pi,7\pi\right\}$
(II) Y	(Q) an arithmetic progression
(III) Z	(R) Not an arithmetic progression
(IV) W	(S) $\supseteq\left\{\dfrac{\pi}{6},\dfrac{7\pi}{6},\dfrac{13\pi}{6}\right\}$
	(T) $\supseteq\left\{\dfrac{\pi}{3},\dfrac{2\pi}{3},\pi\right\}$

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(U) $\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only correct combination?

(a) (II), (R), (S) (b) (I), (P), (R) (c) (II), (Q), (T) (d) (I), (Q), (U)

2. Answer the following by appropriately matching the lists based on the information given in the paragraph
Let f(x) = sin(π cosx) and g(x) = cos(2π sinx) be two functions defined for x > 0. Define the following sets whose element are written in the increasing order:

$$X = \{x : f(\mathrm{x}) = 0\}, \quad Y = \{x : f'(\mathrm{x}) = 0\}$$

$$Z = \{x : g(\mathrm{x}) = 0\}, \quad W = \{x : g'(\mathrm{x}) = 0\}$$

List –I contains the sets X,Y,Z and W. List – II contains some information regarding these sets.

List I	List – II
(I) X	(P) $\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II) Y	(Q) an arithmetic progression
(III) Z	(R) Not an arithmetic progression
(IV) W	(S) $\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
	(T) $\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
	(U) $\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

Which of the following is the only correct combination?

(a) (IV), (Q), (T) (b) (IV), (P), (R), (S) (c) (III), (R), (U) (d) (III), (P), (Q), (U)

Solution:

$$f(\mathrm{x}) = 0 \Rightarrow \sin\left(\pi \cos x\right) = 0$$

$$\Rightarrow \pi \cos x = n\pi \Rightarrow \cos x = n \Rightarrow \cos x = -1, 0, 1$$

$$x = \left\{\mathrm{n}\,\pi, (2\mathrm{n}\,\pi)\,{}^{\pi}\!/_{2}\right\}$$

$$x = \left\{\frac{n\pi}{2}, \mathrm{n} \in \mathrm{I}\right\}$$

$$f'(x)=0 \Rightarrow \cos(\pi \cos x)(-\pi \sin x)=0$$

$$\Rightarrow \pi \cos x=(2n+1)\pi/2 \; or \; x=n\pi$$

$$\Rightarrow \cos x=n+\frac{1}{2} or \; x=n\pi$$

$$\Rightarrow \cos x=\pm\frac{1}{2} or \; x=n\pi$$

$$\therefore y=\left\{2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \frac{2\pi}{3}, n\pi\right\}$$

$$g(x)=0 \Rightarrow \cos(2\pi \sin x)=0$$

$$\Rightarrow 2\pi \sin x=(2n+1)\pi/2$$

$$\Rightarrow \sin x=\frac{2n+1}{4}=\pm\frac{1}{4},\pm\frac{3}{4}$$

$$z=\left\{n\pi \pm \sin^{-1}\frac{1}{4}, n\pi \pm \sin^{-1}\frac{3}{4}, n \in I\right\}$$

$$g'(x)=0 \Rightarrow -\sin(2\pi \sin x)(2\pi \cos x)=0$$

$$\Rightarrow 2\pi \sin x=n\pi \; or \; x=(2n+1)\frac{\pi}{2}$$

$$\Rightarrow \sin x=\frac{n}{2}=0,\pm\frac{1}{2},\pm 1 or \; x=(2n+1)\frac{\pi}{2}$$

$$\Rightarrow W=\left\{n\pi,(2n+1)\frac{\pi}{2}, n\pi \pm \frac{\pi}{6}, n \in I\right\}$$

(1) Option – 3 (2) Option – 2

3. Answer the following by appropriately matching the lists based on the information given in the paragraph

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x - 3)^2 + (y - 4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x - h)^2 + (y - k)^2 = r^2$ satisfies the following conditions:

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x_2 = 8\alpha y$.

