Exercise 10.3

1:

Find the angle between two vectors $\vec{a}$. and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}.\vec{b}=\sqrt{6}$

Solution:

It is given that,

$|\vec{a}|=\sqrt{3}, |\vec{b}|=2 and, \vec{a}.\vec{b}=\sqrt{6}$

Now we know that $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

$\therefore \sqrt{6}=\sqrt{3}\times 2\times\cos\theta$

$\Rightarrow \cos\theta=\dfrac{\sqrt{6}}{\sqrt{3}\times 2}$

$\Rightarrow \cos\theta=\dfrac{1}{\sqrt{2}}$

$\Rightarrow \theta=\dfrac{\pi}{4}$

Hence, the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{4}$

2:

Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$

Solution:

The given vectors are $\vec{a} = \hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$

$|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$

$|\vec{b}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{9+4+1}=\sqrt{14}$

$Now, \vec{a}.\vec{b}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)\left(3\hat{i}-2\hat{j}+\hat{k}\right)$

$=1.3+(-2)(-2)+3.1$

$=3+4+3$

$=10$

Also, we know that $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

$\therefore 10=\sqrt{14}\sqrt{14}\cos\theta$

$\Rightarrow \cos\theta=\dfrac{10}{14}$

$\Rightarrow \theta=\cos^{-1}\left(\dfrac{5}{7}\right)$

3:

Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.

Solution:

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right) = \frac{1}{\sqrt{1+1}}\{1.1 + (-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$$

Hence, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

Question 4:

Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Solution:

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right) = \frac{1}{\sqrt{7^2 + (-1)^2 + 8^2}}\{1(7) + 3(-1) + 7(8)\} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right), \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right), \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right)$$

Also, show that they are mutually perpendicular to each other.

Solution:

Let $\vec{a} = \frac{1}{7}\left(2\hat{i} + 3\hat{j} + 6\hat{k}\right) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$,

$$\vec{b} = \frac{1}{7}\left(3\hat{i} - 6\hat{j} + 2\hat{k}\right) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k},$$

$$\vec{c} = \frac{1}{7}\left(6\hat{i} + 2\hat{j} - 3\hat{k}\right) = \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}.$$

$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a}.\vec{b} = \frac{2}{7} \times \frac{3}{7} + \frac{3}{7} \times \left(\frac{-6}{7}\right) + \frac{6}{7} \times \frac{2}{7} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$$

$$\vec{b}.\vec{c} = \frac{3}{7} \times \frac{6}{7} + \left(\frac{-6}{7}\right) \times \frac{2}{7} + \frac{2}{7} \times \left(\frac{-3}{7}\right) = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$$

$$\vec{c}.\vec{a} = \frac{6}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{3}{7} + \left(\frac{-3}{7}\right) \times \frac{6}{7} = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$$

Hence, the given three vectors are mutually perpendicular to each other

6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Solution:

$$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$$

$$\Rightarrow \vec{a}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}.\vec{b} = 8$$

$$\Rightarrow \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 = 8$$

$$\Rightarrow \left(8\left|\vec{b}\right|\right)^2 - \left|\vec{b}\right|^2 = 8 \qquad \left[\left|\vec{a}\right| = 8\left|\vec{b}\right|\right]$$

$$\Rightarrow 64\left|\vec{b}\right|^2 - \left|\vec{b}\right|^2 = 8$$

$$\Rightarrow 63\left|\vec{b}\right|^2 = 8$$

$$\Rightarrow \left|\vec{b}\right|^2 = \frac{8}{63}$$

$$\Rightarrow \left|\vec{b}\right| = \sqrt{\frac{8}{63}} \qquad \left[\text{Magnitude of a vector is non-negative}\right]$$

$$\Rightarrow \left|\vec{b}\right| = \frac{2\sqrt{2}}{3\sqrt{7}}$$

$$\left|\vec{a}\right| = 8\left|\vec{b}\right| = \frac{8 \times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$$

7:

Evaluate the product. $\left(3\vec{a} - 5\vec{b}\right).\left(2\vec{a} + 7\vec{b}\right)$

