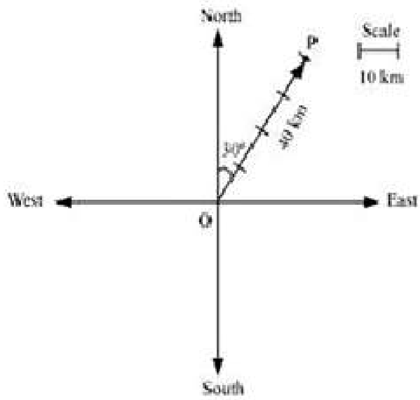
Exercise 10.1

1:

Represent graphically a displacement of 40 km, 30° east of north.

Solution:



Here, vector $\overrightarrow{OP}$ represents the displacement of 40 km, 30° East of North.

2:

Classify the following measures as scalars and vectors.

(i) 10 kg (ii) 2 meters north-west (iii) 40° (iv) 40 watt (v) 10^{-19} coulomb (vi) $20 m/s^2$

Solution:

(i) 10 kg is a scalar quantity because it involves only magnitude.
(ii) 2 meters north-west is a vector quantity as it involves both magnitude and direction.
(iii) 40° is a scalar quantity as it involves only magnitude.
(iv) 40 watts is a scalar quantity as it involves only magnitude.
(v) 10^{-19} Coulomb is a scalar quantity as it involves only magnitude.
(vi) $20 m/s^2$ is a vector quantity as it involves magnitude as well as direction.

3:

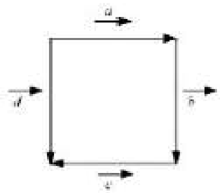
Classify the following as scalar and vector quantities.

(i) time period (ii) distance (iii) force (iv) velocity (v) work done

Solution:

(i) Time period is a scalar quantity as it involves only magnitude.
(ii) Distance is a scalar quantity as it involves only magnitude.
(iii) Force is a vector quantity as it involves both magnitude and direction.
(iv) Velocity is a vector quantity as it involves both magnitude as well as direction.
(v) Work done is a scalar quantity as it involves only magnitude.

4:
In Figure, identify the following vectors.



(i) Co-initial (ii) Equal (iii) Collinear but not equal

Solution:

(i) Vectors $\vec{a}$ and $\vec{d}$ are co-initial because they have the same initial point.

(ii) Vectors $\vec{b}$ and $\vec{d}$ are equal because they have the same magnitude and direction.

(iii) Vectors $\vec{a}$ and $\vec{c}$ are collinear but not equal. This is because although they are parallel, their directions are not the same.

5:
Answer the following as true or false:

(i) $\vec{a}$ and $-\vec{a}$ are collinear.
(ii) Two collinear vectors are always equal in magnitude.
(iii) Two vectors having same magnitude are collinear.
(iv) Two collinear vectors having the same magnitude are equal.

Solution:

(i) True

Vectors $\vec{a}$ and $-\vec{a}$ can be parallel or coinciding vectors. Either way the vectors will have same magnitude but opposite in direction and will be parallel to the same line..

(ii) False
Collinear vectors are those vectors that are parallel to the same line.

(iii) False
It is not necessary for two vectors having the same magnitude to be parallel to the same line.

(iv) False
Two vectors are said to be equal if they have the same **magnitude and direction**, regardless of the positions of their initial points.

Exercise 10.2

1:

Compute the magnitude of the following vectors:

$$\vec{a}=\hat{i}+\hat{j}+\hat{k};\ \vec{b}=2\hat{i}-7\hat{j}-3\hat{k};\ \vec{c}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$$

Solution:

$$\vec{a}=\hat{i}+\hat{j}+\hat{k};\ \vec{b}=2\hat{i}-7\hat{j}-3\hat{k};\ \vec{c}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}-\frac{1}{\sqrt{3}}\hat{k}$$

Magnitude of a vector $\vec{v}=p\hat{i}+q\hat{j}+r\hat{k}$ is given by $\left|\vec{v}\right|=\sqrt{(p)^2+(q)^2+(r)^2}$.

$$\left|\vec{a}\right|=\sqrt{(1)^2+(1)^2+(1)^2}=\sqrt{3}$$

$$\left|\vec{b}\right|=\sqrt{(2)^2+(-7)^2+(-3)^2}$$

$$=\sqrt{4+49+9}$$

$$=\sqrt{62}$$

$$\left|\vec{c}\right|=\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(-\frac{1}{\sqrt{3}}\right)^2}$$

$$=\sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}=1$$

2:

Write two different vectors having same magnitude.

Solution:

Consider $\vec{a}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and $\vec{b}=\left(2\hat{i}-\hat{j}-3\hat{k}\right)$.

It can be observed that $\left|\vec{a}\right|=\sqrt{1^2+2^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$ and

$$\left|\vec{b}\right|=\sqrt{2^2+(-1)^2+(-3)^2}=\sqrt{4+1+9}=\sqrt{14}$$

Hence, $\vec{a}$ and $\vec{b}$ are two different vectors having the same magnitude. The vectors are different because they have different directions.

3:

Write two different vectors having same direction.

Solution:

Consider $\vec{p}=\left(\hat{i}+\hat{j}+\hat{k}\right)$ and $\vec{q}=\left(2\hat{i}+2\hat{j}+2\hat{k}\right)$.

The direction cosines of $\vec{p}$ are given by,

$$l=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}},m=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}},n=\frac{1}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$

The direction cosines of $\vec{q}$ are given by

$$l=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}},m=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}},$$

and $n=\frac{2}{\sqrt{2^2+2^2+2^2}}=\frac{2}{2\sqrt{3}}=\frac{1}{\sqrt{3}}$

The direction cosines of $\vec{p}$ and $\vec{q}$ are the same.
Hence, the two vectors have the same direction.

4:

Find the values of x and y so that the vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ are equal.

Solution:

The two vectors $2\hat{i}+3\hat{j}$ and $x\hat{i}+y\hat{j}$ will be equal if their corresponding components are equal. Hence, the required values of x and y are 2 and 3 respectively.

5:

Find the scalar and vector components of the vector with initial point $(2,1)$ and terminal point $(-5,7)$.

Solution:

The vector with the initial point P $(2,1)$ and terminal point Q $(-5,7)$ can be given by,

$$\vec{PQ}=(-5-2)\hat{i}+(7-1)\hat{j}$$

$$\Rightarrow \vec{PQ}=-7\hat{i}+6\hat{j}$$

Hence, the required scalar components are –7 and 6 while the vector components are $-7\hat{i}+6\hat{j}$.

6:

Find the sum of the vectors $\vec{a}=\hat{i}-2\hat{j}+\hat{k}, \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$.

Solution:

The given vectors are $\vec{a}=\hat{i}-2\hat{j}+\hat{k}, \vec{b}=-2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c}=\hat{i}-6\hat{j}-7\hat{k}$

$$\therefore\ \vec{a}+\vec{b}+\vec{c}=(1-2+1)\hat{i}+(-2+4-6)\hat{j}+(1+5-7)\hat{k}$$
$$=0.\hat{i}-4\hat{j}-1.\hat{k}$$
$$=-4\hat{j}-\hat{k}$$

7:

Find the unit vector in the direction of the vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$.

