Exercise 4.1

1:

Prove that following by using the principle of mathematical induction for all $n \in N$:

$$1+3+3^2+.....+3^{n-1}=\frac{(3^n-1)}{2}$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n):1+3+3^2+.....+3^{n-1}=\frac{(3^n-1)}{2}$$

For $n=1$, we have

$$P(1):=\frac{(3^1-1)}{2}=\frac{3-1}{2}=\frac{2}{2}=1, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k, i.e.,

$$1+3+3^2+....+3^{k-1}=\frac{(3^k-1)}{2} \quad ....(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$1+3+3^2+....+3^{k-1}+3^{(k+1)-1}$$

$$=(1+3+3^2+....+3^{k-1})+3^k$$

$$=\frac{(3^k-1)}{2}+3^k \qquad [\text{Using}(i)]$$

$$=\frac{(3^k-1)+2.3^k}{2}$$

$$=\frac{(1+2)3^k-1}{2}$$

$$=\frac{3.3^k-1}{2}$$

$$=\frac{3^{k+1}-1}{2}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

2:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^3+2^3+3^3+.....+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n):1^3+2^3+3^3+......+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

For $n=1$, we have

$$P(1):1^3=1=\left(\frac{1(1+1)^2}{2}\right)=\left(\frac{1.2}{2}\right)^2=1^2=1$$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$$1^3+2^3+3^3+.....+k^3=\left(\frac{k(k+1)}{2}\right)^2 \quad ....(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$1^3+2^3+3^3+......+k^3+(k+1)^3$$

$$=\left(1^3+2^3+3^3+.....+k^3\right)+(k+1)^3$$

$$=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \qquad [\text{Using}(i)]$$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2\left\{k^2+4(k+1)\right\}}{4}$$

$$=\frac{(k+1)^2\left\{k^2+4k+4\right\}}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

$$=\frac{(k+1)^2(k+1+1)^2}{4}$$

$$=\left(\frac{(k+1)(k+1+1)}{2}\right)^2$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., N.

3:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+.....+\frac{1}{(1+2+3+...n)}=\frac{2n}{(n+1)}$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n):1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{1}{1+2+3+....n}=\frac{2n}{n+1}$$

For $n=1$, we have

$P(1):1=\frac{2.1}{1+1}=\frac{2}{2}=1$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$$1+\frac{1}{1+2}+...+\frac{1}{1+2+3}+....+\frac{1}{1+2+3+.....+k}=\frac{2k}{k+1}\quad ....(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+.....+\frac{1}{1+2+3+.....+k}+\frac{1}{1+2+3+.....+k+(k+1)}$$

$$=\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+....+\frac{1}{1+2+3+.....k}\right)+\frac{1}{1+2+3+....+k+(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+.....+k+(k+1)} \qquad \text{[Using (i)]}$$

$$=\frac{2k}{k+1}+\frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)} \qquad \left[1+2+3....+n=\frac{n(n+1)}{2}\right]$$

$$=\frac{2k}{(k+1)}+\frac{2}{(k+1)(k+2)}$$

$$=\frac{2}{(k+1)}\left(k+\frac{1}{k+2}\right)$$

$$=\frac{2}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$=\frac{2}{(k+1)}\left[\frac{(k+1)^2}{k+2}\right]$$

$$=\frac{2(k+1)}{(k+2)}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., N.

4:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.2.3+2.3.4+....+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n):1.2.3+2.3.4+....+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$$

For $n=1$, we have

$P(1):1.2.3=6=\frac{1(1+1)(1+2)(1+3)}{4}=\frac{1.2.3.4}{4}=6$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1.2.3+2.3.4+.....+k(k+1)(k+2)=\frac{k(k+1)(k+2)(k+3)}{4} \quad ......(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$1.2.3+2.3.4+.....+k(k+1)(k+2)+(k+1)(k+2)(k+3)$$
$$=\{1.2.3+2.3.4+.....+k(k+1)(k+2)\}+(k+1)(k+2)+(k+3)$$
$$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3) \qquad [\text{Using}(i)]$$
$$=(k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$
$$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$$
$$=\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., N.

5:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.3+2.3^2+3.3^3+.....+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n): 1.3+2.3^2+3.3^3+....+n3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

For $n=1$, we have

$P(1): 1.3=3=\frac{(2.1-1)3^{1+1}+3}{4}=\frac{3^2+3}{4}=\frac{12}{4}=3$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$$1.3+2.3^2+3.3^3+.....+k3^k=\frac{(2k-1)3^{k+1}+3}{4} \quad ....(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$1.3+2.3^2+3.3^3+.....+k.3^k+(k+1).3^{k+1}$$

$$=(1.3+2.3^2+3.3^3+.....+k.3^k)+(k+1).3^{k+1}$$

$$=\frac{(2k-1)3^{k+1}+3}{4}+(k+1)3^{k-1} \qquad [\text{Using}(i)]$$

$$=\frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

$$=\frac{3^{k+1}\{2k-1+4(k+1)\}+3}{4}$$

$$=\frac{3^{k+1}\{6k+3\}+3}{4}$$

$$=\frac{3^{k+1}.3\{2k+1\}+3}{4}$$

$$=\frac{3^{(k+1)+1}\{2k+1\}+3}{4}$$

$$=\frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., N.

6:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.2+2.3+3.4+....+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

Solution:

Let the given statement be P(n), i.e.,

$$P(n): 1.2+2.3+3.4+....+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

For n = 1, we have

$P(1):1.2=2=\frac{1(1+1)(1+2)}{3}=\frac{1.2.3}{3}=2$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1.2+2.3+3.4+.....+k.(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right] \quad ....(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$1.2+2.3+3.4+....+k.(k+1)+(k+1).(k+2)$

$=[1.2+2.3+3.4+.....+k.(k+1)]+(k+1).(k+2)$

$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \qquad [\text{Using}(i)]$

$=(k+1)(k+2)\left(\frac{k}{3}+1\right)$

$=\frac{(k+1)(k+2)(k+3)}{3}$

$=\frac{(k+1)(k+1+1)(k+1+2)}{3}$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., N.

7:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.3+3.5+5.7+.....+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

Solution:

Let the given statement be P(n), i.e.,

$$P(n):1.3+3.5+5.7+.....+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

For n = 1, we have

$P(1):1.3=3=\frac{1(4.1^2+6.1-1)}{3}=\frac{4+6-1}{3}=\frac{9}{3}=3$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$$1.3+3.5+5.7+.....+(2k-1)(2k+1)=\frac{k(4k^2+6k-1)}{3} \quad ....(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$(1.3+3.5+5.7+.....+(2k-1)(2k+1))+\{(k+1)-1\}\{2(k+1)+1\}$$

$$=\frac{k(4k^2+6k-1)}{3}+(2k+2-1)(2k+2+1) \quad [\text{Using}(i)]$$

$$=\frac{k(4k^2+6k-1)}{3}+(2k+1)(2k+3)$$

$$=\frac{k(4k^2+6k-1)}{3}+(4k^2+8k+3)$$

$$=\frac{k(4k^2+6k-1)+3(4k^2+8k+3)}{3}$$

$$=\frac{4k^3+6k^2-k+12k^2+24k+9}{3}$$

$$=\frac{4k^3+18k^2+23k+9}{3}$$

$$=\frac{4k^3+14k^2+9k+4k^2+14k+9}{3}$$

$$=\frac{k(4k^2+14k+9)+1(4k^2+14k+9)}{3}$$

$$=\frac{(k+1)(4k^2+14k+9)}{3}$$

$$=\frac{(k+1)\{4k^2+8k+4+6k+6-1\}}{3}$$

$$=\frac{(k+1)\{4(k^2+2k+1)+6(k+1)-1\}}{3}$$

$$=\frac{(k+1)\{4(k+1)^2+6(k+1)-1\}}{3}$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

8:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.2+2.2^2+3.2^2+....+n.2^n=(n-1)2^{n+1}+2$$

Solution:

Let the given statement be P(n), i.e.,

$P(n): 1.2 + 2.2^2 + 3.2^2 + ... + n.2^n = (n-1)2^{n+1} + 2$

For n = 1, we have

$P(1): 1.2 = 2 = (1-1)2^{1+1} + 2 = 0 + 2 = 2$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1.2 + 2.2^2 + 3.2^2 + ... + k.2^k = (k-1)2^{k+1} + 2.....(i)$

We shall now prove that $P(k+1)$ is true.

