Exercise 10.2

1:

Compute the magnitude of the following vectors:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k};\ \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k};\ \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

Solution:

$$\vec{a} = \hat{i} + \hat{j} + \hat{k};\ \vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k};\ \vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$$

Magnitude of a vector $\vec{v} = p\hat{i} + q\hat{j} + r\hat{k}$ is given by $\left|\vec{v}\right| = \sqrt{(p)^2 + (q)^2 + (r)^2}$.

$$|\vec{a}| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

$$\left|\vec{b}\right| = \sqrt{(2)^2 + (-7)^2 + (-3)^2}$$
$$= \sqrt{4 + 49 + 9}$$
$$= \sqrt{62}$$
$$|\vec{c}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(-\frac{1}{\sqrt{3}}\right)^2}$$
$$= \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

2:

Write two different vectors having same magnitude.

Solution:

Consider $\vec{a} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$ and $\vec{b} = \left(2\hat{i} - \hat{j} - 3\hat{k}\right)$.

It can be observed that $|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$ and

$$\left|\vec{b}\right| = \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

Hence, $\vec{a}$ and $\vec{b}$ are two different vectors having the same magnitude. The vectors are different because they have different directions.

3:

Write two different vectors having same direction.

Solution:

Consider $\vec{p} = \left(\hat{i} + \hat{j} + \hat{k}\right)$ and $\vec{q} = \left(2\hat{i} + 2\hat{j} + 2\hat{k}\right)$.

The direction cosines of $\vec{p}$ are given by,

$$l=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}},m=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}},n=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$

The direction cosines of $\vec{q}$ are given by

$$l=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}},m=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}},$$

and $n=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}}$

The direction cosines of $\vec{p}$ and $\vec{q}$ are the same.
Hence, the two vectors have the same direction.

4:
Find the values of x and y so that the vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ are equal.

Solution:
The two vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ will be equal if their corresponding components are equal. Hence, the required values of x and y are 2 and 3 respectively.

5:
Find the scalar and vector components of the vector with initial point $(2, 1)$ and terminal point $(-5, 7)$.

Solution:
The vector with the initial point P $(2, 1)$ and terminal point Q $(-5, 7)$ can be given by,

$$\overrightarrow{PQ}=(-5-2)\hat{i}+(7-1)\hat{j}$$

$$\Rightarrow \overrightarrow{PQ}=-7\hat{i}+6\hat{j}$$

Hence, the required scalar components are –7 and 6 while the vector components are $-7\hat{i}+6\hat{j}$.

6:
Find the sum of the vectors $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$.

Solution:
The given vectors are $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$

$$\therefore\ \vec{a}+\vec{b}+\vec{c}=(1-2+1)\hat{i}+(-2+4-6)\hat{j}+(1+5-7)\hat{k}$$
$$=0.\hat{i}-4\hat{j}-1.\hat{k}$$
$$=-4\hat{j}-\hat{k}$$

7:

Find the unit vector in the direction of the vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$.

Solution:

The unit vector $\hat{a}$ in the direction of vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$

$$|\vec{a}|=\sqrt{1^2+1^2+2^2}=\sqrt{1+1+4}=\sqrt{6}$$

$$\therefore \hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}=\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$$

8:

Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.

Solution:

The given points are P (1, 2, 3) and Q (4, 5, 6).

$$\overrightarrow{PQ}=(4-1)\hat{i}+(5-2)\hat{j}+(6-3)\hat{k}=3\hat{i}+3\hat{j}+3\hat{k}$$

$$\left|\overrightarrow{PQ}\right|=\sqrt{3^2+3^2+3^2}=\sqrt{9+9+9}=\sqrt{27}=3\sqrt{3}$$

Hence, the unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|}=\frac{3\hat{i}+3\hat{j}+3\hat{k}}{3\sqrt{3}}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$$

9:

For given vectors, $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-\hat{k}$, find the unit vector in the direction of the vector $\vec{a}+\vec{b}$

Solution:

