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1:

Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Solution:

Two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13}$$

$$= \frac{48}{169} - \frac{36}{169} - \frac{12}{169}$$

$$= 0$$

Therefore, the lines are perpendicular.

(ii)For the lines with direction cosines, $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ we obtain

$$l_1l_2 + m_1m_2 + n_1n_2 = \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13}$$

$$= \frac{12}{169} - \frac{48}{169} + \frac{36}{169}$$

$$= 0$$

Therefore, the lines are perpendicular.

(iii)For the lines with direction cosines, $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we obtain

$$\left(\frac{3}{13}\right)\times\left(\frac{12}{13}\right)+\left(\frac{-4}{13}\right)\times\left(\frac{-3}{13}\right)+\left(\frac{12}{13}\right)\times\left(\frac{-4}{13}\right)$$
$$=\frac{36}{169}+\frac{12}{169}-\frac{48}{169}$$
$$=0$$

Therefore, the lines are perpendicular.
Thus, all the lines are mutually perpendicular.

2:
Show that the line through the points (1, -1, 2) (3, 4, -2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Solution:
Let AB be the line joining the points, (1, -1, 2) and (3, 4, -2), and CD be the line through the points (0, 3, 2) and (3, 5, 6).
The direction ratios, a_1, b_1, c_1, of AB are (3 -1), (4 – (-1)), and (-2 -2) i.e., 2, 5, and -4.
The direction ratios, a_2, b_2, c_2, of CD are (3 -0), (5 -3), and (6 -2) i.e., 3, 2,and 4.
AB and CD will be perpendicular to each other, if $a_1a_2+b_1b_2+c_1c_2=0$

$a_1a_2+b_1b_2+c_1c_2 = 2\times3+5\times2+(-4)\times4$
= 6 + 10 – 16
= 0
Therefore, AB and CD are perpendicular to each other.

3:
Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (-1, -2, 1), (1, 2, 5).

Solution:
Let AB be the line through the points (4, 7, 8) and (2, 3, 4), CD be the line through the points, (-1, -2, 1) and (1, 2, 5).
The directions ratios, a_1, b_1, c_1, of AB are (2 -4), (3 -7), and (4 -8) i.e., -2, -4, and -4.
The direction ratios, a_2, b_2, c_2, of CD are (1 – (-1)), (2 – (-2)), and (5 -1) i.e., 2, 4, and 4.

AB will be parallel to CD, if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

$$\frac{a_1}{a_2}=\frac{-2}{2}=-1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

4:

Find the equation of the line which passes through point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Solution:

It is given that the line passes through the point A (1, 2, 3). Therefore, the position vector through A is $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$

It is known that the line which passes through point A and parallel to $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda\vec{b}$, where λ is a constant.

$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda\left(3\hat{i} + 2\hat{j} - 2\hat{k}\right)$

This is the required equation of the line.

5:

Find the equation of the line in vector and in Cartesian form that passes through the point with positive vector $2\hat{i} - \hat{j} + 4\hat{k}$ **and is in the direction** $\hat{i} + 2\hat{j} - \hat{k}$ **.**

Solution:

It is given that the line passes through the point with positive vector

$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k} \quad ...(1)$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k} \quad ...(2)$

It is known that a line through a point with positive vector $\vec{a}$ and parallel to $\vec{b}$ is given by the equation,

$\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$

This is the required equation of the line in vector form.

$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$

$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$

Eliminating λ, we obtain the Cartesian form equation as

$$\frac{x-2}{1}=\frac{y+1}{2}=\frac{z-4}{-1}$$

This is the required of the given line in Cartesian form.

6:

Find the Cartesian equation of the line which passes through the point (−2, 4, −5) and parallel to the line given by $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

Solution:

It is given that the line passes through the point (−2, 4, −5) and is parallel to

$$\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$$

The direction ratios of the line, $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$, are 3, 5, and 6.

The required line is parallel to $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$

Therefore, its direction ratios are 3k, 5k, and 6k, when k ≠0

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction ratios, a, b, c, is given by $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$

Therefore the equation of the required line is

$$\frac{x+2}{3k}=\frac{y-4}{5k}=\frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}=k$$

7:

The Cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$.Write its vector form.