Solution:

The unit vector $\hat{a}$ in the direction of vector $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$

$$|\vec{a}|=\sqrt{1^2+1^2+2^2}=\sqrt{1+1+4}=\sqrt{6}$$

$$\therefore \hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}=\frac{1}{\sqrt{6}}\hat{i}+\frac{1}{\sqrt{6}}\hat{j}+\frac{2}{\sqrt{6}}\hat{k}$$

8:

Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.

Solution:

The given points are P (1, 2, 3) and Q (4, 5, 6).

$$\overrightarrow{PQ}=(4-1)\hat{i}+(5-2)\hat{j}+(6-3)\hat{k}=3\hat{i}+3\hat{j}+3\hat{k}$$

$$\left|\overrightarrow{PQ}\right|=\sqrt{3^2+3^2+3^2}=\sqrt{9+9+9}=\sqrt{27}=3\sqrt{3}$$

Hence, the unit vector in the direction of $\overrightarrow{PQ}$ is

$$\frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|}=\frac{3\hat{i}+3\hat{j}+3\hat{k}}{3\sqrt{3}}=\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}$$

9:

For given vectors, $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-\hat{k}$, find the unit vector in the direction of the vector $\vec{a}+\vec{b}$

Solution:

The given vectors are $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=-\hat{i}+\hat{j}-\hat{k}$.

$$\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$$
$$\vec{b}=-\hat{i}+\hat{j}-\hat{k}$$
$$\therefore \vec{a}+\vec{b}=(2-1)\hat{i}+(-1+1)\hat{j}+(2-1)\hat{k}=1\hat{i}+0\hat{j}+1\hat{k}=\hat{i}+\hat{k}$$

$$\left|\vec{a}+\vec{b}\right|=\sqrt{1^2+1^2}=\sqrt{2}$$

Hence, the unit vector in the direction of $\left(\vec{a}+\vec{b}\right)$ is

$$\frac{\left(\vec{a}+\vec{b}\right)}{\left|\vec{a}+\vec{b}\right|}=\frac{\hat{i}+\hat{k}}{\sqrt{2}}=\frac{1}{2}\hat{i}+\frac{1}{\sqrt{2}}\hat{k}$$

10:

Find a vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units.

Solution:

Let $\vec{a}=5\hat{i}-\hat{j}+2\hat{k}$

$$|\vec{a}|=\sqrt{5^2+(-1)^2+2^2}=\sqrt{25+1+4}=\sqrt{30}$$

$$\therefore \hat{a}=\frac{\vec{a}}{|\vec{a}|}=\frac{5\hat{i}-\hat{j}+2\hat{k}}{\sqrt{30}}$$

Hence, the vector in the direction of vector $5\hat{i}-\hat{j}+2\hat{k}$ which has magnitude 8 units is given by

$$8\hat{a}=8\left(\frac{5\hat{i}-\hat{j}+2\hat{k}}{\sqrt{30}}\right)$$

$$=\frac{40}{\sqrt{30}}\hat{i}-\frac{8}{\sqrt{30}}\hat{j}+\frac{16}{\sqrt{30}}\hat{k}$$

11:

Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are collinear.

Solution:

Let $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$ and $\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}$.

It is observed that $\vec{b}=-4\hat{i}+6\hat{j}-8\hat{k}=-2\left(2\hat{i}-3\hat{j}+4\hat{k}\right)-2\vec{a}$

$\therefore\ \vec{b}=\lambda\vec{a}$

Where,

$\lambda=-2$

Hence, the given vectors are collinear.

12:

Find the direction cosines of the vector $\hat{i}+2\hat{j}+3\hat{k}$

Solution:

Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$.

$$|\vec{a}|=\sqrt{1^2+2^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$

Hence, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$

13:

Find the direction cosines of the vector joining the points $A(1,\ 2,-3)$ and $B(-1,\ -2,1)$ directed from A to B.

Solution:

The given points are $A(1,\ 2,-3)$ and $B(-1,\ -2,1)$

$$\therefore \overrightarrow{AB}=(-1-1)\hat{i}+(-2-2)\hat{j}+(1-(-3))\hat{k}$$

$$\Rightarrow \overrightarrow{AB}=-2\hat{i}-4\hat{j}+4\hat{k}$$

$$\therefore \left|\overrightarrow{AB}\right|=\sqrt{(-2)^2+(-4)^2+4^2}=\sqrt{4+16+16}=\sqrt{36}=6$$

Hence, the direction cosines of $\overrightarrow{AB}$ are $\left(-\frac{2}{6},-\frac{4}{6},\frac{4}{6}\right)=\left(-\frac{1}{3},-\frac{2}{3},\frac{2}{3}\right)$

14:

Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined to the axes OX, OY, and OZ.

Solution: Let

= Th$\vec{a}$en, $\hat{i}+\hat{j}+\hat{k}$

$$|\vec{a}|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

Therefore, the direction cosines of $\vec{a}$ are $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

Now, let α, β, and γ be the angles formed by $\vec{a}$ with the positive directions of x, y, and z axes. Then, we have

$$\cos\alpha=\frac{1}{\sqrt{3}},\cos\beta=\frac{1}{\sqrt{3}},\cos\gamma=\frac{1}{\sqrt{3}}.$$

Hence, the given vector is equally inclined to axes OX, OY, and OZ.

Question 15:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are and respectively, in the ration 2:1

(i) internally

(ii) externally

$\hat{i}+2\hat{j}-\hat{k}$ $\quad -\hat{i}+\hat{j}+\hat{k}$

Solution:

The position vector of point R dividing the line segment joining two points P and Q in the ratio m: n is given by:

i. Internally: $\frac{m\vec{b}+n\vec{a}}{m+n}$

ii. Externally: $\frac{m\vec{b}+n\vec{a}}{m-n}$

Position vectors of P and Q are given as:

$\vec{OP} = \hat{i}+2\hat{j}-\hat{k}$ and $\vec{OQ} = -\hat{i}+\hat{j}+\hat{k}$

(i) The position vector of point R which divides the line joining two points P and Q internally in the ratio 2:1 is given by:

$$\vec{OR} = \frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)+1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2+1} = \frac{\left(-2\hat{i}+2\hat{j}+2\hat{k}\right)+\left(\hat{i}+2\hat{j}-\hat{k}\right)}{3}$$

$$= \frac{-\hat{i}+4\hat{j}+\hat{k}}{3} = -\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$$

(ii) The position vector of point R which divides the line joining two points P and Q externally in the ratio 2:1 is given by,

$$\vec{OR} = \frac{2\left(-\hat{i}+\hat{j}+\hat{k}\right)-1\left(\hat{i}+2\hat{j}-\hat{k}\right)}{2-1}$$

$$= \left(-2\hat{i}+2\hat{j}+2\hat{k}\right)-\left(\hat{i}+2\hat{j}-\hat{k}\right)$$

$$= -3\hat{i}+3\hat{k}$$

Question 16:

Find the position vector of the mid-point of the vector joining the points P (2, 3, 4) and Q (4, 1, – 2).