Consider

$\{1.2 + 2.2^2 + 3.2^2 + ... + k.2^k\} + (k+1).2^{k+1}$

$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$

$= 2^{k+1}\{(k-1) + (k+1)\} + 2$

$= 2^{k+1}.2k + 2$

$= k.2^{(k+1)+1} + 2$

$= \{(k+1) - 1\}2^{(k+1)+1} + 2$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

9:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

Solution:

Let the given statement be P(n), i.e.,

$P(n): \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

For $n = 1$, we have

$P(1): \frac{1}{2} = 1 - \frac{1}{2^1} = \frac{1}{2}$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k} = 1 - \frac{1}{2^k}$ ......(i)

We shall now prove that $P(k+1)$ is true.

Consider

$\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^k}\right) + \frac{1}{2^{k+1}}$

$= \left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+1}}$ $\quad [\text{Using}(i)]$

$$1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$$
$$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$$
$$=1-\frac{1}{2^{k+1}}$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

10:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+......+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n):\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+......+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

For $n=1$, we have

$P(1)=\frac{1}{2.5}=\frac{1}{10}=\frac{1}{6.1+4}=\frac{1}{10}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.....+\frac{1}{(3k+1)(3k+2)}=\frac{k}{6k+4} \quad ......(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.....+\frac{1}{(3k+1)(3k+2)}+\frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$
$$=\frac{k}{6k+4}+\frac{1}{(3k+3-1)(3k+3+2)} \qquad [\text{Using}(i)]$$
$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$
$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$
$$=\frac{1}{(3k+2)}\left(\frac{k}{2}+\frac{1}{3k+5}\right)$$
$$=\frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{2(3k+5)}\right)$$

$$=\frac{(k+1)}{6k+10}$$

$$=\frac{(k+1)}{6(k+1)+4}$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e, N.

11:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+.....+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Solution:

Let the given statement be $P(n)$, i.e.,

$$P(n):\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+.....+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

For n = 1, we have

$$P(1):\frac{1}{1\cdot2\cdot3}=\frac{1\cdot(1+3)}{4(1+1)(1+2)}=\frac{1\cdot4}{4\cdot2\cdot3}=\frac{1}{1\cdot2\cdot3}$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+......+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)} \qquad \ldots\ldots \text{(i)}$$

We shall now prove that $P(k+1)$ is true.

Consider

$$\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+......+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \qquad \text{[Using (i)]}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+6k+9\right)+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$$

$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

12:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$a+ar+ar^2+.....+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

Solution:

Let the given statement be P(n), i.e.,

$$P(n): a+ar+ar^2+.....+ar^{n-1}=\frac{a\left(r^n-1\right)}{r-1}$$

For n = 1, we have

$P(1): a=\frac{a\left(r^1-1\right)}{(r-1)}=a$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$a+ar+ar^2+......+ar^{k-1}=\frac{a\left(r^k-1\right)}{r-1}.......(i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$\left\{a+ar+ar^2+.....+ar^{k-1}\right\}+ar^{(k+1)-1}$

$=\dfrac{a\left(r^k-1\right)}{r-1}+ar^k \qquad \left[\text{Using}\,(i)\right]$

$=\dfrac{a\left(r^k-1\right)+ar^k\left(r-1\right)}{r-1}$

$=\dfrac{a\left(r^k-1\right)+ar^{k-1}-ar^k}{r-1}$

$=\dfrac{ar^k-a+ar^{k+1}-ar^k}{r-1}$

$=\dfrac{ar^{k+1}-a}{r-1}$

$=\dfrac{a\left(r^{k+1}-1\right)}{r-1}$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

13:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right).....\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$

Solution:

Let the given statement be P(n), i.e.,

$P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right).....\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$

For n = 1, we have

$P(1):\left(1+\frac{3}{1}\right)=4=(1+1)^2=2^2=4$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right).....\left(1+\frac{(2k+1)}{k^2}\right)=(k+1)^2 ......(1)$

We shall now prove that $P(k+1)$ is true.