The given vectors are $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-\hat{k}$.

$$\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$$
$$\vec{b}=-\hat{i}+\hat{j}-\hat{k}$$
$$\therefore \vec{a}+\vec{b}=(2-1)\hat{i}+(-1+1)\hat{j}+(2-1)\hat{k}=1\hat{i}+0\hat{j}+1\hat{k}=\hat{i}+\hat{k}$$

$$\left|\vec{a}+\vec{b}\right|=\sqrt{1^2+1^2}=\sqrt{2}$$

Hence, the unit vector in the direction of $\left(\vec{a}+\vec{b}\right)$ is

$$\frac{\left(\vec{a}+\vec{b}\right)}{\left|\vec{a}+\vec{b}\right|}=\frac{\hat{i}+\hat{k}}{\sqrt{2}}=\frac{1}{2}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}$$

10:

Find a vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units.

Solution:

Let $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$

$$|\vec{a}|=\sqrt{5^2+(-1)^2+2^2}=\sqrt{25+1+4}=\sqrt{30}$$

$$\therefore \hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{5\hat{i}-\hat{j}+2\hat{k}}{\sqrt{30}}$$

Hence, the vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units is given by

$$8\hat{a}=8\left(\frac{5\hat{i}-\hat{j}+2\hat{k}}{\sqrt{30}}\right)$$

$$=\frac{40}{\sqrt{30}}\hat{i}-\frac{8}{\sqrt{30}}\hat{j}+\frac{16}{\sqrt{30}}\hat{k}$$

11:

Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear.

Solution:

Let $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}$.

It is observed that $\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}=-2\left(2\hat{i}-3\hat{j}+4\hat{k}\right)-2\vec{a}$

$\therefore\ \vec{b}=\lambda\vec{a}$

Where,

$\lambda=-2$

Hence, the given vectors are collinear.

12:

Find the direction cosines of the vector $\hat{i}+2\hat{j}+3\hat{k}$

Solution:

Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$.

$$|\vec{a}|=\sqrt{1^2+2^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$

Hence, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$

13:

Find the direction cosines of the vector joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$ directed from A to B.

Solution:

The given points are $A(1, 2, -3)$ and $B(-1, -2, 1)$

$\therefore \overrightarrow{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + (1-(-3))\hat{k}$

$\Rightarrow \overrightarrow{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

$\therefore \left|\overrightarrow{AB}\right| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{4+16+16} = \sqrt{36} = 6$

Hence, the direction cosines of $\overrightarrow{AB}$ are $\left(-\frac{2}{6},-\frac{4}{6},\frac{4}{6}\right) = \left(-\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)$

14:

Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY, and OZ.

Solution: Let

= Th$\vec{a}$en, $\hat{i} + \hat{j} + \hat{k}$

$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Therefore, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

Now, let α, β, and γ be the angles formed by $\vec{a}$ with the positive directions of x, y, and z axes. Then, we have

$\cos\alpha = \frac{1}{\sqrt{3}}, \cos\beta = \frac{1}{\sqrt{3}}, \cos\gamma = \frac{1}{\sqrt{3}}$.

Hence, the given vector is equally inclined to axes OX, OY, and OZ.

Question 15:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are and respectively, in the ration 2:1

(i) internally

(ii) externally

$\hat{i} + 2\hat{j} - \hat{k}$ $\quad -\hat{i} + \hat{j} + \hat{k}$

Solution:

The position vector of point R dividing the line segment joining two points P and Q in the ratio m: n is given by:

i. Internally: $\frac{m\vec{b}+n\vec{a}}{m+n}$

ii. Externally: $\frac{m\vec{b}+n\vec{a}}{m-n}$

Position vectors of P and Q are given as:

$\vec{OP} = \hat{i}+2\hat{j}-\hat{k}$ and $\vec{OQ} = -\hat{i}+\hat{j}+\hat{k}$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2:1 is given by:

$$\vec{OR} = \frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1} = \frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-\hat{k}\right)}{3}$$

$$= \frac{-\hat{i}+4\hat{j}+\hat{k}}{3} = -\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2:1 is given by,

$$\vec{OR} = \frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)-1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2-1}$$

$$= \left(-2\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-\hat{k}\right)$$

$$= -3\hat{i}+3\hat{k}$$

Question 16:

Find the position vector of the mid-point of the vector joining the points P (2, 3, 4) and Q (4, 1, – 2).