Solution:

The position vector of mid-point R of the vector joining points P (2, 3, 4) and Q (4, 1, – 2) is given by

$$\vec{OR} = \frac{\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) + \left(4\hat{i} + \hat{j} - 2\hat{k}\right)}{2}$$

$$= \frac{(2+4)\hat{i} + (3+1)\hat{j} + (4-2)\hat{k}}{2}$$

$$= \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2}$$

$$= 3\hat{i} + 2\hat{j} + \hat{k}$$

17:

Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$**, respectively form the vertices of a right-angled triangle.**

Solution:

Position vectors of points A, B, and C are respectively given as:

$\vec{a} = 3\hat{i} - 4\hat{j} - 4\hat{k},\ \vec{b} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} - 5\hat{k}$

$$\therefore\ \vec{AB} = \vec{b} - \vec{a} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\vec{BC} = \vec{c} - \vec{b} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\vec{CA} = \vec{a} - \vec{c} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$$

$$\therefore \left|\vec{AB}\right|^2 = (-1)^2 + 3^2 + 5^2 = 1 + 9 + 25 = 35$$

$$\left|\vec{BC}\right|^2 = (-1)^2 + (-2)^2 + (-6)^2 = 1 + 4 + 36 = 41$$

$$\left|\vec{CA}\right|^2 = 2^2 + (-1)^2 + 1^2 = 4 + 1 + 1 = 6$$

$$\therefore \left|\vec{AB}\right|^2 + \left|\vec{CA}\right|^2 = 35 + 6 = 41 = \left|\vec{BC}\right|^2$$

Hence, ABC is a right-angled triangle.

18:

In triangle ABC which of the following is not true:

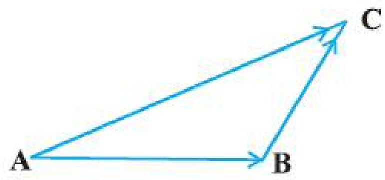


Fig 10.18

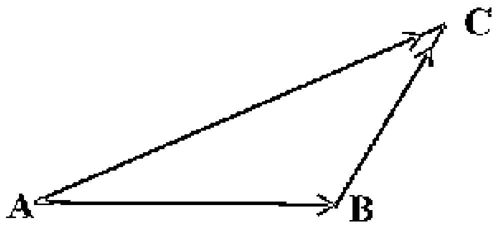
A. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

B. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

C. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

D. $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

Solution:



On applying the triangle law of addition in the given triangle, we have:

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \qquad \ldots(1)$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{CA}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$$

The equation given in (A) is true.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$$

$\therefore$ The equation given in (B) is true.

From equation (2), we have:

$$\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$$

$\therefore$ The equation given in (D) is true.

Now, consider the equation given in (C):

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$$

$$\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{CA} \qquad \ldots(3)$$

From equations (1) and (3), we have:

$$\overrightarrow{AC} = \overrightarrow{CA}$$

$\Rightarrow \overrightarrow{AC} = -\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AC} + \overrightarrow{AC} = \vec{0}$

$\Rightarrow 2\overrightarrow{AC} = \vec{0}$

$\Rightarrow \overrightarrow{AC} = \vec{0}$, which is not true.

Thus, the equation given in (C) is incorrect.
Hence, the correct answer is C.

19:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect:

A. $\vec{b} = \lambda \vec{a}$, for some scalar λ

B. $\vec{a} = \pm \vec{b}$

C. the respective components of $\vec{a}$ and $\vec{b}$ are not proportional

D. both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes

Solution:

If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel.
Therefore, we have:

$\vec{b} = \lambda \vec{a}$ (For some scalar λ)

If $\lambda = \pm 1$, then $\vec{a} = \pm \vec{b}$

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$\vec{b} = \lambda\vec{a}$.

$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$

$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

$\Rightarrow b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$

$\Rightarrow \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$

Thus, the respective components of $\vec{a}$ and $\vec{b}$ are proportional.
However, vectors $\vec{a}$ and $\vec{b}$ can have different directions.
Thus, the statement given in D is incorrect.

Hence, he correct answer is D.

Exercise 10.3

1:

Find the angle between two vectors $\vec{a}$. and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}.\vec{b} = \sqrt{6}$

Solution:

It is given that,

$$|\vec{a}| = \sqrt{3}, |\vec{b}| = 2 and, \vec{a}.\vec{b} = \sqrt{6}$$

Now we know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\therefore \sqrt{6} = \sqrt{3} \times 2 \times \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\sqrt{6}}{\sqrt{3}\times 2}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$

2:

Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$

Solution:

The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$Now, \vec{a}.\vec{b} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right)\left(3\hat{i} - 2\hat{j} + \hat{k}\right)$$

$$= 1.3 + (-2)(-2) + 3.1$$

$$= 3 + 4 + 3$$

$$= 10$$

Also, we know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$\therefore 10 = \sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

3:

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

Solution:

Let $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{i}+\hat{j}$

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right)=\frac{1}{\sqrt{1+1}}\{1.1+(-1)(1)\}=\frac{1}{\sqrt{2}}(1-1)=0$$

Hence, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

Question 4:

Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.

Solution:

Let $\vec{a}=\hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b}=7\hat{i}-\hat{j}+8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right)=\frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{1(7)+3(-1)+7(8)\}=\frac{7-3+56}{\sqrt{49+1+64}}=\frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right),\ \frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right),\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)$$

Also, show that they are mutually perpendicular to each other.

Solution:

Let $\vec{a}=\frac{1}{7}\left(2\hat{i}+3\hat{j}+6\hat{k}\right)=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}$,

$$\vec{b}=\frac{1}{7}\left(3\hat{i}-6\hat{j}+2\hat{k}\right)=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k},$$

$$\vec{c}=\frac{1}{7}\left(6\hat{i}+2\hat{j}-3\hat{k}\right)=\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}.$$

$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a}.\vec{b} = \frac{2}{7} \times \frac{3}{7} + \frac{3}{7} \times \left(\frac{-6}{7}\right) + \frac{6}{7} \times \frac{2}{7} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$$

$$\vec{b}.\vec{c} = \frac{3}{7} \times \frac{6}{7} + \left(\frac{-6}{7}\right) \times \frac{2}{7} + \frac{2}{7} \times \left(\frac{-3}{7}\right) = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$$

$$\vec{c}.\vec{a} = \frac{6}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{3}{7} + \left(\frac{-3}{7}\right) \times \frac{6}{7} = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$$

Hence, the given three vectors are mutually perpendicular to each other

6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Solution:

$(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = 8$

$\Rightarrow \vec{a}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}.\vec{b} = 8$

$\Rightarrow \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2 = 8$

$\Rightarrow \left(8\left|\vec{b}\right|\right)^2 - \left|\vec{b}\right|^2 = 8 \qquad \left[\left|\vec{a}\right| = 8\left|\vec{b}\right|\right]$

$\Rightarrow 64\left|\vec{b}\right|^2 - \left|\vec{b}\right|^2 = 8$

$$\Rightarrow 63\left|\vec{b}\right|^2 = 8$$

$$\Rightarrow \left|\vec{b}\right|^2 = \frac{8}{63}$$

$$\Rightarrow \left|\vec{b}\right| = \sqrt{\frac{8}{63}} \qquad \text{[Magnitude of a vector is non-negative]}$$

$$\Rightarrow \left|\vec{b}\right| = \frac{2\sqrt{2}}{3\sqrt{7}}$$

$$\left|\vec{a}\right| = 8\left|\vec{b}\right| = \frac{8\times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$$

7:

Evaluate the product. $(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$

Solution:

$$(3\vec{a}-5\vec{b}).(2\vec{a}+7\vec{b})$$

$$= 3\vec{a}.2\vec{a} + 3\vec{a}.7\vec{b} - 5\vec{b}.2\vec{a} - 5\vec{b}.7\vec{b}$$

$$= 6\vec{a}.\vec{a} + 21\vec{a}.\vec{b} - 10\vec{a}.\vec{b} - 35\vec{b}.\vec{b}$$

$$= 6\left|\vec{a}\right|^2 + 11\vec{a}.\vec{b} - 35\left|\vec{b}\right|^2$$

8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$

Solution:

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

It is given that $\left|\vec{a}\right| = \left|\vec{b}\right|, \vec{a}.\vec{b} = \frac{1}{2}$, and $\theta = 60^o$ …(1)

We know that $\vec{a}.\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta$.

