Exercise 13.1

1:

Given that E and F are events such that

$P(E)=0.6,\ P(F)=0.3$ and, find $P(E|F)$ and $P(F|E)$.

Solution:

It is given that $P(E)=0.6,\ P(F)=0.3$, and $P(E\cap F)=0.2$

$\Rightarrow P(E|F)=\dfrac{P(E\cap F)}{P(F)}=\dfrac{0.2}{0.3}=\dfrac{2}{3}$

$\Rightarrow P(F|E)=\dfrac{P(E\cap F)}{P(F)}=\dfrac{0.2}{0.6}=\dfrac{1}{3}$

2:

Compute $P(A|B)$, if $P(B)=0.5$ and $P(A\cap B)=0.32$

Solution:

It is given that $P(B)=0.5$ and $P(A\cap B)=0.32$

$\Rightarrow P\left(\dfrac{A}{B}\right)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{0.32}{0.5}=\dfrac{16}{25}$

3:

If $P(A)=0.8,\ P(B)=0.5$ and $P(B|A)=0.4$, find

(i)

(ii) $P(A|B)$

(iii) $P(A\cup B)$

Solution:

It is given that $P(A)=0.8,\ P(B)=0.5$ and $P(B|A)=0.4$

(i) $P(A|B)=0.4$

$\therefore \dfrac{P(A\cap B)}{P(A)}=0.4$

$\Rightarrow \dfrac{P(A\cap B)}{0.8}=0.4$

$\Rightarrow P(A\cap B)=0.32$

$(ii)\text{P(A|B)} = \dfrac{P(A\cap B)}{P(B)}$

$\Rightarrow \text{P(A|B)}=\dfrac{0.32}{0.5}=0.64$

$(iii)P\left(A\cup B\right)=P\left(A\right)+P\left(B\right)-P(A\cap B)$

$\Rightarrow P\left(A\cup B\right)=0.8+0.5-0.32=0.98$

4:

Evaluate $P(A\cup B)$**, if** $2\text{P}\left(\text{A}\right)=\text{P}\left(\text{B}\right)=\dfrac{5}{13}$ **and** $\text{P}\left(\text{A}\,|\,\text{B}\right)=\dfrac{2}{5}$

Solution:

It is given that, $2\text{P}\left(\text{A}\right)=\text{P}\left(\text{B}\right)=\dfrac{5}{13}$

$\Rightarrow P(A)=\dfrac{5}{26}\ and P(B)=\dfrac{5}{13}$

$\text{P(A|B)} = \dfrac{2}{5}$

$\Rightarrow \dfrac{P(A\cap B)}{P(B)}=\dfrac{2}{5}$

$\Rightarrow P(A\cap B)=\dfrac{2}{5}\times p(B)=\dfrac{2}{5}\times\dfrac{5}{13}=\dfrac{2}{13}$

It is known that, $P(A\cup B)=P(A)+P\left(B\right)-P(A\cap B)$

$\Rightarrow P(A\cup B)=\dfrac{5}{26}+\dfrac{5}{13}-\dfrac{2}{13}$

$\Rightarrow P(A\cup B)=\dfrac{5+10-4}{26}$

$\Rightarrow P(A\cup B)=\dfrac{11}{26}$

5:

If $\dfrac{6}{11}, P(B)=\dfrac{5}{11}$ and $\text{P(A}\cup\text{B)}=\dfrac{7}{11}$, find

(i) $P(A\cap B)$ (ii) P(A|B) (iii) P(B|A)

Solution:

It is given that $\dfrac{6}{11}, P(B)=\dfrac{5}{11}$ and $\text{P(A}\cup\text{B)}=\dfrac{7}{11}$

(i) $P(A \cup B) = \frac{7}{11}$

$\therefore P(A)+P(B) - P(A \cap B) = \frac{7}{11}$

$\Rightarrow \frac{6}{11} + \frac{5}{11} - P(A \cap B) = \frac{7}{11}$

$\Rightarrow P(A \cap B) = \frac{11}{11} - \frac{7}{11} = \frac{4}{11}$

(ii)It is known that, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A|B) = \frac{\frac{4}{11}}{\frac{5}{11}} = \frac{4}{5}$

(iii)It is known that, $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$\Rightarrow P(B|A) = \frac{\frac{4}{11}}{\frac{6}{11}} = \frac{4}{6} = \frac{2}{3}$

6:

A coin is tossed three times, where

- **(i) E: head on third toss, F: heads on first two tosses**
- **(ii) E: at least two heads, F: at most two heads**
- **(iii) E: at most two tails, F: at least one tail.**

Solution:

If a coin is tossed three times, then the sample space S is
S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
It can be seen that the sample space has 8 elements.

