MATHEMATICS

STANDARD

X

PART - 2

Government of Kerala
DEPARTMENT OF EDUCATION

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State Council of Educational Research and Training (SCERT) KERALA
2011

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata.
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha.
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he!

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give respect to my parents, teachers and all elders and treat everyone with courtesy.

I pledge my devotion to my country and my people. In their well-being and prosperity alone lies my happiness.

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Dear Children,

In farms and factories,
Up the sky and in the mind,
Mathematics blooms.
Roots deep in history;
Numbers, Equations,
Geometrical Shapes;
Forking branches.
To know a little of all this
A small book.
Fruit of knowledge - a mellow mind
Right in thought, True in word

With regards,

Prof. M. A. Khader
Director
SCERT

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CONTENTS

CONSTITUTION OF INDIA
Part IV A

FUNDAMENTAL DUTIES OF CITIZENS

ARTICLE 51 A

Fundamental Duties- It shall be the duty of every citizen of India:

(a) to abide by the Constitution and respect its ideals and Institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practice derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wildlife and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievements;

(k) who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between age of six and fourteen years.

7 Mathematics of chance

Chance as a number

There are ten beads in a box - nine black and one white. If you take one out (without peeking)...

It is very likely to be black; there's an outside chance of getting white also.

In another box are five black beads and five white beads and you take one from this. It can be black or white; apart from this, we can't say anything much.

Let's put it this way: there's a high chance of drawing a black bead from the first box or in other words, there's a very low chance of getting a white. But from the second box, there's an equal chance of getting black or white. We use numbers to make this more precise.

In the first box, $\frac{9}{10}$ of the beads are black and only $\frac{1}{10}$ of the beads are white. So, we say that the *probability* of getting a black bead from this box is $\frac{9}{10}$; and the probability of getting a white bead is $\frac{1}{10}$.

What about the second box? Since $\frac{5}{10}=\frac{1}{2}$, we say that in this case, the probability of getting a black and the probability of getting a white are both equal to $\frac{1}{2}$.

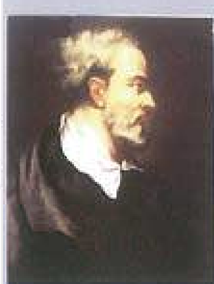
Let's look at another problem: We write the numbers 1 to 25 in paper slips and put them all in a box. One slip is drawn from this. What is the probability of the number to be a multiple of 3?

Among the numbers in the box, only the eight numbers 3, 6, 9, 12, 15, 18, 21, 24 are multiples of 3. So, the probability of getting such a number is $\frac{8}{25}$.

Dicey math

Haven't you played dice-games like Snakes and Ladders? Such games were played from very old times. The picture shows a dice from India during the Indus Valley Civilization, dated about 2500 BC.

We can't predict what number will turn up on rolling a die. The first book on the mathematics of this was written by the Gerolamo Cardano, who lived in Italy during the sixteenth century AD.



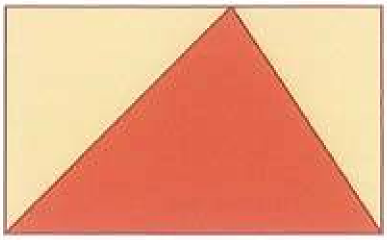
It is basically a guide to gamblers. In it, he has evaluated as numbers, the chances of getting various sums on rolling a pair of dice together.

What is the probability of getting a multiple of 4 from this box?

And the probability of getting an even number?

An odd number?

One more problem: see this picture



A figure like this is cut out and without looking, we mark a dot on it with a pencil. What's the probability that it falls within the red triangle?

What fraction of the rectangle is the red triangle? (Remember the section, **Rectangle and triangle** of the lesson **Areas** in the Class 9 textbook)

So, the probability is $\frac{1}{2}$. In other words, the probability of the dot falling within the triangle and outside it are equal.

Now try these problems:

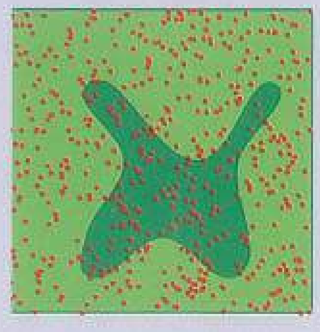
- A box contains 4 white balls and 6 black balls and another one, 3 white and 5 black. We can choose one box and take a ball. If we want a black ball, which box is the better choice?

- You ask someone to say a (natural) number less than 10. What is the probability that the number is a prime? What if the number asked is to be less than 100?

- A box contains paper slips with numbers written on them - 4 odd and 5 even. Two more paper slips, one with an odd number and another with an even number are put in. Does the probability of getting an odd number increase or decrease? What about the probability of getting an even number?

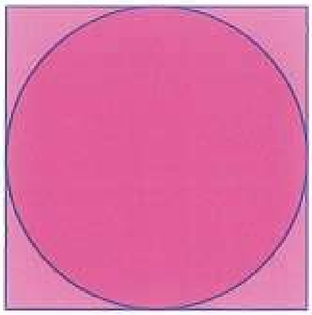
Probability and area

We can use probability to estimate the area of complicated figures. The figure is drawn within a square and then a large number of dots are marked within the square, without any order or scheme.



The number of dots falling within our figure, divided by the total number of dots gives an approximation of the area of the figure divided by the area of the square. And this approximation gets better, as we increase the number of dots. Both the geometric operation of marking the dots and the arithmetic operation of division can be done very fast, using computers. This is called the Monte Carlo Method.

• A point was marked in the picture below, without looking.



What's the probability that it is within the circle? What's the probability that it is outside the circle? Calculate up to two decimal places.

Two at a time

Two slips of paper marked 1 and 2, are put in a box and three slips market 1, 2, 3 are put in another. One slip from each box is drawn. What is the probability that both show odd numbers?

Drawing one slip from each box, we get a pair of numbers. What are the possibilities? It can be 1 from the first box, 2 from the second; or 1 from both boxes; there are several, right? Let's write down all possible pairs:

(1, 1) (1, 2) (1, 3)

(2, 1) (2, 2) (2, 3)

Six pairs altogether. Our interest is in those pairs in which both numbers are odd. How many such pairs are there among these?

Only two, isn't it?

So, what is the probability of this happening? $\frac{2}{6}=\frac{1}{3}$

What is the probability of getting one odd number and one even number?

A problem

The famous scientist Galileo writes about a problem asked by a gambler friend. He had computed that when three dice are rolled together, 9 or 10 can occur as the sum in 6 different ways:

	9	10
1.	1 + 2 + 6	1 + 3 + 6
2.	1 + 3 + 5	1 + 4 + 5
3.	1 + 4 + 4	2 + 2 + 6
4.	2 + 2 + 5	2 + 3 + 5
5.	2 + 3 + 4	2 + 4 + 4
6.	3 + 3 + 3	3 +3+4

But then in actual experience, he found 10 occurring more often than 9 as the sum. He wanted an explanation of this.

In the list above (1,2,6) for example, stands for 1 coming up in some die, 2 in some other die and 6 in yet an other die. Galileo argued instead of this, he must denote by the triple (1,2,6), the occurrence of 1 in the first die, 2 in the second die, 6 in the third die; by the triple (1,6,2), the occurrence of 1 in the first die, 6 in the second die and 2 in the third die and so on. This gives six different triples, (1,2,6),(1,6,2),(2,1,6), (2,6,1),(6,1,2),(6,2,1) all denoting the occurrence of the same numbers 1, 2, 6 in the three dice. Expanding other triples also like this, Galileo shows that the sum 9 can occnr in 25 different ways, while 10 can ocur in 27 different ways. (Try it)

How about increasing the number of slips? Suppose one box contains numbers from 1 to 5 and the other contains numbers from 1 to 10. What can you say about the probabilities we have seen earlier, in this new set up?

How many number pairs are possible in this case? It is a bit tedious to write out all the possibilities, as in the first example (and there is no charm in it). Can we compute this number?

Let's think about it like this: how many pairs are possible with the first number (that is, the number from the first box) 1? How many with this 2?

In short, there are 5 possibilities for the first number; and in each of these, there are 10 possibilities for the second. (You may find it helpful to imagine all these pairs written out in rows and columns, as in the first example: a row of 10 pairs with 1 as the first number; below it, another row of ten pairs with 2 as the first number and so on, giving 5 rows, each containing 10 pairs.)

So, 50 pairs in all. How many of these pairs have both numbers odd?

For such pairs, the first number should one of the three numbers 1, 3, 5. And the second number?

Thus we can see that there are $3 \times 5 = 15$ such pairs. (Do you understand this? Picture these in rows and columns, if you want.)

So, the probability of getting two odd numbers in this case is $\frac{15}{50} = \frac{3}{10}$

Can you find like this, the probability of getting both even and also the probability of getting one odd and one even?

Let's look at another problem: It's about a game played by two children. Both raise some of their fingers. If the total number of fingers raised by both is odd, the first player wins; if it is even, the second player wins. Who has more chance of winning?

In this game, the number of fingers each raises can be any number from 1 to 10. So, if we take the possible pairs of the number of fingers each can raise, how many different pairs do we get?

Theory and reality

When we toss a coin, we may get head or tail. And mathematically, it is logical to take the probability of each as $\frac{1}{2}$.

But this does not mean, if we toss a coin twice we would get head once and tail once. Nor are we sure to get five heads and five tails exactly, if we toss it ten times. It only means that if we toss a coin a large number of times, the number of heads and the number of tails would be more or less the same. For example, in 1000 tosses, we may get 510 heads and 490 tails.

Likewise, if we roll a die 1200 times, each number may not turn up exactly 200 times (this is more likely not to occur). One number may come up 220 times, another number 180 times and so on.

Out of these 100 possible pairs (how did we get this hundred?), in how many do we get an odd sum?

For the sum to be odd, one number must be odd and the other even.

How many pairs are there with the first odd and the second even? $5 \times 5 = 25$ (how is that?) And how many, the other way round?

Thus there are $25 + 25 = 50$ pairs with the sum odd. So, the possibility of the odd-player winning is $\frac{50}{100} = \frac{1}{2}$

We can now say without any actual listing that the probability of the even - player winning is the same (how?)

One more problem: there are 50 mangoes in a basket, in which 20 are not ripe; in another basket, there are 40 mangoes with 15 of them not ripe. One mango is taken from each basket. What is the probability of getting at least one ripe mango?

In how many different ways can we take two mangoes, one from each basket? (You can think of the mangoes in the first basket to be numbered 1 to 50 and the mangoes in the other numbered 1 to 40, and all possible mango pairs arranged in rows and columns, if that helps).

We can classify these 2000 mango pairs as follows:

(i) both unripe

(ii) both ripe

(iii) one ripe and the other unripe

How many pairs are possible with both unripe?

$20 \times 15 = 300$, right?

What about the pairs with both ripe? In the first basket, $50 - 20 = 30$ are ripe, and in the second, $40 - 15 = 25$ are ripe; so $30 \times 25 = 750$ pairs are possible with both ripe.

Now with the first mango (that is, from the first basket) ripe and the second unripe, $30 \times 15 = 450$ pairs are possible. What about the other way round? With the first mango unripe and the second ripe, $20 \times 25 = 500$ pairs are possible. So how many pairs in all in the third group? $450 + 500 = 950$.

Probability and frequency

We said that when a coin is tossed a large number of times, the number of heads and tails are almost equal, and so we can take the probability of each coming up as $\frac{1}{2}$. But due to some reason, such as a manufacturing defect in the coin, it may happen that the probability of head coming up is higher. How do we recognize this?

We suspect such a case, if in a large number of tosses, one side comes up very much more than the other. Then we toss the coin more and more times and tabulate the number of times each side comes up. For example, see this table:

Tosses	Heads	Tails
10	6	4
100	58	42
1000	576	424
10000	5865	4135

This shows that, instead of taking the probability of each face as 0.5, it is better to take the probability of head as 0.6 and the probability of tail as 0.4.

There are mathematical methods for making such assignments of probabilities more accurate, which we will see in the further study of the branch of mathematics called *Probability Theory*.

The pairs with at least one mango ripe are those in the second or third groups; and there are 750 + 950 = 1700 pairs in these groups put together. So, the probability of getting at least one ripe mango is

$$\frac{1700}{2000} = \frac{17}{20}$$

We can write this as 0.85 also.

Instead of finding the number of pairs in each of the three groups, we could have computed this probability from just the number of pairs in the first group alone. How?

Now some problems for you:

- There are two boxes, each containing slips numbered 1 to 5. One slip is drawn from each box and their numbers added. What are the possible sums? Compute the probability of each sum.
- In the finger-raising game, which number has the maximum probability of occurrence as the sum? What is this probability?
- Suppose you ask someone to say a two-digit number.
  - What is the probability of this number having both digits the same?
  - What is the probability of the first digit being larger than the second?
  - What is the probability of the first digit being smaller than the first?

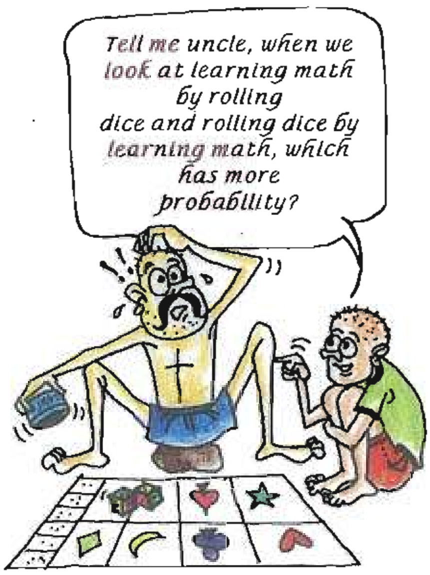
Measuring uncertainty

Have you noticed that calendars show the time of sunrise and sunset for each day? It is possible to compute these, since the earth and sun move according to definite mathematical laws.

Because of this, we can also predict the months of rain and shine. But we may not be able to predict a sudden shower during summer. It is the largeness of the number of factors affecting rainfall and the complexity of their inter-relations that makes such predictions difficult.

But in such instances also, we can analyse the context mathematically and compute probabilities. This is why weather predictions are often given as possibilities. And unexpected changes in the circumstances sometimes make these predictions wrong.

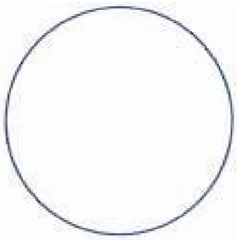
If we look at such situations rationally, we can see that such probability predictions are more reliable than predictions sounding exact, but made without any scientific basis.



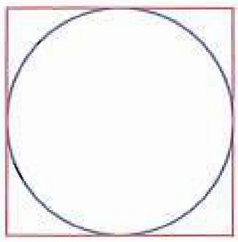
8 Tangents

Around a circle

Draw a circle of any radius you please:

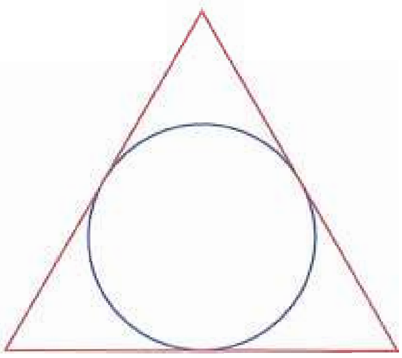


Now draw a square around it as shown below.



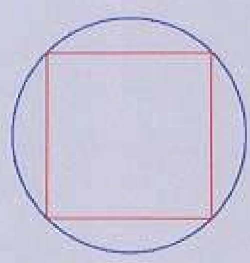
How did you draw the sides?

Next draw a circle and this time, draw an equilateral triangle around it as shown below.

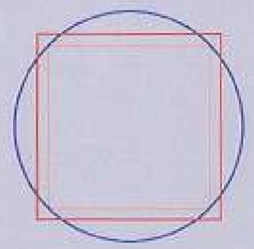


Growing square

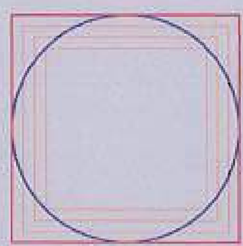
It is not difficult to draw a square inside a circle like this, is it?



We can slightly increase the lengths of the sides and draw like this:



If we keep on increasing the lengths of the sides little by little, we get a square like this:



Can we push out any square within the circle in this manner?

Not that easy, is it?

In the picture of the square and in the picture of the triangle, each side passes through how many points of the circle?

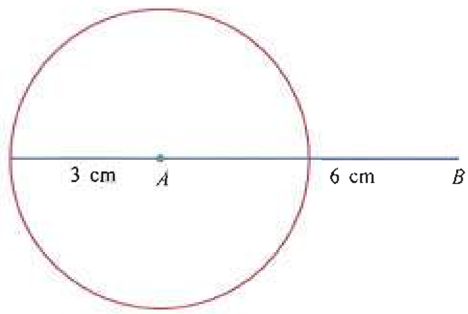
Let's look at such relations between lines and circles in detail.

Lines and circles

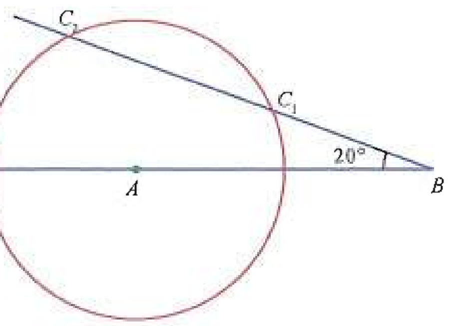
Have you seen before, instances where a line passes through a single point on a circle?

Look at this example. We want to draw triangle ABC with AB of length 6 centimetres, AC is of length 3 centimetres and the angle at B of $20°$. (Do you remember a similar problem done in Class 8?)

We start by drawing AB of length 6 centimetres. We want C to be 3 centimetres away from A; this means C must be a point on the circle of radius 3, centred at A.

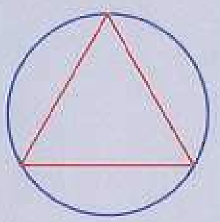


What next? Since the angle at B is to be $20°$, let's draw a line of this slant through B:



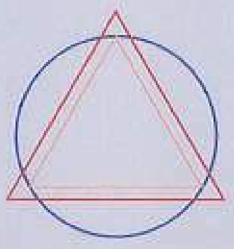
Growing triangles

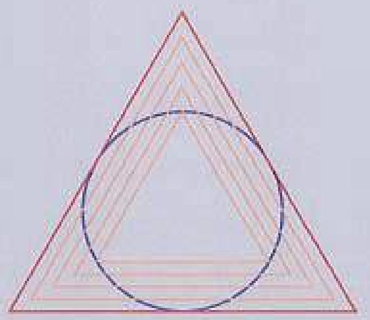
Can you draw an equilateral triangle within a circle, as shown below?



(Recall the section **Arcs, angles and chords,** of the lesson **Circles**)

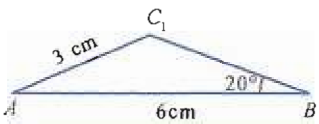
We can enlarge this triangle little by little, as in the case of a square:

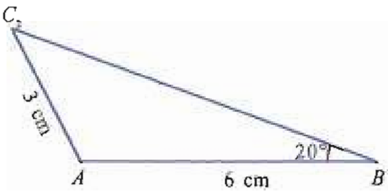




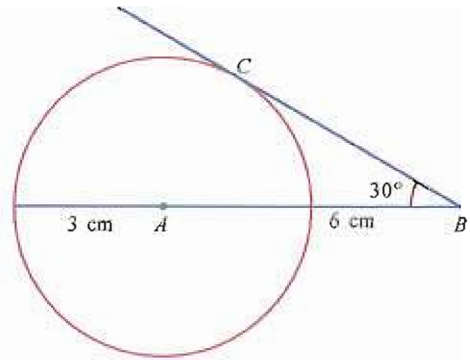
How long should we make the sides to get the triangle outside?

Thus we get two triangles with the given specifications:

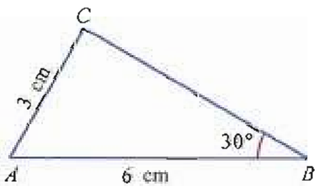




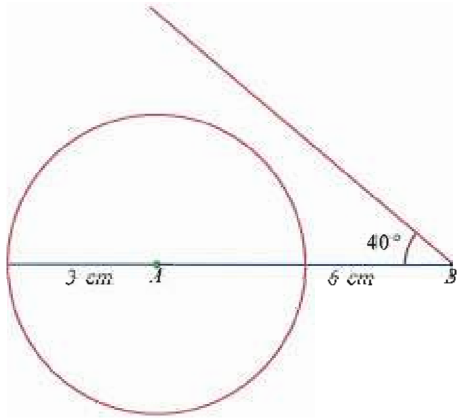
Now suppose we want the angle at B to be 30°.



We get only one triangle:



And if we make it 40°?

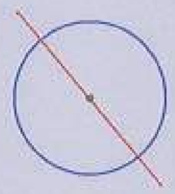


We see the 20° line cutting the circle at two points; the 40° line has nothing to do with the circle.

What about the 30° line? It just touches the circle. Such a line is called a *tangent* to the circle.

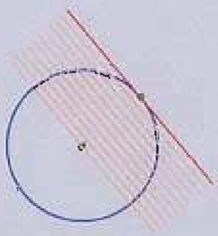
Sliding line

See this picture:

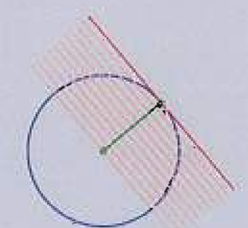


A circle and a line through its centre. Suppose we slide the line up a bit:

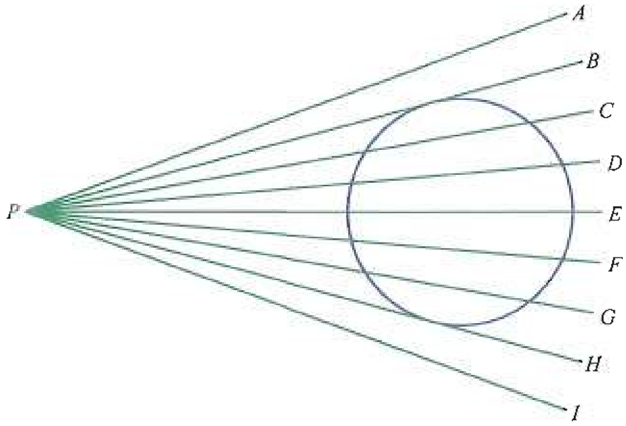
As we go on sliding the line slowly, we get a line which passes through a single point of the circle, right?



