Exercise 7.4

1.

Integrate $\frac{3x^2}{x^6+1}$

Solution:

Let $x^3 = t$

$3x^2\,dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$

$= \tan^{-1} t + C$

$= \tan^{-1}(x^3) + C$

where C is an arbitrary constant.

∴

2.

Integrate $\frac{1}{\sqrt{1+4x^2}}$

Solution:

Let $2x = t$

$2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right]+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$\therefore \quad = \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$$

where C is an arbitrary constant.

3.

Integrate $\frac{1}{\sqrt{(2-x)^2+1}}$

Solution:

Let $2 - x = t$

$\Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right|+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right|+C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$$

where C is an arbitrary constant.

4.

Integrate $\frac{1}{\sqrt{9-25x^2}}$

Solution:

Let 5x = t

5dx = dt

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}}dx = \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}dt$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right)+C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$$

where C is an arbitrary constant.

5.

Integrate $\frac{3x}{1+2x^4}$

Solution:

Let $\sqrt{2}x^2 = t$

$\therefore 2\sqrt{2}xdx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4}dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1}t\right]+C$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right)+C$$

where C is an arbitrary constant.

6.

Integrate $\frac{x^2}{1-x^6}$

Solution:

Let x^3 = t

$3x^2$ dx = dt

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

where C is an arbitrary constant.

7.

Integrate $\frac{x-1}{\sqrt{x^2-1}}$

Solution:

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad ...(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2-1 = t \Rightarrow 2xdx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{x}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$
$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

where C is an arbitrary constant.

8.

Integrate $\frac{x^2}{\sqrt{x^6+a^6}}$

Solution:

Let $x^3 = t$

$\Rightarrow 3x^2\, dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6+a^6}}dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2+\left(a^3\right)^2}}$$

$$= \frac{1}{3}\log\left|t+\sqrt{t^6+a^6}\right|+C$$

$$= \frac{1}{3}\log\left|x^3+\sqrt{x^6+a^6}\right|+C$$

where C is an arbitrary constant.

9.

Integrate $\dfrac{\sec^2 x}{\sqrt{\tan^2 x+4}}$

Solution:

Let tan x = t

$\therefore \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x+4}}dx = \int \frac{dt}{\sqrt{t^2+2^2}}$$

$$= \log\left|t+\sqrt{t^2+4}\right|+C$$

$$= \log\left|\tan x+\sqrt{\tan^2 x+4}\right|+C$$

where C is an arbitrary constant.

10.

Integrate $\dfrac{1}{\sqrt{x^2+2x+2}}$

Solution:

$$\int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{(x+1)^2+(1)^2}}\,dx$$

Let x+1=t

$\therefore$ dx=dt

$$\Rightarrow \int \frac{1}{\sqrt{x^2+2x+2}}dx = \int \frac{1}{\sqrt{t^2+1}}\,dt$$

$$= \log\left|t+\sqrt{t^2+1}\right|+C$$

$$= \log\left|(x+1)+\sqrt{(x+1)^2+1}\right|+C$$

$$=\log\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

where C is an arbitrary constant.

11.

Integrate $\frac{1}{\sqrt{9x^2+6x+5}}$

Solution:

$$\int\frac{1}{\sqrt{9x^2+6x+5}}dx=\int\frac{1}{\sqrt{(3x+1)^2+2^2}}dx$$

$Let\ (3x+1)=t$

$\therefore 3dx=dt$

$$\int\frac{1}{\sqrt{(3x+1)^2+2^2}}dx=\frac{1}{3}\int\frac{1}{\sqrt{t^2+2^2}}dt$$

$$=\frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$=\frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

where C is an arbitrary constant.

12.

Integrate $\frac{1}{\sqrt{7-6x-x^2}}$

Solution:

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$

Therefore,

$$7-\left(x^2+6x+9-9\right)$$

$$=16-\left(x^2+6x+9\right)$$

$$=16-(x+3)^2$$

$$=(4)^2-(x+3)^2$$

$$\therefore\int\frac{1}{\sqrt{7-6x-x^2}}dx=\int\frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

$Let\ x+3=1$

$\Rightarrow dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

where C is an arbitrary constant.