(i) E = {HHH, HTH, THH, TTH}
F = {HHH, HHT}
$\therefore E \cap F$ = {HHH}

$P(F) = \frac{2}{8} = \frac{1}{4}$ and $P(E \cap F) = \frac{1}{8}$

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{4}{8} = \frac{1}{2}$

(ii) E = {HHH, HHT, HTH, THH}
F = {HHT, HTH, HTT, THH, THT, TTH, TTT}
$\therefore E \cap F$ = {HHT, HTH, THH}

Clearly, $P(E\cap F)=\frac{3}{8}$ and $P(F)=\frac{7}{8}$

$$P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{3}{8}}{\frac{7}{8}}=\frac{3}{7}$$

(iii) E = {HHH, HHT, HTT, HTH, THH, THT, TTH}
F = {HHT, HTT, HTH, THH, THT, TTH, TTT}
$\therefore E\cap F$= {HHT, HTT, HTH, THH, THT, TTH}

$P(F)=\frac{7}{8}$ and $P(E\cap F)=\frac{6}{8}$

Therefore, $P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{6}{8}}{\frac{7}{8}}=\frac{6}{7}$

7:

Two coins are tossed once, where

(i) E: tail appears on one coin, F: one coin shows head

(ii) E: not tail appears, F: no head appears

Solution:

If two coins are tossed once, then the sample space S is
S = {HH, HT, TH, TT}

(i) E = {HT, TH}
F = {HT, TH}
$\therefore E\cap F$ = {HT, TH}

$P(F)=\frac{2}{8}=\frac{1}{4}$

$P(E\cap F)=\frac{2}{8}=\frac{1}{4}$

$\therefore P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{2}{2}=1$

(ii) E = {HH}
F = {TT}
$\therefore E\cap F=\phi$

$P(F)=1$ and $P(E\cap F)=0$

$\therefore P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{0}{1}=0$

8:
A die is thrown three times,
E: 4 appears on the third toss, F: 6 and 5 appears respectively on first two tosses

Solution:
If a die is thrown three times, then the number of elements in the sample space will be $6\times6\times6=216$

$$E=\begin{Bmatrix}(1,1,4),(1,2,4),....(1,6,4)\\(2,1,4),(2,2,4),....(2,6,4)\\(3,1,4),(3,2,4),....(3,6,4)\\(4,1,4),(4,2,4),....(4,6,4)\\(5,1,4),(5,2,4),....(5,6,4)\\(6,1,4),(6,2,4),....(6,6,4)\end{Bmatrix}$$

F = {(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5),(6,5,6)}
$\therefore E\cap F = \{(6,5,4)\}$

$$P(F)=\frac{6}{216} \text{ and } P(E\cap F)=\frac{1}{216}$$

$$\therefore P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{1}{216}}{\frac{6}{216}}=\frac{1}{6}$$

9:
Mother, father and son line up at random for a family picture
E: son on one end, F: father in middle

Solution:
If mother (M), father (F), and son (S) line up for the family picture, then the sample space will be
S = {MFS, MSF, FMS, FSM, SMF, SFM}
$\Rightarrow$E = {MFS, FMS, SMF, SFM}
F = {MFS, SFM}
$\therefore E\cap F = \{MFS, SFM\}$

$$P(E\cap F)=\frac{2}{6}=\frac{1}{3}$$

$$P(F)=\frac{2}{6}=\frac{1}{3}$$

$$\therefore P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{1}{3}}{\frac{1}{3}}=1$$

10:

A black and a red dice are rolled.

a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.

b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Solution:

Let the first observation be from the black die and second from the red die.

When two dice (one black and another red) are rolled, the sample space S has $6 \times 6 = 36$ number of elements.

(a) Let

A: Obtaining a sum greater than 9= {(4, 6), (5, 5), (5, 6), (6, 4), (6, 5), (6, 6)}

B: Black die results in a 5.

= {(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)}

$\therefore A \cap B = \{(5, 5), (5, 6)\}$

The conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5, is given by $P(A | B)$.

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{6}{35}} = \frac{2}{6} = \frac{1}{3}$$

(b) E: Sum of the observations is 8.