Consider

$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)....\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$

$=(k+1)^2\left(1+\dfrac{2(k+1)+1}{(k+1)^2}\right) \qquad \left[\text{Using}\,(1)\right]$

$$=(k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$$
$$=(k+1)^2+2(k+1)+1$$
$$=\{(k+1)+1\}^2$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

14:

Prove the following by using principle of mathematical induction for all $n \in N$:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right).....\left(1+\frac{1}{n}\right)=(n+1)$$

Solution:

Let the given statement be P(n), i.e.,

$$P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right).....\left(1+\frac{1}{n}\right)=(n+1)$$

For n = 1, we have

$P(1):\left(1+\frac{1}{1}\right)=2=(1+1)$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$P(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)....\left(1+\frac{1}{k}\right)=(k+1) \;.......(1)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right).....\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$
$$=(k+1)\left(1+\frac{1}{k+1}\right) \qquad [\text{Using}(1)]$$
$$=(k+1)\left[\frac{(k+1)+1}{(k+1)}\right]$$
$$=(k+1)+1$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

15:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^2+3^2+5^2+.....+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Solution:

Let the given statement be P(n), i.e.,

$$P(n):1^2+3^2+5^2+.....+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

For $n=1$, we have

$$P(1)=1^2=1=\frac{1(2.1-1)(2.1+1)}{3}=\frac{1.1.3}{3}=1, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$P(k)=1^2+3^2+5^2+....+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3} \quad ......(1)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$\left\{1^2+3^2+5^2+.....+(2k-1)^2\right\}+\left\{2(k+1)-1\right\}^2$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+2-1)^2 \qquad [\text{Using}(1)]$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

$$=\frac{2(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$=\frac{(2k+1)\{k(2k-1)+3(2k+1)\}}{3}$$

$$=\frac{(2k+1)\{2k^2-k+6k+3\}}{3}$$

$$=\frac{(2k+1)\{2k^2+5k+3\}}{3}$$

$$=\frac{(2k+1)\{2k^2+2k+3k+3\}}{3}$$

$$=\frac{(2k+1)\{2k(k+1)+3(k+1)\}}{3}$$

$$=\frac{(2k+1)(k+1)(2k+3)}{3}$$

$$=\frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

16:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+....+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

Solution:

Let the given statement be P(n), i.e.,

$$P(n): \frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+....+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

For n = 1, we have

$$P(1)=\frac{1}{1.4}=\frac{1}{3.1+1}=\frac{1}{4}+\frac{1}{1.4},$$ which is true.

Let P(k) be true for some positive integer k, i.e.,

$$P(k)=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+.....+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \quad ......(1)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+....+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)} \qquad \text{[Using (1)]}$$

$$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)}{3(k+1)+1}$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

17:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+......+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Solution:

Let the given statement be P(n), i.e.,

$$P(n):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+......+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

For n = 1, we have

$P(1):\frac{1}{3.5}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$P(k):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+....+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \quad ....(1)$$

We shall now prove that $P(k+1)$ is true. Consider

$$\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+....+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

$$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \qquad \text{[Using (1)]}$$

$$=\frac{1}{(2k+3)}\left[\frac{k}{3}+\frac{1}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$$

$$=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)}{3\{2(k+1)+3\}}$$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

18:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$1+2+3+....+n<\frac{1}{8}(2n+1)^2$

Solution:

Let the given statement be P(n), i.e.,

$P(n):1+2+3+....+n<\frac{1}{8}(2n+1)^2$

It can be noted that P(n) is true for n = 1 since

$1<\frac{1}{8}(2.1+1)^2=\frac{9}{8}$

Let P(k) be true for some positive integer k, i.e.,

$1+2+....+k<\frac{1}{8}(2k+1)^2$ ......(1)

We shall now prove that $P(k+1)$ is true whenever P(k) is true.

