Exercise 11.4 Page: 262

In each of the Exercises 1 to 6, find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbolas.

1. $\frac{x^2}{16}-\frac{y^2}{9}=1$

Solution:

The given equation is $\frac{x^2}{16}-\frac{y^2}{9}=1$ or $\frac{x^2}{4^2}-\frac{y^2}{3^2}=1$.

On comparing this equation with the standard equation of hyperbola i.e., $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we obtain a = 4 and b = 3.

We known that $a^2 = b^2 + c^2$.

$\therefore c^2 = 4^2 + 3^2 = 25$

$\Rightarrow c = 5$

Therefore,

The coordinates of the foci are (±5, 0).

The coordinates of the vertices are (±4, 0).

Eccentricity, $e = \frac{c}{a} = \frac{5}{4}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{4} = \frac{9}{2}$

2. $\frac{y^2}{9}-\frac{x^2}{27}=1$

Solution:

The given equation is - =1 or $\frac{y^2}{3^2} - \frac{x^2}{\left(\sqrt{27}\right)^2} = 1$

On comparing this equation with the standard equation of hyperbola i.e., $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we obtain a = 3 and b = $\sqrt{27}$.

We known that $a^2 = b^2 + c^2$.

$\therefore c^2 = 3^2 + \left(\sqrt{27}\right)^2 = 9 + 27 = 36$

$\Rightarrow c = 6$

The coordinates of the foci are (0, ±6).

The coordinates of the vertices are (0, ±3).

Eccentricity, $e = \frac{c}{a} = \frac{6}{3} = 2$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 27}{3} = 18$

3:
$9y^2 - 4x^2 = 36$

Solution:
The given equation is $9y^2 - 4x^2 = 36$.
It can be written as
$9y^2 - 4x^2 = 36$

Or, $\frac{y^2}{4} - \frac{x^2}{9} = 1$

Or, $\frac{y^2}{2^2} - \frac{x^2}{3^2} = 1$ ...(1)

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{y^2}{a^2} - \frac{x^2}{b^2}$, we obtain

a = 2 and b = 3.
We know that $a^2 + b^2 = c^2$.
$\therefore c^2 = 4 + 9 = 13$
$\Rightarrow c = \sqrt{13}$
Therefore,
The coordinates of the foci are $\left(0, \pm\sqrt{13}\right)$.

The coordinates of the vertices are $(0, \pm 2)$.

Eccentricity, $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{2} = 9$

4:
Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $16x^2 - 9y^2 = 576$

Solution:
The given equation is $16x^2 - 9y^2 = 576$.
It can be written as
$16x^2 - 9y^2 = 576$

$\Rightarrow \frac{x^2}{36} - \frac{y^2}{64} = 1$

$\Rightarrow \frac{x^2}{6^2} - \frac{y^2}{8^2} = 1$ ...(1)

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we obtain a = 6 and b = 8.

We know that $a^2 + b^2 = c^2$.

$\therefore c^2 = 36+64=100$

$\Rightarrow c = 10$

Therefore,

The coordinates of the foci are (±10, 0).

The coordinates of the vertices are (±6, 0).

Eccentricity, $e = \frac{c}{a} = \frac{10}{6} = \frac{5}{3}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 64}{6} = \frac{64}{3}$

5:

Find the coordinates of the foci and the vertices, the eccentricity, and the length of the latus rectum of the hyperbola $5y^2 - 9x^2 = 36$

Solution:

The given equation is $5y^2 – 9x^2 = 36$.

$$\Rightarrow \frac{y^2}{\left(\frac{36}{5}\right)} - \frac{x^2}{4} = 1$$

$$\Rightarrow \frac{y^2}{\left(\frac{6}{\sqrt{5}}\right)} - \frac{x^2}{2^2} = 1 \quad ...(1)$$

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, we obtain a = $\frac{6}{\sqrt{5}}$ and b = 2.

We know that $a^2 + b^2 = c^2$.

$\therefore c^2 = \frac{36}{5}+4=\frac{56}{5}$

$\Rightarrow c = \sqrt{\frac{56}{5}} = \frac{2\sqrt{14}}{\sqrt{5}}$

Therefore, the coordinates of the foci are $\left(0, \pm\frac{2\sqrt{14}}{\sqrt{5}}\right)$.

