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1:

$$\int_a^b x dx$$

Solution:

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\dots+f\left(a+(n-1)h\right)\right] \text{ where } h=\frac{b-a}{n}$$

$Here, a=a, b=b,$ and $f(x)=x$

$$\therefore \int_a^b x dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)\dots(a+2h)\dots a+(n-1)h\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{a+a+a+\dots+a}_{n\,times}\right)+\left(h+2h+3h+\dots+(n-1)h\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(1+2+3+\dots+(n-1)\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$
$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$
$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$
$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$
$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$
$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$
$$=\frac{(b-a)(b+a)}{2}$$
$$=\frac{1}{2}\left(b^2-a^2\right)$$

2:

$$\int (x+1)\,dx$$

Solution:

Let I = $\int_0^b (x+1)\,dx$

It is known that,

$$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right],\ \text{where } h=\frac{b-a}{n}$$

$Here, a=0, b=5,\ \text{and } f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)dx=(5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+...f\left((n-1)\frac{5}{n}\right)\right]$$
$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+...\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$
$$=5\lim_{n\to\infty}\frac{1}{n}\left[\left(\underset{n\ times}{1+1+1...1}\right)+\left[\frac{5}{n}+2.\frac{5}{n}+3.\frac{5}{n}+...(n-1)\frac{5}{n}\right]\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3...(n-1)\}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}.\frac{(n-1)n}{2}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$= 5\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= 5\left[1+\frac{5}{2}\right]$$

$$= 5\left[\frac{7}{2}\right]$$

$$= \frac{35}{2}$$

3:

$$\int_2^3 x^2 dx$$

Solution:

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)...+f\{a+(n-1)h\}\right], \text{ where } h=\frac{b-a}{n}$$

Here, $a=2, b=3$, and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore \int_2^3 x^2 dx = (3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)...f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+...\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2.2.\frac{1}{n}\right\}+...+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2.2.\frac{(n-1)}{n}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\left(2^2+\underset{n\ times}{...}+2^2\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+...+\left(\frac{n-1}{n}\right)^2\right\}+2.2.\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+...+\frac{(n-1)}{n}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2...+(n-1)^2\right\}+\frac{4}{n}\{1+2+...+(n-1)\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[4n + \frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\} + \frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[4n + \frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6} + \frac{4n-4}{2}\right]$$

$$= \lim_{n\to\infty}\left[4 + \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + 2 - \frac{2}{n}\right]$$

$$= 4 + \frac{2}{6} + 2$$

$$= \frac{19}{3}$$

4:

$$\int_1^4 \left(x^2 - x\right) dx$$

Solution:

Let $I = \int_1^4 \left(x^2 - x\right) dx$

$$= \int_1^4 x^2 dx - \int_1^4 x dx$$

Let $I = I_1 - I_2$, where $I_1 = \int_1^4 x^2 dx$ and $I_2 = \int_1^4 x dx$ ...(1)

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a) + f(a+h) + f(a+(n-1)h)\right], \text{ where } h = \frac{b-a}{n}$$

For, $I_1 = \int_1^4 x^2 dx,$

a = 1, b = 4, and f (x) = x^2

$$\therefore h = \frac{4-1}{n} = \frac{3}{n}$$

$$I_1 = \int_1^4 x^2 dx = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1) + f(1+h) + \ldots + f(1+(n-1)h)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2 + \left(1+\frac{3}{n}\right)^2 + \left(1+2.\frac{3}{n}\right)^2 + \ldots\left(1+\frac{(n-1)3}{n}\right)^2\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1^2 + \left\{1^2 + \left(\frac{3}{n}\right)^2 + 2.\frac{3}{n}\right\} + \ldots + \left\{1^2 + \left(\frac{(n-1)3}{n}\right)^2 + \frac{2.(n-1).3}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(1^2 + \underset{n\,times}{\ldots} + 1^2\right) + \left(\frac{3}{n}\right)^2\left\{1^2 + 2^2 + \ldots + (n-1)^2\right\} + 2.\frac{3}{n}\left\{1 + 2 + \ldots + (n-1)\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$

$$=3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$

$$=3[1+3+3]$$

$$=3[7]$$

$$I_1=21 \qquad \ldots(2)$$

For $I_2=\int_1^4 x dx,$

$a=1, b=4,$ and $f(x)=x$

$$\Rightarrow h=\frac{4-1}{n}=\frac{3}{n}$$

$$\therefore I_2=(4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots f(a+(n-1)h)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+(1+h)+\ldots+(1+(n-1)h)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+\ldots+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+\ldots+1}_{n\ times}\right)+\frac{3}{n}(1+2+\ldots+(n-1))\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{3}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=3\left[1+\frac{3}{2}\right]$$

$$=3\left[\frac{5}{2}\right]$$

$$I_2=\frac{15}{2} \qquad \ldots(3)$$

From equations (2) and (3), we obtain

$$I=I_1-I_2=21-\frac{15}{2}=\frac{27}{2}$$

5:

$$\int_{-1}^{1} e^x dx$$

Solution:

Let $I=\int_{-1}^{1} e^x dx \qquad ...(1)$

It is known that,

$Here, a=-1, b=1,$ and $f(x)=e^x$

$\therefore h=\frac{1+1}{n}=\frac{2}{n}$

$\therefore I=(1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2.\frac{2}{n}\right)+...+f\left(-1+\frac{(n-1)2}{n}\right)\right]$

$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2.\frac{2}{n}\right)}+...e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$

$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$

$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}}-1}{e^{\frac{2}{n}}-1}\right]$

$=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2-1}{e^{\frac{2}{n}}-1}\right]$

$=\frac{e^{-1}\times 2\left(e^2-1\right)}{\lim_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}}-1}{\frac{2}{n}}\right)\times 2}$

$=e^{-1}\left[\frac{2\left(e^2-1\right)}{2}\right] \qquad \left[\lim_{h\to 0}\left(\frac{e^h-1}{h}\right)=1\right]$

$=\frac{e^2-1}{e}$

$=\left(e-\frac{1}{e}\right)$

6:

$\int_0^4\left(x+e^{2x}\right)dx$

Solution:

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\cdots+f\left(a+(n-1)h\right)\right] \text{ where } h=\frac{b-a}{n}$$

$$Here, a=0, b=4, \text{ and } f(x)=x+e^{2x}$$

$$\therefore h=\frac{4-0}{n}=\frac{4}{n}$$

$$\Rightarrow \int_0^4\left(x+e^{2x}\right)dx=(4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+\ldots+f\left((n-1)h\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^0\right)+\left(h+e^{2h}\right)+\left(2h+e^{2.2h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+\ldots+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+\ldots(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}.\frac{(n-1)n}{2}+\left(\frac{e^8-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$=4(2)+4\lim_{n\to\infty}\frac{\left(e^8-1\right)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$=8+\frac{4.\left(e^8-1\right)}{8} \qquad \left(\lim_{x\to0}\frac{e^x-1}{x}=1\right)$$

$$=8+\frac{e^8-1}{2}$$

$$=\frac{15+e^8}{2}$$