Solution:

The Cartesian equation of the line is

$$\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}. \qquad \ldots(1)$$

The given line passes through the point (5, −4, 6). The positive vector of this point is

$\vec{a}=5\hat{i}-4\hat{j}+6\hat{k}$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of the vector, $\vec{b}=3\hat{i}+7\hat{j}+2\hat{k}$

It is known that the line through positive vector $\vec{a}$ and in the direction of the vector $\vec{b}$ is given by the equation, $\vec{r}=\vec{a}+\lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r} = \left(5\hat{i} - 4\hat{j} + 6\hat{k}\right) + \lambda\left(3\hat{i} + 7\hat{j} + 2\hat{k}\right)$$

This is the required equation of the given line in vector form.

8:

Find the vector and the Cartesian equations of the lines that passes through the origin and (5, −2, 3).

Solution:

The required line passes through the origin. Therefore, its positive vector is given by,

$\vec{a} = \vec{0}$ …(1)

The direction ratios of the line through origin and (5, −2, 3) are

(5 −0) = 5, (−2 −0) = −2, (3 – 0) = 3

The line is parallel to the vector given by the equation, $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$

The equation of the line in vector form through a point with position vector $\vec{a}$ and parallel to $\vec{b}$ is,

$\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$\Rightarrow \vec{r} = \vec{0} + \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$

$\Rightarrow \vec{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$

The equation of the line through the point (x_1, y_1, z_1) and direction ratios a, b, c is given by,

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Therefore, the equation of the required line in the Cartesian form is

$$\frac{x - 0}{5} = \frac{y - 0}{-2} = \frac{z - 0}{3}$$

$$\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

9:

Find the vector and the Cartesian equation of the line that passes through the point (3, −2, −5), (3, −2, 6).

Solution:

Let the line passing through the points, P (3, −2, −5) and Q (3, −2, 6), be PQ.

Since PQ passes through P (3, −2, −5), its positive vector is given by,

$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$

The direction ratios of PQ are given by,

(3 − 3) = 0, (−2 + 2) = 0, (6 + 5) = 11

The equation of the vector in the direction of PQ is

$\vec{b} = 0.\hat{i} - 0.\hat{j} + 11\hat{k} = 11\ \hat{k}$

The equation of PQ in vector form is given by, $\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$\Rightarrow \vec{r} = \left(3\hat{i} - 2\hat{j} - 5\hat{k}\right) + 11\lambda\hat{k}$

The equation of PQ in Cartesian form is

$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ i.e., $\frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{11}$

10:

Find the angle between the following pairs of lines:

(i) $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda\left(3\hat{i} - 2\hat{j} + 6\hat{k}\right)$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

(ii) $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + 1\lambda\left(\hat{i} - \hat{j} - 2\hat{k}\right)$ and

$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$

Solution:

(i) Let Q be the angle between the given lines.

The angle between the given pairs of lines is given by, cos Q = $\left|\frac{\bar{b}_1.\bar{b}_2}{\left|\bar{b}_1\right|\left|\bar{b}_2\right|}\right|$

The given lines are parallel to the vectors, $\bar{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\bar{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$, respectively.

$\therefore \left|\bar{b}_1\right| = \sqrt{3^2 + 2^2 + 6^2} = 7$

$\left|\bar{b}_2\right| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$

$\bar{b}_1.\bar{b}_2 = \left(3\hat{i} - 2\hat{j} + 6\hat{k}\right).\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

$= 3 \times 1 + 2 \times 2 + 6 \times 2$

$= 3 + 4 + 12$

$= 19$

$\Rightarrow \cos Q = \frac{19}{7 \times 3}$

$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$

(ii)The given line are parallel to the vectors, $\bar{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\bar{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$\therefore \left|\bar{b}_1\right| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

$\left|\bar{b}_2\right| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$

$\bar{b}_1.\bar{b}_2 = \left(\hat{i} - \hat{j} - 2\hat{k}\right).\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$

$= 1.3 - 1(-5) - 2(-4)$

$= 3 + 5 + 8$

$= 16$

$\cos Q = \left|\dfrac{\bar{b}_1.\bar{b}_2}{|\bar{b}_1||\bar{b}_2|}\right|$

$\Rightarrow \cos Q = \dfrac{16}{\sqrt{6}.5\sqrt{2}} = \dfrac{16}{\sqrt{2}.\sqrt{3}.5\sqrt{2}} = \dfrac{16}{10\sqrt{3}}$

$\Rightarrow \cos Q = \dfrac{8}{5\sqrt{3}}$

$\Rightarrow Q = \cos^{-1}\left(\dfrac{8}{5\sqrt{3}}\right)$

11:

Find the angle between the following pairs of lines:

(i) $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$ and $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$

(ii) $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z}{1}$ and $\dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$

Solution:

Let $\bar{b}_1$ and $\bar{b}_2$ be the vectors parallel to the pair of lines,

$\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$ and $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$, respectively.

