Exercise 10.4

Page: 454

1:

Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Solution:

We have,

$$\vec{a}=\hat{i}-7\hat{j}+7\hat{k} \text{ and } \vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}\left(-14+14\right)-\hat{j}\left(2-21\right)+\hat{k}\left(-2+21\right)=19\hat{j}+19\hat{k}$$

$$\therefore\left|\vec{a}\times\vec{b}\right|=\sqrt{(19)^2+(19)^2}=\sqrt{2\times(19)^2}=19\sqrt{2}$$

2:

Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, **where** $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

Solution:

We have,

$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$\therefore \vec{a}+\vec{b}=4\hat{i}+4\hat{j},\ \vec{a}-\vec{b}=2\hat{i}+4\hat{k}$

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|=\sqrt{16^2+(-16)^2+(-8)^2}$$
$$=\sqrt{2^2\times 8^2+2^2\times 8^2+8^2}$$
$$=8\sqrt{2^2+2^2+1}$$
$$=8\sqrt{9}=8\times 3=24$$

Hence, the unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by the relation,

$$\pm\frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|}=\pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}$$
$$=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}$$
$$=\pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}$$

3:

If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$

Solution:

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Since $\vec{a}$ is a unit vector, $|\vec{a}|=1$

Also, it is given that $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$. Then, we have:

$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$

$\Rightarrow \frac{1}{2} = a_1 \qquad \left[|\vec{a}| = 1\right]$

$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$

$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \qquad \left[|\vec{a}| = 1\right]$

Also, $\cos\theta = \frac{a_3}{|\vec{a}|}$

$\Rightarrow a_3 = \cos\theta$

Now,

$|\vec{a}| = 1$

$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$

$\Rightarrow \frac{3}{4} + \cos^2\theta = 1$

$\Rightarrow \cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$

Hence, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

4:
Show that

$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$

Solution:

$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right)$

$= \left(\vec{a} - \vec{b}\right) \times \vec{a} + \left(\vec{a} - \vec{b}\right) \times \vec{b} \qquad$ [By distributivity of vector product over addition]

$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$ [Again, by distributivity of vector product over addition]

$=\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}$

$=2\vec{a}\times\vec{b}$

5:

Find λ **and** μ **if** $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$

Solution:

$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$

$\Rightarrow\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 27\\ 1 & \lambda & \mu\end{vmatrix}=0\hat{i}+0\hat{j}+0\hat{k}$

$\Rightarrow\hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)=0\hat{i}+0\hat{j}+0\hat{k}$

On comparing the corresponding components, we have:

$6\mu-27\lambda=0$

$2\mu-27=0$

$2\lambda-6=0$

Now,

$2\lambda-6=0\Rightarrow\lambda=3$

$2\mu-27=0\Rightarrow\mu=\frac{27}{2}$

Hence, $\lambda=3$ and $\mu=\frac{27}{2}$

6:

Given that $\vec{a}.\vec{b}=0$ and $\vec{a}\times\vec{b}=0$

What can you conclude about the vector $\vec{a}$ and $\vec{b}$ **?**

Solution:

$\vec{a}.\vec{b}=0$

Then,

(i) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, *or* $\vec{a}\perp\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

$\vec{a}\times\vec{b}=0$

(ii) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, *or* $\vec{a}$ *is parallel to* $\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Hence, $|\vec{a}|=0$ or $|\vec{b}|=0$.