And the line joining the centre and this final point is perpendicular to all these lines:



Now look at this picture:

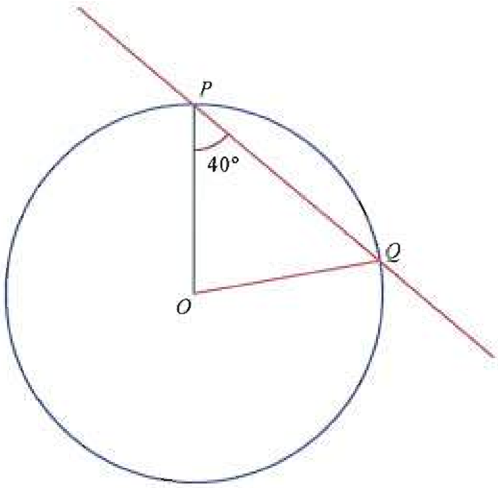


Among all the lines, only two are tangents to the circle. Which are they?

Now take a look at the triangles we drew earlier. In the case of two triangles, the top angle of one of these is greater than a right angle and in the other, it is less than a right angle. Is there any relation between these two angles? See how we got the top vertices of these triangles.

What is the top angle, in the case of a single triangle?

Let's draw another picture.



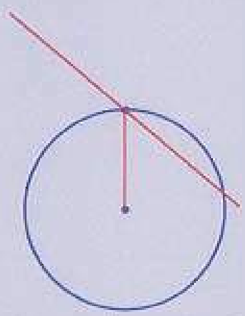
How much is $\angle OQP$ in this?

Draw some more pictures like this with the angle at P increased to 50°, 60° and so on, by shifting the position of Q. What do you see?

As the angle at P increases, the point Q gets closer to P; and ΔPOQ becomes thinner.

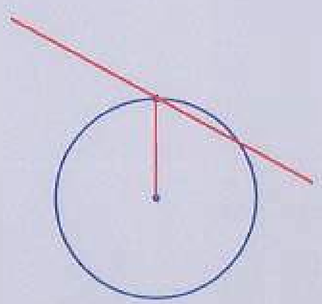
Rotating line

Look at this picture:

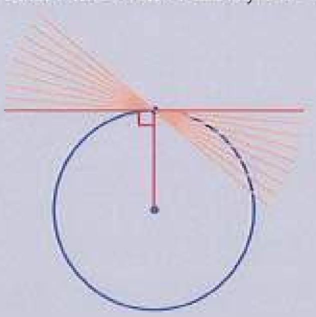


A circle, a radius and a slanted line through its end.

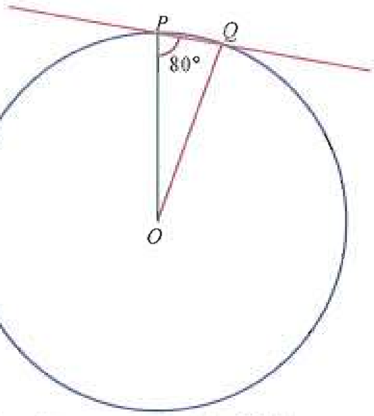
What if we rotate this line a bit, about the top point?



As we go on rotating slowly, we get to a stage when the line is perpendicular to the radius, don't we?



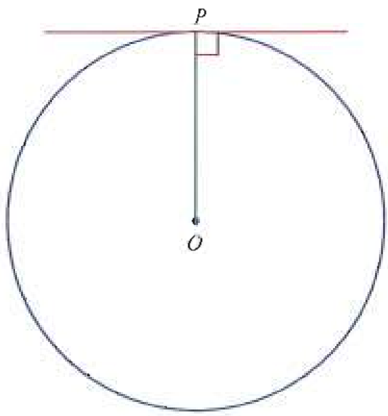
This line goes through how many points of the circle?



What happens when the angle at P is 90°?

Would this line meet the circle at any other point? If so, the angle at Q would also have to be 90°. How can there be two right angles in a triangle?

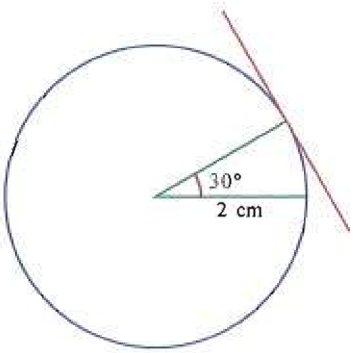
So, there is no other point common to this line and the circle; that is, it is a tangent to the circle.



What general principle do we get from this?

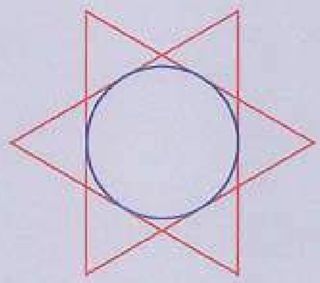
A line drawn through any point of a circle, perpendicular to the radius through that point, is a tangent to the circle.

Now try drawing the pictures below according to the given specifications.

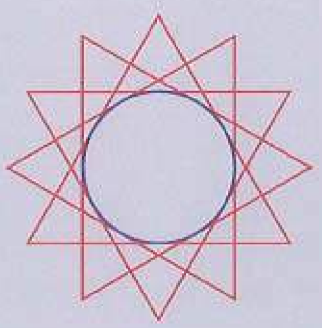


Circle from lines

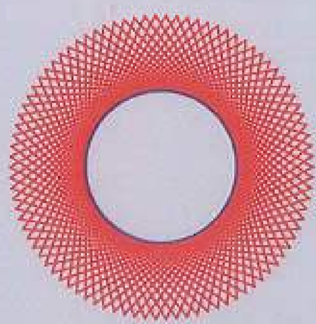
See this picture of a star made by six tangents to a circle:

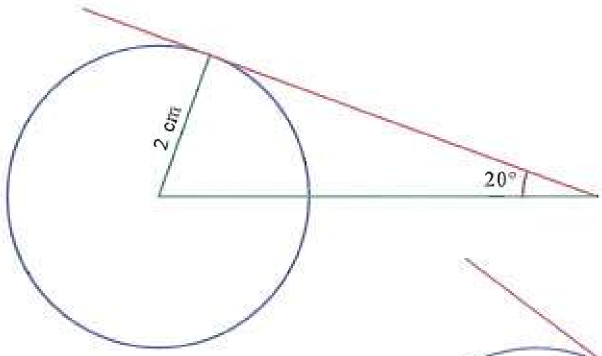


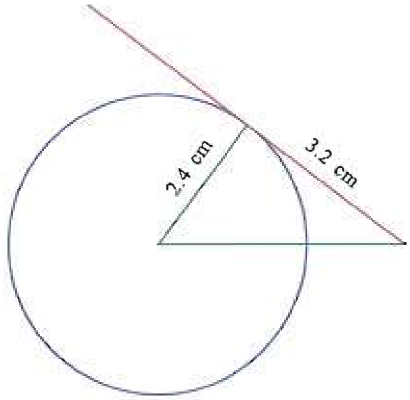
We can increase the number of tangents to 12:



And this is a picture drawn by a computer, using 90 tangents:







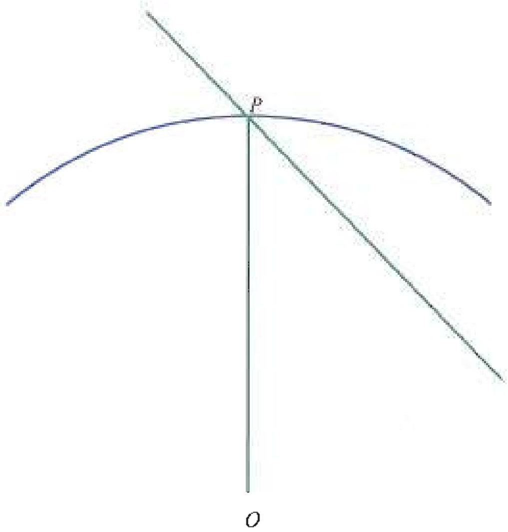
Draw a diameter AB in a circle. Prove that the tangets at A and B do not intersect.

Theory and application

We saw that we can draw tangents by drawing perpendiculars to radius. Are all tangents like this? In other words, is every tangent perpendicular to the radius through the point of contact?

To answer this, first draw a circle and a radius, and then draw a line through the end of the radius, not perpendicular to the radius. You can see that it cuts the circle at another point. What is the position of this second point? Can you specify it without seeing the whole circle?

See this picture:

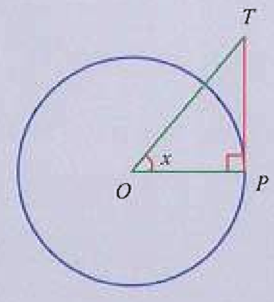


Name and meaning

The word tangent comes from the Latin root *tangere*, meaning to touch.

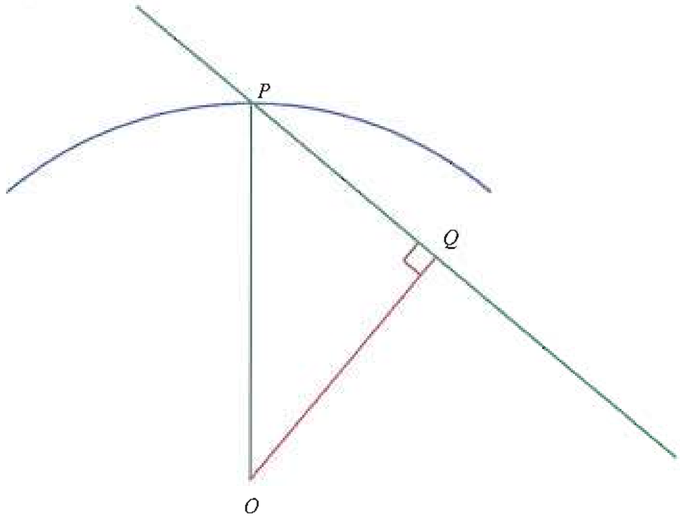
The tan measure used in trigonometry is also an abbreviation of the word tangent, right? What is the connection between this measure of an angle and a line touching a circle?

See this picture:

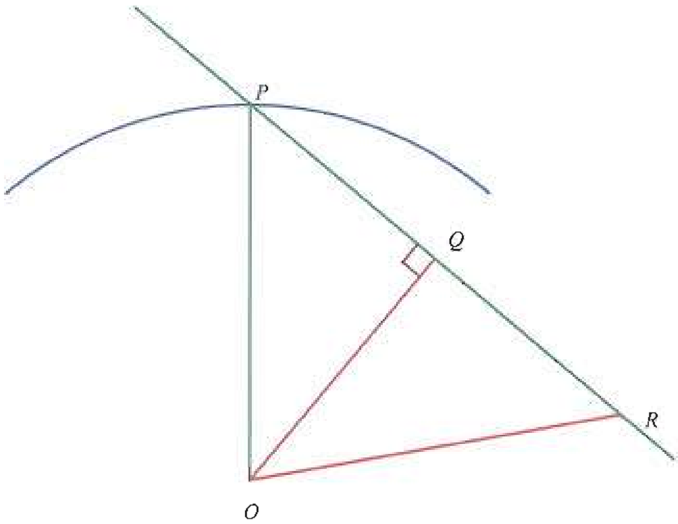


If we take the radius of the circle as 1, then the length of the tangent PT is indeed $\tan x$, isn't it?

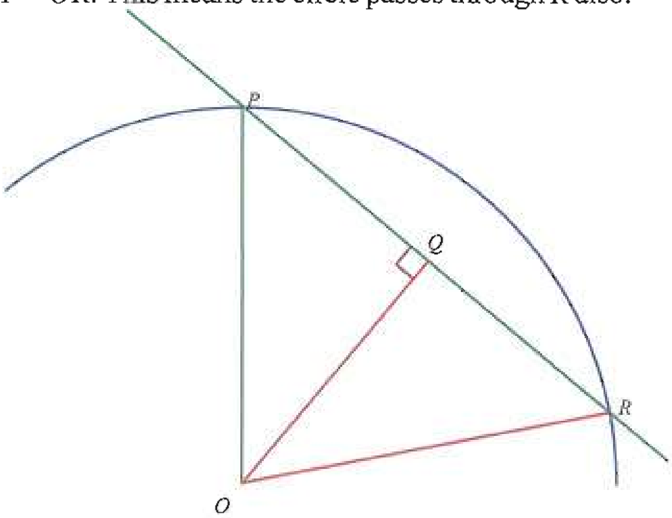
The line through P is not perpendicular to the radius OP; so we can draw a perpendicular to this line from O.



Now mark R ahead of Q at the same distance from P.

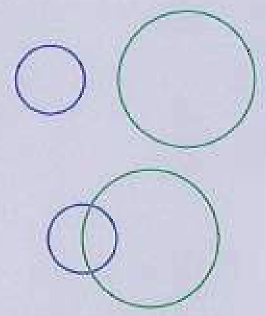


Now the triangles OPQ and ORQ are congruent. (Why?) So, $OP = OR$. This means the circle passes through R also.

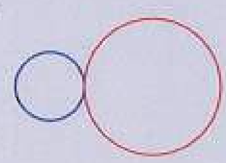


Touching circles

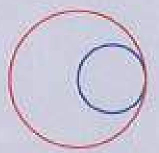
Like a circle and a line, two circles may not intersect, or intersect at two points:



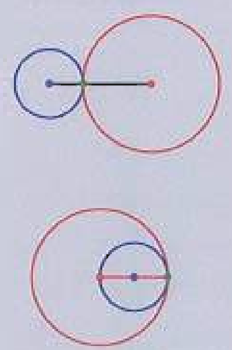
It may happen that two circles only touch:



Instead of tocuhing *externally* as in the picture above, two circles may touch *internally* like this:

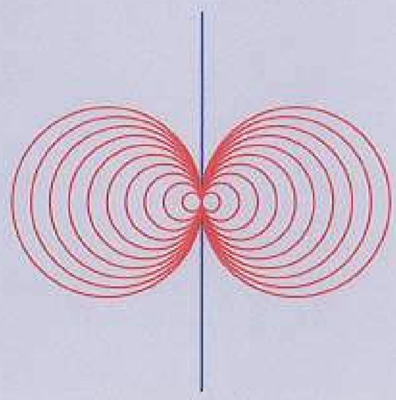


Euclid has proved that, however they touch, the point of contact and the centres of the circles are on the same line:

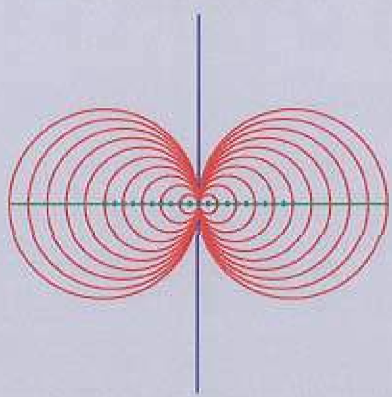


Bunch of circles

There's only one tangent at a specific point on a circle. But there are several circles touching a line at a specific point. See this picture:



These circles all touch one another. So, their centres are all on the same line, And the common tangent is perpendicular to this line.



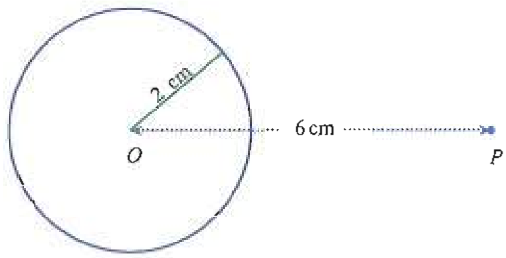
What did we see here? If a line through P is not perpendicular to the radius OP, then it cuts the circle at another point also; on the other hand, the tangent at P does not meet the circle at any other point. So, the tangent at P has to be perpendicular to the radius OP.

Let's write this as a general principle:

Any tangent to a circle is perpendicular to the radius through the point of contact.

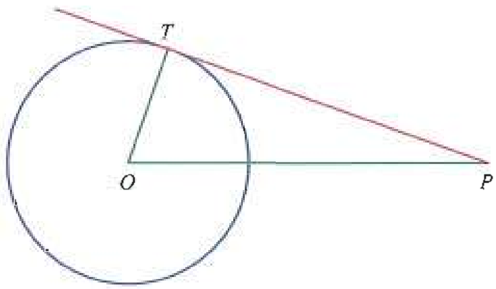
Let's look at an application of this.

Draw a circle of radius 2 centimetres and mark a point 6 centimetres away from its centre.



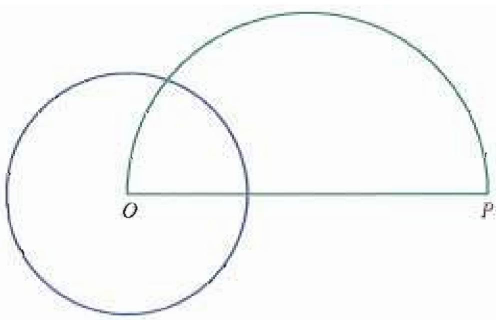
How do we draw a tangent to the circle, passing through this point?

First let's draw a rough sketch to see how we go about doing this:

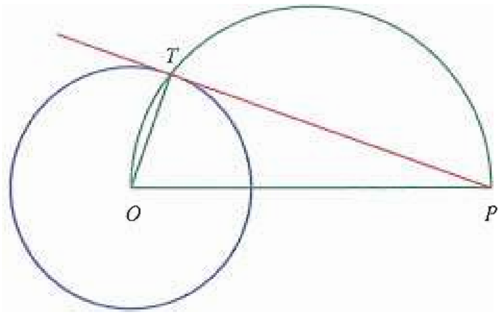


Since what we want is a tangent, the angle at the top should be a right angle. So, what we want is a right angled triangle with the bottom line as hypotenuse. But this can be done by drawing a semicircle, right? (Surely, you haven't forgotten what you have seen in the lesson on circles?)

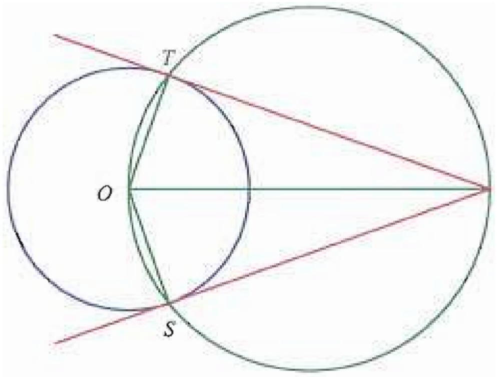
So, let's get down to the actual business of drawing.



Joining any point on this semicircle with O and P, we get a right angled triangle with OP as the hypotenuse. But in the right angled triangle we need, the third vertex should be a point on the small circle also. So, we take the point of intersection of this circle and the newly drawn semicircle.



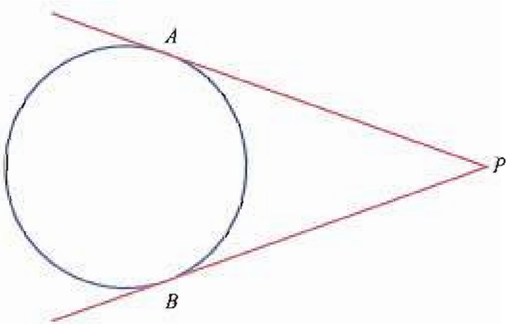
Our job is done, once we draw the line joining P and T. But we can think about another point—won't we also get a tangent by drawing a semicircle downwards?



So, from P, we can draw two tangents.

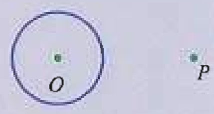
Not only this, but we can also see from the figure above that the lines PT and PS are equal. These we can call the *length of the tangents from P*. So, the lengths of the tangents to a circle from a point outside are equal. How do we prove this?

In the figure below, the lengths of the tangents from P to the circle are PA and PB.

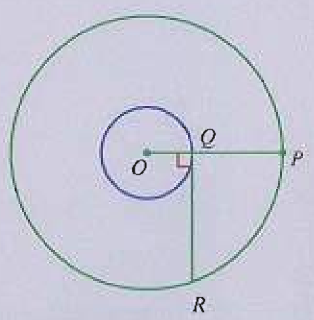


Another method

Euclid uses another method to draw tangents to a circle from an external point:

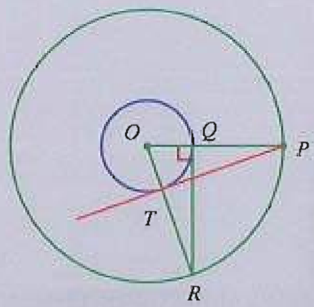


Join OP and draw another circle with this length as radius, centred at O. Draw the perpendicular to OP from the point where it cuts the original circle, and extend it to meet the second circle:



Join OR and join the point where it cuts the original circle with P.

This gives the tangent:



Can you prove it is so? Can you draw the other tangent from P likewise?

