Miscellaneous Exercise

1: Prove that: $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

Solution:

L.H.S. $=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$

$=2\cos\frac{\pi}{3}\cos\frac{9\pi}{13}+2\cos\left(\frac{\frac{3\pi}{13}+\frac{5\pi}{13}}{2}\right)\cos\left(\frac{\frac{3\pi}{13}-\frac{5\pi}{13}}{2}\right)$

$\left[\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right]$

$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\left(\frac{-\pi}{13}\right)$

$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$

$=2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$

$=2\cos\frac{\pi}{13}\left[\cos\frac{9\pi}{13}+\cos\frac{4\pi}{13}\right]$

$=2\cos\frac{\pi}{13}\left[2\cos\left(\frac{\frac{9\pi}{13}+\frac{4\pi}{13}}{2}\right)\cos\frac{\frac{9\pi}{13}-\frac{4\pi}{13}}{2}\right]$

$=2\cos\frac{\pi}{13}\left[2\cos\frac{\pi}{2}\cos\frac{5\pi}{26}\right]$

$=2\cos\frac{\pi}{13}\times2\times0\times\cos\frac{5\pi}{26}$

$=0=$ R.H.S.

2:

Prove that: $(\sin3x+\sin x)\sin x+(\cos3x-\cos x)\cos x=0$

Solution:

$=\cos2x-\cos2x$

L.H.S. $=(\sin3x+\sin x)\sin x+(\cos3x-\cos x)\cos x$

$=\sin3x\sin x+\sin^2x+\cos3x\cos x-\cos^2x$

$=\cos3x\cos x+\sin3x\sin x-(\cos^2-\sin^2x)$

$=\cos(3x-x)-\cos2x$ $\quad\left[\cos(A-B)=\cos A\cos B+\sin A\sin B\right]$

= 0

= R.H.S.

3:

Prove that: $(\cos x+\cos y)^2+(\sin x-\sin y)^2=4\cos^2\frac{x+y}{2}$

Solution:

L.H.S. $=(\cos x+\cos y)^2+(\sin x-\sin y)^2$

$=\cos^2 x+\cos^2 y+2\cos x\cos y+\sin^2 x+\sin^2 y-2\sin x\sin y$

$=(\cos^2 x+\sin^2 x)+(\cos^2 y+\sin^2 y)+2(\cos x\cos y-\sin x\sin y)$

$=1+1+2\cos(x+y)$ $\quad[\cos(A+B)=(\cos A\cos B-\sin A\sin B)]$

$=2+2\cos(x+y)$

$=2[1+\cos(x+y)]$

$=2\left[1+2\cos^2\left(\frac{x+y}{2}\right)-1\right]$ $\quad[\cos 2A=2\cos^2 A-1]$

$=4\cos^2\left(\frac{x+y}{2}\right)=$ R.H.S

4:

Prove that: $(\cos x-\cos y)^2+(\sin x-\sin y)^2=4\sin^2\frac{x-y}{2}$

Solution:

L.H.S. $(\cos x-\cos y)^2+(\sin x-\sin y)^2$

$=\cos^2 x+\cos^2 y-2\cos x\cos y+\sin^2 x+\sin^2 y-2\sin x\sin y$

$=(\cos^2 x+\sin^2 x)+(\cos^2 y+\sin^2 y)-2[\cos x\cos y+\sin x\sin y]$

$=1+1-2[\cos(x-y)]$ $\quad[\cos(A-B)=\cos A\cos B+\sin A\sin B]$

$=2[1-\cos(x-y)]$

$=2\left[1-\left\{1-2\sin^2\left(\frac{x-y}{2}\right)\right\}\right]$ $\quad[\cos 2A=1-2\sin^2 A]$

$=4\sin^2\left(\frac{x-y}{2}\right)=$ R.H.S

5:

Prove that: $\sin x+\sin 3x+\sin 5x+\sin 7x=4\cos x\cos 2x\sin 4x$

Solution:

It is known that $\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$

L.H.S. $=\sin x+\sin 3x+\sin 5x+\sin 7x$

$=(\sin x+\sin 5x)+(\sin 3x+\sin 7x)$

$=2\sin\left(\frac{x+5x}{2}\right).\cos\left(\frac{x-5x}{2}\right)+2\sin\left(\frac{3x+7x}{2}\right)\cos\left(\frac{3x-7x}{2}\right)$

$$= 2\sin 3x\cos(-2x) + 2\sin 5x\cos(-2x)$$
$$= 2\sin 3x\cos 2x + 2\sin 5x\cos 2x$$
$$= 2\cos 2x[\sin 3x + \sin 5x]$$
$$= 2\cos 2x\left[2\sin\left(\frac{3x+5x}{2}\right).\cos\left(\frac{3x-5x}{2}\right)\right]$$
$$= 2\cos 2x[2\sin 4x.\cos(-x)]$$
$$= 4\cos 2x\sin 4x\cos x = \text{R.H.S.}$$