$\therefore \bar{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\bar{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$

$|\bar{b}_1| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$

$|\bar{b}_2| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$

$\bar{b}_1.\bar{b}_2 = (2\hat{i} + 5\hat{j} - 3\hat{k}).(-\hat{i} + 8\hat{j} + 4\hat{k})$

$= 2(-1) + 5 \times 8 + (-3).4$

$= -2 + 40 - 12$

$= 26$

The angle, Q, between the given pair of lines is given by the relation,

$\cos Q = \left|\dfrac{\bar{b}_1.\bar{b}_2}{|\bar{b}_1||\bar{b}_2|}\right|$

$\Rightarrow \cos Q = \dfrac{26}{9\sqrt{38}}$

$$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) Let $\bar{b}_1, \bar{b}_2$ be the vectors parallel to the given pair of lines,

$\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$, respectively.

$\bar{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$

$\bar{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$

$\therefore |\bar{b}_1| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$

$|\bar{b}_2| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$

$\bar{b}_1.\bar{b}_2 = \left(2\hat{i} + 2\hat{j} + \hat{k}\right).\left(4\hat{i} + \hat{j} + 8\hat{k}\right)$

$= 2\times4 + 2\times1 + 1\times8$

$= 8 + 2 + 8$

$= 18$

If Q is the angle between the given pair of lines, then cos $Q = \left|\frac{\bar{b}_1.\bar{b}_2}{|\bar{b}_1||\bar{b}_2|}\right|$

$$\Rightarrow \cos Q = \frac{18}{3\times9} = \frac{2}{3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$$

12:

Find the values of p so the $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ **are at right line angles.**

Solution:

The given equations can be written in the standard form as

$$\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2} \text{ and } \frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

The direction ratios of the lines are -3, $\frac{2p}{7}$, 2 and $\frac{-3p}{7}$, 1, -5 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\therefore (-3).\left(\frac{-3p}{7}\right) + \left(\frac{2p}{7}\right).(1) + 2.(-5) = 0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}=10$$
$$\Rightarrow 11p=70$$
$$\Rightarrow p=\frac{70}{11}$$

Thus, the value of p = $\frac{70}{11}$

13:

Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ **are perpendicular to each other.**

Solution:

The equation of the given lines are $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

The direction ratios of the given lines are 7, -5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1,b_1,c_1 and a_2,b_2,c_2 are perpendicular to each other, if

$$a_1a_2+b_1b_2+c_1c_2=0$$
$$\therefore 7\times 1+(-5)\times 2+1\times 3$$
$$=7-10+3$$
$$=0$$

Therefore, the given lines are perpendicular to each other.

14:

Find the shortest distance between the lines

$\vec{r}|\ /\hat{i}\ \ 22\hat{j}\ \ 2k|\ \varsigma 2\ (\hat{i}\ \ 4\hat{j}\ 2\hat{k})$ and

$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

Solution:

The equations of the given lines are

$\vec{r}|\ /\hat{i}\ \ 22\hat{j}\ \ 2k|\ \varsigma 2\ (\hat{i}\ \ 4\hat{j}\ 2\hat{k})$ and

$\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

t is known that the shortest distance between the lines $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right| \qquad \ldots(1)$$

Comparing the given equations, we obtain

$\vec{a}_1=\hat{i}+2\hat{j}+\hat{k}$

$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$
$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$
$\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$
$\vec{a}_2 - \vec{a}_1 = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - 3j + k$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = (-2 - 1)\hat{i} - (2 - 2)\hat{j} + (1 + 2)k = -3\hat{i} + 3k \Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right|$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-3)^2 + (3)^2} = \sqrt{+9+9} = \sqrt{18} = 3\sqrt{2}$$

Substituting all the values in equation (1) , we obtain

$$d = \left|\frac{\left(-3\hat{i} + 3\hat{k}\right).\left(\hat{i} - 3\hat{j} - 2\hat{k}\right)}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \left|\frac{-3.1 + 3(-2)}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \left|\frac{-9}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

15:

Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ **and** $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Solution:

The given lines are $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ **and** $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$, is given by,