$\therefore \frac{1}{2} = |\vec{a}||\vec{b}| 60^o$ [Using (1)]

$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$

$\Rightarrow |\vec{a}|^2 = 1$

$\Rightarrow |\vec{a}| = |\vec{b}| = 1$

9:

Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 12$

Solution:

$(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 12$

$\Rightarrow \vec{x}.\vec{x} + \vec{x}.\vec{a} - \vec{a}.\vec{x} - \vec{a}.\vec{a} = 12$

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$

$\Rightarrow |\vec{x}|^2 - 1 = 12$ $\left[|\vec{a}| = 1 \text{ as } \vec{a} \text{ is a unit vector}\right]$

$\Rightarrow |\vec{x}|^2 = 13$

$\therefore |\vec{x}| = \sqrt{13}$

10:

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ .

Solution:

The given vectors are $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$.

Now,

$\vec{a} + \lambda\vec{b} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-\hat{i} + 2\hat{j} + \hat{k}\right) = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$

If $\left(\vec{a} + \lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then

$\left(\vec{a} + \lambda\vec{b}\right).\vec{c} = 0$

$\Rightarrow \left[(2+\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}\right].\left(3\hat{i} + \hat{j}\right) = 0$

$\Rightarrow (2+\lambda)+3+(2+2\lambda)1+(3+\lambda)0 = 0$

$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$

$\Rightarrow -\lambda + 8 = 0$

$\Rightarrow \lambda = 8$

11:

Show that:

$|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$

For any two nonzero vectors $\vec{a}$ and $\vec{b}$

Solution:

$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right).\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$

$=|\vec{a}|^2\vec{b}.\vec{b}-|\vec{a}||\vec{b}|\vec{b}.\vec{a}+|\vec{b}||\vec{a}|\vec{a}.\vec{b}-|\vec{b}|^2\vec{a}.\vec{a}$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2$

$=0$

Hence, $\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right) and.\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$ are perpendicular to each other

12:

If $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$, then what can be concluded above the vector $\vec{b}$?

Solution:

It is given that $\vec{a}.\vec{a}=0$ and $\vec{a}.\vec{b}=0$

Now, $\vec{a}.\vec{a}=0\Rightarrow|\vec{a}|^2=0\Rightarrow|\vec{a}|=0$

$\therefore\vec{a}$ is a zero vector.

Hence, vector $\vec{b}$ satisfying $\vec{a}.\vec{b}=0$ can be any vector.

13:

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

Solution:

$|\vec{a}+\vec{b}+\vec{c}|^2=\left(\vec{a}+\vec{b}+\vec{c}\right).\left(\vec{a}+\vec{b}+\vec{c}\right)=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$

Since, $\vec{a}+\vec{b}+\vec{c}=0$, we have

$\Rightarrow 0=1+1+1+2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)$

$\Rightarrow\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}\right)=\frac{-3}{2}$

14:

If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}.\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

Solution:

Consider $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$.

Then,

$\vec{a}\,.\,\vec{b} = 2.3 + 4.3 + 3(-6) = 6 + 12 - 18 = 0$

We now observe that:

$\vec{a} = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$

$\therefore \vec{a} \neq \vec{0}$

$|\vec{b}| = \sqrt{3^2 + 3^2 + (-6)^2} = 54$

$\therefore \vec{b} \neq \vec{0}$

Hence, the converse of the given statement need not be true.

15:

If the vertices A, B, C of a triangle ABC are $(1, 2, 3)$, $(-1, 0, 0)$, $(0, 1, 2)$, respectively, then find $\angle ABC$. [$\angle ABC$ is the angle between the vectors $\vec{BA}$ and $\vec{BC}$]

Solution:

The vertices of ΔABC are given as A$(1, 2, 3)$, B $(-1, 0, 0)$, and C $(0, 1, 2)$,

Also, it is given that $\angle ABC$ is the angle between the vectors $\vec{BA}$ and $\vec{BC}$.

$\vec{BA} = \{1-(-1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{BC} = \{0-(-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$

$\therefore \vec{BA}.\vec{BC} = (2\hat{i} + 2\hat{j} + 3\hat{k}).(\hat{i} + \hat{j} + 2\hat{k})$

$= 2\times1 + 2\times1 + 3\times2 = 2 + 2 + 6 = 10$

$|\vec{BA}| = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{4+4+9} = \sqrt{17}$

$|\vec{BC}| = \sqrt{1+1+2^2} = \sqrt{6}$

We know that: $\vec{BA}.\vec{BC} = |\vec{BA}||\vec{BC}|\cos(\angle ABC)$

$$\therefore 10 = \sqrt{17}\times\sqrt{6}\cos(\angle ABC)$$

$$\Rightarrow \cos(\angle ABC) = \frac{10}{\sqrt{17}\times\sqrt{6}}$$

$$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

16:

Show that the points A $(1, 2, 7)$, B $(2, 6, 3)$ and C $(3, 10, -1)$ are collinear.

Solution:

The given points are A $(1, 2, 7)$, B $(2, 6, 3)$ and C $(3, 10, -1)$

$\therefore \vec{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$

$\vec{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$

$\vec{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$

$\left|\vec{AB}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$

$\left|\vec{BC}\right| = \sqrt{1^2 + 4^2 + (-4)^2} = \sqrt{1+16+16} = \sqrt{33}$

$\left|\vec{AC}\right| = \sqrt{2^2 + 8^2 + 8^2} = \sqrt{4+64+64} = 2\sqrt{33}$

$\therefore \left|\vec{AC}\right| = \left|\vec{AB}\right| + \left|\vec{BC}\right|$

Hence, the given points A, B, and C are collinear.

17:

Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle.

Solution:

Let vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} + 4\hat{j} - 4\hat{k}$ be position vectors of points A, B, and C respectively.

i.e., $\vec{OA} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

Now, vectors $\vec{AB}$, $\vec{BC}$, and $\vec{AC}$ represent the sides of ΔABC.

i.e., $\vec{OA} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{OB} = \hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{OC} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

$\therefore \vec{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\vec{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{AC} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$

$\left|\vec{AB}\right| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{1+4+36} = \sqrt{41}$

$\left|\vec{BC}\right| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

$\left|\vec{AC}\right| = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{1+9+25} = \sqrt{35}$

$\therefore \left|\vec{BC}\right|^2 + \left|\vec{AC}\right|^2 = 6 + 35 = 41 = \left|\vec{AB}\right|^2$

Hence, ΔABC is a right-angled triangle.

