Exercise 7.ρπ

1:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Solution:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Let $x^2+1=t \Rightarrow 2xdx=dt$

When x = 0, t = 1 and when x = 1, t = 2

$$\therefore \int_0^1 \frac{x}{x^2+1}dx=\frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$=\frac{1}{2}\left[\log|t|\right]_1^2$$

$$=\frac{1}{2}[\log 2-\log 1]$$

$$=\frac{1}{2}\log 2$$

2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$$

Solution:

Let

$$I=\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$$

Also, let $\sin\phi=t \Rightarrow \cos\phi d\phi=dt$

When $\phi=0, t=0$ and when $\phi=\frac{\pi}{2}, t=1$

$$\therefore I=\int_0^1 \sqrt{t}(1-t^2)^2 dt$$

$$=\int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$

$$=\int_0^1\left[t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right]dt$$

$$=\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{11}{2}}}{\frac{11}{2}}-\frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$=\frac{2}{3}+\frac{2}{11}-\frac{4}{7}$$

$$=\frac{154+42-132}{231}$$

$$= \frac{64}{231}$$

3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Solution:

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let x = tanθ ⇒ dx = $\sec^2$ θ dθ

When x = 0, θ = 0 and when x = 1, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

$$= \int_0^{\frac{\pi}{4}} 2\theta.\sec^2\theta d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} \theta.\sec^2\theta d\theta$$

Taking θ as first function and $\sec^2$ θ as second function and integrating by parts, we obtain

$$I = 2\left[\theta\int \sec^2\theta d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int \sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta - \int \tan\theta d\theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log|\cos 0|\right]$$

$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$

$$= \frac{\pi}{2} - \log 2$$

4:

$\int_0^2 x\sqrt{x+2}\left(\text{Put x+2=t}^2\right)$

Solution:

$\int_0^2 x\sqrt{x+2}dx$

Let $x + 2 = t^2 \Rightarrow dx = 2t\, dt$

When $x = 0$, $t=\sqrt{2}$ and when $x = 2$, $t = 2$

$\therefore \int_0^2 x\sqrt{x+2}dx = \int_{\sqrt{2}}^2 (t^2-2)\sqrt{t^2}\, 2t dt$

$= 2\int_{\sqrt{2}}^2 (t^2-2)t^2 dt = 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2 = \frac{16(2+\sqrt{2})}{15}$

$= \frac{16\sqrt{2}(\sqrt{2}+1)}{15}$

5:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Solution:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

When $x = 0$, $t = 1$ and when $x=\frac{\pi}{2}, t=0$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{dt}{1+t^2}$

$= -\left[\tan^{-1} t\right]_1^0$

$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$

$= -\left[-\frac{\pi}{4}\right]$

$= \frac{\pi}{4}$

6:

$\int_0^2 \frac{dx}{x+4-x^2}$

Solution:

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let x-$\frac{1}{2}=t$ so dx=dt

when x=0, t=-$\frac{1}{2}$ and when x=2, t=$\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$$

$$= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}} - \frac{\log \frac{\sqrt{17}}{2}-\frac{1}{2}}{\log \frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\sqrt{17}+3}{\sqrt{17}-3} - \log \frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$= \frac{1}{\sqrt{17}} \log \frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{\left(5+\sqrt{17}\right)\left(5+\sqrt{17}\right)}{25-17}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$=\frac{1}{\sqrt{17}}\log\left(\frac{21+5\sqrt{17}}{4}\right)$$

7:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}$$

Solution:

$$\int_{-1}^{1}\frac{dx}{x^2+2x+5}=\int_{-1}^{1}\frac{dx}{(x^2+2x+1)+4}=\int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2}$$

Let x + 1 = t ⇒ dx = dt

When x = −1, t = 0 and when x = 1, t = 2

$$\int_{-1}^{1}\frac{dx}{(x+1)^2+(2)^2}=\int_{0}^{2}\frac{dx}{t^2+2^2}$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$=\frac{1}{2}\tan^{-1}1-\frac{1}{2}\tan^{-1}0$$

$$=\frac{1}{2}\left(\frac{\pi}{4}\right)=\frac{\pi}{8}$$

8:

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Solution:

$$\int_{1}^{2}\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx$$

Let 2x = t ⇒ 2dx = dt

When x = 1, t = 2 and when x = 2, t = 4

$$\int_1^2\left(\frac{1}{x}-\frac{1}{2x^2}\right)e^{2x}dx=\frac{1}{2}\int_2^4\left(\frac{2}{t}-\frac{2}{t^2}\right)e^t dt$$

Let $\frac{1}{t}=f(t)$

Then, $f'(t)=-\frac{1}{t^2}$

$$=\int_2^4\left(\frac{1}{t}-\frac{1}{t^2}\right)e^t dt=\int_2^4(f(t)+f'(t))e^t dt$$

$$=[e^t f(t)]_2^4$$

$$=\left[e^t\cdot\frac{1}{t}\right]_2^4$$

$$=\left[\frac{e^t}{t}\right]_2^4$$

$$=\frac{e^4}{4}-\frac{e^2}{2}$$

$$=\frac{e^2\left(e^2-2\right)}{4}$$

Chose the correct answer in Exercises 21 and 22.

9:

The value of the integral $\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$ is

A. 6
B. 0
C. 3
D. 4

Solution:

Let $I=\int_{\frac{1}{3}}^{1}\frac{\left(x-x^3\right)^{\frac{1}{3}}}{x^4}dx$

Also, let x = sin $\theta \Rightarrow$ dx = cos θ d θ

When $x=\frac{1}{3},\theta=\sin^{-1}\left(\frac{1}{3}\right)$ and when x=1, $\theta=\frac{\pi}{2}$

$$\Rightarrow I=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta-\sin^3\theta)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}\left(1-\sin^{2}\theta\right)^{\frac{1}{3}}}{\sin^{4}\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^{4}\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^{2}\theta\sin^{2}\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\cos\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}}\cos ec^{2}\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}(\cot\theta)^{\frac{5}{3}}\cos ec^{2}\theta d\theta$$

Let $\cot\theta = t \Rightarrow -\text{cosec}^2\,\theta\, d\theta = dt$

When $\theta=\sin^{-1}\left(\frac{1}{3}\right), t=2\sqrt{2}$ and when $\theta=\frac{\pi}{2}, t=0$

$$\therefore I=-\int_{2\sqrt{2}}^{0}(t)^{\frac{5}{3}}dt$$

$$=-\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=-\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=\frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$=\frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$=\frac{3}{8}[16]$$

$$=3\times 2$$

$$=6$$

Hence, the correct Answer is A.

10:

If $f(x)=\int_{0}^{x}t\sin t dt,$ then $f'(x)$ is

A. cos x + x sin x

B. x sin x

C. x cos x

D. sin x + x cos x

Solution:

$f(x)=\int_0^x t\sin t dt$

Integrating by parts, we obtain

$f(x)=t\int_0^x \sin t dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int \sin t dt\right\}dt$

$=\left[t(-\cos t)\right]_0^x-\int_0^x(-\cot t)dt$

$=\left[-t\cos t+\sin t\right]_0^x$

$=-x\cos x+\sin x$

$\Rightarrow f'(x)=-\left[\{x(-\sin x)\}+\cos x\right]+\cos x$

$=x\sin x-\cos x+\cos x$

$=x\sin x$

Hence, the correct Answer is B.