Consider

$(1+2+....+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$ [Using (1)]

$<\frac{1}{8}\left\{(2k+1)^2+8(k+1)\right\}$

$<\frac{1}{8}\left\{4k^2+4k+1+8k+8\right\}$

$<\frac{1}{8}\left\{4k^2+12k+9\right\}$

$<\frac{1}{8}(2k+3)^2$

$<\frac{1}{8}\left\{2(k+1)+1\right\}^2$

Hence, $(1+2+3+....+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

19:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$n(n+1)(n+5)$ is a multiple of 3.

Solution:

Let the given statement be P(n), i.e.,

$P(n):n(n+1)(n+5)$, which is a multiple of 3.

It can be noted that $P(n)$ is true for $n=1$ since $1(1+1)(1+5)=12$, which is a multiple of 3.

Let P(k) be true for some positive integer k, i.e.,

$k(k+1)(k+5)$ is a multiple of 3.

$\therefore k(k+1)(k+5)=3m$, where $m \in \mathbf{N}$ ....(1)

We shall now prove that $P(k+1)$ is true whenever P(k) is true.

Consider

$(k+1)\{(k+1)+1\}\{(k+1)+5\}$

$=(k+1)(k+2)\{(k+5)+1\}$

$=(k+1)(k+2)(k+5)+(k+1)(k+2)$

$=\{k(k+1)(k+5)+2(k+1)(k+5)\}+(k+1)(k+2)$

$=3m+(k+1)\{2(k+5)+(k+2)\}$

$=3m+(k+1)\{2k+10+k+2\}$

$=3m+(k+1)\{3k+12\}$

$=3m+3(k+1)\{k+4\}$

$=3\{m+(k+1)(k+4)\}=3\times q$, where $q=\{m+(k+1)(k+4)\}$ is some natural number.

Therefore, $(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of 3.

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

20:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$10^{2n-1}+1$ is divisible by 11.

Solution:

Let the given statement be P(n), i.e.,

$P(n): 10^{2n-1}+1$ is divisible by 11.

It can be observed that $P(n)$ is true for n = 1

Since $P(1)=10^{2.1-1}+1=11$, which is divisible by 11.

Let $P(k)$ be true for some positive integer k,

i.e., $10^{2k-1}+1$ is divisible by 11.

$\therefore 10^{2k-1}+1=11m$, where

$m \in \mathbf{N}$ ....(1)

We shall now prove that $P(k+1)$ is true whenever P(k) is true.

Consider

$10^{2(k+1)-1}+1$

$=10^{2k+2-1}+1$

$=10^{2k+1}+1$
$=10^2\left(10^{2k-1}+1-1\right)+1$
$=10^2\left(10^{2k-1}+1\right)-10^2+1$
$=10^2.11m-100+1$ [Using (1)]
$=100\times 11m-99$
$=11(100m-9)$
$=11r$, where $r=(100m-9)$ is some natural number

Therefore, $10^{2(k+1)-1}+1$ is divisible by 11.
Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

21:
Prove the following by using the principle of mathematical induction for all $n \in N$:
$x^{2n}-y^{2n}$ is divisible by $x+y$.

Solution:
Let the given statement be $P(n)$, i.e.,
$P(n): x^{2n}-y^{2n}$ is divisible by $x+y$.
It can be observed that $P(n)$ is true for n = 1.
This is so because $x^{2\times1}-y^{2\times1}=x^2-y^2=(x+y)(x-y)$ is divisible by $(x+y)$.
Let $P(k)$ be true for some positive integer k, i.e.,
$x^{2k}-y^{2k}$ is divisible by $x+y$.
$\therefore$ Let $x^{2k}-y^{2k}=m(x+y)$, where $m \in \mathbf{N}$ .....(1)
We shall now prove that $P(k+1)$ is true whenever P(k) is true.
Consider
$x^{2(k+1)-y^{2(k+1)}}$
$=x^{2k}\cdot x^2-y^{2k}\cdot y^2$
$=x^2\left(x^{2k}-y^{2k}+y^{2k}\right)-y^{2k}\cdot y^2$
$=x^2\left\{m(x+y)+y^{2k}\right\}-y^{2k}\cdot y^2$ [Using (1)]
$=m(x+y)x^2+y^{2k}\cdot x^2-y^{2k}\cdot y^2$
$=m(x+y)x^2+y^{2k}\left(x^2-y^2\right)$
$=m(x+y)x^2+y^{2k}(x+y)(x-y)$
$=(x+y)\left\{mx^2+y^{2k}(x-y)\right\}$, which is a factor of $(x+y)$.
Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