= {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

F: Red die resulted in a number less than 4

$$= \begin{Bmatrix} (1,1),(1,2),(1,3),(2,1),(2,2),(2,3), \\ (3,1),(3,2),(3,3),(4,1),(4,2),(4,3), \\ (5,1),(5,2),(5,3),(6,1),(6,2),(6,3) \end{Bmatrix} \therefore E \cap F = \{(5,3),(6,2)\}$$

$\therefore E \cap F = \{(5,3),(6,2)\}$

$P(F) = \frac{18}{36}$ *and* $P(E \cap F) = \frac{2}{36}$

The conditional probability of obtaining the sum equal to 8, given that the red die resulted in a number less than 4, is given by $P(E | F)$.

$$\text{Therefore, } P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{18}{36}} = \frac{2}{18} = \frac{1}{9}$$

11:

A fair die is rolled. Consider events $E=\{1, 3, 5\}$, $F=\{2, 3\}$, and $G=\{2, 3, 4, 5\}$

Find

(i) $P(E|F)$ and $P(F|E)$ (ii) $P(E|G)$ and $P(G|E)$

(ii) $P((E\cup F)|G)$ and $P((E\cap G)|G)$

Solution:

When a fair die is rolled, the sample space S will be

$S\{=1 ,2 ,3 ,4 ,5 ,6\}$

It is given that $E=\{1, 3, 5\}$, $F=\{2, 3\}$, and $G=\{2, 3, 4, 5\}$

$$\therefore P(E)=\frac{3}{6}=\frac{1}{2}$$

$$P(F)=\frac{2}{6}=\frac{1}{3}$$

$$P(G)=\frac{4}{6}=\frac{2}{3}$$

(i) $E\cap F=\{3\}$

$$\therefore P(E\cap F)=\frac{1}{6}$$

$$\therefore P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{1}{6}}{\frac{1}{3}}=\frac{1}{2}$$

$$P(F|E)=\frac{P(E\cap F)}{P(E)}=\frac{\frac{1}{6}}{\frac{1}{2}}=\frac{1}{3}$$

(ii) $E\cap G = \{3, 5\}$

$$\therefore P(E\cap G)=\frac{2}{6}=\frac{1}{3}$$

$$P(E|G)=\frac{P(E\cap G)}{P(G)}=\frac{\frac{1}{3}}{\frac{2}{3}}=\frac{1}{2}$$

$$P(G|E)=\frac{P(E\cap G)}{P(E)}=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}$$

(iii) $E\cup F = \{1, 2, 3, 5\}$

$$(E\cup F)\cap G = \{1, 2, 3, 5\}\cap\{2, 3, 4, 5\}=\{2, 3, 5\}$$

$$\mathrm{E}\cap F=\{3\}$$

$$(\mathrm{E}\cap F)\cap G=\{3\}\ \cap\{2,\ 3,\ 4,\ 5\}=\{3\}$$

$$\therefore \mathrm{P}(\mathrm{E}\cap G)=\frac{4}{6}=\frac{2}{3}$$

$$\therefore \mathrm{P}(\mathrm{E}\cup G)=\frac{4}{6}=\frac{2}{3}$$

$$\therefore \mathrm{P}((\mathrm{E}\cup F)\cap G)=\frac{3}{6}=\frac{1}{2}$$

$$P(E\cap F)=\frac{1}{6}$$

$$P((E\cap F)\cap G)=\frac{1}{6}$$

$$\therefore \mathrm{P}((\mathrm{E}\cup F)|G)=\frac{P((\mathrm{E}\cup F)\cap G)}{P(G)}$$

$$=\frac{\frac{1}{2}}{\frac{2}{3}}=\frac{1}{2}\times\frac{3}{2}=\frac{3}{4}$$

$$\therefore \mathrm{P}((\mathrm{E}\cap F)|G)=\frac{P((\mathrm{E}\cap G)\cap G)}{P(G)}$$

$$=\frac{\frac{1}{6}}{\frac{2}{3}}=\frac{1}{6}\times\frac{3}{2}=\frac{1}{4}$$

12:

Assume that each born child is equally like to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl?

Solution:

Let b and g represent the boy and girl child respectively. If a family has two children, the sample space will be

Let A be the event that both children are girls.

$$\therefore\ \mathrm{A}=\{(\mathrm{g},\ \mathrm{g})\}$$

(i) Let B be the event that the youngest child is a girl.

$$\therefore\ \mathrm{B}=[(\mathrm{b},\ \mathrm{g}),\ (\mathrm{g},\ \mathrm{g})]$$

$\Rightarrow A \cap B = \{(g, g)\}$

$\therefore P(B) = \frac{2}{4} = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4}$

The conditional probability that both are girls, given that the youngest child is a girl, is given by $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

Therefore, the required probability is $\frac{1}{2}$.