There are some expression given in the List – I whose values are given in List – II below:

List – I	**List – II**
(I) $2h + k$	(P) 6
(II) $\dfrac{Length\ of\ ZW}{Length\ of\ XY}$	(Q) $\sqrt{6}$
(III) $\dfrac{Area\ of\ triangle\ MZN}{Area\ of\ triangle\ ZMW}$	(R) $\dfrac{5}{4}$
(IV) α	(S) $\dfrac{21}{5}$
	(T) $2\sqrt{6}$
	(U) $\dfrac{10}{3}$

Which of the following is the only INCORRECT combination?

(a) (IV), (S) (b) (IV), (U) (c) (III), (R) (d) (I), (P)

Solution:

(ii) Equation of line zw

$C_1 = C_2$

$\Rightarrow 3x + 4y = 9$

$\Rightarrow Dis\tan ce\, of zw\ from\, (0,0)$

$$\left|\frac{-9}{\sqrt{3^2+4^2}}\right| = \frac{9}{5}$$

Length of xy = $2\sqrt{9-\left(\frac{9}{5}\right)^2} = \frac{24}{5}$

Distance of zw from c

$$\frac{\left|\frac{3\times 9}{5}+4\times\frac{12}{5}-9\right|}{\sqrt{3^2+4^2}} = \frac{6}{5}$$

Length of zw = $2\sqrt{6^2-\frac{6^2}{5^2}} = \frac{24\sqrt{6}}{5}$

$$\frac{length\,of\;zw}{length\,of\;xy} = \sqrt{6}$$

(iii) Area of $\Delta mzN = \frac{1}{2}\cdot Nm\cdot\left(\frac{1}{2}zw\right) = \frac{72\sqrt{6}}{5}$

Area of $\Delta zmw = \frac{1}{2}\cdot zw\cdot(om+op) = \frac{1}{2}.\frac{24\sqrt{6}}{5}.\left(3+\frac{9}{5}\right) = \frac{288\sqrt{6}}{25}$

$$\therefore \frac{\text{Area of } \Delta mzN}{\text{Area of } \Delta zmw} = \frac{5}{4}$$

(iv) Slope of tangent to C_1 at $m = \frac{-1}{4/3} = -\frac{3}{4}$

Equation of Tangent $y = mx - 2\sqrt{1+m^2}$

$$y = \frac{-3x}{4} - 3\sqrt{1+\frac{9}{16}}$$

$$y = \frac{-3x}{4}\frac{-15}{4}$$

$$\Rightarrow x = \frac{-4y}{3} - 5 \qquad \rightarrow 1$$

Tangent to $x^2 = 4(2d)y$ is $x = m'y + \frac{2d}{m^1} \qquad \rightarrow 2$

Compare 1 and 2

$$m' = \frac{-4}{3} \; and \; \frac{2\propto}{m^1} = -5 \qquad \Rightarrow \propto = \frac{10}{3}$$

4. Answer the following by appropriately matching the lists based on the information given in the paragraph
Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x - 3)^2 + (y - 4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x - h)^2 + (y - k)^2 = r^2$ satisfies the following conditions:
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(IV) α	(S) $\frac{21}{5}$
	(T) $2\sqrt{6}$
	(U) $\frac{10}{3}$

Which of the following is the only INCORRECT combination?
(a) (II), (T) (b) (I), (S) (c) (I), (U) (d) (II), (Q)

Solution:

$$2r = MN = 3 + \sqrt{3^2 + 4^2} + 4 = 12$$

$$\Rightarrow r = 6$$

Centre c of circle c_3 lies on $y = \frac{4}{3}x$

Let $c\left(h, \frac{4}{3}h\right)$

$$OC = MC - OM = \frac{12}{2} - 3 = 3$$

$$\sqrt{h^2 + \frac{16}{9}h^2} = 3 \Rightarrow h = \frac{9}{5}$$

$$k = \frac{4}{3}h = \frac{12}{5} \Rightarrow 2h + k = 6$$