Miscellaneous Exercise

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1:

Show that the line joining the origin to the point $(2, 1, 1)$ is perpendicular to the line determined by the points $(3, 5, -1), (4, 3, -1)$.

Solution:

Let OA be the line joining the origin, O (0, 0, 0), and the points, A$(2, 1, 1)$.

Also, let BC be the line joining the points, B $(3, 5, -1)$ and C $(4, 3, -1)$.

The direction ratios of OA are 2, 1, and 1 and of BC are

$(4-3)=1, (3-5) = -2,$ and $(-1+1)=0$

OA is perpendicular to BC, if $a_1a_2+b_1b_2+c_1c_2=0$

$\therefore a_1a_2+b_1b_2+c_1c_2=2\times1+1(-2)+1\times0=2-2=0$

Thus, OA is perpendicular to BC.

2:

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines. Show that the direction cosines of the perpendicular to both of these are

$m_1n_2-m_2n_1, n_1l_2-n_2l_1, l_1m_2-l_2m_1$.

Solution:

It is given that l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines. Therefore,

$l_1l_2+m_1m_2+n_1n_2=0 \quad ...(1)$

$l_1^2+m_1^2+n_1^2=1 \quad ...(2)$

$l_2^2+m_2^2+n_2^2=1 \quad ...(3)$

Let l, m, n be the direction cosines of the line which is perpendicular to the line with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2.

$\therefore ll_1+mm_1+nn_1=0$

$ll_2+mm_2+nn_2=0$

$$\therefore \frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$

$$\Rightarrow \frac{l^2}{(m_1n_2-m_2n_1)^2}=\frac{m^2}{(n_1l_2-n_2l_1)^2}=\frac{n^2}{(l_1m_2-l_2m_1)^2}$$

$$=\frac{l^2+m^2+n^2}{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2} \quad ...(4)$$

l, m, n are the direction cosines of the line.

$\therefore\ l^2+m^2+n^2=1 \quad ....(5)$

It is known that,

$(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)$

$=(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2$

From (1, (2), and (3), we obtain

$\Rightarrow 1.1-0=\left(m_1n_2-m_2n_1\right)^2+\left(n_1l_2-n_2l_1\right)^2+\left(l_1m_2-l_2m_1\right)^2$

$\therefore \therefore \left(m_1n_2-m_2n_1\right)^2+\left(n_1l_2-n_2l_1\right)^2+\left(l_1m_2-l_2m_1\right)^2=1 \qquad \ldots(6)$

Substituting the values from equation (5) and (6) in equation (4), we obtain

$$\frac{l^2}{\left(m_1n_2-m_2n_1\right)^2}=\frac{m^2}{\left(n_1l_2-n_2l_1\right)^2}=\frac{n^2}{\left(l_1m_2-l_2m_1\right)^2}=1=1$$

$\Rightarrow 1=m_1n_2-m_2n_1,\ m=n_1l_2-n_2l_1, n=l_1m_2-l_2m_1$

Thus, the direction cosines of the required line are $m_1n_2-m_2n_1$, $n_1l_2-n_2l_1$, $l_1m_2-l_2m_1$.

3:

Find the angle between the lines whose direction ratios a $a,\ b,\ c$ and $b-c,\ c-a,\ a-b$.

Solution:

The angle Q between the lines with direction cosines $a,\ b,\ c$ and $b-c,\ c-a,\ a-b$ is given by,

$$\cos Q=\left|\frac{a(b-c)+b(c-a)+c(a-b)}{\sqrt{a^2+b^2+c^2}+\sqrt{(b-c)^2+(c-a)^2+(a-b)^2}}\right|$$

$\Rightarrow \cos Q=0$

$\Rightarrow Q=\cos^{-1}0$

$\Rightarrow Q=90^o$

Thus, the angle between the lines is 90°

4:

Find the equation of a line parallel to x-axis and passing through the origin.

Solution:

The line parallel to x-axis and passing through the origin is x-axis itself.

Let A be a point on x-axis. Therefore, the coordinates of A are given by (a, 0, 0), where $a\in R$.