Solution:

$$\left(3\vec{a} - 5\vec{b}\right).\left(2\vec{a} + 7\vec{b}\right)$$

$$= 3\vec{a}.2\vec{a} + 3\vec{a}.7\vec{b} - 5\vec{b}.2\vec{a} - 5\vec{b}.7\vec{b}$$

$$= 6\vec{a}.\vec{a} + 21\vec{a}.\vec{b} - 10\vec{a}.\vec{b} - 35\vec{b}.\vec{b}$$

$$= 6\left|\vec{a}\right|^2 + 11\vec{a}.\vec{b} - 35\left|\vec{b}\right|^2$$

8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$

Solution:

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

It is given that $\left|\vec{a}\right| = \left|\vec{b}\right|, \vec{a}.\vec{b} = \frac{1}{2}$, and $\theta = 60^o$ …(1)

We know that $\vec{a}.\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$.

$$\therefore \frac{1}{2}=|\vec{a}||\vec{b}|60^o \qquad [\text{Using } (1)]$$

$$\Rightarrow \frac{1}{2}=|\vec{a}|^2\times\frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2=1$$

$$\Rightarrow |\vec{a}|=|\vec{b}|=1$$

9:

Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=12$

Solution:

$$(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=12$$

$$\Rightarrow \vec{x}.\vec{x}+\vec{x}.\vec{a}-\vec{a}.\vec{x}-\vec{a}.\vec{a}=12$$

$$\Rightarrow |\vec{x}|^2-|\vec{a}|^2=12$$

$$\Rightarrow |\vec{x}|^2-1=12 \qquad [|\vec{a}|=1 \text{ as } \vec{a} \text{ is a unit vector}]$$

$$\Rightarrow |\vec{x}|^2=13$$

$$\therefore\ |\vec{x}|=\sqrt{13}$$

10:

If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ .

Solution:

The given vectors are $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$.

Now,

$$\vec{a}+\lambda\vec{b}=(2\hat{i}+2\hat{j}+3\hat{k})+\lambda(-\hat{i}+2\hat{j}+\hat{k})=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$$

If $(\vec{a}+\lambda\vec{b})$ is perpendicular to $\vec{c}$, then

$$(\vec{a}+\lambda\vec{b}).\vec{c}=0$$

$$\Rightarrow \left[(2+\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right].(3\hat{i}+\hat{j})=0$$

$$\Rightarrow (2+\lambda)+3+(2+2\lambda)1+(3+\lambda)0=0$$

$$\Rightarrow 6-3\lambda+2+2\lambda=0$$

$$\Rightarrow -\lambda+8=0$$

$$\Rightarrow \lambda=8$$

11:

Show that:

$|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$

For any two nonzero vectors $\vec{a}$ and $\vec{b}$

Solution:

$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$

$=|\vec{a}|^2\vec{b}.\vec{b}-|\vec{a}||\vec{b}|\vec{b}.\vec{a}+|\vec{b}||\vec{a}|\vec{a}.\vec{b}-|\vec{b}|^2\vec{a}.\vec{a}$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2$

$=0$

Hence, $\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right) and. \left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$ are perpendicular to each other

12:

If $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$, then what can be concluded above the vector $\vec{b}$?

Solution:

It is given that $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$

Now, $\vec{a}.\vec{a}=0\Rightarrow|\vec{a}|^2=0\Rightarrow|\vec{a}|=0$

$\therefore \vec{a}$ is a zero vector.

Hence, vector $\vec{b}$ satisfying $\vec{a}.\vec{b}=0$ can be any vector.

13:

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

Solution:

$|\vec{a}+\vec{b}+\vec{c}|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$

Since, $\vec{a}+\vec{b}+\vec{c}=0$, we have

$\Rightarrow 0=1+1+1+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$

$\Rightarrow\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\frac{-3}{2}$

14:

If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}.\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

Solution:

Consider $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$.

Then,

$\vec{a}\,.\,\vec{b} = 2.3 + 4.3 + 3(-6) = 6 + 12 - 18 = 0$

We now observe that:

$\vec{a} = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$

$\therefore \vec{a} \neq \vec{0}$

$\left|\vec{b}\right| = \sqrt{3^2 + 3^2 + (-6)^2} = 54$

$\therefore \vec{b} \neq \vec{0}$

Hence, the converse of the given statement need not be true.