Solution:

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, – 2) is given by

$$\vec{OR} = \frac{\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) + \left(4\hat{i} + \hat{j} - 2\hat{k}\right)}{2}$$

$$= \frac{(2+4)\hat{i} + (3+1)\hat{j} + (4-2)\hat{k}}{2}$$

$$= \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2}$$

$$= 3\hat{i} + 2\hat{j} + \hat{k}$$

17:

Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ **and** $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$**, respectively form the vertices of a right-angled triangle.**

Solution:

Position vectors of points A, B, and C are respectively given as:

$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

$$\therefore\ \vec{AB} = \vec{b} - \vec{a} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\vec{BC} = \vec{c} - \vec{b} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\vec{CA} = \vec{a} - \vec{c} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\therefore \left|\vec{AB}\right|^2 = (-1)^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35$$

$$\left|\vec{BC}\right|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41$$

$$\left|\vec{CA}\right|^2 = 2^2 + (-1)^2 + 1^2 = 4 + 1 + 1 = 6$$

$$\therefore \left|\vec{AB}\right|^2 + \left|\vec{CA}\right|^2 = 35 + 6 = 41 = \left|\vec{BC}\right|^2$$

Hence, ABC is a right-angled triangle.

18:

In triangle ABC which of the following is not true:

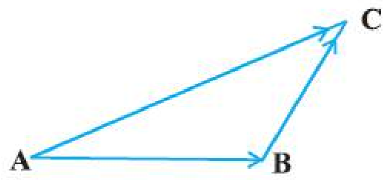


Fig 10.18

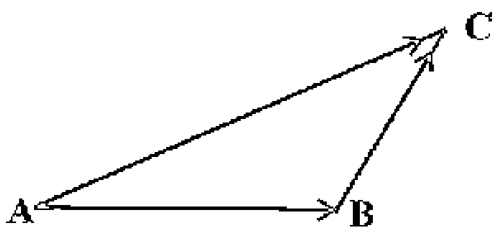
A. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

B. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

C. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

D. $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

Solution:



On applying the triangle law of addition in the given triangle, we have:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \quad \ldots(1)$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

The equation given in (A) is true.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$$

$\therefore$ The equation given in (B) is true.

From equation (2), we have:

$$\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$$

$\therefore$ The equation given in (D) is true.

Now, consider the equation given in (C):

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CA} \quad \ldots(3)$$

From equations (1) and (3), we have:

$$\overrightarrow{AC} = \overrightarrow{CA}$$

$\Rightarrow \overrightarrow{AC} = -\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AC} + \overrightarrow{AC} = \vec{0}$

$\Rightarrow 2\overrightarrow{AC} = \vec{0}$

$\Rightarrow \overrightarrow{AC} = \vec{0}$, which is not true.

Thus, the equation given in (C) is incorrect.
Hence, the correct answer is C.

19:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

A. $\vec{b} = \lambda\vec{a}$, for some scalar λ

B. $\vec{a} = \pm\vec{b}$

C. the respective components of $\vec{a}$ and $\vec{b}$ are not proportional

D. both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes

Solution:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel.
Therefore, we have:

$\vec{b} = \lambda\vec{a}$ (For some scalar λ)

If $\lambda = \pm 1$, then $\vec{a} = \pm\vec{b}$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$\vec{b} = \lambda\vec{a}$.

$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$

$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

$\Rightarrow b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$

$$\Rightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Thus, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.

However, vectors $\vec{a}$ and $\vec{b}$ can have different directions.
Thus, the statement given in D is incorrect.

Hence, he correct answer is D.