Direction ratios of OA are $(a-0)=a,\ 0,\ 0$

The equation of OA is given by,

$$\frac{x-0}{a}=\frac{y-0}{0}=\frac{z-0}{0}$$

$$\Rightarrow \frac{x}{1}=\frac{y}{0}=\frac{z}{0}=a$$

Thus, the equation of line parallel to x -axis and passing origin is $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

5:

If the coordinates of the points A, B, C, D be $(1,\ 2,\ 3),\ (4,\ 5,\ 7),\ (-4,3,-6)$ and $(2,9,2)$ respectively, then find the angle between the lines AB and CD.

Solution:

The coordinates of A, B, C, and D are $(1,\ 2,\ 3)$, $(4,\ 5,\ 7)$, $(-4,\ 3,-6)$ and $(2,\ 9,\ 2)$ respectively.

The direction ratios of AB are $(4-1)=3,\ (5-2)=3,$ and $(7-3)=4$

The direction ratios of CD are $(2-(-4))=6,\ (9-3)=6,$ and $(2-(-6))=8$

It can be seen that, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=\frac{1}{2}$

Therefore, AB is parallel to CD.

Thus, the angle between AB and CD is either 0^o or 180^o.

6:

If the line $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular, find the value of k.

Solution:

The direction of ratios of the line, $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ **and** $\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$, are $-3, 2k, 2$ and $3k, 1, -5$ respectively.

It is known that two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$

$\therefore -3(3k)+2k\times 1+2(-5)=0$

$\Rightarrow -9k+2k-10=0$

$\Rightarrow 7k=-10$

$\Rightarrow k=\frac{-10}{7}$

Therefore, for $k=-\frac{10}{7}$, the given lines are perpendicular to each other.

7:

Find the vector equation of the plane passing through (1, 2, 3) and perpendicular to the plane $\vec{r}.(\hat{i}+2\hat{j}-5\hat{k})+9=0$

Solution:

The position vector of the point (1, 2, 3) is $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})$

The direction ratios of the normal to the plane, $\vec{r}.(\hat{i}+2\hat{j}-5\hat{k})+9=0$, are 1, 2, and -5

and the normal vector is $\vec{N}=\hat{i}+2\hat{j}-5\hat{k}$

The equation of a line passing through a point and perpendicular to the given plane is given by,

$\vec{l}=\vec{r}+\lambda\vec{N}, \lambda\in R$

$\Rightarrow \vec{l}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}+2\hat{j}-5\hat{k})$

8:

Find the equation of the plane passing through $(a,\ b,\ c)$ **and parallel to the plane** $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=2$

Solution:

Any plane parallel to the plane, $\vec{r}_1.(\hat{i}+\hat{j}+\hat{k})=2$, is of the form

$\vec{r}.(\hat{i}+\hat{j}+\hat{k})=\lambda$ …(1)

The plane passes through the point (a, b, c). Therefore, the positive vector $\vec{r}$ of this point is $\vec{r}=a\hat{i}+b\hat{j}+c\hat{k}$

Therefore, equation (1) becomes

$\left(a\hat{i}+b\hat{j}+c\hat{k}\right).\left(\hat{i}+\hat{j}+\hat{k}\right)=\lambda$

$\Rightarrow a+b+c=\lambda$

Substituting $\lambda=a+b+c$ in equation (1), we obtain

$\vec{r}_1.(\hat{i}+\hat{j}+\hat{k})=a+b+c$ …(2)

This is the vector equation of the required plane.

Substituting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in equation (2), we obtain

$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).(\hat{i}+\hat{j}+\hat{k})=a+b+c$

$\Rightarrow x+y+z=a+b+c$

9:

Find the shortest distance between lines $\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda\left(\hat{i}-2\hat{j}+2\hat{k}\right)$ and $\vec{r}=-4\hat{i}-\hat{k}+\mu\left(3\hat{i}-2\hat{j}-2\hat{k}\right)$.