13.

Integrate $\frac{1}{\sqrt{(x-1)(x-2)}}$

Solution:

(x – 1) (x – 2) can be written as x^2-3x+2.

Therefore,

$$x^2-3x+2$$

$$= x^2-3x+\frac{9}{4}-\frac{9}{4}+2$$

$$= \left(x-\frac{3}{2}\right)^2-\frac{1}{4}$$

$$= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$$Let\ x-\frac{3}{2}=t$$

$$\therefore dx = dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx = \int \frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$= \log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

where C is an arbitrary constant.

14.

Integrate $\frac{1}{\sqrt{8+3x-x^2}}$

Solution:

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$

Therefore,

$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$

$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$

$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx = \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$

$Let\ x-\frac{3}{2}=t$

$\therefore dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx = \int \frac{1}{\sqrt{\left(\frac{41}{4}\right)-t^2}}dt$

$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$

$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$

$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$

where C is an arbitrary constant.

15.

Integrate $\frac{1}{\sqrt{(x-a)(x-b)}}$

Solution:

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$$x^2-(a+b)x+ab$$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}}}dx$$

$$Let\; x-\left(\frac{a+b}{2}\right)=t$$

$$\therefore dx=dt$$

$$\int\frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

16:

Integrate $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Solution:

Let $2x^2 + x - 3 = t$

$\therefore (4x + 1)\, dx = dt$

$$\Rightarrow\int\frac{4x+1}{\sqrt{2x^2+x-3}}dx=\int\frac{1}{\sqrt{t}}dt$$

$$=2\sqrt{t}+C$$

$$=2\sqrt{2x^2+x-3}+C$$

where C is an arbitrary constant.

17.

Integrate $\frac{x+2}{\sqrt{x^2-1}}$

Solution:

Let x + 2 = $A\frac{d}{dx}\left(x^2-1\right)+B$ ...(1)

$\Rightarrow x+2=A(2x)+B$

Equating the coefficients of x and constant terms on both sides, we obtain

$2A=1\Rightarrow A=\frac{1}{2}$

$B=2$

From (1), we obtain

$(x+2)=\frac{1}{2}(2x)+2$

Then, $\int\frac{x+2}{\sqrt{x^2-1}}dx=\int\frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$

$=\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx+\int\frac{2}{\sqrt{x^2-1}}dx$ ...(2)

In $\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx$ let $x^2-1=t\Rightarrow 2xdx=dt$

$\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx=\frac{1}{2}\int\frac{dt}{\sqrt{t}}$

$=\frac{1}{2}\left[2\sqrt{t}\right]$

$=\sqrt{t}$

$=\sqrt{x^2-1}$

Then, $\int\frac{2}{\sqrt{x^2-1}}dx=2\int\frac{1}{\sqrt{x^2-1}}dx=2\log\left|x+\sqrt{x^2-1}\right|$

From equation (2), we obtain

$\int\frac{x+2}{\sqrt{x^2-1}}dx=\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+C$

where C is an arbitrary constant.

18.

Integrate $\frac{5x-2}{1+2x+3x^2}$

Solution:

Let $5x-2$ = $A\frac{d}{dx}\left(1+2x+3x^2\right)+B$

$\Rightarrow 5x-2=A(2+6x)+B$

Equating the coefficient of x and constant term on both sides, we obtain

$$5=6A \Rightarrow A=\frac{5}{6}$$

$$2A+B=-2 \Rightarrow B=-\frac{11}{3}$$

$$\therefore 5x-2=\frac{5}{6}(2+6x)+\left(-\frac{11}{3}\right)$$

$$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}dx=\int \frac{\frac{5}{6}(2+6x)-\frac{11}{3}}{1+2x+3x^2}dx$$

$$=\frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx-\frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$