(ii) Let C be the event that at least one child is a girl.

$\therefore C = \{(b, g), (g, b), (g, g)\}$

$\Rightarrow A \cap C = \{g, g\}$

$\Rightarrow P(C) = \frac{3}{4}$

$P(A \cap C) = \frac{1}{4}$

The conditional probability that both are girls, given that at least one child is a girl, is given by $P(A|C)$.

Therefore, $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

13:

An instructor has a question bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

Solution:

The given data can be tabulated as

	True / False	Multiple choice	Total
Easy	300	500	800
Difficult	200	400	600
Total	500	900	1400

Let us denote E = easy questions, M = multiple choice questions, D = difficult questions, and T = True/False questions

Total number of questions $= 1400$
Total number of multiple choice questions $= 900$
Therefore, probability of selecting an essay multiple choice question is

$$P(E\cap M)=\frac{500}{1400}=\frac{5}{14}$$

Probability of selecting a multiple choice question, P(M), is $\frac{900}{1400}=\frac{9}{14}$

$P\ (E|M)$ Represents the probability that a random selected question will be an easy question, given that it is a multiple choice question.

$$\therefore\ P(E|M)\ =\frac{P(E\cap M)}{P(M)}=\frac{\frac{5}{14}}{\frac{9}{14}}=\frac{5}{9}$$

Therefore, the required probability is $\frac{5}{9}$.

14:
Given that the two numbers appearing on throwing the two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.

Solution:
When dice is thrown, number of observations in the sample space $= 6\times 6 = 36$
Let A be the event that the sum of the numbers on the dice is 4 and B be the event that the two numbers appearing on throwing the two dice are different.

$$\therefore\ A=\{(1,\ 3),\ (2,\ 2),\ (3,\ 1)\}$$

$$B=\begin{Bmatrix}(1,2),(1,3),(1,4),(1,5),(1,6)\\(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)\\(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\\(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\\(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\\(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{Bmatrix}$$

$$A\cap B=\{(1,3),(3,1)\}$$

$$\therefore P(B)=\frac{30}{36}=\frac{5}{6}\text{ and }P(A\cap B)=\frac{2}{36}=\frac{1}{18}$$

Let $P(A|B)$ represents the probability that the sum of the numbers on the dice is 4, given that the two numbers appearing on throwing the two dice are different.

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{18}}{\frac{5}{6}}=\frac{1}{15}$$

Therefore, the required probability is $\frac{1}{15}$.

15:

Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows as 3'.

Solution:

The outcome of the given experiment can be represented by the following tree diagram.

The sample space of the experiment is,

$$S=\begin{Bmatrix}(1,H),(1,T),(2,H),(2,T),(3,1),(3,2),(3,3)(3,4),(3,5),(3,6),\\(4,H),(4,T),(5,H),(5,T),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\end{Bmatrix}$$

Let A be the event that the coin shows a tail and B be the event that at least one die shows 3.

$\therefore A=\{(1,T),(2,T),(4,T),(5,T)\}$

$B=\{(3,1),(3,2),(3,3)(3,4),(3,5),(3,6),(6,3)\}$

$\Rightarrow A\cap B=\phi$

$\therefore P(A\cap B)=0$

Then, $P(B)=P(\{3,1\})+P(\{3,2\})+P(\{3,3\})+P(\{3,4\})+P(\{3,5\})+P(\{3,6\})+P(\{6,3\})$

$=\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}+\frac{1}{20}$

$=\frac{7}{20}$

Probability of the event that the coin shows a tail, given that at least one die shows 3, is given by $P(A|B)$.

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{0}{\frac{7}{20}}=0$$

16:

If $P(A)=\frac{1}{2},\ P(B)=0,$ then $P(A|B)$ is

A. 0

B. (B) $\frac{1}{2}$

C. Not defined

D. 1

Solution:

It is given that $P(A)=\frac{1}{2}$ and $P(B)=0$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{0}$$

Therefore, $P(A|B)$ is not defined.

Thus, the correct answer is c.

17:

If A and B are events such that $P(A|B) = P(B|A)$, then

(A) $A \subset B$ but $A \neq B$

(B) $A = B$

(C) $A \cap B = \phi$

(D) $P(A) = P(B)$

Solution:

It is given that, $P(A|B) = P(B|A)$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow P(A) = P(B)$$

Thus, the correct answer is D.