Solution:

The given lines are

$r=6\hat{i}+2\hat{j}+2k\left(\lambda+\hat{i}-2\hat{j}+2\hat{k}\right)$ …(1)

$r-=4\hat{i}-k\left(\mu+3\hat{i}-2\hat{j}-2\hat{k}\right).$ …(2)

It is known that the shortest distance between two lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$, is given by

$$d=\left|\frac{\left(\vec{b}_1\times\vec{b}_2\right).(\vec{a}_2-\vec{a}_1)}{\left|\vec{b}\times\vec{b}_2\right|}\right| \quad \ldots(3)$$

Comparing, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}_2$ to equation (1) and (2), we obtain

$\vec{a}_1=6\hat{i}+2\hat{j}+2\hat{k}$

$\vec{b}_1=\hat{i}-2\hat{j}+2\hat{k}$

$\vec{a}_2=-4\hat{i}-\hat{k}$

$\vec{b}_2 = 3\hat{i} - 2\hat{j} - 2\hat{k}$

$\Rightarrow \vec{a}_2 - \vec{a}_1 = \left(-4\hat{i} - \hat{k}\right) - \left(6\hat{i} + 2\hat{j} + 2\hat{k}\right) = -10\hat{i} - 2\hat{j} - 3\hat{k}$

$$\Rightarrow \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k} = 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$\left(\vec{b}_1 \times \vec{b}_2\right).(\vec{a}_2 - \vec{a}_1) = \left(8\hat{i} + 8\hat{j} + 4\hat{k}\right).\left(-10\hat{i} - 2\hat{j} - 3\hat{k}\right) = -80 - 16 - 12 = -108$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{-108}{12}\right| = 9$$

Therefore, the shortest distance between the two given lines is 9 units.

10:

Find the coordinates of the point where the line through $(5,\ 1,\ 6)$ and $(3,\ 4,\ 1)$ crosses the YZ-plane.

Solution:

It is known that the equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

The line passing through the point, $(5,\ 1,\ 6)$ and $(3,\ 4,\ 1)$ is given by,

$$\frac{x-5}{3-5} = \frac{y-1}{4-1} = \frac{z-6}{1-6}$$

$$\Rightarrow \frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} = k(say)$$

$\Rightarrow x = 5 - 2k, y = 3k + 1, z = 6 - 5k$

Any point on the line is of the form $(5 - 2k,\ 3k + 1,\ 6 - 5k)$.

The equation of YZ-plane is $x = 0$.

Since the line passes through YZ-plane,

$5 - 2k = 0$

$$\Rightarrow k = \frac{5}{2}$$

$$\Rightarrow 3k + 1 = 3 \times \frac{5}{2} + 1 = \frac{17}{2}$$

$$6 - 5k = 6 - 5 = \frac{5}{2} = \frac{-13}{2}$$

Therefore, the required point is $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$.

11:

Find the coordinates of the point where the line through $(5,\ 1,\ 6)$ and $(3,\ 4,\ 1)$ **crosses the ZX-plane.**

Solution:

It is known that the equation of the line passing through the points,

(x_1, y_1, z_1) and (x_2, y_2, z_2), is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

The line passing through the points, $(5,\ 1,\ 6)$ and $(3,\ 4,\ 1)$, is given by,

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6}$$

$$\Rightarrow \frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=k(say)$$

$$\Rightarrow x=5-2k, y=3k+1, z=6-5k$$

Any point on the line is of the form $(5-2k,\ 3k+1,\ 6-5k)$.

Since the line passes through ZX-plane,

$$3k+1=0$$

$$\Rightarrow k=-\frac{1}{3}$$

$$\Rightarrow 5-2k=5-2\left(-\frac{1}{3}\right)=\frac{17}{3}$$

$$6-5k=6-5=\left(-\frac{1}{3}\right)=\frac{23}{2}$$

Therefore, the required point is $\left(\frac{17}{3},0,\frac{23}{2}\right)$.

12:

Find the coordinates of the point where the line through $(3,-4,-5)$ and $(2,-3,1)$ **crosses the plane** $(2x+y+z=7)$.

Solution:

It is known that the equation of the line through the point, (x_1, y_1, z_1) and (x_2, y_2, z_2), is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Since the line passes through the points, $(3,-4,-5)$ and $(2,-3,1)$, its equation is given by,

$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5}$$

$$\Rightarrow \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=k(say)$$
$$\Rightarrow x=3-k, y=k-4, z=6k-5$$

Therefore, any point on the line is of the form $(3-k,\ k-4,\ 6k-5)$.