Let $I_1=\int \frac{2+6x}{1+2x+3x^2}dx$ and $I_2=\int \frac{1}{1+2x+3x^2}dx$

$$\therefore \int \frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}I_1-\frac{11}{3}I_2 \qquad ...(1)$$

$$I_1=\int \frac{2+6x}{1+2x+3x^2}dx$$

Let $1+2x+3x^2=t$

$$\Rightarrow (2+6x)dx=dt$$

$$\therefore I_1=\int \frac{dt}{t}$$

$$I_1=\log|t|$$

$$I_1=\log\left|1+2x+3x^2\right| \qquad ...(2)$$

$$I_2=\int \frac{1}{1+2x+3x^2}dx$$

$1+2x+3x^2$ can be written as $1+3\left(x^2+\frac{2}{3}x\right)$

Therefore,

$$1+3\left(x^2+\frac{2}{3}x\right)$$

$$=1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)$$

$$=1+3\left(x+\frac{1}{3}\right)^2-\frac{1}{3}$$

$$=\frac{2}{3}+3\left(x+\frac{1}{3}\right)^2$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\frac{2}{9}\right]$$

$$=3\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2=\frac{1}{3}\int\frac{1}{\left[\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$=\frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$=\frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int\frac{5x-2}{1+2x+3x^2}dx=\frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right]-\frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]+C$$

$$=\frac{5}{6}\log\left|1+2x+3x^2\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)+C$$

where C is an arbitrary constant.

19.

Integrate $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

Solution:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}}=\frac{6x+7}{\sqrt{x^2-9x+20}}$$

Let $6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$

$\Rightarrow 6x+7=A(2x-9)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}=\int\frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$=3\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx+34\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore\int\frac{6x+7}{\sqrt{x^2-9x+20}}=3I_1+34I_2 \qquad (1)$$

Then,

$$I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

Let $x^2-9x+20=t$

$$\Rightarrow(2x-9)\,dx=dt$$

$$\Rightarrow I_1=\frac{dt}{\sqrt{t}}$$

$$I_1=2\sqrt{t}$$

$$I_1=2\sqrt{x^2-9x+20} \qquad ...(2)$$

and $I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2=\int\frac{1}{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$$

$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

where C is an arbitrary constant.

20.

Integrate $\frac{x+2}{\sqrt{4x-x^2}}$

Solution:

Let x + 2 = $A\frac{d}{dx}\left(4x-x^2\right)+B$

$\Rightarrow x+2=A(4-2x)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$-2A=1\Rightarrow A=-\frac{1}{2}$

$4A+B=2\Rightarrow B=4$

$\Rightarrow (x+2)=-\frac{1}{2}(4-2x)+4$

$$\therefore \int\frac{x+2}{\sqrt{4x-x^2}}dx=\int\frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$$

$$=-\frac{1}{2}\int\frac{4-2x}{\sqrt{4x-x^2}}dx+4\int\frac{1}{\sqrt{4x-x^2}}dx$$

$Let\ I_1=\int\frac{4-2x}{\sqrt{4x-x^2}}dx\ and\ I_2\int\frac{1}{\sqrt{4x-x^2}}dx$

$\therefore \int\frac{x+2}{\sqrt{4x-x^2}}dx=-\frac{1}{2}I_1+4I_2$ .... (1)

Then, $I_1=\int\frac{4-2x}{\sqrt{4x-x^2}}dx$

$Let\ 4x-x^2=t$

$\Rightarrow (4-2x)dx=dt$

$\Rightarrow I_1=\int\frac{dt}{\sqrt{t}}=2\sqrt{t}=2\sqrt{4x-x^2}$ ...(2)

$I_2=\int\frac{1}{\sqrt{4x-x^2}}dx$

$\Rightarrow 4x-x^2=-\left(-4x+x^2\right)$

$=\left(-4x+x^2+4-4\right)$

$=4-(x-2)^2$

$=(2)^2-(x-2)^2$

$\therefore I_2=\int\frac{1}{\sqrt{(2)^2-(x-2)^2}}dx=\sin^{-1}\left(\frac{x-2}{2}\right)$ ...(3)

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right)+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

$$= -\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

Where C is an arbitrary constant

21.