How to write a proof

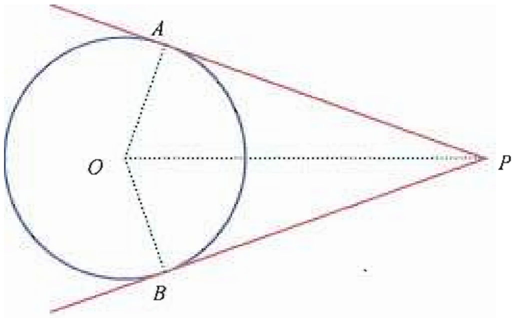
We have said much about the master geometer Euclid and his work named Elements. (See the section **Circles and triangles,** of the lesson **Math Drawing** in the Class 7 textbook.)

The method used in this work is to start from some basic assumptions, prove some simple facts using these, and go on to prove more and more complex theorems using these. (An online version of this work is available at http: //aleph0.clarku.edu/~djoyce/java/elements/elements.html)

This method is now used not only in geometry, but in all branches of mathematics. Even in other sciences, we can see this method being used more or less.

Whatever be the way we discover mathematical theorems, the current practice is writing proofs the Euclidean way, giving each conclusion concisely, each a logical consequence of the earlier one.

We want to prove that $PA = PB$. For this, join P, A, B to the center O of the circle.



The line AP is a tangent at the point A on the circle, and the line OA is the radius through A, so that $\angle OAP = 90°$.

Thus ΔOAP is a right angled triangle and so by Pythagoras Theorem,

$$PA = \sqrt{OP^2 - OA^2}$$

Likewise, since BP is the tangent to the circle at B and BO is the radius through B, we have $\angle OBP = 90°$ and so from the right angled triangle OBP, we get

$$PB = \sqrt{OP^2 - OB^2}$$

Now since OA and OB are radii of the circle, we have

$$OA = OB$$

From the three equations above, we get

$$PA = \sqrt{OP^2 - OA^2} = \sqrt{OP^2 - OB^2} = PB$$

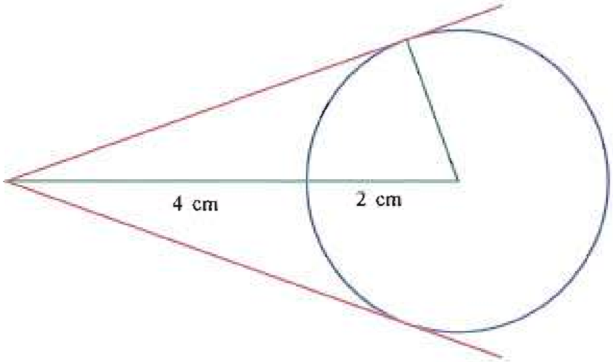
Let's write this as a general principle:

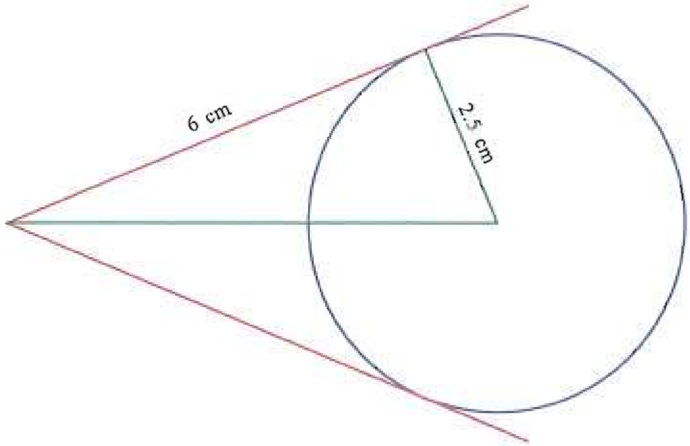
From any point outside a circle, we can draw two tangents; and the lengths of these tangents are equal.

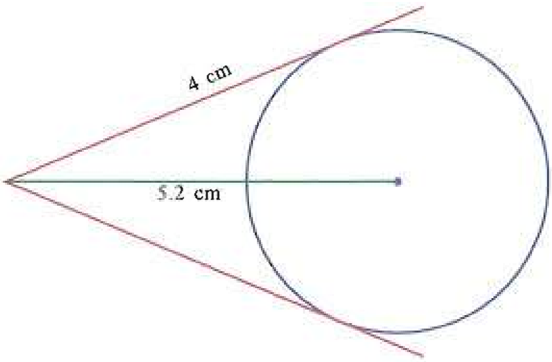
Now try your hand at these problems:

- Circles centred at A and B cut at P. Prove that if AP is a tangent to the circle centred at B, then BP is tangent to the circle centred at A.

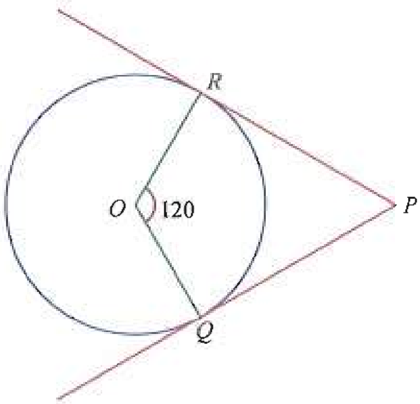
• Draw the figures below according to the specifications given.





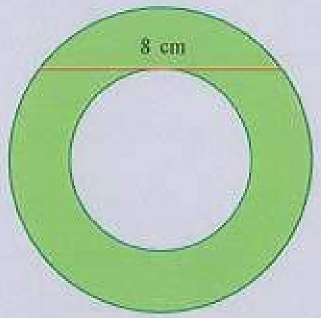


• In the picture below, the radius of the circle is 15 centimetres. Compute the lengths of the tangents PQ and PR



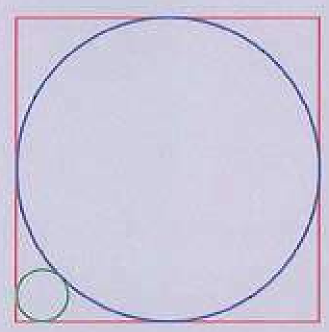
Area problem

What is the area of the green region in the figure below:



Corner problem

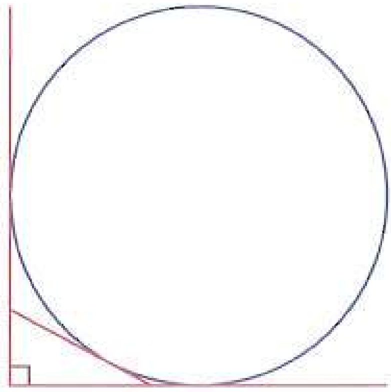
In the figure below, the large circle touches all four sides of the square and the small circle touches two sides of the square and the large circle:



What is the ratio of the radii of the two circles?

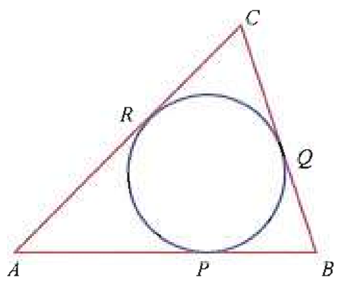
- In the circle centred at O, the tangents at A and B intersect at P. Prove the following:
  - the point P is equidistant from A and B
  - the line OP bisects the line AB and the angle APB
  - if the line OP cuts the line AB at Q, then $OQ \times OP = r^2$, where r is the radius of the circle

- The circle in the figure below touches all the three lines.



Prove that the perimeter of the right angled triangle is equal to the diameter of the circle.

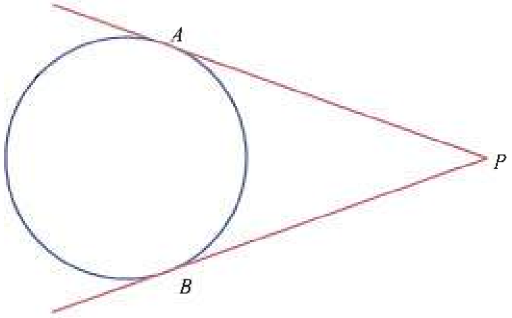
- In the figure below, the lines AB, BC, CA touch the circle at P, Q, R.



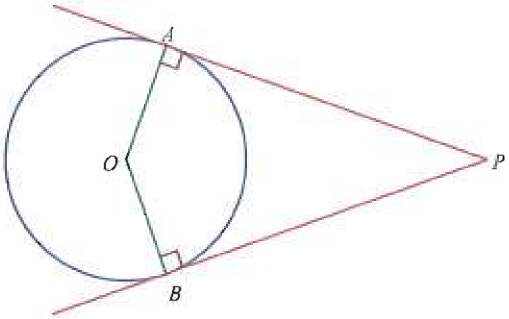
Prove that the perimeter of the triangle is $2(AP + BQ + CR)$.

Tangents and angles

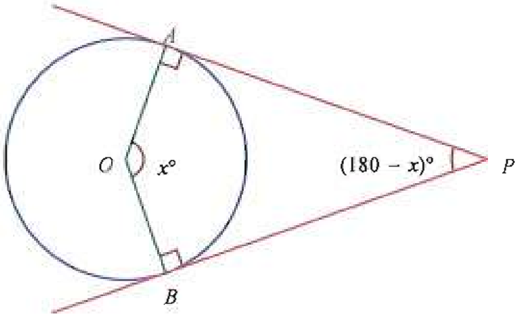
Suppose that the tangents to a circle at two points intersect at a point.



Look at the central angle of the small arc from A to B and the angle between the tangents at P.

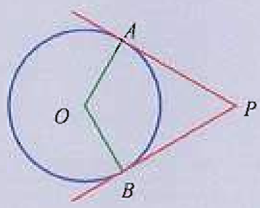


Two angles of the quadrilateral $OAPB$ are right and so their sum is 180°; this means the sum of the other two angles is also 180°. (What is the sum of all four angles of a quadrilateral?)

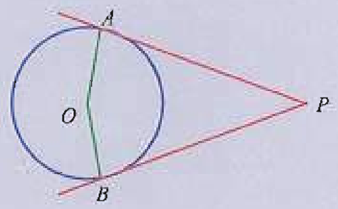


Near and afar

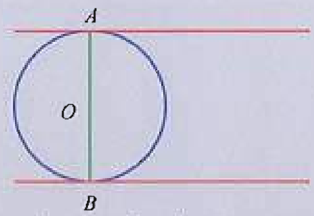
See this figure:



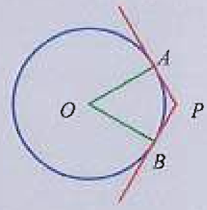
As $\angle AOB$ becomes larger, $\angle APB$ becomes smaller; moreover, P moves further away from O:



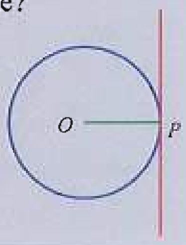
Finally what happens when AB becomes a diameter?



On the other hand, what happens as $\angle AOB$ becomes smaller?



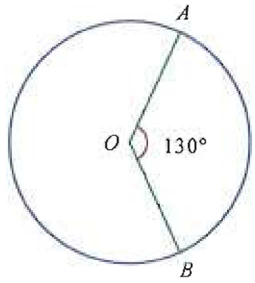
And finally, what happen when A and B coincide?

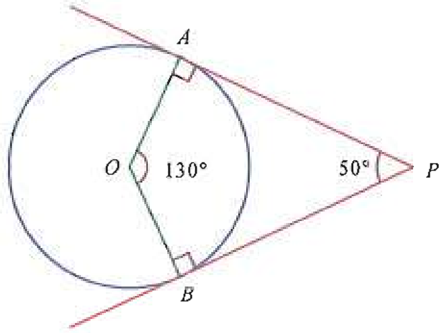


What do we see here?

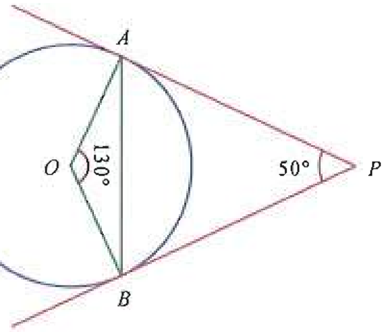
The central angle of the smaller arc between two points on a circle and the angle between the tangents at these points are supplementary.

For example, if we are asked to draw two tangents to a circle with the angle between them equal to 50°, we need only draw tangents at the ends of an arc of central angle 130°.





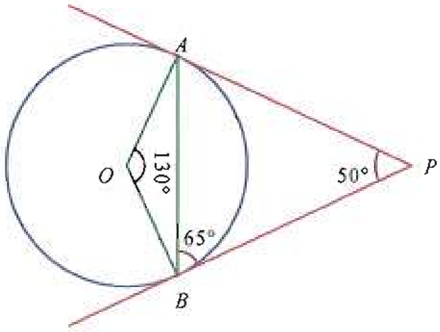
Suppose in this picture, we draw the chord AB also.



What is the angle between this chord and the tangents?

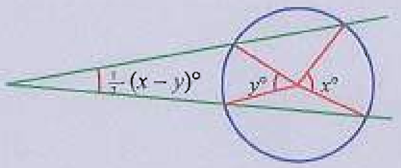
In the isosceles triangle OAB, the smaller angles are 25° each.

So, $\angle ABP = 90° - 25° = 65°$.

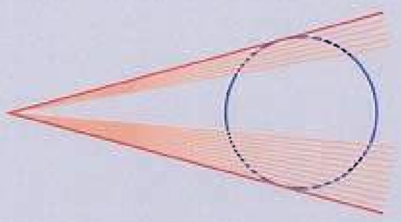


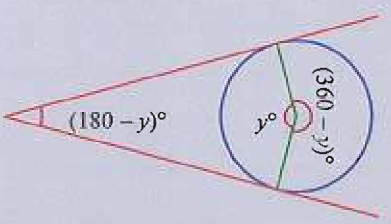
Chord and tangent

We have discussed the angle between two chords intersecting outside a circle in the section **Outside a circle,** of the lesson **Circles**:



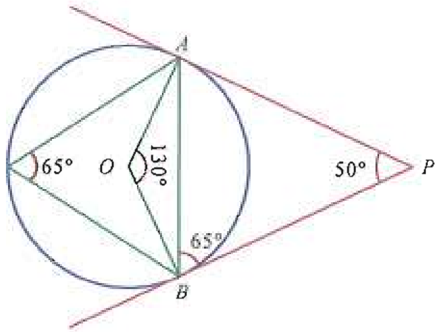
What happens if these chords rotate about O and become tangents?





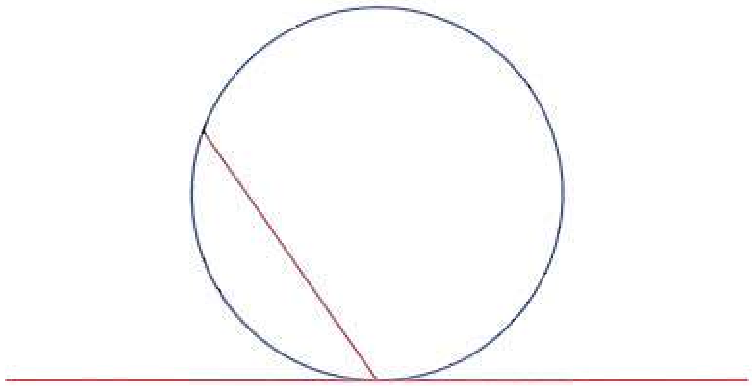
That is, half of 130°. But this is the angle in the larger segment cut off by the chord AB.

See this picture:

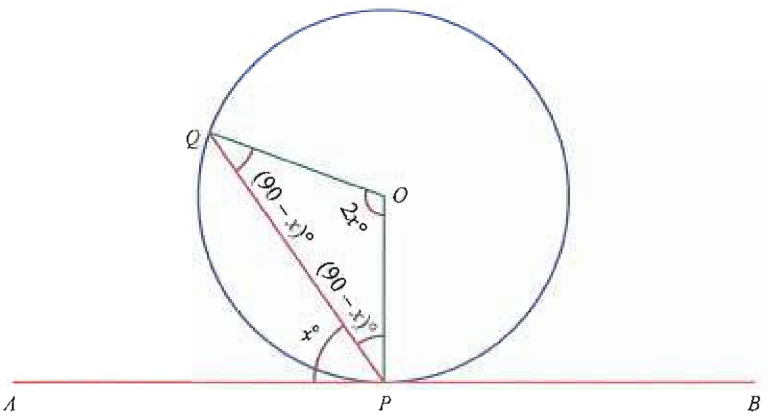


Is this true for all tangents and chords?

Let's draw a chord and the tangent at one of its ends.



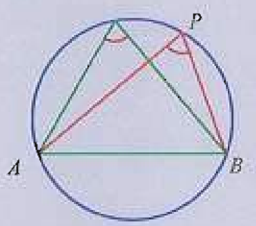
If we take one angle between the chord and the tangent as $x°$, then we can see from the figure below that the central angle of the smaller arc is $2x$. (Can you explain?)



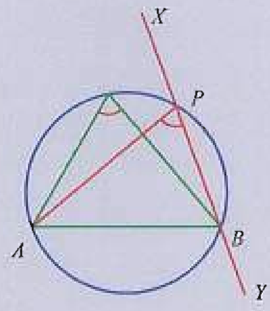
So, the angle made by the chord PQ in the larger segment is also $x°$.

Unchanging angle

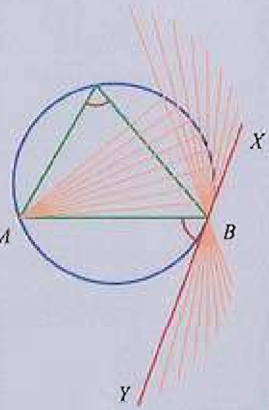
We have seen that angles in the same segment are equal:



Let's extend the line PB.



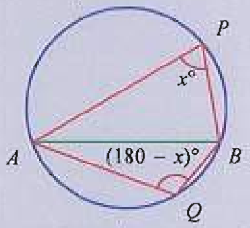
Now what happens as P moves along the circle to B?



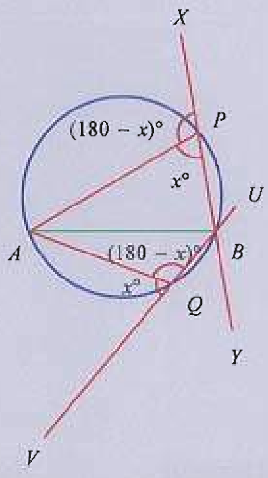
The line XY becomes the tangent at B; and the angle doesn't change.

Flip-flop angles

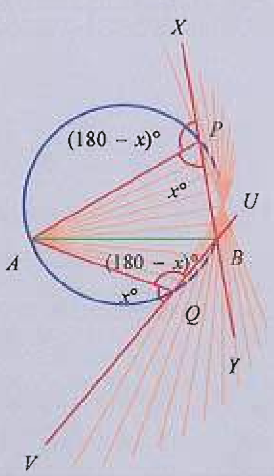
We have seen that angles in opposite segments of a circle are supplementary:



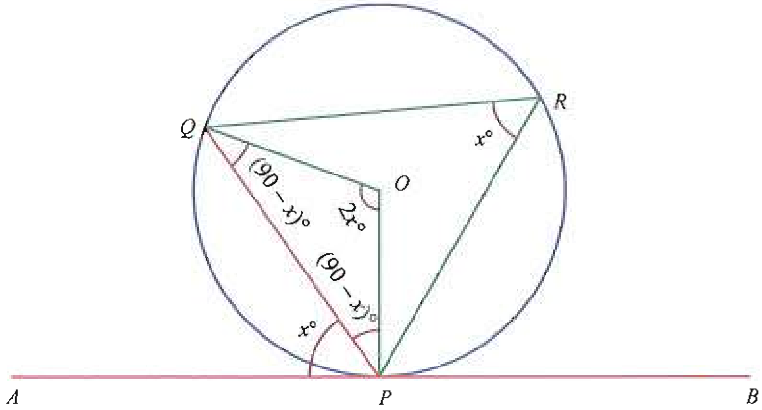
Let's extend the lines as before:



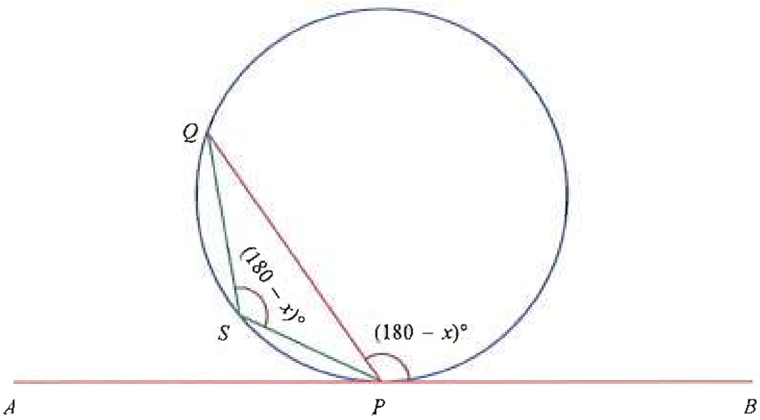
Suppose P moves along the circle to Q:



The angle below AP is $x°$ and the angle above AP is $(180 - x)°$. And this is so throughout the motion.



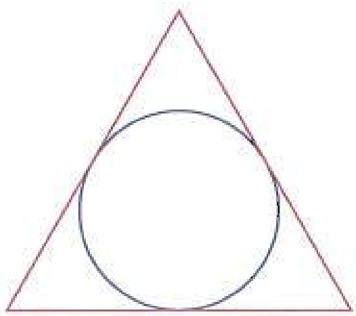
Not only this, but we can also see that the larger angle between the chord and the tangent and the angle in the smaller segment cut off by the chord are both equal to $(180 - x)°$.



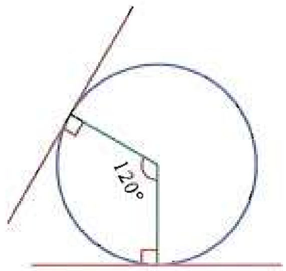
Let's write what we have seen now also as a general principle:

Each angle between a chord and the tangent at one of its ends in a circle is equal to the angle in the segment on the other side of the chord.