This point lies on the plane, $2x+y+z=7$

$\therefore\ 2(3-k)+(k-4)+(6k-5)=7$

$=5k-3=7$

$\Rightarrow k=2$

Hence, the coordinates of the required point are $(3-2,\ 2-4,\ 62-5)$ i.e., $(1-2,\ 7)$.

13:

Find the equation of the plane passing through the points $(-1,\ 3,\ 2)$ and perpendicular to each of the planes $x+2y+3z=5$ and $3x+3y+z=0$.

Solution:

The equation of the plane passing through the point $(-1,\ 3,\ 2)$

$a(x+1)+b\ (y-3)+\ c\ (z-2)=0 \quad \ldots(1)$

Where a, b, c are direction ratios of normal to the plane.

It is known that two planes, $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are perpendicular, if $a_1a_2+b_1b_2+c_1c_2=0$

Plane (1) is perpendicular to the plane, $x+2y+3z=5$

$\therefore\ a.1\ +\ b.2\ +c.3\ =0$

$\Rightarrow\ a+2b+3c=0 \quad \ldots(2)$

Also, plane (1) is perpendicular to the plane, $3x+3y+z=0$

$\therefore\ a.3+b.3+c.1=0$

$\Rightarrow 3a+3b+c=0 \quad \ldots(3)$

From equation (2) and (3), we obtain

$$\frac{a}{2\times1-3\times3}=\frac{b}{3\times3-1\times1}=\frac{c}{1\times3-2\times3}$$
$$\Rightarrow \frac{a}{-7}=\frac{b}{8}=\frac{c}{-3}=k(say)$$
$$\Rightarrow a=-7k,\ b=8k, c=-3k$$

Substituting the values of a, b, and c in equation (1), we obtain

$$-7k(x+1)+8k(y-3)-3k(z-2)=0$$
$$\Rightarrow(-7x-7)+(8y-24)-3z+6=0$$
$$\Rightarrow -7x+8y-3z-25=0$$
$$\Rightarrow 7x-8y+3z+25=0$$

This is the required equation of the plane.

14:

If the points $(1,\ 1,\ \text{p})$ and $(-3,\ 0,\ 1)$ **be equidistant from the plane** $\vec{r} = \left(3\hat{i} + 4\hat{j} - 12\hat{k}\right) + 13 = 0$, then find the value of p.

Solution:

The position vector through the point $(1,\ 1,\ \text{p})$ is $\vec{a}_1 = \hat{i} + \hat{j} + p\hat{k}$

Similarly, the positive vector through the point $(-3,\ 0,\ 1)$ is $\vec{a}_2 = -4\hat{i} + \hat{k}$

The equation of the given plane is $\vec{r} = \left(3\hat{i} + 4\hat{j} - 12\hat{k}\right) + 13 = 0$

It is known that the perpendicular distance between a point whose vector is $\vec{a}$ and the plane, $\vec{r}.\vec{N} = d$, is given by, $D = \frac{\left|\vec{a}.\vec{N} - d\right|}{\left|\vec{N}\right|}$

Here, $\vec{N} = 3\hat{i} + 4\hat{j} - 12\hat{k}$ and $\text{d} = -13$

Therefore, the distance between the point $(1,\ 1,\ \text{p})$ and the given plane is

$$D_1 = \frac{\left|\left(\hat{i} + \hat{j} + p\hat{k}\right).\left(3\hat{i} + 4\hat{j} - 12\hat{k}\right) + 13\right|}{\left|3\hat{i} + 4\hat{j} - 12\hat{k}\right|}$$

$$\Rightarrow D_1 = \frac{|3 + 4 - 12p + 13|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$

$$\Rightarrow D_1 = \frac{|20 - 12p|}{13} \quad \ldots\ldots(1)$$

Similarly, the distance between the point $(-3,\ 0,\ 1)$ and the given plane is

$$D_2 = \frac{\left|\left(-3\hat{i} + \hat{k}\right).\left(3\hat{i} + 4\hat{j} - 12\hat{k}\right) + 13\right|}{\left|3\hat{i} + 4\hat{j} - 12\hat{k}\right|}$$

$$\Rightarrow D_2 = \frac{|-9 - 12 + 13|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$

$$\Rightarrow D_2 = \frac{8}{13} \quad \ldots(2)$$

It is given that the distance between the required plane and the points, $(1,\ 1,\ \text{p})$ and $(-3,\ 0,\ 1)$ is equal.