6:

Prove that: $\dfrac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)} = \tan 6x$

Solution:

It is known that

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right), \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)$$

$$\text{L.H.S.} = \frac{(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)}{(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)}$$
$$= \frac{\left[2\sin\left(\frac{7x+5x}{2}\right).\cos\left(\frac{7x-5x}{2}\right)\right] + \left[2\sin\left(\frac{9x+3x}{2}\right).\cos\left(\frac{9x-3x}{2}\right)\right]}{\left[2\cos\left(\frac{7x+5x}{2}\right).\cos\left(\frac{7x-5x}{2}\right)\right] + \left[2\cos\left(\frac{9x+3x}{2}\right).\cos\left(\frac{9x-3x}{2}\right)\right]}$$

$= \text{R.H.S.}$

$$= \frac{[2\sin 6x.\cos x] + [2\sin 6x.\cos 3x]}{[2\cos 6x.\cos x] + [2\cos 6x.\cos 6x]}$$
$$= \frac{2\sin 6x[\cos x + \cos 3x]}{2\cos 6x[\cos x + \cos 3x]}$$
$$= \tan 6x$$

7:

Prove that: $\sin 3x + \sin 2x - \sin x = 4\sin x\cos\dfrac{x}{2}\cos\dfrac{3x}{2}$

Solution:

$$\text{L.H.S.} = \sin 3x + \sin 2x - \sin x$$
$$= \sin 3x + \left[2\cos\left(\frac{2x+x}{2}\right)\sin\left(\frac{2x-x}{2}\right)\right] \qquad \left[\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$
$$= \sin 3x + \left[2\cos\left(\frac{3x}{2}\right)\sin\left(\frac{x}{2}\right)\right]$$

$$= \sin 3x + 2\cos\frac{3x}{2}\sin\frac{x}{2}$$

$$= 2\sin\frac{3x}{2}.\cos\frac{3x}{2} + 2\cos\frac{3x}{2}\sin\frac{x}{2} \qquad [\sin 2A = 2\sin A.\cos B]$$

$$= 2\cos\left(\frac{3x}{2}\right)\left[\sin\left(\frac{3x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]$$

$$= 2\cos\left(\frac{3x}{2}\right)\left[2\sin\left\{\frac{\left(\frac{3x}{2}\right)+\left(\frac{x}{2}\right)}{2}\right\}\cos\left\{\frac{\left(\frac{3x}{2}\right)-\left(\frac{x}{2}\right)}{2}\right\}\right]$$

$$\left[\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\right]$$

$$= 2\cos\left(\frac{3x}{2}\right).2\sin x\cos\left(\frac{x}{2}\right)$$

$$= 4\sin x\cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right) = \text{R.H.S.}$$

8:

Find $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$, if $\tan x = \frac{-4}{3}, x$ in quadrant II

Solution:

Here, x is in quadrant II.

i.e., $\frac{\pi}{2} < x < \pi$

$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

There, $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$

are lies in first quadrant.

It is given that $\tan x = -\frac{4}{3}$

$$\sec^2 x = 1 + \tan^2 x = 1 + \left(\frac{-4}{3}\right)^2 = 1 + \frac{16}{9} = \frac{25}{9}$$

$$\therefore \cos^2 x = \frac{9}{25}$$

$$\Rightarrow \cos x = \pm\frac{3}{5}$$

As x is in quadrant II, $\cos x$ is negative.

$\cos x = \frac{-3}{5}$

Now, $\cos x = 2\cos^2\frac{x}{2} - 1$

$\Rightarrow \frac{-3}{5} = 2\cos^2\frac{x}{2} - 1$

$\Rightarrow 2\cos^2\frac{x}{2} = 1 - \frac{3}{5}$

$\Rightarrow 2\cos^2\frac{x}{2} = \frac{2}{5}$

$\Rightarrow \cos^2\frac{x}{2} = \frac{1}{5}$

$\Rightarrow \cos\frac{x}{2} = \frac{1}{\sqrt{5}} \qquad \left[\because \cos\frac{x}{2} \text{ is positive}\right]$

$\therefore \cos\frac{x}{2} = \frac{\sqrt{5}}{5}$

$\sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1$

$\Rightarrow \sin^2\frac{x}{2} + \left(\frac{1}{\sqrt{5}}\right)^2 = 1$

$\Rightarrow \sin^2\frac{x}{2} = 1 - \frac{1}{5} = \frac{4}{5}$

$\Rightarrow \sin^2\frac{x}{2} = \frac{2}{\sqrt{5}} \qquad \left[\because \sin\frac{x}{2} \text{ is positive}\right]$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\left(\frac{2}{\sqrt{5}}\right)}{\left(\frac{1}{\sqrt{5}}\right)} = 2$$

Thus, the respective values of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}$, and 2.