18:

If $\vec{a}$ is a non-zero vector of magnitude 'a' and λ a non-zero scalar, then $\lambda\ \vec{a}$ is unit vector if

(A) $\lambda = 1$	(B) $\lambda = -1$
(C) $a = \lvert\lambda\rvert$	(D) $a = \dfrac{1}{\lvert\lambda\rvert}$

Solution:

Vector $\lambda\vec{a}$ is a unit vector if $\lvert\lambda\vec{a}\rvert = 1$.

Now,

$$\lvert\lambda\vec{a}\rvert = 1$$

$$\Rightarrow \lvert\lambda\rvert\lvert\vec{a}\rvert = 1$$

$$\Rightarrow \lvert\vec{a}\rvert = \frac{1}{\lvert\lambda\rvert} \qquad [\lambda \neq 0]$$

$$\Rightarrow a = \frac{1}{\lvert\lambda\rvert} \qquad [\lvert\vec{a}\rvert = a]$$

Hence, vector $\lambda\vec{a}$ is a unit vector if $a = \dfrac{1}{\lvert\lambda\rvert}$

The correct answer is D

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1:

Find $\left|\vec{a}\times\vec{b}\right|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

Solution:

We have,

$$\vec{a}=\hat{i}-7\hat{j}+7\hat{k} \text{ and } \vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}\left(-14+14\right)-\hat{j}\left(2-21\right)+\hat{k}\left(-2+21\right)=19\hat{j}+19\hat{k}$$

$$\therefore\left|\vec{a}\times\vec{b}\right|=\sqrt{\left(19\right)^2+\left(19\right)^2}=\sqrt{2\times\left(19\right)^2}=19\sqrt{2}$$

2:

Find a unit vector perpendicular to each of the vector $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ **, where** $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

Solution:

We have,

$\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$\therefore \vec{a}+\vec{b}=4\hat{i}+4\hat{j},\ \vec{a}-\vec{b}=2\hat{i}+4\hat{k}$

$$\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix}=\hat{i}(16)-\hat{j}(16)+\hat{k}(-8)=16\hat{i}-16\hat{j}-8\hat{k}$$

$$\begin{aligned}\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right| &= \sqrt{16^2+(-16)^2+(-8)^2}\\ &= \sqrt{2^2\times8^2+2^2\times8^2+8^2}\\ &= 8\sqrt{2^2+2^2+1}\\ &= 8\sqrt{9}=8\times3=24\end{aligned}$$

Hence, the unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by the relation,

$$\begin{aligned}\pm\frac{\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)}{\left|\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}-\vec{b}\right)\right|} &= \pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24}\\ &= \pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3}\\ &= \pm\frac{2}{3}\hat{i}\mp\frac{2}{3}\hat{j}\mp\frac{1}{3}\hat{k}\end{aligned}$$

3:

If a unit vector $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\vec{a}$

Solution:

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Since $\vec{a}$ is a unit vector, $|\vec{a}|=1$

Also, it is given that $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$. Then, we have:

$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$

$\Rightarrow \frac{1}{2} = a_1 \qquad [|\vec{a}| = 1]$

$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$

$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \qquad [|\vec{a}| = 1]$

Also, $\cos\theta = \frac{a_3}{|\vec{a}|}$

$\Rightarrow a_3 = \cos\theta$

Now,

$|\vec{a}| = 1$

$\Rightarrow \sqrt{a_1^2 + a_2^2 + a_3^2} = 1$

$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$

$\Rightarrow \frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$

$\Rightarrow \frac{3}{4} + \cos^2\theta = 1$

$\Rightarrow \cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$

Hence, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

4:
Show that

$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$

Solution:

$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right)$

$= \left(\vec{a} - \vec{b}\right) \times \vec{a} + \left(\vec{a} - \vec{b}\right) \times \vec{b} \qquad$ [By distributivity of vector product over addition]

$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$ [Again, by distributivity of vector product over addition]

$= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$

$= 2\vec{a} \times \vec{b}$

5:

Find λ and μ if $\left(2\hat{i} + 6\hat{j} + 27\hat{k}\right) \times \left(\hat{i} + \lambda\hat{j} + \mu\hat{k}\right) = \vec{0}$

Solution:

$\left(2\hat{i} + 6\hat{j} + 27\hat{k}\right) \times \left(\hat{i} + \lambda\hat{j} + \mu\hat{k}\right) = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$\Rightarrow \hat{i}\left(6\mu - 27\lambda\right) - \hat{j}\left(2\mu - 27\right) + \hat{k}\left(2\lambda - 6\right) = 0\hat{i} + 0\hat{j} + 0\hat{k}$

On comparing the corresponding components, we have:

$6\mu - 27\lambda = 0$

$2\mu - 27 = 0$

$2\lambda - 6 = 0$

Now,

$2\lambda - 6 = 0 \Rightarrow \lambda = 3$

$2\mu - 27 = 0 \Rightarrow \mu = \dfrac{27}{2}$

Hence, $\lambda = 3$ and $\mu = \dfrac{27}{2}$

6:

Given that $\vec{a}.\vec{b} = 0$ and $\vec{a} \times \vec{b} = 0$

What can you conclude about the vector $\vec{a}$ and $\vec{b}$?

Solution:

$\vec{a}.\vec{b} = 0$

Then,

(i) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0, or\ \vec{a} \perp \vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

$\vec{a} \times \vec{b} = 0$

(ii) Either $|\vec{a}| = 0$ or $|\vec{b}| = 0, or\ \vec{a}$ *is parallel to* $\vec{b}$ (in case $\vec{a}$ and $\vec{b}$ are non-zero)

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Hence, $|\vec{a}| = 0$ or $|\vec{b}| = 0$.

7:

Let the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k}, b_1\hat{i}+b_2\hat{j}+b_3\hat{k}, c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

Then show that $=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Solution:

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

$$\left(\vec{b}+\vec{c}\right)=(b_1+c_1)\hat{i}+(b_2+c_2)\hat{j}+(b_3+c_3)\hat{k}$$

Now, $\vec{a}\times\left(\vec{b}+\vec{c}\right)\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$

$$=\hat{i}\left[a_2(b_3+c_3)-a_3(b_2+c_2)\right]-\hat{j}\left[a_1(b_3+c_3)-a_3(b_1+c_1)\right]$$
$$+\hat{k}\left[a_1(b_2+c_2)-a_2(b_1+c_1)\right]$$

$$=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3-a_1c_3+a_3b_1+a_3c_1\right]$$
$$+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right] \quad ...(1)$$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[b_1a_3-a_1b_3\right]+\hat{k}\left[a_1b_2-a_2b_1\right] \quad ...(2)$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_1c_2-a_2c_1\right] \quad ...(3)$$

On adding (2) and (3), we get:

$$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[b_1a_3+a_3c_1-a_1b_3-a_1c_3\right]$$
$$+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right] \quad ...(4)$$

Now, from (1) and (4), we have:

$$=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

Hence, the given result is proved.

8:

If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$.