$$\therefore D_1 = D_2$$

$$\Rightarrow \frac{|20 - 12p|}{13} = \frac{8}{13}$$

$\Rightarrow 20-12p=8$ or $-(20-12p)=8$

$\Rightarrow 12p=12$ or $12p=28$

$\Rightarrow p=1$ or $p=\frac{7}{3}$

15:
Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}.(2\hat{i}+3\hat{j}-\hat{k})+4=0$ and parallel to x-axis.

Solution:
The given planes are

$\vec{r}.(\hat{i}+\hat{j}+\hat{k})=1$

$\Rightarrow \vec{r}.(\hat{i}+\hat{j}+\hat{k})-1=0$

$\vec{r}.(2\hat{i}+3\hat{j}-\hat{k})+4=0$

The equation of any plane passing through the line of intersection of these planes is

$\left[\vec{r}.(\hat{i}+\hat{j}+\hat{k})-1\right]+\lambda\left[(2\hat{i}+3\hat{j}-\hat{k})+4\right]=0$

$\vec{r}.\left[(2\lambda+1)\hat{i}+(3\lambda+1)\hat{j}+(1-\lambda)\hat{k}\right]+(4\lambda+1)0 \quad ...(1)$

Its direction ratios are $(2\lambda+1)$, $(3\lambda+1)$, and $(1-\lambda)$.

The required plane is parallel to x-axis. Therefore, its normal is perpendicular to x-axis.
The direction ratios of x-axis are 1, 0, and 0.

$\therefore 1.(2\lambda+1)+0(3\lambda+1)+0(1-\lambda)=0$

$\Rightarrow 2\lambda+1=0$

$\Rightarrow \lambda=-\frac{1}{2}$

Substituting $\lambda=-\frac{1}{2}$ in equation (1), we obtain

$\Rightarrow \vec{r}.\left[-\frac{1}{2}\hat{j}+\frac{3}{2}\hat{k}\right]+(-3)=0$

$\Rightarrow \vec{r}(\hat{j}-3\hat{k})+6=0$

Therefore, its Cartesian equation is $y-3z+6=0$

This equation of the required plane.

16:
If O be the origin and the coordinates of P be $(1,\ 2,-3)$, then find the equation of the plane passing through P and perpendicular to OP.

Solution:

The coordinates of points, O and P, are $(0,\ 0,\ 0)$ and $(1,\ 2,-3)$ respectively.

Therefore, the direction ratios of OP are $(1-0)=1,\ (2-0)=2,$ and $(-3-0)=-3$

It is known that the equation of the plane passing through the point (x_1, y_1, z_1) is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ Where, a, b, and c are the direction ratios of normal.

Here, the direction ratios of normal are 1, 2, and -3 and the point P is $(1,\ 2,-3)$

Thus, the equation of the required plane is

$$1(x-1)+2(y-2)-3(z+3)=0$$

$$\Rightarrow x+2y-3z-14=0$$

17:

Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0,\ \vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ and which is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$.

Solution:

The equation of the given planes are

$$\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0 \qquad \ldots(1)$$

$$\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0 \qquad \ldots(2)$$

The equation of the plane passing through the line intersection of the plane given in equation (1) and equation (2) is

$$\left[\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4\right]+\lambda\left[\vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5\right]=0$$

$$\vec{r}.\left[(2\lambda+1)\hat{i}+(\lambda+2)\hat{j}+(3-\lambda)\hat{k}\right]+(5\lambda-4)=0 \qquad \ldots(3)$$

The plane in equation (3) is perpendicular to the plane, $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

$$\therefore 5(2\lambda+1)+3(\lambda+2)-6(3-\lambda)=0$$

$$\Rightarrow 19\lambda-7=0$$

$$\Rightarrow \lambda=\frac{7}{19}$$

Substituting $\lambda=\frac{7}{19}$ in equation (3), we obtain

$$\Rightarrow \vec{r}.\left[\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right]\frac{-41}{19}=0$$

$$\Rightarrow \vec{r}.\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0 \qquad \ldots(4)$$

This is the vector equation of the required plane.