Using what we have seen about the angle between the tangents, we can solve our first problem about the equilateral triangle covering a circle.



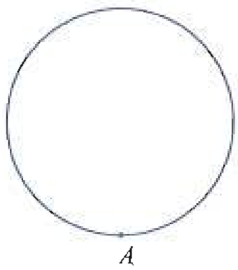
Here, the sides of the triangle are tangents to the circle, right? What is the angle between two of them? So, then what is the central angle of the arc between them?



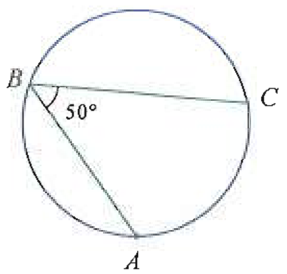
Now can't we draw the triangle? Can we draw nonequilateral triangles touching the circle like this? Try!

In this, we used the centre of the circle to draw the triangle. Can we do it without using the centre? (Suppose that we have a circle in which the centre is not marked.)

First let's see how we can draw a tangent without using the centre.

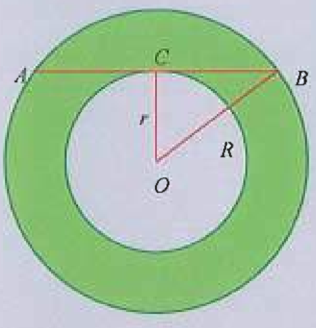


In the circle shown above, we want to draw the tangent at the point A. First draw two lines as shown below.



Now join AC and draw the line PQ through A, making an angle of 50° with AC.

Area problem—solution

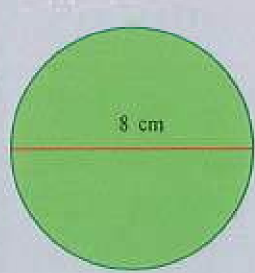


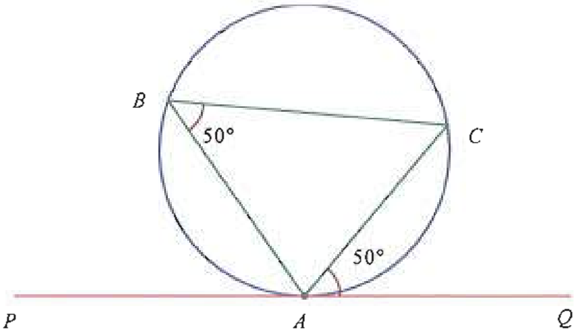
If we take the radius of the small circle as r and the radius of the large circle as R, then the area of the green region is $\pi(R^2 - r^2)$, right?

In the picture, AB is a tangent to the small circle and so it is perpendicular to the radius OC. Since AB is also a chord of the large circle, we also get $AC = BC$ (how?) So, from the right angled triangle OCB, we get $R^2 - r^2 = 4^2 = 16$.

Thus the area of the green region is seen to be 16π square centimetres.

That this area is also equal to the area of the circle with diameter AB, is another point of interest:



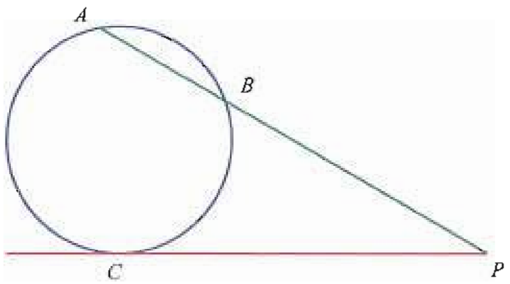


This is the tangent at A, isn't it?

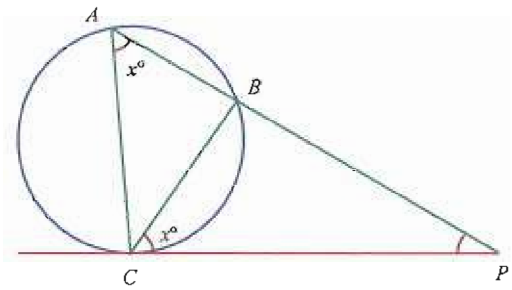
Can't we take any angle instead of 50° here?

Using the theorem about the angle between chord and tangent, we can form another general principle:

See this picture:



Join AC and BC. Taking $\angle BCP = x°$, we also get $\angle BAC = x°$.



That is, the angle at A in ΔAPC and the angle at C of ΔBPC are equal. Also, the angle at P is the same for both triangles. So, their third angles must also be equal. This means the pairs of sides opposite equal angles must have the same ratio. By a proper choice of such pairs, we get

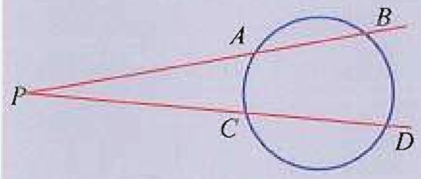
$$\frac{PA}{PC} = \frac{PC}{PB}$$

We can rewrite this as

$$PA \times PB = PC^2$$

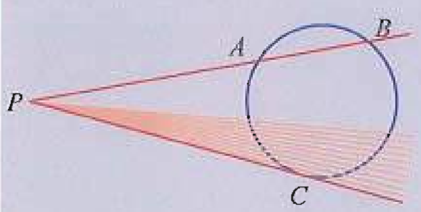
Unchanging relation

See this picture:



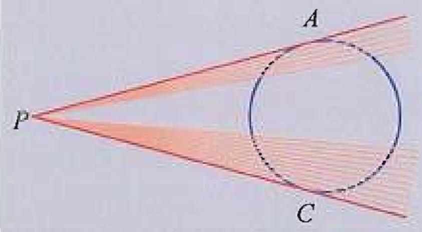
We know that $AP \times PB = CP \times PD$.

What if the bottom line turns to become the tangent at C?



PD and PC become the same; and the relation between lengths becomes $AP \times PB = CP^2$

Suppose now the top line also turns to become the tangent at A:

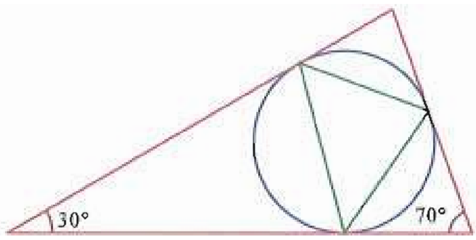


The relation becomes, $PA^2 = PC^2$; that is $PA = PC$.

We have already seen that the lengths of the tangents from a point are equal, haven't we?

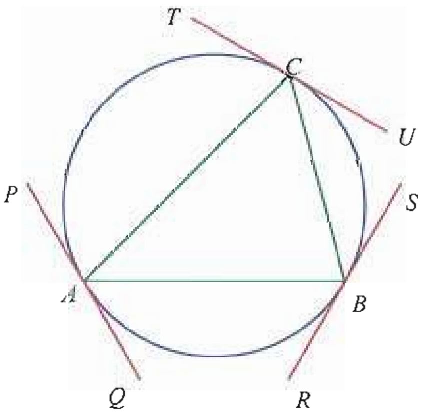
Now try these problems on your own:

- Draw a circle of radius 3 centimetres and draw a rhombus with one angle 40°, all four sides touching the circle.
- Draw a circle of radius 4 centimetres and draw a regular pentagon with all its sides touching the circle.
- Prove that in any circle, the tangents at two points make equal angles with the chord joining the points of contact.
- In the figure below, all the vertices of the small triangle are on the circle and all the sides of the larger triangle touch the circle at these points.



Find all angles of the small triangle.

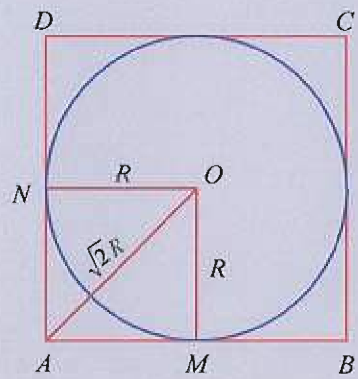
- In the picture below, PQ, RS, TU are tangents to the circle at A, B, C.



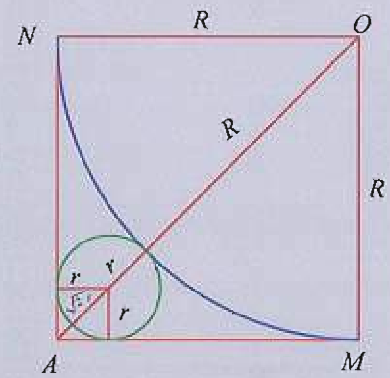
How many pairs of equal angles are there in it?

Corner problem—solution

Let's take the radius of the large circle as R. Drawing perpendiculars from its centre to the sides of the square gives the figure below:



We can do this with the small circle also. Let's take its radius as r. To see things clearly, we show below an enlarged view of a portion of the figure:



If we compute the length of OA from the two pictures, we get

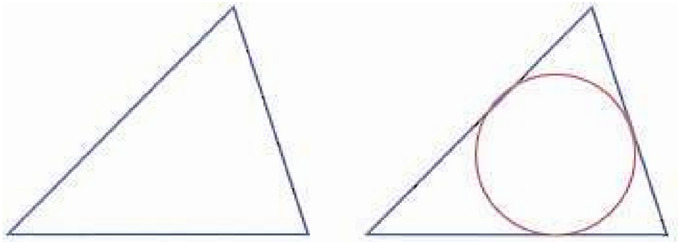
$$\sqrt{2}R = \sqrt{2}r + r + R$$

From this we get

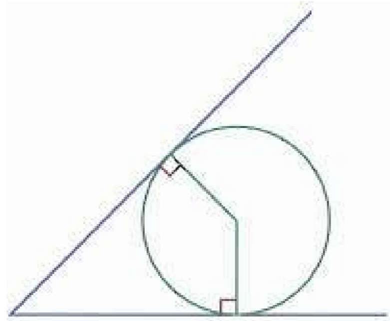
$$\frac{r}{R} = \frac{\sqrt{2}-1}{\sqrt{2}+1}$$

Inner circle

We have seen how we can draw a triangle with its sides touching a circle. Now let's see how we can draw a circle touching the sides of a triangle.

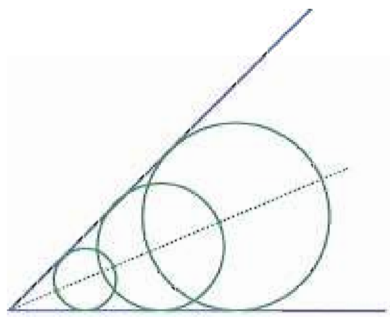


The circle should touch all three sides of the triangle. We can draw several circles touching a single side, what about circles touching a pair of sides?



In the picture, the radii are perpendicular to the pair of sides . In other words, the centre of the circle must be equidistant from these sides. So, it must be on the bisector of the angle between these sides (the section, **Equidistant bisector** of the lesson **Congruent triangles** in the Class 8 textbook)

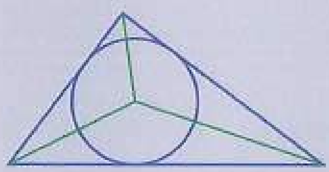
We can draw a circle touching the two sides, centred at any point on the angle bisector.



The circle we seek must touch the third side also. What do we do to get it?

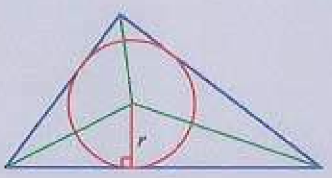
Radius of the incircle

We can compute the radius of the incircle of a triangle from the lengths of its sides. See this picture:

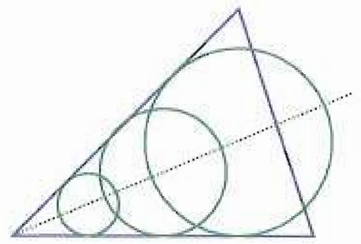


By joining the incentre with the three vertices of the triangle, we can divide the triangle into three smaller ones. The area of our triangle is the sum of the areas of these small triangles.

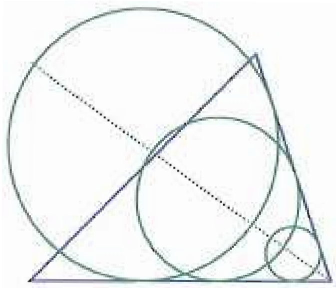
Now look at this picture:



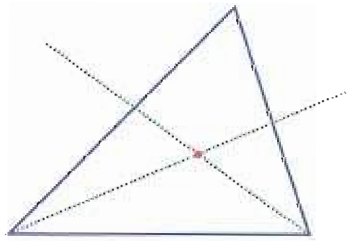
If we denote the inradius by r and the length of the bottom side of the triangle by a, the area of the small triangle at the bottom is $\frac{1}{2}ar$. Similarly, if the lengths of the other two sides are denoted b and c, the areas of the other two small triangles can be found as $\frac{1}{2}br$ and $\frac{1}{2}cr$. So, the area of our original triangle is $\frac{1}{2}(a + b + c)r$. So, the area of the triangle divided by half the perimeter gives the inradius.



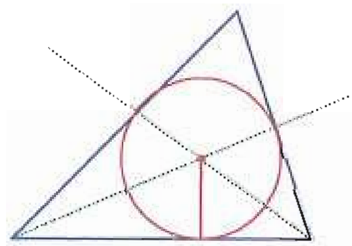
By taking points on the bisector of the angle between the bottom and right sides of the triangle as centres, we can draw circles touching these two sides.



So, how about taking the point on both these two angle bisectors? That is, the point of intersection of these bisectors.



From this point, the distances to all three sides are equal, right? So, what about the circle with this length as radius, centred on this point?

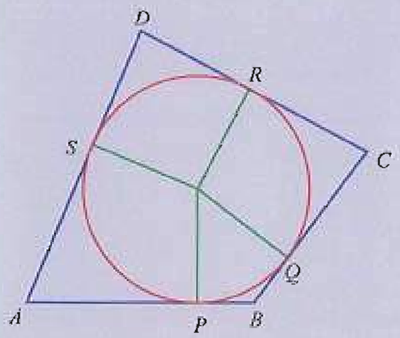


This circle is called the *incircle* of the triangle.

We can note another fact here. Since the incircle touches the left and right sides of the triangle, the perpendicular distances from these sides to the centre of the circle are equal; this means the incentre is the bisector of the angle between these sides also.

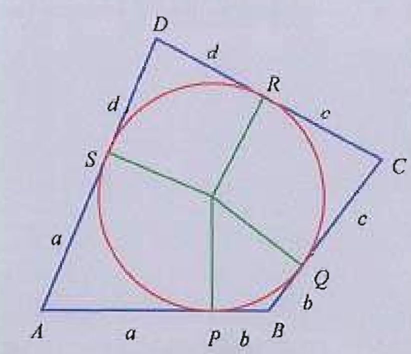
Quadrilateral and circle

See this picture:



The quadrilateral $ABCD$ has an incircle. P, Q, R, S are the feet of the perpendiculars from its centre to the sides of the quadrilateral.

Using the fact that tangents intersect at a point equidistant from the points of contact, we can mark lengths as below:



From this, we can see that

$$AB + CD = a + b + c + d = AD + BC$$

That is, if a quadrilateral has an incircle, then the sums of its opposite sides are equal.

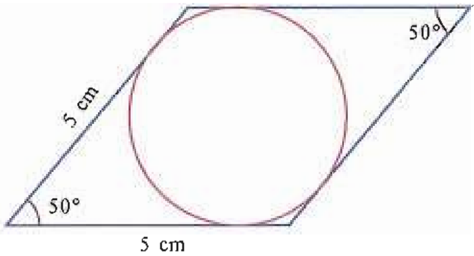
On the other hand, we can prove that any quadrilateral with the sums of the opposite sides equal, has an incircle (Try it!)

And this is true for any triangle, isn't it?

In any triangle, the angle bisectors meet at a point.

Now some problems for you:

- Draw a triangle of sides 4, 5, 6 centimetres and draw its incircle.
- Draw an equilateral triangle of sides 6 centimetres and draw its incircle and circumcircle.
- Prove that in an equilateral triangle, the circumcentre and incentre are the same. What is the ratio of the circumradius and inradius?
- Draw a square of sides 5 centimetres and draw its circumcircle and incircle.
- Draw the figure below according to the given specifications:

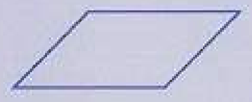


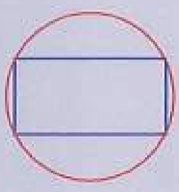
Project

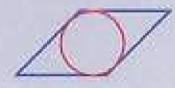
- Find different methods of drawing line segments of lengths $\sqrt{2}, \sqrt{3}, \sqrt{5}$ using the following ideas:
    - Pythagoras Theorem
    - Theorems on chords of circles
    - Theorems on tangents to circles

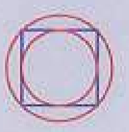
Circumcircle and incircle

We can draw a circumcircle and an incircle for any triangle. But when we come to quadrilaterals, some may have neither, some may have one and not the other, and some may have both:

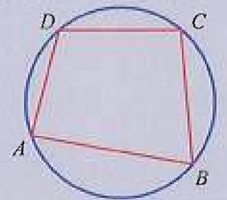






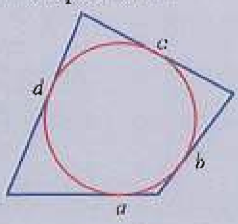


We have seen that in a quadrilateral which has a circumcircle, the sum of the opposite angles is 180°. In other words, both pairs of opposite angles have the same sum:



$$\angle A + \angle C = \angle B + \angle D$$

What about quadrilaterals which have incircles? Both pairs of opposite sides have equal sum:



$$a + c = b + d$$

9 Polynomials

New equations

Is 7 a factor of 315?

We'll have to divide and see.

$$315 \div 7 = 45$$

So, 7 is a factor of 315.

From the division above, we get

$$315 = 45 \times 7$$

Is 7 a factor of 316?

Division gives the remainder 1; so it is not a factor. We can write

$$316 = (45 \times 7) + 1$$

Now what about dividing the polynomial $x^2 - 1$ by the polynomial $x - 1$?

We know that

$$x^2 - 1 = (x - 1)(x + 1)$$

This means, we can divide $x^2 - 1$ by $x - 1$ without leaving a remainder. In other words

$$\frac{x^2 - 1}{x - 1} = x + 1$$

So, we can say that the polynomial $x - 1$ is a factor of the polynomial $x^2 - 1$.

Similarly $x + 1$ is a factor of $x^2 - 1$.

Now let's see whether $x - 1$ is a factor of $x^2 + 1$.

We can write

$$x^2 + 1 = (x - 1)(x + 1) + 2$$

which means $x^2 + 1$ leaves the remainder 2 on division by $x - 1$.

So, $x - 1$ is not a factor of $x^2 + 1$.

Meaning of factor

The idea of factor which we have seen for natural numbers, can be extended to all integers. For example, since $-12 = 3 \times (-4)$, we can say that -4 is a factor of -12.

What about rational numbers? If we take any two non-zero rational numbers, we can multiply one of these by a suitable rational number to get the other. For example, taking $\frac{2}{3}$ and $\frac{5}{7}$, we can write $\frac{2}{3} = \frac{14}{15} \times \frac{5}{7}$.

(What if one of the numbers is zero?)

So, if we consider the collection of all rational numbers, the idea of factor is not of much use.

In the case of polynomials also, we talk about factors only with respect to the collection of polynomials and not in terms of all algebraic expressions. We have

$$x^2 + 1 = (x - 1)\left(x + 1 + \frac{2}{x - 1}\right)$$

But we don't consider $x - 1$ a factor of $x^2 + 1$ because of this.

Polynomials and numbers

In formulating general principles on polynomials, we often have to include numbers also. For example, the sum of two polynomials may be a number as in

$$(x^2+x+1)+(-x^2-x+1)=2$$

Again, the quotient of a polynomial by another may be a number, like

$$\frac{2x+4}{x+2}=2$$

It is inconvenient to say "polynomial or number" everytime. So, we consider numbers also as polynomials. (We can write $2 = 2x^0$, right?).

All non-zero numbers are said to be zero-degree polynomials. The number 0 itself is treated as a polynomial without degree. This is because, any polynomial multiplied by 0 gives 0 itself so that the general rule, "the degree of a product is the sum of the degrees of the factors" will not hold, whatever number we take as the degree of the zero polynomial.

Now how do we check whether $x - 1$ is a factor of $x^3 - 1$?

We have to divide and see whether there is a remainder. Since we are dividing by the first degree polynomial $x - 1$, the remainder must be a number. What about the quotient?

As we did in Class 9 let us write

$$x^3 - 1 = (x - 1)(ax^2 + bx + c) + d$$

and find a, b, c, d.

How do we do the multiplication on the right of the above equation? Multiply each term of the first polynomial by each term of the second polynomial and add, right? Thus we get

$$x^3 - 1 = ax^3 + (b - a)x^2 + (c - b)x + (d - c)$$

For this to hold, we need only take

$$a = 1$$
$$b - a = 0$$
$$c - b = 0$$
$$d - c = -1$$

That is,

$$a = 1$$

$$b = 1$$

$$c = 1$$

$$d = 0$$

From this we get

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

Since there is no remainder, we can see that $x - 1$ is indeed a factor of $x^3 - 1$.

But then another question arises: is the polynomial $2x - 2$ a factor of $x^3 - 1$?

We can write

$$2x - 2 = 2(x - 1)$$

which gives

$$x - 1 = \frac{1}{2}(2x - 2)$$

So, we can rewrite the equation $x^3 - 1 = (x - 1)(x^2 + x + 1)$ as

$$x^3 - 1 = \frac{1}{2}(2x - 2)(x^2 + x + 1)$$
$$= (2x - 2)\frac{1}{2}(x^2 + x + 1)$$
$$= (2x - 2)\left(\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2}\right)$$

What can we say then?