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{\left(b_1c_2 - b_2c_1\right)^2 + \left(c_1a_2 - c_2a_1\right)^2 + \left(a_1b_2 - a_2b_1\right)^2}} \quad \ldots(1)$$

Comparing the equations, we obtain

$x_1 = -1, \quad y_1 = -1, \quad z_1 = -1$
$a_1 = 7, \quad b_1 = -6, \quad c_1 = 1$
$x_2 = 3, \quad y_2 = 5, \quad z_2 = 7$
$a_2 = 1, \quad b_2 = -2, \quad c_2 = 1$

Then, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$= 4(-16+2) - 6(7-1) + 8(-14+6)$
$= -16 - 36 - 64$
$= -116$

$\Rightarrow \sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} = \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2}$
$= \sqrt{16+36+64}$
$= \sqrt{116}$
$= 2\sqrt{29}$

Substituting all the values in equation (1), we obtain

$$d = \frac{-116}{2\sqrt{29}} = \frac{-58}{\sqrt{29}} = \frac{-2\times 29}{\sqrt{29}} = -2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is $2\sqrt{29}$ units.

16:

Find the shortest distance between the lines whose vector equations are

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$

$and\ \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

Solution:

The given lines are $\hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| \qquad \ldots(1)$$

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we obtain

$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$$\vec{a}_2-\vec{a}_1=\left(4\hat{i}+5\hat{j}+6\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=3\hat{i}+3\hat{j}+3\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ 2 & 3 & 1\end{vmatrix}=(-3-6)\hat{i}-(1-4)\hat{j}+(3+6)\hat{k}=-9\hat{i}+3\hat{j}+9\hat{k}$$

$$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(-9)^2+(3)^2+(9)^2}=\sqrt{81+9+81}=\sqrt{171}=3\sqrt{19}$$

$$\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_2-\vec{a}\right)=\left(-9\hat{i}+3\hat{j}+9\hat{k}\right).\left(3\hat{i}+3\hat{j}+3\hat{k}\right)$$

$$=-9\times3+3\times3+9\times3$$

$$=9$$

Substituting all the values in equation (1), we obtain

$$d=\left|\frac{9}{3\sqrt{19}}\right|=\frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

17:

Find the shortest distance between the lines whose vector equations are

$\vec{r}\,(=1-t)i\,(+t-2)\hat{j}\,(+3-2t)\hat{k}$ and

$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$

Solution:

The given lines are

$\vec{r}\,(=1-t)i\,(+t-2)\hat{j}\,(+3-2t)\hat{k}$

$$\Rightarrow\vec{r}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)+t\left(-\hat{i}+\hat{j}-2\hat{k}\right)\qquad\ldots(1)$$

$$\vec{r}=(s+1)\hat{i}+(2s-1)\hat{j}-(2s+1)\hat{k}$$

$$\vec{r}=\left(\hat{i}-\hat{j}-\hat{k}\right)+s\left(\hat{i}+2\hat{j}-2\hat{k}\right)\qquad\ldots(2)$$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).\left(\vec{a}_2-\vec{a}_1\right)}{\left|\vec{b}_1\times\vec{b}_2\right|}\right|\qquad\ldots(3)$$

For the given equations,

$\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k}$

$\vec{b}_1=-\hat{i}+\hat{j}-2\hat{k}$

$\vec{a}_2=\hat{i}-\hat{j}-\hat{k}$

$\vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$

$$\vec{a}_2-\vec{a}_1=\left(\hat{i}-\hat{j}-\hat{k}\right)-\left(\hat{i}+2\hat{j}+3\hat{k}\right)=\hat{j}-4\hat{k}$$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & -2\\ 1 & 2 & -2\end{vmatrix}=(-2+4)\hat{i}-(2+2)\hat{j}+(-2-1)\hat{k}=2\hat{i}-4\hat{j}-3\hat{k}$$

$$\Rightarrow\left|\vec{b}_1\times\vec{b}_2\right|=\sqrt{(2)^2+(-4)^2+(-3)^2}=\sqrt{4+16+9}=\sqrt{29}$$

$$\therefore\left(\vec{b}_1\times\vec{b}_2\right).(\vec{a}_2-\vec{a})=\left(2\hat{i}-4\hat{j}-3\hat{k}\right).\left(\hat{j}-4\hat{k}\right)=-4+12=8$$

Substituting all the values in equation (3), we obtain

$$d=\left|\frac{8}{\sqrt{29}}\right|=\frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.