Is the converse true? Justify your answer with an example.

Solution:

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$.

Let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}$.

Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} = \hat{i}(24-24) - \hat{j}(16-16) + \hat{k}(12-12) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

It can now be observed that:

$$|\vec{a}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$$

$$\therefore \vec{a} \neq \vec{0}$$

$$|\vec{b}| = \sqrt{4^2 + 6^2 + 8^2} = \sqrt{116}$$

$$\therefore \vec{b} \neq \vec{0}$$

Hence, the converse of the given statement need not be true.

9:

Find the area of the triangle with vertices A $(1, 1, 2)$, B $(2, 3, 5)$ and C $(1, 5, 5)$.

Solution:

The vertices of triangle ABC are given as A $(1, 1, 2)$, B $(2, 3, 5)$ and C $(1, 5, 5)$.

The adjacent sides $\vec{AB}$ and $\vec{BC}$ of ΔABC are given as:

$$\vec{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$$

Area of $\Delta ABC = \frac{1}{2}\left|\vec{AB} \times \vec{BC}\right|$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \left|\vec{AB} \times \vec{BC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

10

Find the area of the parallelogram whose adjacent sides are determined by the vector

$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ **and** $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$

Solution:

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$.

Adjacent sides are given as:

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=\hat{i}(-1+21)-\hat{j}(1-6)+\hat{k}(-7+2)=20\hat{i}+5\hat{j}-5\hat{k}$$

$$\therefore \left|\vec{a}\times\vec{b}\right|=\sqrt{20^2+5^2+5^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units

11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution:

It is given that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$

We know that $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}$ is a unit vector if $\left|\vec{a}\times\vec{b}\right|=1$

$$\left|\vec{a}\times\vec{b}\right|=1$$

$$\Rightarrow \left||\vec{a}||\vec{b}|\sin\theta\hat{n}\right|=1$$

$$\Rightarrow |\vec{a}||\vec{b}||\sin\theta|=1$$

$$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$$

$$\Rightarrow \sin\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

Hence, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

The correct answer is B.

12:

Area of a rectangle having vertices A, B, C, and D with position vectors

$-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Solution:

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$\overrightarrow{OA}=-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overrightarrow{OB}=\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$

$\overrightarrow{OC}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}\ \ \overrightarrow{OD}=-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as:

$$\overrightarrow{AB}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{BC}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=\hat{k}(-2)=-2\hat{k}$$

$$\therefore \left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(-2)^2}=2$$

Now, it is known that the area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Hence, the area of the given rectangle is $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=2$ square units.

The correct answer is C.

Miscellaneous Exercise

1:Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

Solution:

If $\vec{r}$ is a unit vector in the XY-plane, then $\vec{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$.

Here, θ is the angle made by the unit vector with the positive direction of the x-axis.

Therefore, for $\theta = 30^\circ$.

$$\vec{r} = \cos 30^o \hat{i} + \sin 30^o \hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

Hence, the required unit vector is $\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$

2:

Find the scalar components and magnitude of the vector joining the points $\mathrm{P}(x_1, y_1, z_1)$ and $\mathrm{Q}(x_2, y_2, z_2)$.

Solution:

The vector joining the points $\mathrm{P}(x_1, y_1, z_1)$ and $\mathrm{Q}(x_2, y_2, z_2)$ can be obtained by,

$\overrightarrow{\mathrm{PQ}} =$ Position vector of Q-Position vector of P

$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$$\left|\overrightarrow{\mathrm{PQ}}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Hence, the scalar components and the magnitude of the vector joining the given points are respectively $\{(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)\}$ and $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

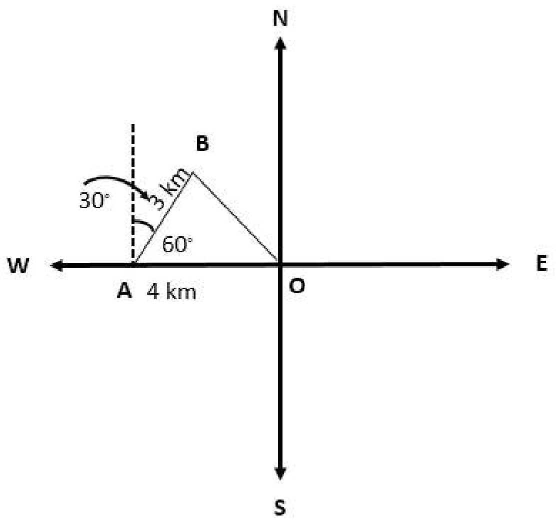
3:

A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Solution 3:

Let O and B be the initial and final positions of the girl respectively.

Then, the girl's position can be shown as:



Now, we have:

$\vec{OA} = -4\hat{i}$

$$\vec{AB} = \left|\vec{AB}\right|\cos 60^o\,\hat{i} + \left|\vec{AB}\right|\sin 60^o\,\hat{j}$$

$$= 3\times\frac{1}{2}\hat{i} + 3\times\frac{\sqrt{3}}{2}\hat{j}$$

$$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

By the triangle law of vector addition, we have:

$$\vec{OB} = \vec{OA} + \vec{AB}$$

$$= \left(-4\hat{i}\right) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right)$$

$$= \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$= \left(\frac{-8+3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

$$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

Hence, the girl's displacement from her initial point of departure is

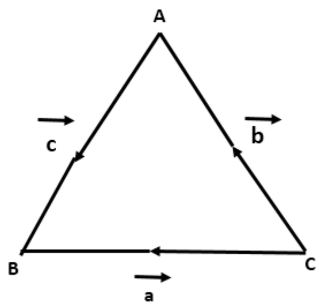
$$\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$$

4:

If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$?Justify your answer.

Solution:

In ΔABC, let $\vec{CB} = \vec{a}$, $\vec{CA} = \vec{b}$, and $\vec{AB} = \vec{c}$ (as shown in the following figure).



Now, by the triangle law of vector addition, we have $\vec{a} = \vec{b} + \vec{c}$.

It is clearly known that $|\vec{a}|$, $|\vec{b}|$ and $|\vec{c}|$ represent the sides of ΔABC.

Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$\therefore |\vec{a}| < |\vec{b}| + |\vec{c}|$

Hence, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$

5:

Find the value of x for which $x\left(\hat{i}+\hat{j}+\hat{k}\right)$ is a unit vector

Solution:

$x\left(\hat{i}+\hat{j}+\hat{k}\right)$ is a unit vector if $\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right| = 1$.

Now,

$\left|x\left(\hat{i}+\hat{j}+\hat{k}\right)\right| = 1$

$\Rightarrow \sqrt{x^2+x^2+x^2} = 1$

$\Rightarrow \sqrt{3x^2} = 1$

$\Rightarrow \sqrt{3}x = 1$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$

Hence, the required value of x is $\pm\frac{1}{\sqrt{3}}$.

6:

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors

$\boldsymbol{a} = 2\hat{i} + 3\hat{j} - k$ ***and*** $\boldsymbol{b} = \hat{i} - 2\hat{j} + \hat{k}$

Solution:

We have,

Let $\vec{c}$ be the resultant of $\vec{a}, \vec{b}$ *and* $\vec{c}$.