The polynomial $2x - 2$ is also a factor of $x^3 - 1$.

So, what about $3x - 3$?

And $\frac{2}{3}x - \frac{2}{3}$?

What about $1 - x$?

Now in each of the pairs of polynomials given below, check whether the first is a factor of the second:

- $x + 1, x^3 - 1$
- $x - 1, x^3 + 1$
- $x + 1, x^3 + 1$
- $x^2 - 1, x^4 - 1$
- $x - 1, x^4 - 1$
- $x + 1, x^4 - 1$
- $x - 2, x^2 - 5x + 1$
- $x + 2, x^2 + 5x + 6$
- $\frac{1}{3}x - \frac{2}{3}, x^2 - 5x + 6$
- $1.3x - 2.6,\ x^2 - 5x + 6$

First degree factors

How do we check whether the polynomial $x - 2$ is a factor of $4x^3 - 3x^2 + x - 1$?

Let's divide and see whether the remainder is zero or not. Since the quotient would be a second degree polynomial and the remainder a number, we write

$$4x^3 - 3x^2 + x - 1 = (x - 2)(ax^2 + bx + c) + d$$

and find a, b, c, d.

Polynomial factors

When we consider numbers also as polynomials, every non-zero number is a factor of every polynomial.

For example,

$$x^2 - 2x + 3 = 2\left(\frac{1}{2}x^2 - x + \frac{3}{2}\right)$$

$$2x^3 + 5x + 7 = \frac{1}{5}\left(10x^3 + 25x + 35\right)$$

and so on.

Moreover, we can multiply any factor of a polynomial by numbers to get new factors. In general, if the polynomial $p(x)$ is a factor of the polynomial $q(x)$, then for any non-zero number a, the polynomial $ap(x)$ is also a factor of $q(x)$.

Meaningful math

We have noted that the various kinds of numbers, natural numbers, rational numbers, irrational numbers, were created to indicate various types of measures; and that the very instances where these numbers are used, determine the way these numbers are operated upon.

In trying to divide 14 sweets equally among 3 children, we end up with 2 sweets which cannot be given whole; and in trying to cut a 14 metre long string into 3 metre pieces, we get a 2 metre piece which is not of the required length. These are some of the instances leading to the mathematical idea that 14 divided by 3 leaves the remainder 2.

Now consider this question:

What is the remainder on dividing −14 by −3?

In view of the above remarks, what is the meaning of this question?

For checking whether $x-2$ is a factor, do we need to compute all these? Isn't it enough to find just the remainder?

How do we get d alone from the right side of the above equation? We must make the other terms zero. We know that this equation holds, whatever be the number we take as x.

For example, taking $x = 1$, this equation gives,

$$(4 \times 1^3) - (3 \times 1^2) + 1 - 1 = (1 - 2)(a \times 1^2 + b \times 1 + c) + d$$

and reading this in reverse, we get

$$-(a + b + c) + d = 1$$

What if we take $x = 2$?

$$(4 \times 2^3) - (3 \times 2^2) + 2 - 1 = (2 - 2)((a \times 2^2) + (b \times 2) + c) + d$$

which means

$$0 \times (4a + 2b + c) + d = 21$$

That is, $d = 21$

What does this mean? On dividing the poynomial $4x^3 - 3x^2 + x - 1$ by $x - 2$, we get the remainder 21; and so $x - 2$ is not a factor of $4x^3 - 3x^2 + x - 1$.

Let's now check, whether the first degree polynomial $x - 3$ is a factor of $2x^3 - 5x^2 - 4x + 3$.

Since we are not interested in the quotient, let's write it in short as simply $q(x)$; and the remainder as r. Thus

$$2x^3 - 5x^2 - 4x + 3 = (x - 3)\, q(x) + r$$

Now we need only check whether r is zero. To get r, all we need to do is to take $x = 3$ in the above equation, right? This gives

$$(2 \times 3^3) - (5 \times 3^2) - (4 \times 3) + 3 = (3 - 3)\, q(3) + r$$

From this, we get

$$0 \times q(3) + r = 54 - 45 - 12 + 3$$

That is, $r = 0$

What does this mean?

The polynomial $x - 3$ is a factor of $2x^3 - 5x^2 - 4x + 3$.

Let's see how we can write this as a general principle. We want to check whether the first degree polynomial $x - a$ is a factor of the polynomial $p(x)$.

We write the quotient polynomial on dividing $p(x)$ by $x - a$ as $q(x)$ and the remainder as r. Then we get the identity

$$p(x) = (x - a)q(x) + r$$

This holds for all numbers x. In particular, if we take $x = a$, then this gives.

$$p(a) = (a - a)q(a) + r$$

and this means

$$p(a) = r$$

Thus we have this general result:

The remainder on dividing the polynomial $p(x)$ by the polynomial $x - a$ is $p(a)$.

Now what happens if $p(a) = 0$? This means the remainder on dividing $p(x)$ by $x - a$ is zero; that is, $x - a$ is a factor of $p(x)$. On the other hand, what if $p(a) \neq 0$? Since the remainder is not zero, $x - a$ is not a factor of $p(x)$.

For the polynomial $p(x)$, and for the number a, if we have $p(a) = 0$, then $x - a$ is a factor of $p(x)$; if $p(a) \neq 0$, then the polynomial $x - a$ is not a factor of $p(x)$.

The first principle is called the *Remainder Theorem* and the second is called the *Factor Theorem*.

Let's look at some examples:

- Is the polynomial $x - 2$ a factor of the polynomial $x^4 - x^3 - x^2 - x - 2$?

  The Factor Theorem says that to check this, we only need to put $x = 2$ in $x^4 - x^3 - x^2 - x - 2$ and check whether we get zero.

  $$2^4 - 2^3 - 2^2 - 2 - 2 = 16 - 8 - 4 - 2 - 2 = 0$$

  So, $x - 2$ is indeed a factor of $x^4 - x^3 - x^2 - x - 2$.

Meaning of remainder

To extend the idea of remainder to all integers, we must first interpret this idea *in purely mathematical terms* for natural numbers.

We say that when the natural number a is divided by the natural number b, the quotient is q and remainder is r, if q and r satisfy the following conditions:

1. $a = qb + r$
2. q and r are natural numbers or zero
3. $r < b$

We can extend this definition to all integers with some minor modifications:

We say that when the integer a is divided by the integer b, the quotient is q and remainder is r, if q and r satisfy the following conditions:

1. $a = qb + r$
2. q and r are integers
3. $r = 0$ or $0 < r < |b|$

For example, taking the numbers -14 and -3, we find

1. $-14 = 5 \times (-3) + 1$
2. 5 and 1 are integers
3. $0 < 1 < |-3|$

So, we say that on dividing -14 by -3, the quotient is 5 and the remainder is 1.

Polynomial division

Once we include numbers also as polynomials, we can extend to polynomials, the definition of quotient and remainder for integers to in much the same way.

We say that when the the polynomial $a(x)$ is divided by the polynomial $b(x)$, the quotient is $q(x)$ and remainder is $r(x)$, if $q(x)$ and $r(x)$ satisfy the following conditions:

1. $a(x) = q(x)b(x) + r(x)$
2. $q(x)$ and $r(x)$ are polynomials.
3. $r(x) = 0$ or $\deg r(x) < \deg b(x)$

In this, we denote by deg, the degree of a polynomial.

- Is $x + 3$ a factor of $2x^2 + 3x - 5$?

  The Factor Theorem talks about factors of the type $x - a$; but here what we want to check is $x + 3$.

  Can't we write $x + 3$ in this form also?

  $$x + 3 = x - (-3)$$

  So, we need only check whether $x = -3$ in $2x^2 + 3x - 5$ gives zero:

  $$(2 \times (-3)^2) + (3 \times (-3)) - 5 = 18 - 9 - 5 = 4$$

  Since we don't get zero, we find that the polynomial $x + 3$ is not a factor of $2x^2 + 3x - 5$.

- Is the polynomial $2x - 3$ a factor of the polynomial $2x^2 - x - 3$?

  How do we rewrite $2x - 3$ in a form suitable for the application of the Factor Theorem?

  $$2x - 3 = 2\left(x - \frac{3}{2}\right)$$

  Now we check whether the polynomial $x - \frac{3}{2}$ is a factor of $2x^2 - x - 3$ (Would that do?)

  For this, we take $x = \frac{3}{2}$ in $2x^2 - x - 3$ and find

  $$2 \times \left(\frac{3}{2}\right)^2 - \frac{3}{2} - 3 = \left(2 \times \frac{9}{4}\right) - \frac{3}{2} - 3 = \frac{9}{2} - \frac{3}{2} - 3 = 0$$

  Thus $x - \frac{3}{2}$ is a factor of $2x^2 - x - 3$ and so $2x - 3$ is also a factor of $2x^2 - x - 3$ (why?)

Now try these problems on your own:

- Check whether each of the polynomials listed below is a factor of $3x^3 - 2x^2 - 3x + 2$; if not, find the remainder.
  - $x - 1$
  - $3x - 2$
  - $2x - 3$

- $x + 1$ • $3x + 2$ • $2x + 3$

- What is the remainder on dividing the polynomial $p(x)$ by $ax + b$? What is the condition under which $ax + b$ is a factor of the polynomial $p(x)$?
- Is $x - 1$ a factor of $x^{100} - 1$? What about $x + 1$?
- Is $x - 1$ a factor of $x^{101} - 1$? What about $x + 1$?
- Prove that $x - 1$ is a factor of $x^n - 1$ for every natural number n.
- Prove that $x + 1$ is a factor of $x^n - 1$ for every even number n.
- Prove that $x + 1$ is not a factor of $x^n - 1$ for every odd number n.
- What number added to $3x^3 - 2x^2 + 5x$ gives a polynomial for which $x - 1$ is a factor?
- What first degree polynomial added to $3x^3 - 2x^2$ gives a polynomial for which both $x - 1$ and $x + 1$ are factors?

Factorization

How do we find the factors of the polynomial, $x^2 + x - 12$?

It is easy to check whether a given polynomial such as $x - 2$ or $2x + 1$ is a factor of $x^2 + x - 12$. Instead, how do we directly find a factor of $x^2 + x - 12$?

According to the Factor Theorem, to find the first degree factors of $x^2 + x - 12$, we need only find those numbers x which make $x^2 + x - 12$ zero.

In other words, we need only solve the second degree equation

$$x^2 + x - 12 = 0$$

That we know:

$$x = \frac{-1 \pm \sqrt{1+48}}{2} = \frac{-1 \pm 7}{2} = 3 \text{ or } -4$$

Sum of powers

We noted that $x - 1$ is a factor of $x^n - 1$, for every natural number n. What is the quotient here?

We have seen that

when $n = 2$, $\frac{x^2-1}{x-1} = x + 1$

and when $n = 3$, $\frac{x^3-1}{x-1} = x^2 + x + 1$

In the same manner, it's not difficult to see that

$$\frac{x^4-1}{x-1} = x^3 + x^2 + x + 1$$

In general, for any natural number n,

$$\frac{x^n-1}{x-1} = x^{n-1} + x^{n-2} + ... + x^2 + x + 1$$

Reading this in reverse,

$$1 + x + x^2 + ... + x^{n-1} = \frac{x^n-1}{x-1}$$

This equation is true for all numbers except 1. For example, taking $x = 2$ in this,

$$1 + 2 + 2^2 + ... + 2^{n-1}$$

$$= \frac{2^n-1}{2-1} = 2^n - 1$$

as we have seen in the section **Growing triangles** of the lesson **Arithmetic Sequences.** Similarly

$$1 + 3 + 3^2 + ... + 3^{n-1}$$

$$= \frac{3^n-1}{3-1} = \frac{3^n-1}{2}$$

and

$$1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}}$$

$$= \frac{\frac{1}{2^n}-1}{\frac{1}{2}-1} = 2\left(1-\frac{1}{2^n}\right)$$

Thus if we take $x = 3$ or if we take $x = -4$, we can make $x^2 + x - 12$ zero. So, by the Factor Theorem, $x - 3$ and $x - (-4) = x + 4$ are factors of the polynomial $x^2 + x - 12$.

If we multiply these two factors, we get

$$(x - 3)(x + 4) = x^2 + x - 12$$

which is the polynomial we started with.

How do we find the factors of $3x^2 + 5x + 2$ like this?

As before, we first solve the equation

$$3x^2 + 5x + 2 = 0$$

We find

$$x = \frac{-5 \pm \sqrt{25-24}}{6} = \frac{-5 \pm 1}{6} = -\frac{2}{3} \text{ or } -1$$

Again, as in the first problem, using the Factor Theorem we see that $x + \frac{2}{3}$ and $x + 1$ are factors of $3x^2 + 5x + 2$.

$$\left(x+\frac{2}{3}\right)(x + 1) = x^2 + \frac{5}{3}x + \frac{2}{3}$$

This is not the original polynomial $3x^2 + 5x + 2$ we started with. However, we can write

$$x^2 + \frac{5}{3}x + \frac{2}{3} = \frac{1}{3}(3x^2 + 5x + 2)$$

Thus

$$\left(x+\frac{2}{3}\right)(x + 1) = \frac{1}{3}(3x^2 + 5x + 2)$$

From this we find

$$3x^2 + 5x + 2 = 3\left(x+\frac{2}{3}\right)(x + 1) = (3x + 2)(x + 1)$$

Next, let's see how $6x^2 - 7x - 3$ is split into factors.

Another way

Some polynomials of the form $x^2 + ax + b$ can be easily factorized. Take $x^2 + 5x + 6$, for example. If we take its factors as $x + a$ and $x + b$, then we have

$$x^2 + 5x + 6 = (x + a)(x + b)$$
$$= x^2 + (a + b)x + ab$$

For this to hold, we need only have

$$a + b = 5$$
$$ab = 6$$

In other words, we want to find two numbers with sum 5 and product 6. A moment's thought will give these as 2 and 3. Thus

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

See if you can factorize $x^2 + 10x + 24$ like this.

What about $x^2 - 10x + 24$?

First we solve

$$6x^2 - 7x - 3 = 0$$

(Why?)

This gives

$$x = \frac{7 \pm \sqrt{49+72}}{12} = \frac{7 \pm 11}{12} = \frac{3}{2} \text{ or } -\frac{1}{3}$$

Next we find the product of $x - \frac{3}{2}$ and $x + \frac{1}{3}$

$$\left(x - \frac{3}{2}\right)\left(x + \frac{1}{3}\right) = x^2 + \left(\frac{1}{3} - \frac{3}{2}\right)x - \left(\frac{3}{2} \times \frac{1}{3}\right)$$

$$= x^2 - \frac{7}{6}x - \frac{1}{2}$$

$$= \frac{1}{6}(6x^2 - 7x - 3)$$

Now we need only write this in reverse:

$$6x^2 - 7x - 3 = 6\left(x - \frac{3}{2}\right)\left(x + \frac{1}{3}\right)$$

$$= 2\left(x - \frac{3}{2}\right) \times 3\left(x + \frac{1}{3}\right)$$

$$= (2x - 3)(3x + 1)$$

Let's look at one more example. How do we factorize $x^2 - 2x - 1$?

Solving the equation

$$x^2 - 2x - 1 = 0$$

we get

$$x = 1 \pm \sqrt{2}$$

In other words, the solutions of this equation are $1 + \sqrt{2}$ and $1 - \sqrt{2}$.

Let's multiply the polynomials formed by subtracting each of these from x:

$$(x - (1 + \sqrt{2}))(x - (1 - \sqrt{2}))$$

$$= x^2 - \big((1 + \sqrt{2}) + (1 - \sqrt{2})\big)x + (1 + \sqrt{2})(1 - \sqrt{2})$$

$$= x^2 - 2x + \big(1^2 - (\sqrt{2})^2\big)$$

$$= x^2 - 2x - 1$$

Factorization and solution

We saw that to factorize a polynomial $p(x)$, we need only solve the equation $p(x) = 0$. On the other hand, if we are able to factorize a polynomial $p(x)$, then we get the solutions of the equation $p(x) = 0$ as well.

For example, look at the equation

$$x^2 + 5x + 6 = 0$$

Once we find

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

we can write the equation we started with as

$$(x + 2)(x + 3) = 0$$

For this to hold, we must find numbers x such that the product of the numbers $x + 2$ and $x + 3$ is zero.

For a product to be zero, we need only have one of the factors zero. Thus we need only find x such that either $x + 2$ or $x + 3$ is zero. That is,

$$x + 2 = 0 \text{ or } x + 3 = 0$$

which gives

$$x = -2 \text{ or } x = -3$$

Third degree polynomials

Consider the polynomial $x^3-6x^2+11x-6$? How do we factorize it? To use the Factor Theorem, we must solve the equation

$$x^3 - 6x^2 + 11x - 6 = 0$$

But we haven't seen any general technique for doing this.

We can check some possibilities. If we take $x = 1$ in this polynomial, we get $1 - 6 + 11 - 6 = 0$. So, $x - 1$ is a factor. How do we find the other factors?

If we divide $x^3 - 6x^2 + 11x - 6$ by $x - 1$, we get $x^2 - 5x + 6$ (try!) Also, for $x^2 - 5x + 6 = 0$, we must have $x = 2$ or $x = 3$. So, what do we have now?

$$x^3 - 6x^2 + 11x - 6$$
$$= (x - 1)(x^2 - 5x + 6)$$
$$= (x - 1)(x - 2)(x - 3)$$

Can you factorize $x^3 - 4x^2 + x + 6$ like this?

Thus

$$x^2 - 2x - 1 = (x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$$

Can we factorize all polynomials like this?

Take the polynomial $x^2 + 1 = 0$, for example. If it has first degree factors, then the equation $x^2 + 1 = 0$ must have solutions; since it has none (why?), this polynomial does not have first degree factors.

Now try these problems:

- Write each of the polynomials listed below as a product of two first degree polynomials:
  - $2x^2 + 5x + 3$
  - $x^2 + 2x - 1$
  - $x^2 + 3x + 2$
  - $x^2 - 2$
  - $4x^2 + 20x + 25$
  - $x^2 - x - 1$
- Prove that none of the polynomials listed below has first degree factors:
  - $x^2 + x + 1$
  - $x^4 + 1$
  - $x^2 - x + 1$
  - $x^4 + x^2 + 1$

Project

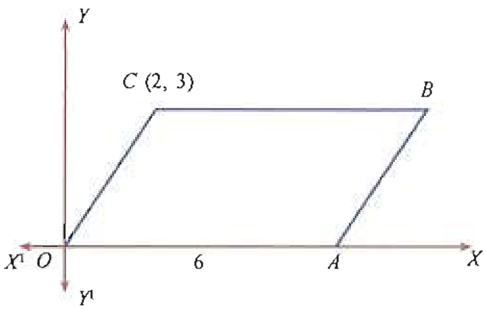
- Find separately the speciality of the coefficients of polynomials for which $x - 1$, $x + 1$ or $x^2 - 1$ is a factor.

10 Geometry and Algebra

Distance

We saw that by choosing a pair of perpendicular lines and a unit to measure length, we can represent all points in a plane as pairs of numbers.

In the figure below, $OABC$ is a parallelogram.

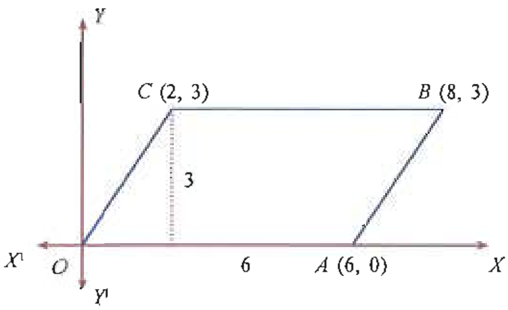


Can you find the coordinates of the vertices A and B?

The vertex A is on the x-axis itself and its distance from the origin is 6. So what are its coordinates?

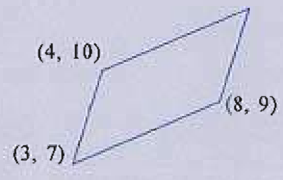
What about B? The line BC is parallel to the x-axis; and the point C on it has y-coordinate 3. So, what's the y-coordinate of B?

Now the length of BC is also 6. (How do we get that?) So, what's the x-coordinate of B?

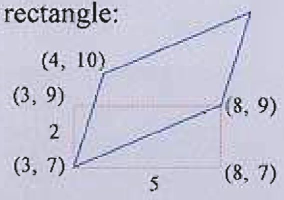


Fourth vertex

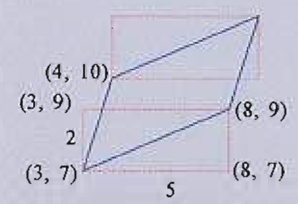
What are the coordinates of the fourth vertex of the parallelogram shown below?



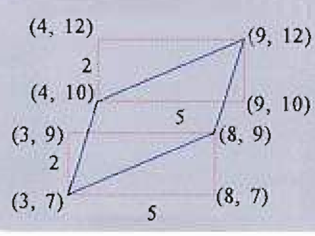
We can draw lines parallel to the axes through the bottom vertices and make a rectangle:



Similarly we can draw a rectangle through the top vertices also:

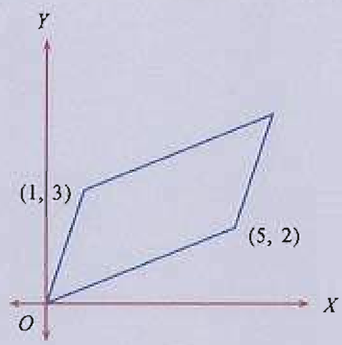


Its width and height are the same as those of the bottom rectangle (why?)

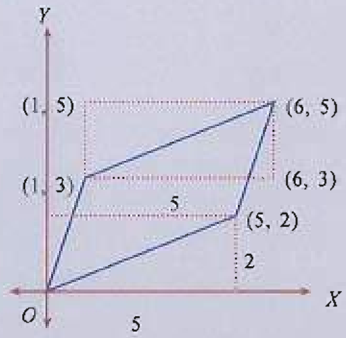


Parallelogram addition

What are the coordinates of the fourth vertex of this parallelogram?