9:

Find, $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ for $\cos x = -\frac{1}{3}, x$ in quadrant III

Solution:

Here, x is in quadrant III.

i.e., $\pi < x < \frac{3\pi}{2}$

$\Rightarrow \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$

Therefore, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are negative, where $\sin\frac{x}{2}$ as is positive.

It is given that $\cos x = -\frac{1}{3}$.

$$\cos x = 1 - 2\sin^2\frac{x}{2}$$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{1-\cos x}{2}$$

$$\Rightarrow \sin^2\frac{x}{2} = \frac{1-\left(-\frac{1}{3}\right)}{2} = \frac{\left(1+\frac{1}{3}\right)}{2} = \frac{4/3}{2} = \frac{2}{3}$$

$$\Rightarrow \sin\frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \qquad \left[\because \sin\frac{x}{2} \text{ is positive}\right]$$

$$\therefore \sin\frac{x}{2} = \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

Now $\cos x = 2\cos^2\frac{x}{2} - 1$

$$\Rightarrow \cos^2\frac{x}{2} = \frac{1+\cos x}{2} = \frac{1+\left(-\frac{1}{3}\right)}{2} = \frac{\left(\frac{3-1}{3}\right)}{2} = \frac{\left(\frac{2}{3}\right)}{2} = \frac{1}{3}$$

$$\Rightarrow \cos\frac{x}{2} = -\frac{1}{\sqrt{3}} \qquad \left[\because \cos\frac{x}{2} \text{ is negative}\right]$$

$$\therefore \cos\frac{x}{2} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)}{\left(\frac{-1}{\sqrt{3}}\right)} = -\sqrt{2}$$

Thus, the respective values of $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ are $\frac{\sqrt{6}}{3}, \frac{-\sqrt{3}}{3}$, and $-\sqrt{2}$.

10:

Find $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$ *for* $\sin x = \frac{1}{4}$, x in quadrant II

Solution:

Here, x is in quadrant II.

i.e., $\frac{\pi}{2} < x < \pi$

$$\Rightarrow \frac{\pi}{4} < \frac{x}{2} < \frac{\pi}{2}$$

Therefore, $\sin\frac{x}{2}, \cos\frac{x}{2}, \tan\frac{x}{2}$ are all positive.

It is given that $\sin x = \frac{1}{4}$

$$\cos^2 x = 1 - \sin^2 x = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\Rightarrow \cos x = -\frac{\sqrt{15}}{4} \text{ [cos x is negative in quadrant II]}$$

$$\sin^2\frac{x}{2} = \frac{1-\cos x}{2} - \frac{1-\left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4+\sqrt{15}}{8}$$

$$\Rightarrow \sin\frac{x}{2} = \sqrt{\frac{4+\sqrt{15}}{8}} \qquad \left[\because \sin\frac{x}{2} \text{ is negative}\right]$$

$$= \sqrt{\frac{4+\sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \sqrt{\frac{8+2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8+2\sqrt{15}}}{4}$$

$$\cos^2\frac{x}{2} = \frac{1+\cos x}{2} = \frac{1+\left(-\frac{\sqrt{15}}{4}\right)}{2} = \frac{4-\sqrt{15}}{8}$$

$$\Rightarrow \cos\frac{x}{2} = \sqrt{\frac{4-\sqrt{15}}{8}} \qquad \left[\because \cos\frac{x}{2} \text{ is positve}\right]$$

$$= \sqrt{\frac{4+\sqrt{15}}{8} \times \frac{2}{2}}$$

$$= \sqrt{\frac{8-2\sqrt{15}}{16}}$$

$$= \frac{\sqrt{8-2\sqrt{15}}}{4}$$

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\left(\frac{\sqrt{8+2\sqrt{15}}}{4}\right)}{\frac{\sqrt{8-2\sqrt{15}}}{4}} = \frac{\sqrt{8+2\sqrt{15}}}{\sqrt{8-2\sqrt{15}}}$$

$$= \sqrt{\frac{8+2\sqrt{15}}{8-2\sqrt{15}} \times \frac{8+2\sqrt{15}}{8+2\sqrt{15}}}$$

$$= \sqrt{\frac{\left(8+2\sqrt{15}\right)^2}{64-60}} = \frac{8+2\sqrt{15}}{2} = 4+\sqrt{15}$$

Thus, the respective values are $\sin\frac{x}{2}, \cos\frac{x}{2}$ and $\tan\frac{x}{2}$

are $\frac{\sqrt{8+2\sqrt{15}}}{4}, \frac{\sqrt{8-2\sqrt{15}}}{4}$ and $4+\sqrt{15}$