Then,

$\vec{c} = \vec{a}+\vec{b} = (2+1)\hat{i} + (3+2)\hat{j} + (-1+1)\hat{k} = 3\hat{i} + \hat{j}$

$\therefore |\vec{c}| = \sqrt{3^2+1^2} = \sqrt{9+1} = \sqrt{10}$

$\therefore \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{\left(3\hat{i}+\hat{j}\right)}{\sqrt{10}}$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$\pm 5.\hat{c} = \pm 5.\frac{1}{\sqrt{10}}\left(3\hat{i}+\hat{j}\right) = \pm\frac{3\sqrt{10}\hat{i}}{2} \pm \frac{\sqrt{10}}{2}\hat{j}$

7:

If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$,$\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ *and* $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$,find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

Solution:

We have,

$\vec{a}=\hat{i}+\hat{j}+\hat{k}$,$\vec{b}=2\hat{i}-\hat{j}+3\hat{k}$ *and* $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$

$2\vec{a}-\vec{b}+3\vec{c}=2\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(2\hat{i}-\hat{j}+3\hat{k}\right)+3\left(\hat{i}-2\hat{j}+\hat{k}\right)$

$=2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+j-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}$

$=3\hat{i}-3\hat{j}+2\hat{k}$

$\left|2\vec{a}-\vec{b}+3\vec{c}\right|=\sqrt{3^2+(-3)^2+2^2}=\sqrt{9+9+4}=\sqrt{22}$

Hence, the unit vector along $2\vec{a}-\vec{b}+3\vec{c}$ is

$$\frac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|}=\frac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}}=\frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}.$$

8:

Show that the points $A\ (1,\ -2,\ -8)$, $B\ (5,\ 0,\ -2)$ and $C\ (11,\ 3,\ 7)$ **are collinear, and find the ratio in which B divides AC.**

Solution:

The given points are $A\ (1,\ -2,\ -8)$, $B\ (5,\ 0,\ -2)$ and $C\ (11,\ 3,\ 7)$

$\therefore\ \overrightarrow{AB}=(5-1)\hat{i}+(0+2)\hat{j}+(-2+8)\hat{k}=4\hat{i}+2\hat{j}+6\hat{k}$

$\overrightarrow{BC}=(11-5)\hat{i}+(3-0)\hat{j}+(7+2)\hat{k}=6\hat{i}+3\hat{j}+9\hat{k}$

$\overrightarrow{AC}=(11-1)\hat{i}+(3+2)\hat{j}+(7+8)\hat{k}=10\hat{i}+5\hat{j}+15\hat{k}$

$\left|\overrightarrow{AB}\right|=\sqrt{4^2+2^2+6^2}=\sqrt{16+4+36}=\sqrt{56}=2\sqrt{14}$

$\left|\overrightarrow{BC}\right|=\sqrt{6^2+3^2+9^2}=\sqrt{36+9+81}=\sqrt{126}=3\sqrt{14}$

$\left|\overrightarrow{AC}\right|=\sqrt{10^2+5^2+15^2}=\sqrt{100+25+225}=\sqrt{350}=5\sqrt{14}$

$\therefore\ \left|\overrightarrow{AC}\right|=\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|$

Thus, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda:1$. Then, we have:

$$\overrightarrow{OB}=\frac{\lambda\ \overrightarrow{OC}+\overrightarrow{AA}}{(\lambda+1)}$$

$$\Rightarrow 5\hat{i}-2\hat{k}=\frac{\lambda\left(11\hat{i}+3\hat{j}+7\hat{k}\right)+\left(\hat{i}-2\hat{j}-8\hat{k}\right)}{\lambda+1}$$

$$\Rightarrow(\lambda+1)\left(5\hat{i}-2\hat{k}\right)=11\lambda\hat{i}+3\lambda\hat{j}+7\lambda\hat{k}+\hat{i}-2\hat{j}-8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i}-2(\lambda+1)\hat{k}=(11\lambda+1)\hat{i}+(3\lambda-2)\hat{j}+(7\lambda-8)\hat{k}$$

On equating the corresponding components, we get:

$$5(\lambda+1)=11\lambda+1$$

$$\Rightarrow 5\lambda+1=11\lambda+1$$

$$\Rightarrow 6\lambda=4$$

$$\Rightarrow \lambda=\frac{4}{6}=\frac{2}{3}$$

Hence, point B divides AC in the ratio $2:3$.

9:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\left(2\vec{a}+\vec{b}\right)$ and $\left(\vec{a}\text{-}3\vec{b}\right)$ externally in the ratio $1:2$. Also, show that P is the midpoint of the line segment RQ.

Solution:

It is given that $\overrightarrow{OP}=2\vec{a}+\vec{b}, \overrightarrow{OQ}=\vec{a}\text{-}3\vec{b}$.

It is given that point R divides a line segment joining two points P and Q externally in the ratio $1:2$. Then, on using the section formula, we get:

$$\overrightarrow{OR}=\frac{2\left(2\vec{a}+\vec{b}\right)-\left(\vec{a}-3\vec{b}\right)}{2-1}$$

$$=\frac{4\vec{a}+2\vec{b}-\vec{a}+3\vec{b}}{1}$$

$$=3\vec{a}+5\vec{b}$$

Therefore, the position vector of point R is $3\vec{a}+5\vec{b}$

Position vector of the mid-point of RQ = $\dfrac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$=\frac{\left(\vec{a}\text{-}3\vec{b}\right)+\left(3\vec{a}\text{+}5\vec{b}\right)}{2}$$

$$=2\vec{a}+\vec{b}$$

$$=\overrightarrow{OP}$$

Hence, P is the mid-point of the line segment RQ.

10:

The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$.
Find the unit vector parallel to its diagonal. Also, find its area.

Solution:
Adjacent sides of a parallelogram are given as: $\vec{a}=2\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}-3\hat{k}$
Then, the diagonal of a parallelogram is given by $\vec{a}+\vec{b}$

$$\vec{a}+\vec{b}=(2+1)\hat{i}+(-4-2)\hat{j}+(5-3)\hat{k}=3\hat{i}-6\hat{j}+2\hat{k}$$

Thus, the unit vector parallel to the diagonal is

$$\frac{\vec{a}+\vec{b}}{\left|\vec{a}+\vec{b}\right|}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{3^2+(-6)^2+2^2}}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4}}=\frac{3\hat{i}-6\hat{j}+2\hat{k}}{7}=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}.$$

$\therefore$ Area of parallelogram ABCD = $\left|\vec{a}+\vec{b}\right|$

$$\vec{a}+\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -4 & 5\\ 1 & -2 & -3\end{vmatrix}$$

$$=\hat{i}(12+10)-\hat{j}(-6-5)+\hat{k}(-4+4)$$

$$=22\hat{i}+11\hat{j}$$

$$=11\left(2\hat{i}+\hat{j}\right)$$

$$\therefore\ \left|\vec{a}+\vec{b}\right|=11\sqrt{2^2+1^2}=11\sqrt{5}$$

Hence, the area of the parallelogram is $11\sqrt{5}$ square units

11:
Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$.