7:

Let the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, b_1\hat{i}+b_2\hat{j}+b_3\hat{k}, c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

Then show that $=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Solution:

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

$\left(\vec{b}+\vec{c}\right)=(b_1+c_1)\hat{i}+(b_2+c_2)\hat{j}+(b_3+c_3)\hat{k}$

Now, $\vec{a}\times\left(\vec{b}+\vec{c}\right)\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$

$=\hat{i}\left[a_2(b_3+c_3)-a_3(b_2+c_2)\right]-\hat{j}\left[a_1(b_3+c_3)-a_3(b_1+c_1)\right]$

$+\hat{k}\left[a_1(b_2+c_2)-a_2(b_1+c_1)\right]$

$=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3-a_1c_3+a_3b_1+a_3c_1\right]$

$+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]$ ...(1)

$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

$=\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[b_1a_3-a_1b_3\right]+\hat{k}\left[a_1b_2-a_2b_1\right]$ ...(2)

$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$

$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_1c_2-a_2c_1\right]$ ...(3)

On adding (2) and (3), we get:

$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[b_1a_3+a_3c_1-a_1b_3-a_1c_3\right]$

$+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right]$ ...(4)

Now, from (1) and (4), we have:

$=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Hence, the given result is proved.

8:

If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$.

Is the converse true? Justify your answer with an example.

Solution:

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$.

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k},\vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$.

Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}=\hat{i}(24-24)-\hat{j}(16-16)+\hat{k}(12-12)=0\hat{i}+0\hat{j}+0\hat{k}$$

It can now be observed that:

$$|\vec{a}|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

$$\therefore \vec{a}\neq\vec{0}$$

$$|\vec{b}|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$$

$$\therefore \vec{b}\neq\vec{0}$$

Hence, the converse of the given statement need not be true.

9:

Find the area of the triangle with vertices A $(1, 1, 2)$, B $(2, 3, 5)$ and C $(1, 5, 5)$.

Solution:

The vertices of triangle ABC are given as A $(1, 1, 2)$, B $(2, 3, 5)$ and C $(1, 5, 5)$.

The adjacent sides $\vec{AB}$ and $\vec{BC}$ of ΔABC are given as:

$$\vec{AB}=(2-1)\hat{i}+(3-1)\hat{j}+(5-2)\hat{k}=\hat{i}+2\hat{j}+3\hat{k}$$

$$\vec{BC}=(1-2)\hat{i}+(5-3)\hat{j}+(5-5)\hat{k}=-\hat{i}+2\hat{j}$$

$$\text{Area of } \Delta ABC=\frac{1}{2}\left|\vec{AB}\times\vec{BC}\right|$$

$$\vec{AB}\times\vec{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ -1 & 2 & 0\end{vmatrix}=\hat{i}(-6)-\hat{j}(3)+\hat{k}(2+2)=-6\hat{i}-3\hat{j}+4\hat{k}$$

$$\therefore \left|\vec{AB}\times\vec{BC}\right|=\sqrt{(-6)^2+(-3)^2+4^2}=\sqrt{36+9+16}=\sqrt{61}$$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

10

Find the area of the parallelogram whose adjacent sides are determined by the vector

$\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$

Solution:

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Adjacent sides are given as:

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=\hat{i}(-1+21)-\hat{j}(1-6)+\hat{k}(-7+2)=20\hat{i}+5\hat{j}-5\hat{k}$$

$$\therefore |\vec{a}\times\vec{b}|=\sqrt{20^2+5^2+5^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units

11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution:

It is given that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$

We know that $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}$ is a unit vector if $|\vec{a}\times\vec{b}|=1$

$$|\vec{a}\times\vec{b}|=1$$

$$\Rightarrow \left||\vec{a}||\vec{b}|\sin\theta\hat{n}\right|=1$$

$$\Rightarrow |\vec{a}||\vec{b}||\sin\theta|=1$$

$$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$$

$$\Rightarrow \sin\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

Hence, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

The correct answer is B.

12:

Area of a rectangle having vertices A, B, C, and D with position vectors

$-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Solution:

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$\overrightarrow{OA}=-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overrightarrow{OB}=\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$

$\overrightarrow{OC}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}\ \ \overrightarrow{OD}=-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as:

$$\overrightarrow{AB}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{BC}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=\hat{k}(-2)=-2\hat{k}$$

$$\therefore \left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(-2)^2}=2$$

Now, it is known that the area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Hence, the area of the given rectangle is $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=2$ square units.

The correct answer is C.