The coordinates of the vertices are $\left(0, \pm\frac{6}{\sqrt{5}}\right)$.

$$\text{Eccentricity, e} = \frac{c}{a} = \frac{\left(\frac{2\sqrt{14}}{\sqrt{5}}\right)}{\left(\frac{6}{\sqrt{5}}\right)} = \frac{\sqrt{14}}{3}$$

$$\text{Length of latus rectum} = \frac{2b^2}{a} = \frac{2\times 4}{\left(\frac{6}{\sqrt{5}}\right)} = \frac{4\sqrt{5}}{3}$$

6:

$49y^2 - 16x^2 = 784$

Solution:

The given equation is $49y^2 - 16x^2 = 784$.

It can be written as $49y^2 - 16x^2 = 784$

Or, $\frac{y^2}{16} - \frac{x^2}{49} = 1$

Or, $\frac{y^2}{4^2} - \frac{x^2}{7^2} = 1 \quad ...(1)$

On comparing equation (1) with the standard equation of hyperbola i.e., $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, we obtain a = 4 and b = 7.

We know that $a^2 + b^2 = c^2$.

$\therefore c^2 = 16 + 49 = 65$

$\Rightarrow c = \sqrt{65}$

Therefore,

The coordinates of the foci are $\left(0, \pm\sqrt{65}\right)$.

The coordinates of the vertices are (0, ±4).

$$\text{Eccentricity, e} = \frac{c}{a} = \frac{\sqrt{65}}{4}$$

$$\text{Length of latus rectum} = \frac{2b^2}{a} = \frac{2\times 49}{4} = \frac{49}{2}$$

In each of the Exercises 7 to 15, find the equations of the hyperbola satisfying the given conditions.

7:

Vertices (±2, 0), foci (±3, 0)

Solution:

Vertices (±2, 0), foci (±3, 0)

Here, the vertices are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Since the vertices are (±2, 0), a = 2.
Since the foci are (±3, 0), c = 3.
We know that $a^2 + b^2 = c^2$.
$\therefore 2^2 + b^2 = 3^2$
$b^2 = 9 - 4 = 5$

Thus, the equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{5} = 1$

8:
Vertices (0, ±5), foci (0, ±8)

Solution:
Vertices (0, ±5), foci (0, ±8)
Here, the vertices are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Since the vertices are (0, ±5), a = 5.
Since the foci are (0, ±8), c = 8.
We know that $a^2 + b^2 = c^2$.
$\therefore 5^2 + b^2 = 8^2$
$b^2 = 64 - 25 = 39$

Thus, the equation of the hyperbola is $\frac{y^2}{25}$ - $\frac{x^2}{39}$ = 1.

9:
Vertices (0, ±3), foci (0, ±5)

Solution:
Vertices (0, ±3), foci (0, ±5)
Here, the vertices are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

Since the vertices are (0, ±3), a = 3.
Since the foci are (0, ±5), c = 5.
We know that $a^2 + b^2 = c^2$.
$\therefore 3^2 + b^2 = 52$
$\Rightarrow b^2 = 25 - 9 = 16$

Thus, the equation of the hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

10:

Foci (±5, 0), the transverse axis is of length 8.

Solution:

Foci (±5, 0), the transverse axis is of length 8.

Here, the foci are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

Since the foci are (±5, 0), c = 5.

Since the length of the transverse axis is 8, 2a = 8 ⇒ a = 4.

We know that $a^2 + b^2 = c^2$.

$\therefore 4^2 + b^2 = 52$

$\Rightarrow b^2 = 25 - 16 = 9$

Thus, the equation of the hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$

11:

Foci (0, ±13), the conjugate axis is of length 24.

Solution:

Foci (0, ±13), the conjugate axis is of length 24.

Here, the foci are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1.$

Since the foci are (0, ±13), c = 13.

Since the length of the conjugate axis is 24, 2b = 24 ⇒ b = 12.

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + 12^2 = 13^2$

$\Rightarrow a^2 = 169 - 144 = 25$

Thus, the equation of the hyperbola is $\frac{y^2}{25}-\frac{x^2}{144}=1.$

12: $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1.$

Foci , the latus rectum is of length 8. $\left(\pm 3\sqrt{5},0\right)$

Solution:

Foci $\left(\pm 3\sqrt{5},0\right)$, the latus rectum is of length 8.