22:

Prove the following by using the principle of mathematical induction for all $n \in N : 3^{2n+2} - 8n - 9$ is divisible by 8.

Solution:

Let the given statement be P(n), i.e.,

$P(n): 3^{2n+2} - 8n - 9$ is divisible by 8.

It can be observed that P(n) is true for n = 1

Since $3^{2\times1+2} - 8\times1 - 9 = 64$, which is divisible by 8.

Let P(k) be true for some positive integer

k, i.e., $3^{2k+2} - 8k - 9$ is divisible by 8.

$\therefore 3^{2k+2} - 8k - 9 = 8m;$ where $m \in N ....(1)$

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true.

Consider

$3^{2(k+1)+2} - 8(k+1) - 9$

$3^{2k+2} \cdot 3^2 - 8k - 8 - 9$

$= 3^2\left(3^{2k+2} - 8k - 9 + 8k + 9\right) - 8k - 17$

$= 3^2\left(3^{2k+2} - 8k - 9\right) + 3^2(8k+9) - 8k - 17$

$= 9.8m + 9(8k+9) - 8k - 17$

$= 9.8m + 72k + 81 - 8k - 17$

$= 9.8m + 64k + 64$

$= 8(9m + 8k + 8)$

$= 8r$, where $r = (9m + 8k + 8)$ is a natural number

Therefore, $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8.

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all numbers i.e., N.

23:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$41^n - 14^n$ is a multiple of 27.

Solution:

Let the given statement be $P(n)$, i.e.,

$P(n): 41^n - 14^n$ is a multiple of 27.

It can be observed that $P(n)$ is true for n = 1

Since $41^1 - 14^1 = 27$, which is a multiple of 27.

Let P(k) be true for some positive integer k, i.e.,

$41^k - 14^k$ is a multiple of 27

$\therefore 41^k - 14^k = 27m,\ m \in \mathbf{N}$......(1)

We shall now prove that $P(k+1)$ is true whenever P(k) is true.

Consider

$41^{k+1} - 14^{k+1}$

$= 41^k.41 - 14^k.14$

$= 41\left(41^k - 14^k + 14^k\right) - 14^k \cdot 14$

$= 41.27m + 14^k(41-14)$

$= 41.27m + 27.14^k$

$= 27\left(41m - 14^k\right)$

$= 27 \times r$, where $r = \left(41m - 14^k\right)$ is a natural number.

Therefore, $41^{k+1} - 14^{k+1}$ is a multiple of 27.

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.

24:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$(2n+7) < (n+3)^2$

Solution:

Let the given statement be $P(n)$, i.e.,

$P(n): (2n+7) < (n+3)^2$

It can be observed that $P(n)$ is true for n = 1

Since $2.1 + 7 = 9 < (1+3)^2 = 16$, which is true.

Let $P(k)$ be true for some positive integer k, i.e.,

$(2k+7) < (k+3)^2$ ......(1)

We shall now prove that $P(k+1)$ is true whenever $P(k)$ is true.

Consider

$\{2(k+1)+7\} = (2k+7)+2$

$\therefore \{2(k+1)+7\} = (2k+7)+2 < (k+3)^2 + 2$ [Using (1)]

$2(k+1)+7 < k^2 + 6k + 9 + 2$

$2(k+1)+7 < k^2 + 6k + 11$

Now, $k^2+6k+11<k^2+8k+16$

$\therefore\ 2(k+1)+7<(k+4)^2$

$2(k+1)+7<\{(k+1)+3\}^2$

Thus, $P(k+1)$ is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., N.