The Cartesian equation of this plane can be obtained by substituting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in equation (3).

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right).\left(33\hat{i}+45\hat{j}+50\hat{k}\right)-41=0$$
$$\Rightarrow 33x+45y+50z-41=0$$

18:

Find the distance of the point $\left(-1,-5,-10\right)$ from the point of intersection of the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)$ and the plane $\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5$

Solution:

The equation of the given line is

$$\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right) \qquad \ldots(1)$$

The equation of the given plane is

$$\vec{r}.\left(\hat{i}-\hat{j}+\hat{k}\right)=5 \qquad \ldots(2)$$

Substituting the value of $\vec{r}$ from equation (1) in equation (2), we obtain

$$\left[2\hat{i}-\hat{j}+2\hat{k}+\lambda\left(3\hat{i}+4\hat{j}+2\hat{k}\right)\right].\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$
$$\Rightarrow\left[(3\lambda+2)\hat{i}+(4\lambda-1)\hat{j}+(2\lambda+2)\hat{k}\right].\left(\hat{i}-\hat{j}+\hat{k}\right)=5$$
$$\Rightarrow(3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5$$
$$\Rightarrow\lambda=0$$

Substituting this value in equation (1), we obtain the equation of the line as $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$

This means that the position vector of the point of intersection of the line and plane is $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}$

This shows that the point of intersection of the given line and plane is given by the coordinates, $\left(2,-1,\ 2\right)$. The point is $\left(-1,-5,-10\right)$

The distance d between the points $\left(2,-1,\ 2\right)$ and $\left(-1,-5,-10\right)$ is

$$d=\sqrt{(-1-2)^2+(-5+1)^2+(-10-2)^2}=\sqrt{9+16+144}=\sqrt{169}=13$$

19:

Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes

$\vec{r}\left(=\hat{i}\ -\hat{j}\ +2k=\right)$and $\vec{r}.\left(3\hat{i}\ +\hat{j}\ +\hat{k}\ =\right)6$

Solution:

Let the required line be parallel to vector $\vec{b}$ given by,

$\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

The position vector of the point $(1,\ 2,\ 3)$ is $\hat{a}=\hat{i}+2\hat{j}+3\hat{k}$

The equation of line passing through $(1,\ 2,\ 3)$ and parallel to $\vec{b}$ is given by,

$$\vec{r}=\vec{a}+\lambda\vec{b}$$

$\Rightarrow \vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$ ...(1)

The equation of the given planes are

$\vec{r}.\left(\hat{i}-\hat{j}+2\hat{k}\right)=5$ ...(2)

$\vec{r}.\left(3\hat{i}+\hat{j}+\hat{k}\right)=6$ ...(3)

The line in equation (1) and plane in equation (2) are parallel. Therefore, the normal to the plane of equation (2) and the given line are perpendicular.

$\Rightarrow\left(\hat{i}-\hat{j}+2\hat{k}\right).\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)=0$

$\Rightarrow\lambda\left(b_1-b_2+2b_3\right)=0$

$\Rightarrow b_1-b_2+2b_3=0$ ...(4)

Similarly, $\left(3\hat{i}+\hat{j}+\hat{k}\right).\,\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)=0$

$\Rightarrow\lambda\left(3b_1+b_2+b_3\right)=0$

$\Rightarrow 3b_1+b_2+b_3=0$ ...(5)

From equation (4) and (5), we obtain

$$\frac{b_1}{(-1)\times1-1\times2}=\frac{b_2}{2\times3-1\times1}=\frac{b_3}{1\times1-3(-1)}$$

$$\Rightarrow\frac{b_1}{-3}=\frac{b_2}{5}=\frac{b_3}{4}$$

Therefore, the direction ratios of $\vec{b}$ are $-3,\ 5,$ and 4.