Let's draw rectangles as before:

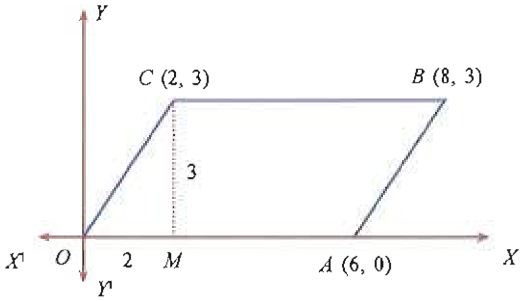


Try this with other points instead of (1, 3) and (5, 2). Do you see any relation between the coordinates we start with and the coordinates of the fourth vertex?

Try with the starting points as (x_1, y_1) and (x_2, y_2).

Now another question: what's the length of the other side of this parallelogram?

See this picture:



How do we get $OM = 2$?

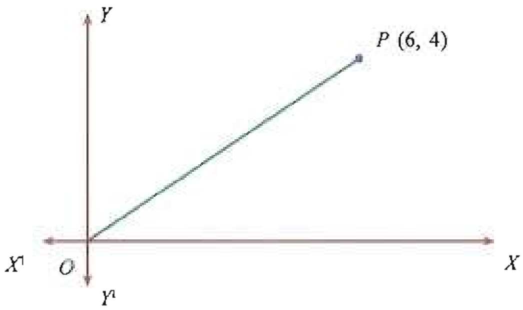
Now from the right angled triangle *COM*, can't we find *OC*?

$$OC^2 = OM^2 + MC^2 = 13$$

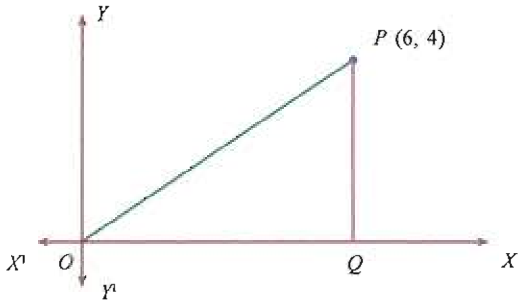
This gives the length of the other side of the parallelogram as $\sqrt{13}$.

Here we used only the coordinates of *C* to compute the length *OC*, right?

Like this, can you find the length *OP* in the figure below?



Draw the perpendicular from *P* to the *x*-axis.

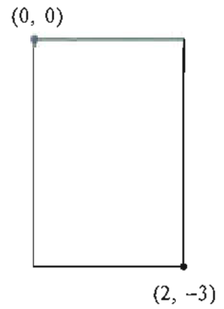


What are the lengths of the perpendicular sides of the right angled triangle *OPQ*? So, can't you find *OP*?

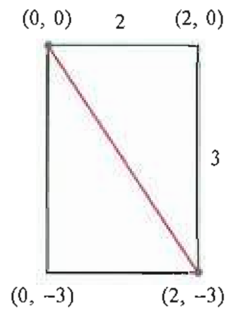
Now suppose the coordinates of a point are given in a figure in which the axes are not shown. Can you find the distance of this point from the origin?

For example, what is the distance between the origin and the point (2, −3)?

For this, we draw a rectangle with its sides parallel to the axes, as shown below:



What are its other vertices? And the lengths of its sides? So, can't we find the length of the diagonal?

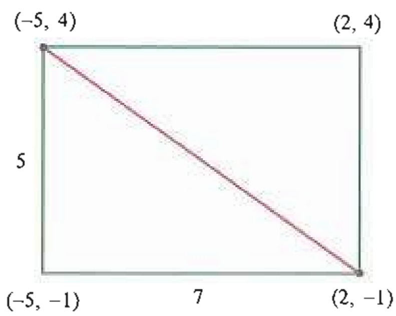


Thus the distance we seek is

$$\sqrt{4+9}=\sqrt{13}$$

Can we find the distance between any two points like this?

For example, let's take (2, −1) and (−5, 4) . In this case, we can draw a rectangle like this, with sides parallel to the axes:



Algebra of geometry

We usually state general relations between numbers using algebra. And we have also seen how some such relations between positive numbers can be geometrically described.

Once we have represented all points in a plane as pairs of numbers, we can describe the geometric relations between these points and the geometric figures formed by joining these points, in the language of algebra.

One such example we have already seen: if (x_1, y_1) and (x_2, y_2) are joined to (0, 0) and the figure is completed to a parallelogram, then the fourth vertex is $(x_1 + x_2, y_1 + y_2)$.

The distance we need is the length of the diagonal of this rectangle; which we can find as

$$\sqrt{7^2+5^2}=\sqrt{74}$$

Now let's take points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ and $y_1 \neq y_2$ instead of specific points like (2, −1), (−5, 4). Then also, we can draw a rectangle with these as opposite vertices and sides parallel to the axes. A pair of adjacent sides of this rectangle are the lines joining (x_1, y_1), (x_2, y_1) and (x_1, y_1), (x_1, y_2). The lengths of these lines are $|x_1-x_2|$ and $|y_1-y_2|$. So, the square of the length of its diagonal is

$$|x_1-x_2|^2+|y_1-y_2|^2=(x_1-x_2)^2+(y_1-y_2)^2$$

(Do you remember seeing in the lesson **Real Numbers** of the Class 9 textbook that the square of a number is equal to the square of its absolute value?)

Thus the distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Now a rectangle like this is possible, only if the line joining the points chosen is not parallel to either axis. But we have already seen how we can compute the distance between such points. (The lesson **Coordinates**)

Let's write those also in algebraic terms: .

.

- If the y-coordinates of two points are equal, as in (x_1, y) and (x_2, y), then the line joining them is parallel to the x-axis; and the distance between them is got by subtracting the smaller of x_1, x_2 from the larger, that is $|x_1-x_2|$.
- If the x-coordinates of two points are equal, as in (x, y_1) and (x, y_2), then the line joining them is parallel to the y-axis; and the distance between them is got by subtracting the smaller of y_1, y_2 from the larger, that is $|y_1-y_2|$.

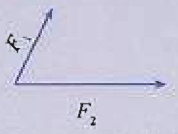
Thus we have three formulas to compute the distance between two points, depending on their position.

Now what do we get if we take $y_1 = y_2$ in the algebraic expression $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$?

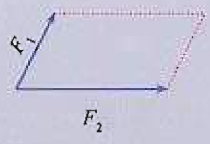
Force parallelogram

We can produce the same effect of two forces acting along different directions on a body, by a single force acting along a definite direction.

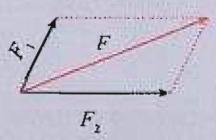
There is a method, recognized through experiments, to find this force and its direction. Draw two lines from a point with their lengths proportional to the forces (such as for example one centimetre for one Newton), along the directions of the forces:



Next draw a parallelogram with these as adjacent sides:



The single force to replace these two forces acts along the diagonal of this parallelogram; and its magnitude is the length of this diagonal, in the scale chosen.



This is known as the Parallelogram Law of Forces.

$$\sqrt{(x_1 - x_2)^2} = |x_1 - x_2|$$

(See the section **Square root and absolute value** of the lesson **Real Numbers**, in the Class 9 textbook.)

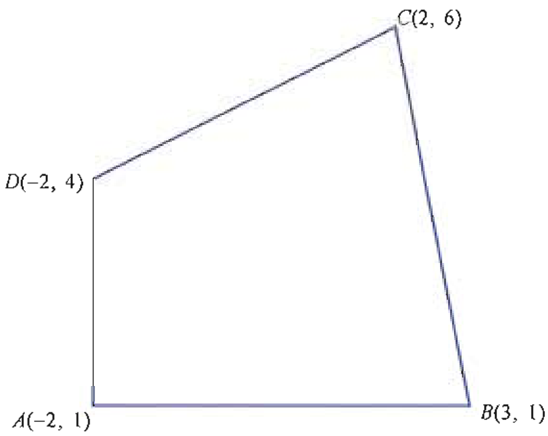
Likewise, if we take $x_1 = x_2$ in the expression, we get $|y_1 - y_2|$.

Thus the distance between two points can be given by a single algebraic expression.

The distance between any two points (x_1, y_1) *and* (x_2, y_2) *is* $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Let's look at some examples: .

- Find the perimeter of the quadrilateral shown below:



Here, the points A and B have the same y-coordinates. So, the length of AB is $3 - (-2) = 5$

The points A and D have the same x-coordinate and so the length of AD is $4 - 1 = 3$.

The points B and C have different x-coordinates and different y-coordinates. So, the length of BC is $\sqrt{(2-3)^2 + (6-1)^2} = \sqrt{26}$

Similarly, the length of CD is $\sqrt{(2-(-2))^2 + (6-4)^2} = \sqrt{20}$

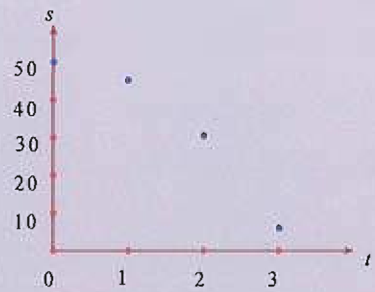
Now we can find the perimeter as $5 + 3 + \sqrt{26} + \sqrt{20} = 8 + \sqrt{26} + 2\sqrt{5}$

Relations in physics

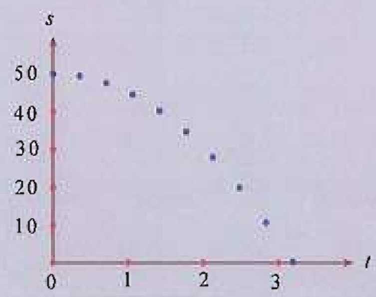
We have seen how algebra is used to describe relations between various physical quantities. For example, if an object falling towards the ground from a height of 50 metres is at a height s metres after t seconds, then

$$s = 50 - 4.9t^2$$

If we take $t = 0, 1, 2, 3$, in this equation, we get $s = 50, 45.1, 30.4, 5.9$: If we draw two perpendicular lines and mark s and t along them using suitable scales, then we can plot the points (0, 50), (1, 45.1), (2, 30.4), (3, 5.9). And we get a picture like this:



Taking more numbers as t and plotting more points, we get a picture like this:



Electronic aid

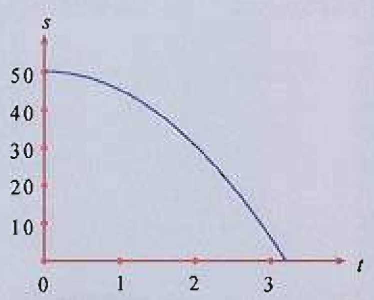
We saw how the relation between the height and time of a falling body is geometrically pictured. In drawing such pictures, it is not easy to compute a large number of coordinates and to plot them.

There are numerous computer applications which can be used for this, such as GeoGebra, Gnuplot, Kmplot. We need only specify the equation of the relation for these to plot the picture.

Given below is the plot of the equation

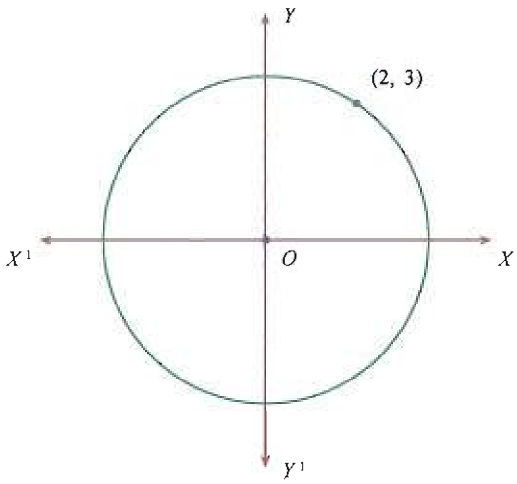
$$s = 50 - 4.9t^2$$

seen earlier, drawn using the PostScript language:



From this picture, we can see such things as how the height decreases with time and when the body would hit the ground.

- The circle shown below is centred at the origin. What is its radius?



O is the centre of the circle and (2, 3) is a point on the circle. So, the radius is the distance between them.

What are the coordinates of O?

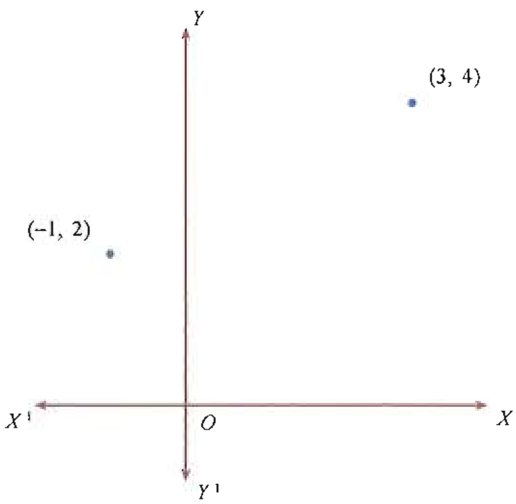
So, the radius is

$$\sqrt{(2-0)^2+(3-0)^2} = \sqrt{13}$$

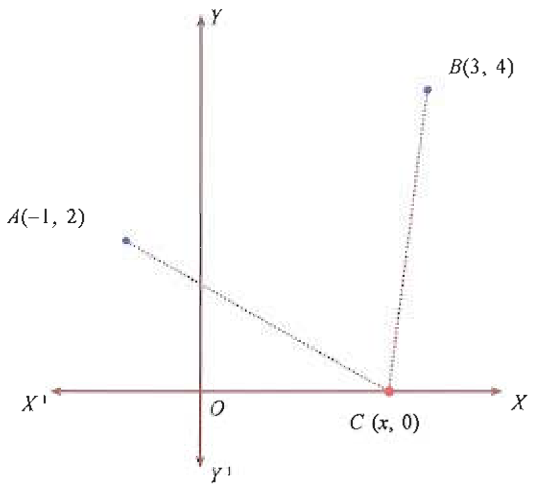
We can note another thing here. The distance between (3, 2) and the origin is also $\sqrt{13}$. (How is that?) So, this point is also on the circle.

Can you quickly find a few more points on the circle? Mark them on the circle.

- In the figure below, find the point on the x-axis, equidistant from the two points marked:



The required point is on the x-axis and so its y - coordinate is zero. If we take its x-coordinate as x, then its coordinates are $(x, 0)$.



Now in the figure, we have $AC = BC$ which means $AC^2 = BC^2$. Thus

$$(x + 1)^2 + (0 - 2)^2 = (x - 3)^2 + (0 - 4)^2$$

From this, we get

$$(x + 1)^2 - (x - 3)^2 = 12$$

Simplifying, we get

$$8x - 8 = 12$$

and this gives $x = \frac{5}{2}$. So, the coordinates of the point we seek are $(\frac{5}{2}, 0)$.

Now some problems for you:

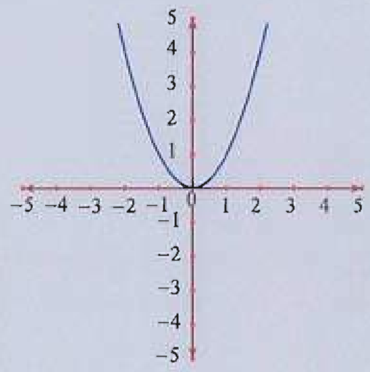
- The centre of a circle is (3, 4) and it passes through the point (2, 5). What is its radius?
- A circle of radius 3 is drawn with centre at (−2, 1). Find out whether the point (4, 1) lies on the circle, within the circle or outside the circle.
- Prove that we get a right angled triangle by joining the points (2, 1), (3, 4), (−3, 6).
- How many points are there on the x-axis, at a distance 5 from the point (1, 3)? What are their coordinates? What about such points on the y-axis?

Pictures of equations

Using computers, we can also plot equations giving relations between numbers, For example, the picture below is the plot of the equation

$$y = x^2$$

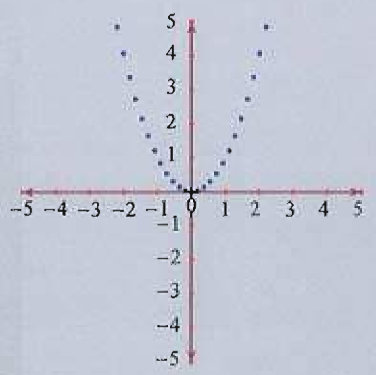
done using PostScript:



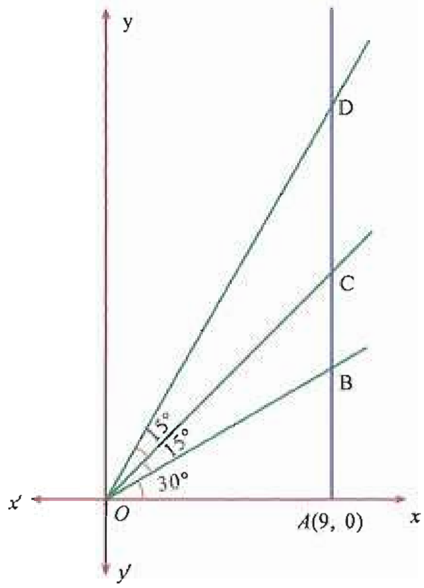
What is the meaning of this picture?

By taking different numbers as x, we can calculate the corresponding numbers x^2. The plot is made by joining a large number of such pairs (x, x^2), such as for example (1.5, 2.25).

The picture above is made up of 50 such points. If only 25 points are plotted, the plot would look like this:



- Find the coordinates of the points B, C, D in the picture below:



Write the lengths AB, BC, CD in the order of their magnitudes.

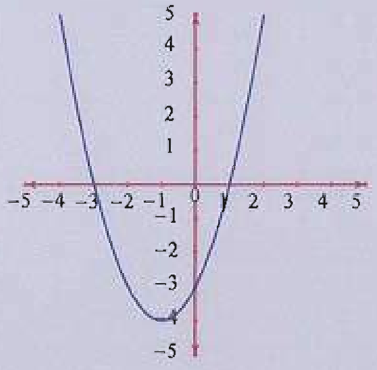
- The circle centred at (2, 3) and of radius 5, intersects the x-axis at A and B. Find the coordinates of A and B and also the length of the chord AB.
- The vertices of a triangle are the points (1, 2), (2, 3), (3, 1). Find the centre and radius of its circumcircle.

Second degree plot

The plot below is that of the equation

$$y = x^2 + 2x - 3$$

done using a computer:



What is the difference between this and the plot of $y = x^2$, seen earlier?

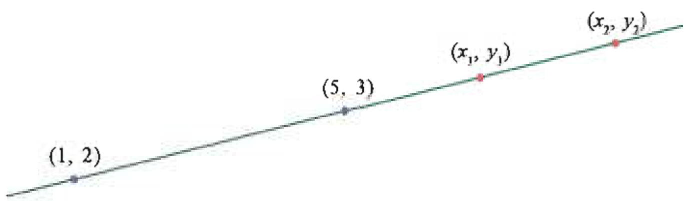
Plot a few more second degree polynomials like this.

Slope of a line

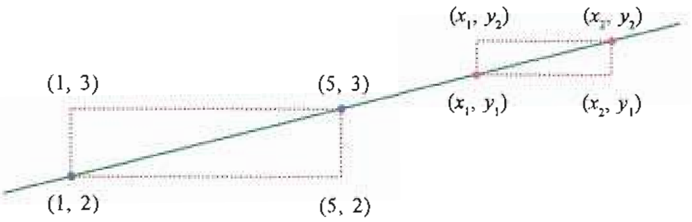
We have seen the peculiarities of the coordinates of points on lines parallel to the axes of coordinates: the y-coordinates of points on a line parallel to the x-axis are equal; and the x-coordinates of points on a line parallel to the y-axis are equal.

What about points on lines not parallel to either axis? Do the coordinates of points on them have any speciality?

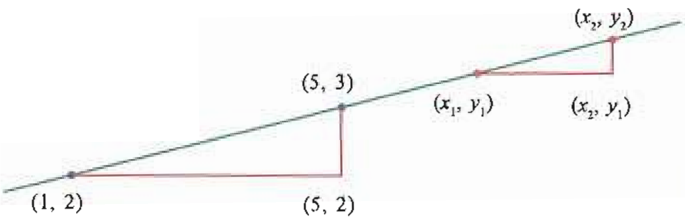
For example, let's look at the line joining the points (1, 2), (5, 3) and two other points on this line:



We can draw rectangles as in this picture:



Let's focus on the right angled triangles below the line:



Their angles are the same (why?) So, the sides opposite equal angles are proportional. That is,

$$\frac{y_2 - y_1}{3-2} = \frac{x_2 - x_1}{5-1}$$

(How do we get this?) From this, we find

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{4}$$

This equation holds wherever on this line we take the points (x_1, y_1) and (x_2, y_2), due to the equality of angles and similarity of triangles as described above.

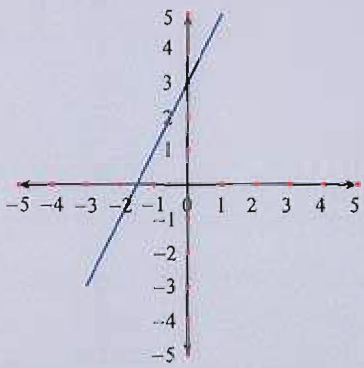
This means for any pair of points on this line, the difference of y-coordinates, divided by the difference of x- coordinates is $\frac{1}{4}$.

What if we start with some other points instead of (1, 2) and (5, 3) as we did?

For example, let's take (6, 2) and (3, 4). Then for any pair of points on this line, the y-difference divided by the x-difference would be $\frac{2-4}{6-3} = -\frac{2}{3}$.