15:

If the vertices A, B, C of a triangle ABC are $(1,\ 2,\ 3)$, $(-1,\ 0,\ 0)$, $(0,\ 1,\ 2)$, respectively, then find $\angle ABC$. [$\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Solution:

The vertices of ΔABC are given as A$(1,\ 2,\ 3)$, B $(-1,\ 0,\ 0)$, and C $(0,\ 1,\ 2)$,

Also, it is given that $\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$\overrightarrow{BA} = \{1-(-1)\}\hat{i} + (2-0)\,\hat{j} + (3-0)\,\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{BC} = \{0-(-1)\}\hat{i} + (1-0)\,\hat{j} + (2-0)\,\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$

$\therefore \overrightarrow{BA}.\overrightarrow{BC} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right).\left(\hat{i} + \hat{j} + 2\hat{k}\right)$

$= 2\times1 + 2\times1 + 3\times2 = 2 + 2 + 6 = 10$

$\left|\overrightarrow{BA}\right| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$

$\left|\overrightarrow{BC}\right| = \sqrt{1+1+2^2} = \sqrt{6}$

We know that: $\overrightarrow{BA}.\overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|\cos(\angle ABC)$

$$\therefore 10 = \sqrt{17}\times\sqrt{6}\cos(\angle ABC)$$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17}\times\sqrt{6}}$$

$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

16:

Show that the points A $(1,\ 2,\ 7)$, B $(2,\ 6,\ 3)$ and C $(3,\ 10,\ -1)$ are collinear.

Solution:

The given points are A $(1,\ 2,\ 7)$, B $(2,\ 6,\ 3)$ and C $(3,\ 10,\ -1)$

$\therefore \overrightarrow{AB} = (2-1)\,\hat{i} + (6-2)\,\hat{j} + (3-7)\,\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$

$$\overrightarrow{BC}=(3-2)\hat{i}+(10-6)\hat{j}+(-1-3)\hat{k}=\hat{i}+4\hat{j}-4\hat{k}$$

$$\overrightarrow{AC}=(3-1)\hat{i}+(10-2)\hat{j}+(-1-7)\hat{k}=2\hat{i}+8\hat{j}-8\hat{k}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{1^2+4^2+(-4)^2}=\sqrt{1+16+16}=\sqrt{33}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{1^2+4^2+(-4)^2}=\sqrt{1+16+16}=\sqrt{33}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{2^2+8^2+8^2}=\sqrt{4+64+64}=2\sqrt{33}$$

$$\therefore \left|\overrightarrow{AC}\right|=\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|$$

Hence, the given points A, B, and C are collinear.

17:

Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Solution:

Let vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}+4\hat{j}-4\hat{k}$ be position vectors of points A, B, and C respectively.

i.e., $\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k},\ \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k},\ \overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

Now, vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$ represent the sides of ΔABC.

i.e., $\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k},\ \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k},\ \overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

$$\therefore \overrightarrow{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$$

$$\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$$

$$\overrightarrow{AC}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$$

$$\left|\overrightarrow{BC}\right|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$$

$$\left|\overrightarrow{AC}\right|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$$

$$\therefore \left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{AC}\right|^2=6+35=41=\left|\overrightarrow{AB}\right|^2$$

Hence, ΔABC is a right-angled triangle.

18:

If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ a non-zero scalar, then $\lambda\ \vec{a}$ is unit vector if

(A) $\lambda = 1$	(B) $\lambda = -1$
(C) $a = \lvert\lambda\rvert$	(D) $a = \frac{1}{\lvert\lambda\rvert}$

Solution:

Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}| = 1$.

Now,

$|\lambda\vec{a}| = 1$

$\Rightarrow |\lambda||\vec{a}| = 1$

$\Rightarrow |\vec{a}| = \frac{1}{|\lambda|} \qquad [\lambda \neq 0]$

$\Rightarrow a = \frac{1}{|\lambda|} \qquad [|\vec{a}| = a]$

Hence, vector $\lambda\vec{a}$ is a unit vector if $a = \frac{1}{|\lambda|}$

The correct answer is D