$\therefore\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}=-3\hat{i}+5\hat{j}+4\hat{k}$

Substituting the value of $\vec{b}$ in equation (1), we obtain

$\vec{r}=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-3\hat{i}+5\hat{j}+4\hat{k}\right)$

This is the equation of the required line.

20:

Find the vector equation of the line passing through the point $(1, 2,-4)$ and perpendicular to the two lines: $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$

Solution:

Let the required line be parallel to the vector $\vec{b}$ given by, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

The position vector of the point $(1,\ 2,-4)$ is $\hat{a}=\hat{i}+2\hat{j}-4\ \hat{k}$

The equation of the line passing through $(1,\ 2,-4)$ and parallel to vector $\vec{b}$ is

$\vec{r}=\vec{a}+\lambda\vec{b}$

$\Rightarrow\vec{r}\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$ ...(1)

The equation of the lines are

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7} \qquad \ldots(2)$$

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \qquad \ldots(3)$$

Line (1) and (2) are perpendicular to each other.

$$\therefore 3b_1-16b_2+7b_3=0 \qquad \ldots(4)$$

Also, line (1) and line (3) are perpendicular to each other.

$$\therefore 3b_1+8b_2-5b_3=0 \qquad \ldots(5)$$

From equation (4) and (5), we obtain

$$\frac{b_1}{(-16)(-5)-8\times 7}=\frac{b_2}{7\times 3-3(-5)}=\frac{b_3}{3\times 8-3(-16)}$$

$$\Rightarrow \frac{b_1}{24}=\frac{b_2}{36}=\frac{b_3}{72}$$

$$\Rightarrow \frac{b_1}{2}=\frac{b_2}{3}=\frac{b_3}{6}$$

Direction ratios of $\vec{b}$ are 2, 3, and 6.

$$\therefore \vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$$

Substituting $\therefore \vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ in equation (1), we obtain

$$\Rightarrow \vec{r}=\left(\hat{i}+2\hat{j}-4\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$$

This is the equation of the required line.

21:

Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$

Solution:

The equation of the plane having intercepts a, b, c with x, y, z axes respectively is given by,

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \ldots(1)$$

The distance (p) of the plane from the origin is given by,

$$p=\left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\left(\frac{1}{a^2}\right)+\left(\frac{1}{b^2}\right)+\left(\frac{1}{c^2}\right)}}\right|$$

$$\Rightarrow p=\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

$$\Rightarrow p^2=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}$$

$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Choose the correct answer in Exercise 22 and 23.

22:

Distance between the two planes: $2x+3y+4z=4$ and $4x+6y+8z=12$ **is**

(A) 2 units (B) 4 units (C) 8 units (D) $\frac{2}{\sqrt{29}}$ units

Solution:

The equation of the planes are

$2x+3y+4z=4$ …(1)

$4x+6y+8z=12$

$\Rightarrow 2x+3y+4z=6$ …(2)

It can be seen that the given planes are parallel.

It is known that the distance between two parallel planes,

$ax+by+cz=d_1$ and $ax+by+cz=d_2$ is given by,

$$D=\left|\frac{d_2-d_1}{\sqrt{a^2+b^2+c^2}}\right|$$

$$\Rightarrow D=\left|\frac{6-4}{\sqrt{(2)^2+(3)^2+(4)^2}}\right|$$

$$D=\frac{2}{\sqrt{29}}$$

Thus, the distance between the lines is $\frac{2}{\sqrt{29}}$ units.

Hence, the correct answer is D.

23:

The planes: $2x-y+4z=5$ and $5x-2.5y+10z=6$ **are**

(A)Perpendicular (B) Parallel (C) intersect y-axis (D) passes through $\left(0,0,\frac{5}{4}\right)$

Solution:

The equation of the planes are

$2x-y+4z=5$ …(1)

$5x-2.5y+10z=6$ …(2)

It can be seen that,

$$\frac{a_1}{a_2}=\frac{2}{5}$$

$$\frac{b_1}{b_2}=\frac{-1}{-2.5}=\frac{2}{5}$$

$$\frac{c_1}{c_2} = \frac{4}{10} = \frac{2}{5}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, the given planes are parallel.

Hence, the correct answer is B.