First degree plot

The plot of $y = 2x + 3$ done by a computer is shown below y:



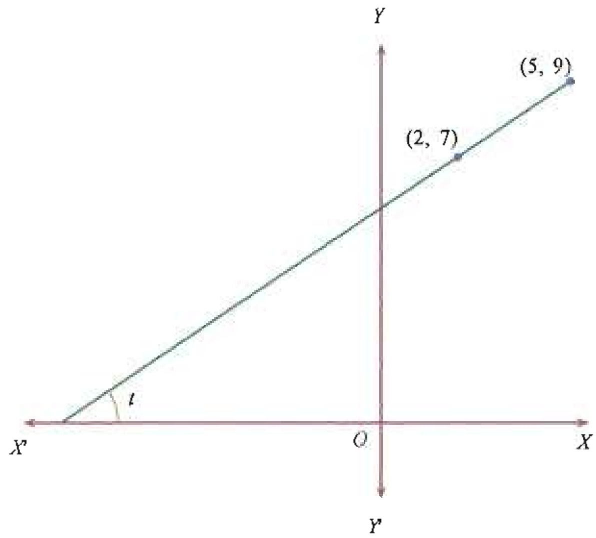
Plot some more first degree polynomials. Do you get a straight line every time?

In general, we have the following:

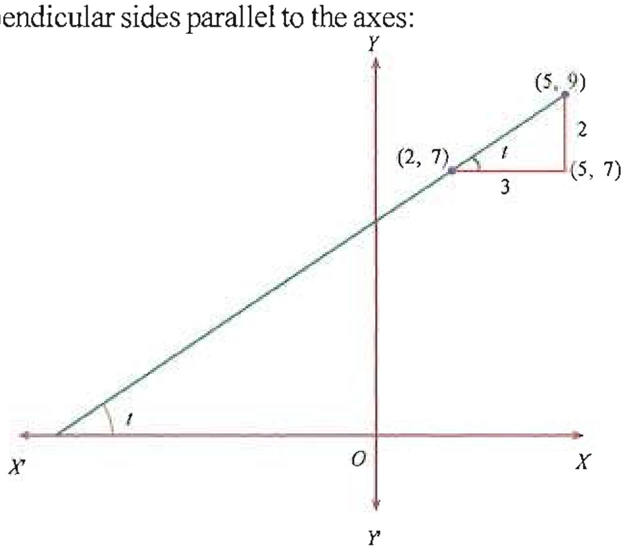
> *For any two points on a line not parallel to the y-axis, the difference of the y-coordinates, divided by the difference of the x-coordinates is the same number*

For points on a line parallel to the x-axis, this number is zero, right? (Why?) For lines parallel to the y-axis, we don't have such a number. (Why?)

Now let's look at this number from another angle. Consider the line joining (2, 7) and (5, 9), for example. Let's take the angle which this line makes with the x-axis as t.



We can draw a small right angled triangle at the top with its perpendicular sides parallel to the axes:



Why is the top angle also t? From this right angled triangle, we get

$$\tan t = \frac{2}{3}$$

Slope and rate and proportion

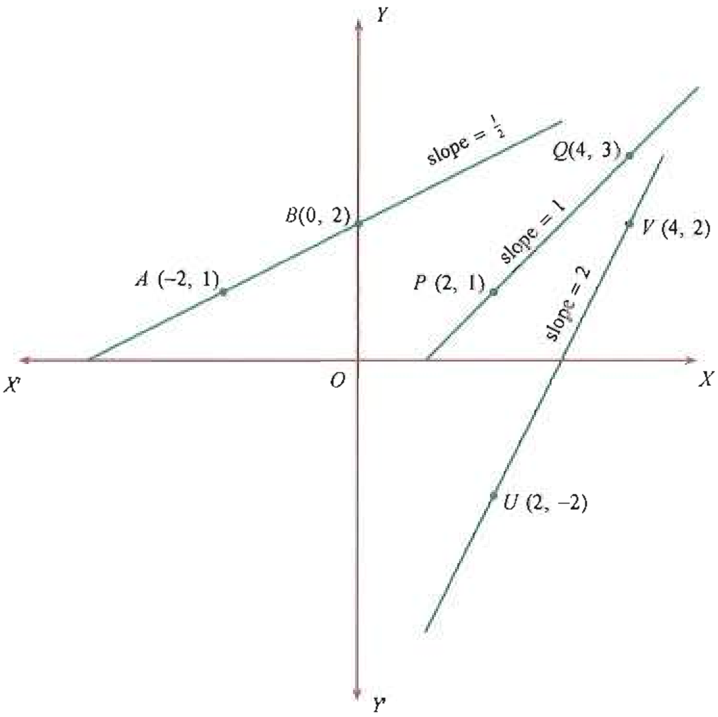
In a line not parallel to the either axis, when we move from one point to another, the x-coordinates and the y-coordinates change.

For example, consider the line joining (2, 7) and (5, 9). As we move from the first point to the second, the x-coordinate increases by 3 and the y-coordinate increases by 2. This means that for points on any other position on this line also, as the x-coordinate changes by 3 units, the y-coordinate changes by 2 units; and we indicate this when we say that the slope of the line is $\frac{2}{3}$.

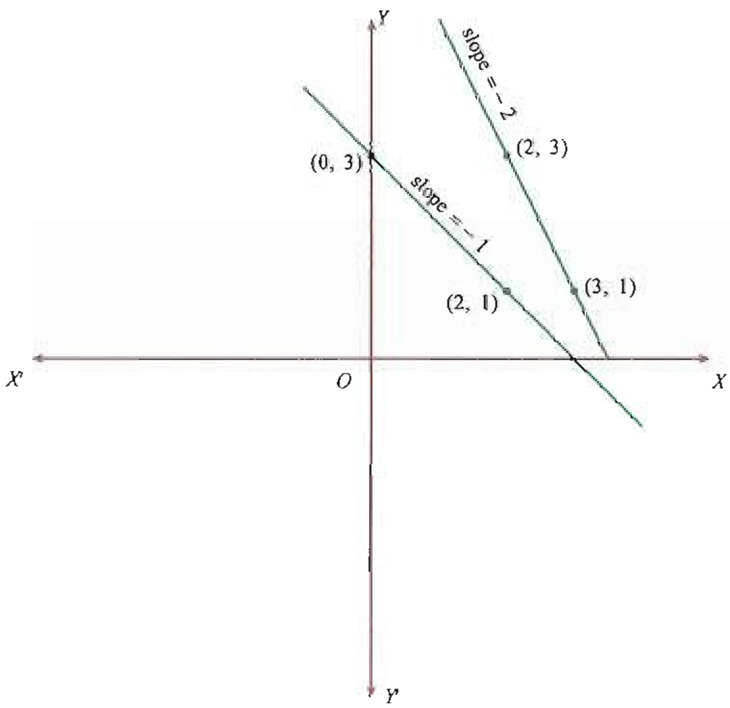
In other words, at any position on this line, as the x-coordinate increases by 1 unit, the y-coordinate increases by $\frac{2}{3}$ unit. That is, the number $\frac{2}{3}$ is the rate at which the y-coordinate changes with respect to the x-coordinate.

We can put it this way also: for points on this line, the difference in y-coordinates is proportional to the difference in x-coordinates; and the constant of proportionality is $\frac{2}{3}$.

Thus for a line not parallel to the y-axis, if we take a pair of points and divide the difference in y-coordinates by the difference in x-coordinates, the number we get is the tan measure of the angle which this line makes with the x-axis. And this number changes from line to line, as this angle changes. So, this number is called the *slope* of the line.

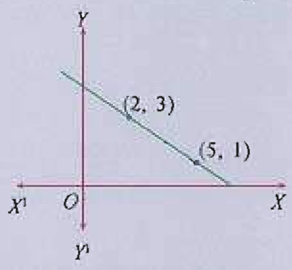


For some lines, the slope is negative, See this picture:



Negative slopes

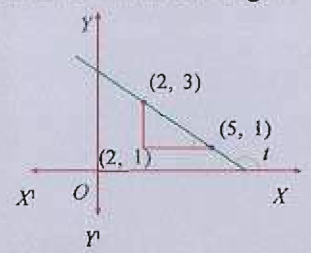
The slopes of some lines are negative. For example, the slope of the line joining (2, 3) and (5, 1) is $-\frac{2}{3}$ right?



This happens, since in this case the y-coordinate decreases as the x- coordinate increases.

Geometrically, such lines make an angle larger than a right angle with the direction OX.

For such lines also, is the slope equal to the tan measure of this angle?



From the pair of similar triangles in the picture above, we get

$$\tan(180 - t) = \frac{2}{3}$$

Since $\tan\ (180 - t) = -\tan t$, by definition, this equation gives

$$-\tan t = \frac{2}{3}$$

and this gives

$$\tan t = -\frac{2}{3}$$

Thus in such cases also, slope gives the tan measure of the angle.

Let's look at some examples: .

- What is the point at which the line joining (3, 1) and (2,–1) meets the x-axis? And the y-axis?

  The slope of this line is

  $$\frac{1-(-1)}{3-2}=2$$

  This means for any two points on this line, the y- difference divided by the x-difference is 2.

  So, if we take the point where this line cuts the x-axis as $(x, 0)$, then

  $$\frac{0-1}{x-3}=2$$

  This gives

  $$x-3=-\frac{1}{2}$$

  and so

  $$x=\frac{5}{2}$$

  Thus the line cuts the x-axis at $\left(\frac{5}{2},0\right)$

  Similarly, we can find the point of intersection with the y-axis as (0,–5). Try!

  See also if you can do this problem without using algebra.

- Prove that the line joining (3, 5) and (1, 7) passes through the point (5, 3).

  The slope of this line is

  $$\frac{5-7}{3-1}=-1$$

  Now what is the slope of the line joining (3, 5) and (5, 3)?

  $$\frac{5-3}{3-5}=-1$$

  Since the slopes are equal, these lines make the same angle with the x-axis. But they both pass through the point (3, 5). So, they must be the same line.

Physics, algebra and geometry

Suppose a body moves such that the distance travelled is 10 metres in the first second, 15 metres in the next second, 20 metres in the second after and it goes on increasing like this. So, its speed also is increasing every second, as 10m/s during the first second, 15m/s during the next second, 20m/s during the second after that and so on.

In other words, the speed increases by 5m/s every second. In the language of physics, the body has an acceleration of 5 metres per second per second (written 5metres/sec/sec or 5 m/s^2)

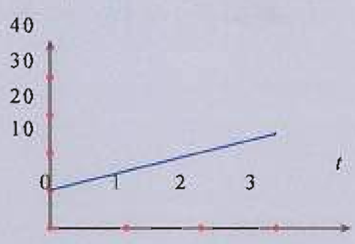
We can use the algebraic equation

$$v=10+5t$$

to calculate the speed v of this body at time t.

Now let's mark the various t and v along a pair of perpendicular lines and plot this relation between time and speed. (see the section,

Relations in physics)



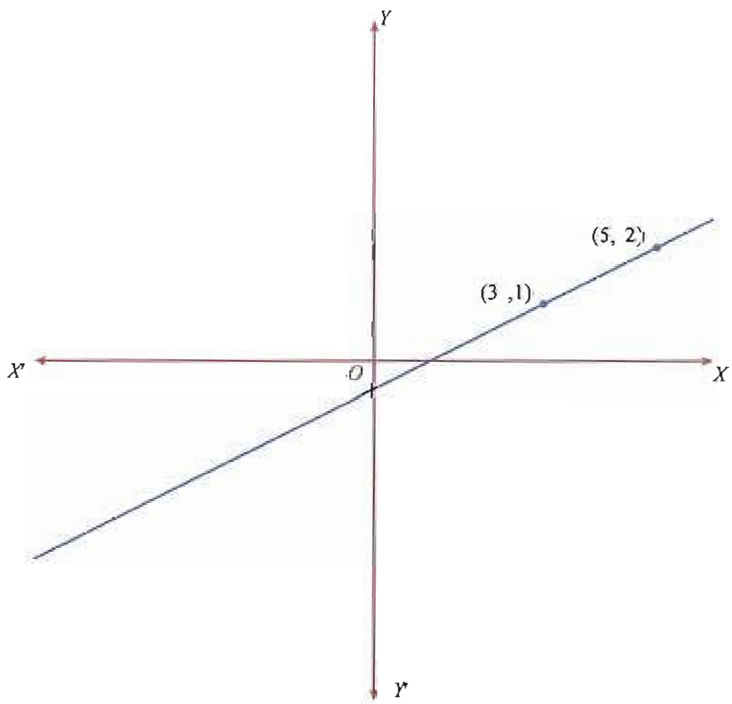
The slope of this line is 5. Here the slope is the rate at which v changes with respect to t; that is, the acceleration.

Now do these problems on your own:

- Does the line joining (2, 3) and (3, −1) pass through the point (5, 6)? What about (5,-9)?
- Prove that the points (1, 4), (4, 1) and $\left(\frac{5}{2}, \frac{5}{2}\right)$ lie on the same line.
- Prove that the points (2, 3), (7, 5), (9, 8), (4, 6) are the vertices of a parallelogram.
- Prove that the line joining the points (2, 1) and (1, 2) and the line joining the points (3, 5) and (4, 7) are not parallel. What are the coordinates of their point of intersection?
- Write down the coordinates of two more points on the line through (1, 3), of slope $\frac{1}{2}$.
- Two lines are drawn through the point (1, 3), one of slope $\frac{1}{2}$ and the other of slope −2. Write the coordinates of one more point on each of these lines. Prove that these lines are perpendicular to each other.

Equation of a line

The figure below shows the line joining (3, 1) and (5, 2):



Its slope is $\frac{1}{2}$, right?

Parallel slopes

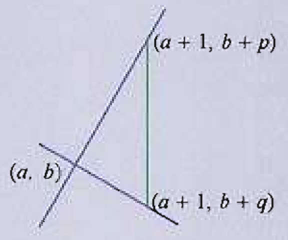
Two lines can have the same slope. For example, the line joining (3, 4), (2, 1) and the line joining (1, 2), (3, 8) are both of slope 3.
If two lines have the same slope, they both make the same angle with the positive direction of the x-axis; and so they must be parallel. On the other hand, parallel lines have the same slope (why?)

Perpendicular slopes

We saw that the slopes of parallel lines are equal. What is the relation between the slopes of perpendicular lines?

Suppose that lines of slopes p and q are perpendicular to each other. Let's take their point of intersection as (a, b).

Then the point $(a + 1, b + p)$ is on the first line and the point $(a + 1, b + q)$ is on the second line (why?)



Since the lines are perpendicular, the points (a, b), $(a + 1, b + p)$, $(a + 1, b + q)$ are the vertices of a right angled triangle. The hypotenuse is the line joining the second and third points. So, the squares of the lengths of the perpendicular sides of this triangle are $p^2 + 1$ and $q^2 + 1$ and the length of the hypotenuse is $|p - q|$. This gives

$$(p^2 + 1) + (q^2 + 1) = (p - q)^2$$

Simplifying, this gives

$$2 = -2pq$$

which means

$$pq = -1$$

Thus, for lines perpendicular to each other, the slope of one is the negative of the reciprocal of the slope of the other.

So, for any point (x, y) on the line, we must have

$$\frac{y-1}{x-3} = \frac{1}{2}$$

From this, we get

$$2(y - 1) = x - 3 ,$$

and this gives

$$x - 2y - 1 = 0$$

Thus the coordinates (x, y) of any point on this line satisfies the above equation.

Let's think in reverse: suppose we take a pair of numbers (x, y) satisfying the above equation. Would the point with coordinates (x, y) lie on this line?

From the equation $x - 2y - 1 = 0$, we get $x = 2y + 1$. So,

$$\frac{y-1}{x-3} = \frac{y-1}{(2y+1)-3} = \frac{y-1}{2y-2} = \frac{1}{2}$$

Thus the line joining the points (x, y) and $(3, 1)$ and that joining $(3, 1)$ and $(5, 2)$ have the same slope; so they make the same angle with the x-axis. And these lines pass through the same point $(3, 1)$. From all these, we see that they are one and the same line. In other words, (x, y) is indeed a point on our line.

Is the point $(3, 4)$ on this line? We have

$$3 - (2 \times 4) - 1 = -6 \neq 0$$

and so this point is not on our line.

How about $(3, 1)$? We have

$$3 - (2 \times 1) - 1 = 0$$

and so this point is on our line.

Let's look at the connection between the line and the equation once more:

- If (x, y) are the coordinates of a point on the line joining $(3, 1)$ and $(5, 2)$, then $x - 2y - 1 = 0$
- If x, y is a pair of numbers with $x - 2y - 1 = 0$, then the point with (x, y) as coordinates is on the line joining $(3, 1)$ and $(5, 2)$.

In short, the collection of the pairs of numbers giving the coordinates of the points on the line joining (3, 1), (5, 2) is the same as the collection of the pairs of numbers satisfying the equation $x - 2y - 1 = 0$.

We shorten this by saying

The equation of the line joining (3, 1) *and* (5, 2) *is* $x - 2y - 1 = 0$

Can you now find the equation of the line joining the points (2, 5) and (−1, 4)?

Let's look at a couple of examples: .

- What is the equation of the line through (2, 5) and of slope $\frac{2}{3}$?

  For any point (x, y) on this line,

  $$\frac{y-5}{x-2} = \frac{2}{3}$$

  isn't it? (Why?) And this is the equation of the line. We can simplify this as

  $$2x - 3y + 11 = 0$$

- What is the slope of the line given by the equation $2x - 3y + 4 = 0$?

  Taking the coordinates of two points on this line as (x_1, y_1) and (x_2, y_2), we get

  $$2x_1 - 3y_1 + 4 = 0$$
  $$2x_2 - 3y_2 + 4 = 0$$

  So, we must have

  $$(2x_1 - 3y_1 + 4) - (2x_2 - 3y_2 + 4) = 0$$

  That is,

  $$2(x_1 - x_2) - 3(y_1 - y_2) = 0$$

  This gives

  $$2(x_1 - x_2) = 3(y_1 - y_2)$$

  and so

  $$\frac{y_1 - y_2}{x_1 - x_2} = \frac{2}{3}$$

  Thus the slope of the line is $\frac{2}{3}$.

Equation of a circle

Suppose a circle is drawn with centre at (1, 2), of radius 4. What is the speciality of the coordinates of the points on it?

The distance from the centre to any point on a circle is equal to the radius, right?

So, if the point (x, y) is on this circle, then

$$(x-1)^2 + (y-2)^2 = 16$$

On the other hand, for any pair of numbers x, y satisfying the above equation, the point (x, y) must be on this circle. Thus

The equation of the circle centred at (1, 2) *and of radius* 4 *is* $(x-1)^2 + (y-2)^2 = 16$

We can think about this in a slightly different way. First we find two points on this line. For this, we need only find two pairs of numbers satisfying the equation $2x - 3y + 4 = 0$. For example, (1, 2), (4, 4). Since these are points on this line, its slope is

$$\frac{4-2}{4-1} = \frac{2}{3}$$

Now do these problems:

- Prove that for all points on the line joining the origin and the point (4, 2), the x-coordinate is double the y-coordinate. What is the equation of this line?
- What is the equation of the line joining the points (1, 3) and (2, 7)? Prove that if (x, y) is a point on this line, so is the point $(x + 1, y + 4)$.
- What is the point at which the line $2x + 4y - 1 = 0$ cuts the x-axis? What about the y-axis?
- Prove that the lines given by the equations $3x + 2y + 5 = 0$ and $3x + 2y - 1 = 0$ are parallel. At what points do they intersect the x-axis? And the y-axis?
- At what point do the lines of equations $3x + 2y + 5 = 0$ and $2x - 3y - 1 = 0$ intersect each other? Find one more point on each of these lines. Prove that these lines are perpendicular.

Mathematical merger

Descartes started the method of studying geometrical figures by converting them to algebraic equations and vice versa, through the device of taking points as pairs of numbers. This united algebra and geometry, which were considered separate branches of mathematics till then. This new geometry is called *Analytic Geometry*.

The branch of mathematics called Calculus, which revolutionized mathematical thought and also other sciences which use mathematics, is based on this new view of geometry. Progress occurs through the synthesis of duals.

11 Statistics

Frequency table and mean

We have seen the use of such numbers as the mean and median in diverse instances—to understand the educational standard of children in a class or to understand the economic standard of people in a locality.

- The table below gives the number of workers doing various jobs in a factory and their daily wages:

Daily Wages (Rupees)	Number of Workers
210	2
225	4
250	6
270	2
300	1

What is the mean wage?

Here, the mean is the total wages divided by the total number of workers. See how the total wages is computed in the table below:

Daily Wages (Rupees)	Number of Workers	Total Wages (Rupees)
210	2	420
225	4	900
250	6	1500
270	2	540
300	1	300
Total	**15**	**3660**

Repeated addition

To find the mean of a collection of numbers, we have to divide their sum by their number. If some numbers in the collection are repeated, we can find their sum by multiplication. For example, suppose that the ages of 10 children are recorded as below:

13	13	14	15	13
15	14	15	13	15

The sum is easily computed as

$$(4 \times 13) + (2 \times 14) + (4 \times 15) = 140$$

From this, we can compute the mean age as

$$\frac{140}{10} = 14$$

Small, large and mean

The mean of two numbers is the number exactly at their middle. In algebraic language, the mean of the numbers a and b is $\frac{1}{2}(a+b)$.

What about three numbers? Let them be a, b, c in ascending order of magnitude. The mean is $\frac{1}{3}(a+b+c)$. Here, b and c are greater than or equal to a and so the number $\frac{1}{3}(a+b+c)$ is greater than or equal to $\frac{1}{3}(a+a+a)=a$. On the other hand, since a and b are less than or equal to c, the number $\frac{1}{3}(a+b+c)$ is less than or equal to $\frac{1}{3}(c+c+c)=c$.

Thus, the mean is between the smallest number a and the largest number c.