Here, the foci are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

Since the foci are $\left(\pm 3\sqrt{5},0\right)$, c = $\pm 3\sqrt{5}$

Length of latus rectum = 8

$\Rightarrow \frac{2b^2}{a} = 8$

$\Rightarrow b^2 = 4a$

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + 4a = 45$

$\Rightarrow a^2 + 4a - 45 = 0$

$\Rightarrow a^2 + 9a - 5a - 45 = 0$

$\Rightarrow (a + 9)(a - 5) = 0$

$\Rightarrow a = -9, 5$

Since a is non-negative, a = 5.

$\therefore b^2 = 4a = 4 \times 5 = 20$

Thus, the equation of the hyperbola is $\frac{x^2}{25} - \frac{y^2}{20} = 1$.

13:

Foci (±4, 0), the latus rectum is of length 12.

Solution:

Foci (±4, 0), the latus rectum is of length 12.

Here, the foci are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Since the foci are (±4, 0), c = 4.

Length of latus rectum = 12

$\Rightarrow \frac{2b^2}{a} = 12$

$\Rightarrow b^2 = 6a$

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + 6a = 16$

$\Rightarrow a^2 + 6a - 16 = 0$

$\Rightarrow a^2 + 8a - 2a - 16 = 0$

$\Rightarrow (a + 8)(a - 2) = 0$

$\Rightarrow a = -8, 2$

Since a is non-negative, a = 2.

$\therefore b^2 = 6a = 6 \times 2 = 12$

Thus, the equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{12} = 1$

14:

Vertices (±7, 0), $e = \frac{4}{3}$

Solution:

Vertices (±7, 0), e =

Here, the vertices are on the x-axis.

Therefore, the equation of the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Since the vertices are (±7, 0), a = 7.

It is given that e =

$$\therefore \frac{c}{a}=\frac{4}{3} \quad \left[e=\frac{c}{a}\right]$$

$$\Rightarrow \frac{c}{7}=\frac{4}{3}$$

$$\Rightarrow c=\frac{28}{3}$$

We know that $a^2 + b^2 = c^2$.

$$\therefore 7^2+b^2=\left(\frac{28}{3}\right)^2$$

$$\Rightarrow b^2=\frac{784}{9}-49$$

$$\Rightarrow b^2=\frac{784-441}{9}=\frac{343}{9}$$

Thus, the equation of the hyperbola is $\frac{x^2}{49}-\frac{y^2}{343}=1$.

15:

Foci $\left(0,\pm\sqrt{10}\right)$, passing through (2, 3)

Solution:

Foci , passing through (2, 3)

Here, the foci are on the y-axis.

Therefore, the equation of the hyperbola is of the form $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

Since the foci are , c = $\sqrt{10}$.

We know that $a^2 + b^2 = c^2$.

$\therefore a^2 + b^2 = 10$

$\Rightarrow b^2 = 10 - a^2$ … (1)

Since the hyperbola passes through point (2, 3),

$$\frac{9}{a^2}-\frac{4}{b^2}=1 \qquad ….(2)$$

From equations (1) and (2), we obtain

$\frac{9}{a^2} - \frac{4}{(10-a)^2} = 1$

$\Rightarrow 9(10-a^2) - 4a^2 = a^2(10-a^2)$

$\Rightarrow 90 - 9a^2 - 4a^2 = 10a^2 - a^2$

$\Rightarrow a^2 - 23a^2 + 90 = 0$

$\Rightarrow a^4 - 18a^2 - 5a^2 + 90 = 0$

$\Rightarrow a^2(a^2 - 18) - 5(a^2 - 18) = 0$

$\Rightarrow (a^2 - 18) - (a^2 - 5) = 0$

$\Rightarrow a^2 = 18$ or 5

In hyperbola, c > a, i.e., $c^2 > a^2$

$\therefore a^2 = 5$

$\Rightarrow b^2 = 10 - a^2 = 10 - 5 = 5$

Thus, the equation of the hyperbola is $\frac{y^2}{5} - \frac{x^2}{5} = 1$.