Integrate $\dfrac{x+2}{\sqrt{x^2+2x+3}}$

Solution:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$Let\ I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx \ and\ I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+\left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right]+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$

$$= \sqrt{x^2+2x+3}+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$

Where C is an arbitrary constant

22:

Integrate $\frac{x+3}{x^2-2x-5}$

Solution:

Let $(x+3)=A\frac{d}{dx}(x^2-2x-5)+B$

$(x+3)=A(2x-2)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$2A=1\Rightarrow A=\frac{1}{2}$

$-2A+B=3\Rightarrow B=4$

$\therefore (x+3)=\frac{1}{2}(2x-2)+4$

$\Rightarrow \int\frac{x+3}{x^2-2x-5}dx=\int\frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$

$=\frac{1}{2}\int\frac{2x-2}{x^2-2x-5}dx+4\int\frac{1}{x^2-2x-5}dx$

Let $I_1=\int\frac{2x-2}{x^2-2x-5}dx$ and $I_2=\int\frac{1}{x^2-2x-5}dx$

$\therefore \int\frac{x+3}{x^2-2x-5}dx=\frac{1}{2}I_1+4I_2 \qquad ...(1)$

Then, $I_1=\int\frac{2x-2}{x^2-2x-5}dx$

Let $x^2-2x-5=\text{t}$

$\Rightarrow (2x-2)dx=dt$

$\Rightarrow I_1=\int\frac{dt}{t}=\log|t|=\log|x^2-2x-5| \qquad ...(2)$

$I_2=\int\frac{1}{x^2-2x-5}dx$

$=\int\frac{1}{(x^2-2x+1)-6}dx$

$=\int\frac{1}{(x-1)^2-(\sqrt{6})^2}dx$

$=\frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right) \qquad ...(3)$

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Where C is an arbitrary constant.

23.

Integrate $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$

Solution:

Let $5x+3 = A\frac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3 = A(2x+4)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 5 \Rightarrow A = \frac{5}{2}$

$4A + B = 3 \Rightarrow B = -7$

$\therefore 5x+3 = \frac{5}{2}(2x+4)-7$

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \qquad \text{....(1)}$$

Then, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10 = t$

$\therefore (2x+4)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \qquad \text{...(2)}$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$=\int \frac{1}{\sqrt{\left(x^2+4x+4\right)+6}}dx$$

$$=\int \frac{1}{(x+2)^2+\left(\sqrt{6}\right)^2}dx$$

$$=\log|(x+2)+\sqrt{x^2+4x+10}| \qquad \ldots (3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx=\frac{5}{2}\left[2\sqrt{x^2+4x+10}\right]-7\log\left|(x+2)\sqrt{x^2+4x+10}\right|+C$$

$$=5\sqrt{x^2+4x+10}-7\log\left|(x+2)\sqrt{x^2+4x+10}\right|+C$$

Where C is an arbitrary constant.

24.

$\int \frac{dx}{x^2+2x+2}$ equals

(A) x $\tan^{-1}$ (x + 1) + C (B) $\tan^{-1}$ (x + 1) + C

(C) (x + 1) $\tan^{-1}$ x + C (D) $\tan^{-1}$x + C

Solution:

$$\int \frac{dx}{x^2+2x+2}=\int \frac{dx}{\left(x^2+2x+1\right)+1}$$

$$=\int \frac{1}{(x+1)^2+(1)^2}dx$$

$$=\left[\tan^{-1}(x+1)\right]+C$$

Hence, the correct Answer is B.

25.

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$

(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ (D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution:

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$= \int \frac{1}{\sqrt{-4\left(x^2 - \frac{9}{4}x\right)}} dx$$

$$= \int \frac{1}{-4\left(x^2 - \frac{9}{4}x + \frac{81}{64} - \frac{81}{64}\right)} dx$$

$$= \int \frac{1}{\sqrt{-4\left[\left(x - \frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]}} dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{x - \frac{9}{8}}{\frac{9}{8}}\right)\right] + C \qquad \left(\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}\frac{y}{a} + C\right)$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x - 9}{9}\right) + C$$

Hence, the correct Answer is B.