This is true for four numbers also, isn't it? Check it out. What if we take more numbers?

So, the mean is

$$3660 \div 15 = 244$$

Now look at this problem:

- The table below shows the classification of 50 persons in a locality according to their daily income:

Daily Income (Rupees)	Number of People
145 - 155	7
155 - 165	9
165 - 175	14
175 - 185	11
185 - 195	7
195 - 205	2

What is the mean daily income?

How do we find the total daily income of these 50 persons from this?

What is the difference between this table and the earlier one?

For example, look at the first line of this table: from this we know only that there are 7 persons whose daily income is between 145 rupees and 155 rupees. We don't know how many persons earn exactly 145 rupees or how many earn exactly 155 rupees. So, how do we compute the total daily income of these 7 persons?

To get the total, we don't need the individual incomes of these 7 persons; it is enough if we have their mean income. Here, the mean must be between 145 and 155 (see the box **Small, large and mean**). Moreover, this mean would be a number around 150. So, we assume the mean to be 150 and proceed with the computation.

Likewise, the mean daily income of the 9 persons earning between 155 rupees and 165 rupees is assumed to be the middle value 160 of 155 and 165.

So, we enlarge the first table as below:

Daily Income (Rupees)	Number of People	Class Average	Total Income
145 - 155	7	150	1050
155 - 165	9	160	1440
165 - 175	14	170	2380
175 - 185	11	180	1980
185 - 195	7	190	1330
195 - 205	2	200	400
Total	**50**		**8580**

Now can't we compute the mean?

$$8580 \div 50 = 171.6$$

Thus we can take the mean daily income as 172 rupees.

Now try these problems:

- The table below classifies the number of days of a month according to the amount of rainfall received in a certain locality.

  Find the mean daily rainfall during this month.

Rainfall (mm)	Number of Days
54	3
56	5
58	6
55	3
50	2
47	4
44	5
41	2

Distribution and mean

Suppose 7 numbers between 145 and 155 are as below:

145, 147, 147, 150, 152, 152, 155

We can compute the mean and it is 149.71.

These numbers are distributed more or less equally on either side of the middle number, 150. And the mean 149.71 does not differ much from 150.

Now what if the numbers are as below?

145, 145, 145, 146, 146, 148, 155

Most of these are near 145. And what about the mean? It is 147.14. What if most of the numbers are near 155?

- The table below classifies the members of a committee according to their ages:

Age	Number of Members
25 - 30	6
30 - 35	14
35 - 40	16
40 - 45	22
45 - 50	5
50 - 55	4
55 - 60	3

Calculate the mean age of the members of this committee.

- The table below shows the number of students in Class 10 of a school, classified according to their heights:

Height(cm)	Number of Students
120 - 125	19
125 - 130	36
130 - 135	23
135 - 140	23
140 - 145	43
145 - 150	21
150 - 155	23
155 - 160	12

Calculate the mean height.

Frequency table and median

We have seen examples of instances where the mean does not give a correct picture of the data. See this table, showing the classification of 25 families in a locality, on the basis of their monthly incomes:

Middle and mean

For a collection numbers in an arithmetic sequence, we have seen that their sum is the product of half the sum of the first and the last, with their number. So for such numbers, the mean is half the sum of the first and the last; that is, the number right in the middle of the first and the last numbers.

In an arithmetic sequence the numbers are distributed in the same fashion on either side of the middle number of the first and the last, isn't it so?

Monthly Income (Rupees)	Number of Families
4000	2
5000	6
6000	7
7000	3
8000	3
9000	2
10000	2
Total	**25**

We get the mean monthly income as Rs. 6520 (Try!) But the table shows that about sixty percent of the families have only a monthly income less than or equal to six thousand rupees. So, the mean is not a very good indicator of this distribution.

How do we compute the median here? Median occurs at the middle, we know. More precisely, here 12 families should have income less than the median income and 12 families should have income more than the median income.

To find this, we need only arrange the incomes in ascending order and find the income of the thirteenth family. From the table, we see that the first two families have an income of 4000 rupees, and the next 6 have an income of 5000 rupees; in other words, when we take the first 8 families, the top income reaches 5000 rupees. What we need is the income of the 13th family in this order. So, we must take the next 5 families. We see that the next 7 families have an income of 6000 rupees. That is, the income of the family numbers 9th to the 15th is 6000 rupees. In particular, this is the income of the 13th family. Thus the median monthly income is Rs. 6000.

Mean and median

We use numbers such as mean and median to get a quick appraisal of data given by a large mass of numbers. (See the section **Ways of statistics** of the lesson, **Statistics** in the Class 9 textbook).

In instances where the distribution has a relatively high frequency at the middle, with frequencies decreasing in more or less the same way on either side, the mean gives a somewhat correct picture of the distribution.

But when the frequency at an end is very high, the mean tends to move towards that part. And it does not give a true picture of the data. In such cases, the median may represent the data better.

Cumulative frequency

We have seen how a table giving classes and their frequencies can be changed to a table giving the frequency upto the upper limit of each class, by adding up the frequencies. These sums of frequencies are called *cumulative frequencies*.

In such instances, we compute the median based on the assumption that at every stage, the change in cumulative frequency is proportional to the change in the numbers given in classes; the median is *defined* as the number corresponding to the cumulative frequency equal to half the total frequency.

Such a notion first arose in connection with probability theory. The problem was to find the age at which one has an equal probability of living or dying, based on a mortality table. For this, the number of people above and below that age must be the same.

To make the computations easier, let's re-write our table as below:

Monthly Income (Rupees)	Number of Families
upto 4000	2
upto 5000	8
upto 6000	15
upto 7000	18
upto 8000	21
upto 9000	23
upto 10000	25

What if the tabulation is done as classes?

See this table, which gives the distribution of heights of children of a certain class in a school:

Height (cm)	Number of Children
135 - 140	4
140 - 145	7
145 - 150	18
150 - 155	11
155 - 160	6
160 - 165	5
Total	**51**

Here also, we first add up the frequencies and make a new table showing the number of children below specific heights:

Height (cm)	Number of Children
below 140	4
below 145	11
below 150	29
below 155	40
below 160	46
below 165	51

In instances such as this, the very definition of median is purely mathematical. For example, in the above example, let's first tabulate the numbers 140, 145, 150, . . . in the first column and 4, 11, 29,... in the second column as below:

x	140	145	150	155	160	165
y	4	11	29	40	46	51

Between the numbers we have taken as x, there are other numbers; and we don't know the numbers y corresponding to these. To compute them, we assume that at every stage, *the change in y is proportional to the corresponding change in x.*

For example, as the variable x changes from 140 to 145, the variable y changes from 4 to 11. So, to compute the y corresponding to $x = 141$, we use our proportionality assumption to get

$$\frac{y-4}{141-140} = \frac{11-4}{145-140}$$

From this, we get

$$y - 4 = \frac{7}{5}$$

and from this

$$y = \frac{27}{5} = 5.4$$

On the other hand, we can use the same technique to find what x must be for a specified value of y. For example, for $y = 41.5$, we must take the x satisfying the equation,

$$\frac{x-155}{160-155} = \frac{41.5-40}{46-40}$$

That is,

$$x = 155 + 5 \times \frac{1.5}{6} = 156.25$$

Now the median. Here the median is, by definition, that x for which

$$y = \frac{51}{2} = 25.5$$

Polynomial of proportionality

If the relation between two quantities is a first degree polynomial of the form $y = ax + b$, then the difference of two numbers taken as x and the difference in the corresponding numbers y would be proportional. For if $y_1 = ax_1 + b$ and $y_2 = ax_2 + b$, then

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{a(x_1 - x_2)}{x_1 - x_2} = a$$

On the other hand, suppose that two quantities, denoted x and y, are related such that the difference of numbers taken as x and the difference of the corresponding numbers y are proportional. Then the algebraic expression giving this relation would be a first degree polynomial. To prove this, let's take the constant of proportionality of this relation as a. Let the number x_1 be related to the number y_1 by this relation. Then for any pair (x, y) of related numbers, we have

$$\frac{y - y_1}{x - x_1} = a$$

That is,

$$y = ax + (y_1 - ax_1)$$

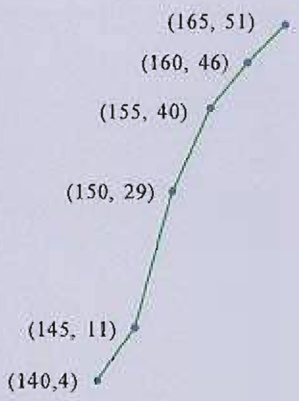
Writing $y_1 - ax_1$ as b, we get

$$y = ax + b$$

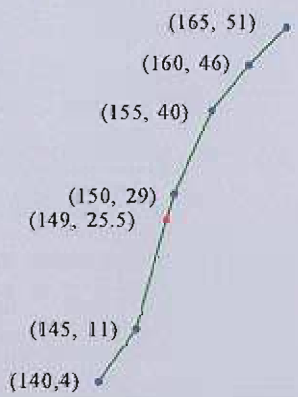
So,we can say that the assumption underlying the computation of the median is that the relation between the quantities and the cumulative frequencies at every stage is a first degree polynomial.

Median picture

If we plot the pairs (x, y) of the height problem as points, and join these by line segments, we get a picture like this:



The median is then the x-coordinate of the point on this with y-coordinate 25.5



The number $y = 25.5$ is between the tabulated numbers $y = 11$ and $y = 29$. And the corresponding numbers are $x = 145$ and $x = 150$. So, as seen in the examples above, corresponding to $y = 25.5$, we must have x such that

$$\frac{x - 145}{150 - 145} = \frac{25.5 - 11}{29 - 11}$$

That is,

$$x = 145 + 5 \times \frac{14.5}{18} \approx 149.03$$

So, in our problem, the median height of the children is 149 centimetres.

Now look at this problem. The table shows the number of workers of a factory classified according to their ages:

Age	Number of Workers
25 - 30	6
30 - 35	8
35 - 40	12
40 - 45	20
45 - 50	16
50 - 55	6
Total	**68**

Let's find the median age. First we make a table showing the numbers of workers below each age:

Age	Number of Workers
below 30	6
below 35	14
below 40	26
below 45	46
below 50	62
below 55	68

Next we look at this as a relation between numbers:

x	30	35	40	45	50	55
y	6	14	26	46	62	68

Here the median is the number x which gives $y = \frac{68}{2} = 34$.

The number $y = 34$ occurs between $y = 26$ and $y = 46$ in the table.

Also, from the table we see that $y = 26$ corresponds to $x = 40$, and $y = 46$ corresponds to $x = 45$. So, as in the first problem, using our proportionality assumption,

$$\frac{x - 40}{45 - 40} = \frac{34 - 26}{46 - 26}$$

and this gives

$$x = 40 + \left(5 \times \frac{8}{20}\right) = 42$$

This means, the median age is 42.

Now try these problems on your own:

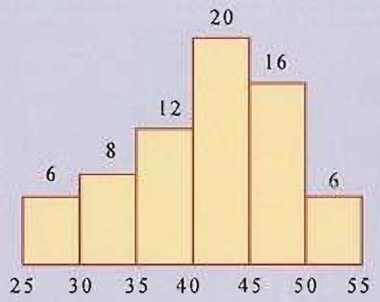
- The table below classifies according to weight, the infants born during a week in a hospital:

Weight (kg)	Number of Infants
2.500	4
2.600	6
2.750	8
2.800	10
3.000	12
3.150	10
3.250	8
3.300	7
3.500	5

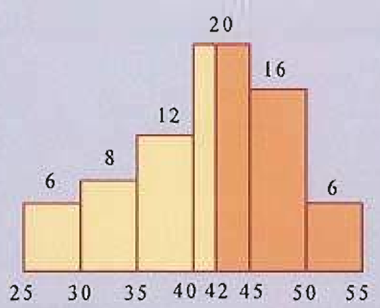
Find the median weight.

Median and area

Remember how we drew histograms of frequency distributions? Here's the histogram of the age problem:



The vertical line through the median splits the picture into two:

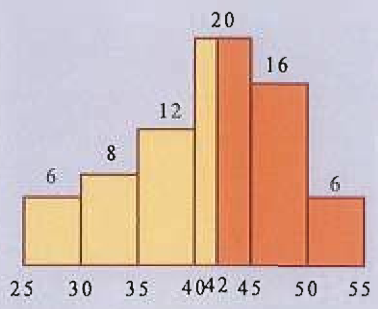


It's not difficult to see that the areas of these two parts are equal. (Try!)

Does the median have this property in all such distributions? Why?

Median and probability

We saw that the vertical line through the median splits the histogram into two parts of equal area.



So, if we mark a point on this picture, the probability of it falling on either part is the same (that is, the probability is $\frac{1}{2}$).

In other words, if we choose a worker of this factory, without any special consideration, the probability of his age to be less than 42, or to be more than 42, is the same.

- The table below shows the number of employees of an office, classified according to the income-tax paid by them:

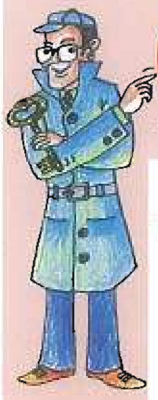
Income Tax (Rupees)	Number of Employees
1000 - 2000	8
2000 - 3000	10
3000 - 4000	15
4000 - 5000	18
5000 - 6000	22
6000 - 7000	8
7000 - 8000	6
8000 - 9000	3

Compute the median income-tax.

- The table below classifies the candidates who took an examination, according to the marks scored by them:

Marks	Number of Candidates
0 - 10	44
10 - 20	40
20 - 30	35
30 - 40	20
40 - 50	12
50 - 60	10
60 - 70	8
70 - 80	6
80 - 90	4
90 - 100	1

Find the median mark.



Glossary

അങ്കഗണിതം (Arithmetic)

അക്കം	digit
അധിസംഖ്യ	positive number
അഭാജ്യസംഖ്യ	prime number
അഭിന്നകസംഖ്യ	irrational number
അംശം	numerator
അംശബന്ധം	ratio
ഇരട്ടസംഖ്യ	even number
ഉയരം, ഉന്നതി	height, altitude
എണ്ണൽസംഖ്യ	natural number, counting number
ഏകകം	unit
ഏകദേശവില	approximate value
ഒറ്റസംഖ്യ	odd number
കൂട്ടുപലിശ	compound interest
കൃതി	power
കൃത്യങ്കം	exponent
കൃതീകരണം	exponentiation
കേവലവില	absolute value
ഗുണനം	multiplication
ഗുണിതം	multiple
ഗുണനഫലം	product
ഘടകം	factor
ഘനസെന്റിമീറ്റർ	cubic centimetre
ചതുർമുഖസംഖ്യകൾ	tetrahedral numbers
ഛേദം	denominator
തുക	sum
ദശാംശരൂപം	decimal form
നഷ്ടം	loss
ന്യൂനസംഖ്യ	negative number
പലിശ	interest
പലിശനിരക്ക്	rate of interest
പൂർണസംഖ്യ	integer
പൂർണവർഗസംഖ്യ	perfect square
പൊതുവ്യത്യാസം	common difference
ഭിന്നകസംഖ്യ	rational number
ഭിന്നസംഖ്യ	fraction
മുടക്ക് മുതൽ	investment
മുതൽ	principal
രേഖീയസംഖ്യ	real number
ലാഭം	profit
വർഗം	square
വർഗമൂലം	square root
വാങ്ങിയവില	cost price
വിറ്റവില	selling price
വ്യുൽക്രമം	reciprocal
വ്യവകലനം	subtraction
ശതമാനം	percentage
ശിഷ്ടം	remainder
ശ്രേണി	sequence
സങ്കലനം	addition
സമചതുരസംഖ്യ	square number
സമാന്തരശ്രേണി	arithmetic sequence, arithmetic progression
സാധാരണപലിശ	simple interest
സംഖ്യ	number
സംഖ്യാശ്രേണി	number sequence
സ്തൂപികാസംഖ്യകൾ	pyramidal numbers
സ്ഥാനവില	place value
ഹരണം	division
ഹരണഫലം	quotient
ഹാരകം	divisor
ഹാര്യം	dividend

ബീജഗണിതം (Algebra)

അജ്ഞാതസംഖ്യ	unknown number
ചരം	variable
പദം	term
ബഹുപദം	polynomial
ബഹുപദത്തിന്റെ കൃത്യങ്കം	degree of a polynomial
ബീജഗണിതവാചകം	algebraic expression
രണ്ടാംകൃതി സമവാക്യം	quadratic equation
ലഘൂകരിക്കുക	simplify
വിവേചകം	discriminant
സമവാക്യം	equation
സൂത്രവാക്യം	formula

ജ്യാമിതി (Geometry)

അഗ്രമുഖം, പാദം	base
അനുപാതം	proportion
അനുപൂരകം	supplementary
ആനുപാതികസ്ഥിരം	constant of proportionality
അന്തർവൃത്തം	incircle
അന്തർവൃത്തകേന്ദ്രം	incentre
അർധഗോളം	hemisphere
അർധവൃത്തം	semicircle
അഷ്ടഭുജം	octagon
ആന്തരികകോൺ	internal angle
ആന്തരസഹകോൺ	co-interior angle
ആരം	radius
ഉപരിതലപരപ്പളവ്	total surface area
ഉയരം	height
എതിർകോണുകൾ	opposite angles
എതിർവശം	opposite side
കർണം	hypotenuse
കീഴ്ക്കോൺ	angle of depression
കേന്ദ്രകോൺ	central angle
കോൺ	angle

കോൺമാപിനി	- protractor
കോൺസമഭാജി	- angle bisector
ഖണ്ഡിക്കുക	- intersect
ഗോളം	- sphere
ഘനരൂപങ്ങൾ	- solids
ചതുരശ്രസെന്റിമീറ്റർ	- square centimetre
ചക്രീയചതുർഭുജം	- cyclic quadrilateral
ചക്രീയഷഡ്ഭുജം	- cyclic hexagon
ചതുരം	- rectangle
ചതുരസ്തംഭം	- rectangular prism
ചതുർഭുജം	- quadrilateral
ചതുർമുഖം	- tetrahedron
ചരിവുയരം	- slant height
ചരിവ്	- slope
ചാപം	- arc
ചാപനീളം	- arc length
ജ്യാമിതിപ്പെട്ടി	- geometry box
ഞാൺ	- chord
തൊടുവര	- tangent
ത്രികോണം	- triangle
ത്രികോണമിതി	- trigonometry
ദശഭുജം	- decagon
നവഭുജം	- nonagon
പഞ്ചഭുജം	- pentagon
പരപ്പളവ്	- area
പരിവൃത്തകേന്ദ്രം	- circumcentre
പരിവൃത്തം	- circumcircle
പാദം	- base
പാർശ്വമുഖം	- lateral face
പാദവക്ക്	- base edge
പാർശ്വവക്ക്	- lateral edge
പൂരകചാപം	- complementary arc
പാർശ്വോന്നതി	- slant height
ബഹുമുഖം	- polyhedron
ബഹുഭുജം	- polygon
ബാഹ്യകോൺ	- exterior angle
ബാഹ്യസഹകോൺ	- co-exterior angle
ബിന്ദു	- point
മട്ടം	- right angle, setsquare
മട്ടത്രികോണം	- right triangle
മധ്യബിന്ദു	- midpoint
മധ്യലംബം,	- perpendicular
മറുകോൺ	- alternate angle
മറുഖണ്ഡം	- alternate segment
മുഖം	- face
മേൽക്കോൺ	- angle of elevation
രേഖീയജോടി	- linear pair
വശം	- side
വക്ക്	- edge
വൃത്തം	- circle
വ്യാസം	- diameter
വികർണം	- diagonal
വൃത്തകേന്ദ്രം	- centre of a circle
വൃത്താംശം	- sector
വൃത്തഖണ്ഡം	- segment of a circle
വര, രേഖ	- line
വൃത്തസ്തൂപിക	- cone
വക്രമുഖം	- curved surface
വൃത്തസ്തംഭം	- cylinder
വക്രതലപരപ്പളവ്	- curved surface area
വ്യാപ്തം	- volume
വൃത്തസ്തൂപികാപീഠം	- frustum of a cone
ലംബം	- perpendicular
ലംബകം	- trapezium
ലംബസമഭാജി	bisector
ശീർഷം	- apex
ശിഷ്ടചാപം, മറുചാപം,	- complementary arc
ഷഡ്ഭുജം	- hexagon
സമാന്തരം	- parallel
സമചതുരം	- square
സർവസമം	- congruent
സർവസമത	- congruence
സപ്തഭുജം	- heptagon
സമഭുജത്രികോണം	- equilateral triangle
സമപാർശ്വത്രികോണം	- isosceles triangle
സഞ്ചാരപാത	- locus
സാമാന്തരികം	- parallelogram
സമഭുജസാമാന്തരികം	- rhombus
സമപാർശ്വലംബകം	- isosceles trapezium
സമാനകോണുകൾ	- corresponding angles
സാദൃശ്യം	- similarity
സദൃശം	- similar
സമചതുരക്കട്ട	- cube
സമീപവശം	- adjacent side
സ്തംഭം	- prism
സ്തൂപികം	- pyramid
സമചതുരസ്തൂപിക	- square pyramid
സമബഹുമുഖം	- regular polyhedron
സമചതുര സ്തൂപികാപീഠം	- frustum of a square pyramid

സ്ഥിതിവിവരക്കണക്ക് (statistics)

ആവൃത്തി	- frequency
ആവൃത്തിപ്പട്ടിക	- frequency table
ആവൃത്തിബഹുഭുജം	- frequency polygon
ചതുരചിത്രം	- histogram
മധ്യമം	- median
മഹിതം	- mode
മാധ്യം	- arithmetic mean
വിഭാഗം	- class
വിഭാഗവിസ്താരം	- class width
സഞ്ചിതാവൃത്തി	- cumulative frequency
സാധ്യത	- probability