Solution:
Let a vector be equally inclined to axes OX, OY, and OZ at angle α.
Then, the direction cosines of the vector are $\cos\alpha$, $\cos\alpha$, and $\cos\alpha$.
Now,

$$\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$$

$$\Rightarrow 3\cos^2\alpha=1$$

$$\Rightarrow \cos\alpha=\frac{1}{\sqrt{3}}$$

Hence, the direction cosines of the vector which are equally inclined to the axes are $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$.

12:

Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$ **and** $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ *and* $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$**.Find a vector** $\vec{d}$ **which is perpendicular to both** $\vec{a}$ **and** $\vec{b}$ **, and** $\vec{c}.\vec{d}=15$

Solution:

Let $\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$

Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

$\vec{d}.\vec{a}=0$

$\Rightarrow d_1+4d_2+2d_3=0 \qquad ...(i)$

$\vec{d}.\vec{b}=0$

$\Rightarrow 3d_1-2d_2+7d_3=0 \qquad ...(ii)$

Also, it is given that:

$\vec{c}.\vec{d}=15$

$\Rightarrow 2d_1-d_2+4d_3=15 \qquad ...(iii)$

On solving (i), (ii), and (iii), we get:

$$d_1=\frac{160}{3}, d_2=-\frac{5}{3}\, and\; d_3=-\frac{70}{3}$$

$$\therefore \vec{d}=\frac{160}{3}\hat{i}-\frac{5}{3}\hat{j}-\frac{70}{3}\hat{k}=\frac{1}{3}\left(160\hat{i}-5\hat{j}-70\hat{k}\right)$$

Hence, the required vector is $\frac{1}{3}\left(160\hat{i}-5\hat{j}-70\hat{k}\right)$

13:

The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

Solution:

$$\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$$

$$=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

Therefore, unit vector along $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$ is given as:

$$\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{4+4\lambda+\lambda^2+36+4}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$$

Scalar product of ($\hat{i}+\hat{j}+\hat{k}$) with this unit vector is 1.

$$\Rightarrow \hat{i}+\hat{j}+\hat{k}.\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$$

$$\Rightarrow \frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$$
$$\Rightarrow \sqrt{\lambda^2+4\lambda+44}=\lambda+6$$
$$\Rightarrow \lambda^2+4\lambda+44=(\lambda+6)^2$$
$$\Rightarrow \lambda^2+4\lambda+44=\lambda^2+12\lambda+36$$
$$\Rightarrow 8\lambda=8$$
$$\Rightarrow \lambda=1$$

Hence, the value of λ is 1.

14:

If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$.

Solution:

Since $\vec{a},\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors, we have

$\vec{a}.\vec{b}=\vec{b}.\vec{c}=\vec{c}.\vec{a}=0$. It is given that: $|\vec{a}|=|\vec{b}|=|\vec{c}|$

Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b}$ and $\vec{c}$ at angles $\theta_1,\theta_2,$ and θ_3 respectively.

Then, we have:

$$\cos\theta_1=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}=\frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$=\frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} \qquad \left[\vec{b}.\vec{a}=\vec{c}.\vec{a}=0\right]$$

$$=\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_2=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}=\frac{\vec{a}.\vec{b}+\vec{b}.\vec{b}+\vec{c}.\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|}$$

$$=\frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|} \qquad \left[\vec{a}.\vec{b}=\vec{c}.\vec{b}=0\right]$$

$$=\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\text{os}\,\theta_3=\frac{(\vec{a}+\vec{b}+\vec{c}).\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}=\frac{\vec{a}.\vec{c}+\vec{b}.\vec{c}+\vec{c}.\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}$$

$= \frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}$ $\quad [\vec{a}.\vec{c} = \vec{b}.\vec{c} = 0]$

$= \frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$

Now as $|\vec{a}| = |\vec{b}| = |\vec{c}|$, $\cos\theta_1 = \cos\theta_2 = \cos\theta_3$

$\therefore \theta_1 = \theta_2 = \theta_3$

Hence, the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}, \vec{b} \, and \, \vec{c}$.

15:

Prove that, $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$

Solution:

$(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$

$\Leftrightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2$ $\quad$ [Distributivity of scalar products over addition]

$\Leftrightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$ $\quad [\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (Scalar product is commutative)$]$

$\Leftrightarrow 2\vec{a}.\vec{b} = 0$

$\Leftrightarrow \vec{a}.\vec{b} = 0$

$\therefore \vec{a}$ and $\vec{b}$ are prerendicular. $\quad [\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$ (Given)$]$

16:

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then only when

(A) $0 < \theta < \frac{\pi}{2}$ $\quad$ (B) $0 \leq \theta \leq \frac{\pi}{2}$

(C) $0 < \theta < \pi$ $\quad$ (D) $0 \leq \theta \leq \pi$

Solution:

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors so that $|\vec{a}|$ and $|\vec{b}|$ be positive

It is known that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\therefore \vec{a}.\vec{b} \geq 0$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$

$\Rightarrow \cos\theta \geq 0 \qquad \left[|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}\right]$

$\Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$

Hence, $\vec{a}.\vec{b} \geq 0$ when $0 \leq \theta \leq \frac{\pi}{2}$

The correct answer is B.

17:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if

(A) $\theta = \frac{\pi}{4}$ (B) $\theta = \frac{\pi}{3}$ (C) $\theta = \frac{\pi}{2}$ (D) $\theta = \frac{2\pi}{3}$

Solution:

Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ be the angle between them.

Then, $|\vec{a}| = |\vec{b}| = 1$

Now, $\vec{a}+\vec{b}$ is a unit vector if $|\vec{a}+\vec{b}| = 1$.

$|\vec{a}+\vec{b}| = 1$

$\Rightarrow (\vec{a}+\vec{b})^2 = 1$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = 1$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1$

$\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$

$\Rightarrow 1 + 2.1.1\cos\theta + 1 = 1$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$

Hence, $\vec{a}+\vec{b}$ is a unit vector if $\theta = \frac{2\pi}{3}$

The correct answer is D.

18:

The value of $\hat{i}.(\hat{j}\times\hat{k}) + \hat{j}.(\hat{i}\times\hat{k}) + \hat{k}.(\hat{i}\times\hat{j})$ is

(A) 0 (B) -1 (C) 1 (D) 3

Solution:

$\hat{i}.\left(\hat{j}\times\hat{k}\right)+\hat{j}.\left(\hat{i}\times\hat{k}\right)+\hat{k}.\left(\hat{i}\times\hat{j}\right)$

$=\hat{i}.\hat{i}+\hat{j}.\left(-\hat{j}\right)+\hat{k}.\hat{k}$

$=1-\hat{j}.\hat{j}+1$

$=1-1+1$

$=1$

The correct answer is C.

19:

If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$ then $\left|\vec{a}.\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$ when θ is equal to

(A) 0 (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

Solution:

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so

That $|\vec{a}|$ and $|\vec{b}|$ are positive.

$\left|\vec{a}.\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$

$\Rightarrow|\vec{a}||\vec{b}|\cos\theta=|\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow\cos\theta=\sin\theta \qquad \left[|\vec{a}| \text{ and } |\vec{b}| \text{ are positive}\right]$

$\Rightarrow\tan\theta=1$

$\Rightarrow\theta=\frac{\pi}{4}$

Hence, $\left|\vec{a}.\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$ when θ is equal to $\frac{\pi}{4}$

The correct answer is B.