Exercise 8.1

Expand each of the expressions in Exercises 1 to 5.

1:

$(1-2x)^5$

Solution:

$$={}^5C_0(1)^5-{}^5C_1(1)^4(2x)+{}^5C_2(1)^3(2x)^2-{}^5C_3(1)^2(2x)^3+{}^5C_4(1)^1(2x)^4-{}^5C_5(2x)^5$$
$$=1-5(2x)+10(4x)^2-10(8x^3)+5(16x^4)-(32x^5)$$
$$=1-10x+40x^2-80x^3+80x^4-32x^5$$

2:

$$\left(\frac{2}{x}-\frac{x}{2}\right)^5$$

Solution:

By using Binomial Theorem, the expression $\left(\frac{2}{x}-\frac{x}{2}\right)^5$ can be expanded as

$$\left(\frac{2}{x}-\frac{x}{2}\right)^5={}^5C_0\left(\frac{2}{x}\right)^5-{}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right)+{}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2-{}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3+{}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4-{}^5C_5\left(\frac{x}{2}\right)^5$$
$$=\frac{32}{x^3}-5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right)+10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right)-10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right)+5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right)-\frac{x^5}{32}$$
$$=\frac{32}{x^5}-\frac{40}{x^3}+\frac{20}{x}-5x+\frac{5}{8}x^3-\frac{x^5}{32}$$

3:

$$(2x-3)^6$$

Solution:

By using Binomial Theorem the expression $(2x-3)^6$ can be expanded as

$$(2x-3)^6={}^6C_0(2x)^6-{}^6C_1(2x)^5(3)+{}^6C_2(2x)^4(3)^2-{}^6C_3(2x)^3(3)^3+{}^6C_4(2x)^2(3)^4-{}^6C_5(2x)(3)^5+{}^6C_6(3)^6$$
$$=64x^6-6(32x^5)(3)+15(16x^4)(9)-20(8x^3)(27)+15(4x^2)(81)-6(2x)(243)+729$$
$$=64x^6-576x^5+2160x^4-4320x^3+4860x^2-2916x+729$$

4:

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5$$

Solution:

By using Binomial Theorem, the expression $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ can be expanded as

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5$$

$$= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$

$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}$$

5:

$$\left(x+\frac{1}{x}\right)^6$$

Solution:

By using Binomial Theorem, the expression $\left(x+\frac{1}{x}\right)^6$ can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$

$$= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

Using binomial theorem, evaluate each of the following:

6: $(96)^3$

Solution:

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $96 = 100 - 4$

$$\therefore (96)^3 = (100-4)^3$$

$$= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - 3C_3(4)^3$$

$$= (100)^3 - 3(100)^2(4) + 3(100)(4)^2 - (4)^3$$

$$= 1000000 - 120000 + 4800 - 64$$

$$= 884736$$

7: $(102)^5$

Solution:

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $102 = 100 + 2$

$\therefore (102)^5 = (100+2)^5$

$= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5$

$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$

$= 11040808032$

8:

$(101)^4$

Solution:

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $101 = 100 + 1$

$\therefore (101)^4 = (100+1)^4$

$= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$

$= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4$

$= 100000000 + 4000000 + 60000 + 400 + 1$

$= 104060401$

9:

$(99)^5$

Solution:

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $99 = 100 - 1$

$\therefore (99)^5 = (100-1)^5$

$= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5$

$= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1$

$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$

$= 10010000500 - 500100001$

$= 9509900499$

10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution:

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ be obtained as

$$(1.1)^{10000} = (1+0.1)^{10000}$$
$$= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms}$$
$$= 1 + 10000 \times 1.1 + \text{Other positive terms}$$
$$= 1 + 11000 + \text{Other positive terms}$$
$$> 1000$$

Hence, $(1.1)^{10000} > 1000$.

11:

Find $(a+b)^4 - (a-b)^4$. Hence, evaluate. $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$

Solution:

Using Binomial Theorem, the expressions, $(a+b)^4$ and $(a-b)^4$, can be expanded as

$$(a+b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4$$
$$(a-b)^4 = {}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4$$
$$\therefore (a+b)^4 - (a-b)^4 = {}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4 - \left[{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4\right]$$
$$= 2\left({}^4C_1a^3b + {}^4C_3ab^3\right) = 2\left(4a^3b + 4ab^3\right)$$
$$= 8ab\left(a^2 + b^2\right)$$

By putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4 = 8(\sqrt{3})(\sqrt{2})\left\{(\sqrt{3})^2 + (\sqrt{2})^2\right\}$$
$$= 8(\sqrt{6})\{3+2\} = 40\sqrt{6}$$

12:

Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate. $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$

Solution:

Using Binomial Theorem, the expression, $(x+1)^6$ and $(x-1)^6$, can be expanded as

$(x+1)^6 = {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6$

$(x-1)^6 = {}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6$

$\therefore (x+1)^6 + (x-1)^6 = 2\left[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6 \right]$

$= 2\left\lfloor x^6 + 15x^4 + 15x^2 + 1 \right\rfloor$

By putting $x = \sqrt{2}$ we obtain

$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2\left\lfloor \left(\sqrt{2}\right)^6 + 15\left(\sqrt{2}\right)^4 + 15\left(\sqrt{2}\right)^2 + 1 \right\rfloor$

$= 2(8 + 15\times4 + 15\times2 + 1)$

$= 2(8 + 60 + 30 + 1)$

$= 2(99) = 198$

13:

Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Solution:

In order to show that $9^{n+1} - 8n - 9$ is divisible by 64, it has to be prove that, $9^{n+1} - 8n - 9 = 64k$, where k is some natural number

By Binomial Theorem,

$(1+a)^m = {}^mC_0 + {}^mC_1a + {}^mC_2a^2 + ... + {}^mC_ma^m$

For $a = 8$ and $m = n+1$, we obtain

$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + ... + {}^{n+1}C_{n+1}(8)^{n+1}$

$\Rightarrow 9^{n+1} = 1 + (n+1)(8) + 8^2\left\lfloor {}^{n+1}C_2 + {}^{n+1}C_3\times8 + ... + {}^{n+1}C_{n+1}(8)^{n-1} \right\rfloor$

$\Rightarrow 9^{n+1} = 9 + 8n + 64\left\lfloor {}^{n+1}C_2 + {}^{n+1}C_3\times8 + ... + {}^{n+1}C_{n+1}(8)^{n-1} \right\rfloor$

$\Rightarrow 9^{n+1} - 8n - 9 = 64k$, where $k = {}^{n+1}C_2 + {}^{n+1}C_3\times8 + ... + {}^{n+1}C_{n+1}(8)^{n-1}$ is a natural number

Thus, $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

14:

Prove that $\sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$

Solution:

By Binomial Theorem,

$\sum_{r=0}^{n} {}^nC_r\, a^{n-r}b^r = (a+b)^n$

By putting $b = 3$ and $a = 1$ in the above equation, we obtain

$\sum_{r=0}^{n} {}^nC_r\,(1)^{n-r}(3)^r = (1+3)^n$

$\Rightarrow \sum_{r=0}^{n} 3^r\ {}^nC_r = 4^n$

Hence proved.