Miscellaneous Exercise on Chapter 7

Page: 352

1:

Integrate $\frac{1}{x-x^3}$

Solution:

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}=\frac{1}{x(1-x)(1+x)}$$

Let $\frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x}$ ...(1)

$$\Rightarrow 1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

−A + B − C = 0

B + C = 0

A = 1

On solving these equations, we obtain

$$A=1,\ B=\frac{1}{2},\text{ and } C=-\frac{1}{2}$$

From equation (1), we obtain

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

$$\Rightarrow \int\frac{1}{x(1-x)(1+x)}dx=\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{(1-x)}dx-\frac{1}{2}\int\frac{1}{(1+x)}dx$$

$$=\log|x|-\frac{1}{2}\log|(1-x)|-\frac{1}{2}\log|(1+x)|$$

$$=\log|x|-\log\left|(1-x)^{\frac{1}{2}}\right|-\log\left|(1+x)^{\frac{1}{2}}\right|$$

$$=\log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right|+C$$

$$=\log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$$

2:

Integrate $\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}$

Solution:

$$\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x-b)}$$

$$=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow\int\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$$

3:

Integrate $\frac{1}{x\sqrt{ax-x^2}}$ $\left[\text{Hint: } x=\frac{a}{t}\right]$

Solution:

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t}\Rightarrow dx=-\frac{a}{t^2}dh$

$$\Rightarrow\int\frac{1}{x\sqrt{ax-x^2}}dx=\int\frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$=-\int\frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$=-\frac{1}{a}\int\frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$= -\frac{2}{a}\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)+C$$

$$= -\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+C$$

4:

Integrate $\dfrac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$

Solution:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we obtain

$$\frac{x^{-3}}{x^2x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}(x^4+1)^{\frac{-3}{4}}}{x^2x^{-3}}$$

$$= \frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5.\left(x^4\right)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4} = t \Rightarrow -\frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}dt$$

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$= -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}} + C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$$

5:

Integrate $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$

$$\left[\text{Hint:} \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)} \text{ Put x = t}^6\right]$$

Solution:

Let $x = t^6 \Rightarrow dx = 6t^5 dt$

$$\int \frac{1}{x^{1/2} + x^{1/3}} dx = \int \frac{6t^5}{t^3 + t^2} dt$$

$$= \int \frac{6t^5}{t^2(1+t)} dt$$

$$= 6\int \frac{t^3}{(1+t)} dt$$

On dividing, we obtain

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx = 6\int \left\{(t^2 - t + 1) - \frac{1}{1+t}\right\} dt$$

$$= 6\left[\left(\frac{t^3}{3}\right) - \left(\frac{t^2}{2}\right) + t - \log|1+t|\right]$$

$$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1 + x^{\frac{1}{6}}\right) + C$$

$$= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1 - x^{\frac{1}{6}}\right) + C$$

6:

Integrate $\frac{5x}{(x+1)(x^2+9)}$

Solution:

Let $\frac{5x}{(x+1)(x^2+9)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+9)}$ ...(1)

$\Rightarrow 5x=A(x^2+9)+(Bx+C)(x+1)$

$\Rightarrow 5x=Ax^2+9A+Bx^2+Bx+Cx+C$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

B + C = 5

9A + C = 0

On solving these equations, we obtain

$A=-\frac{1}{2}, B=\frac{1}{2}, and\ C=\frac{9}{2}$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)}=\frac{-1}{2(x+1)}+\frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int\frac{5x}{(x+1)(x^2+9)}dx=\int\left\{\frac{-1}{2(x+1)}+\frac{(x+9)}{2(x^2+9)}\right\}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{2}\int\frac{x}{x^2+9}dx+\frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\int\frac{2x}{x^2+9}dx+\frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+9|+\frac{9}{2}.\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+9)+\frac{3}{2}\tan^{-1}\frac{x}{3}+C$$

7:

Integrate $\frac{\sin x}{\sin(x-a)}$

Solution:

$\frac{\sin x}{\sin(x-a)}$

Let $x-a=t \Rightarrow dx=dt$

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$

$$= \int \frac{\sin t\cos a + \cos t\sin a}{\sin t}dt$$

$$= \int (\cos a + \cot t\sin a)dt$$

$$= t\cos a + \sin a\log|\sin t| + C_1$$

$$= (x-a)\cos a + \sin a\log|\sin(x-a)| + C_1$$

$$= x\cos a + \sin a\log|\sin(x-a)| - a\cos a + C_1$$

$$= \sin a\log|\sin(x-a)| + x\cos a + C$$

8:

Integrate $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

Solution:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)}$$

$$= e^{2\log x}$$

$$= e^{\log x^2}$$

$$= x^2$$

$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int x^2 dx = \frac{x^3}{3} + C$$

9:

Integrate $\frac{\cos x}{\sqrt{4-\sin^2 x}}$

Solution:

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Let sin x = t ⇒ cos x dx = dt

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}}dx = \int \frac{dt}{\sqrt{(2)^2-(t)^2}}$$

$$= \sin^{-1}\left(\frac{t}{2}\right) + C$$

$$= \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

10:

Integrate $\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}$

Solution:

$$\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^4 x-\cos^4 x\right)}{1-2\sin^2 x+\cos^2 x}$$

$$=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x+\cos^2 x\right)\left(\sin^2 x-\cos^2 x\right)}{1-2\sin^2 x+\cos^2 x}$$

$$=\frac{\left(\sin^4 x+\cos^4 x\right)\left(\sin^2 x-\cos^2 x\right)}{1-2\sin^2 x+\cos^2 x}$$

$$=\frac{-\left(\sin^4 x+\cos^4 x\right)\left(\cos^2 x-\sin^2 x\right)}{\left(\sin^4 x+\cos^4 x\right)}$$

$$=-\cos 2x$$

$$\therefore \int\frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx=\int-\cos 2x dx=-\frac{\sin 2x}{2}+C$$

11:

Integrate $\frac{1}{\cos(x+a)\cos(x+b)}$

Solution:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by sin (a – b), we obtain.

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a).\cos(x+b)-\cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)}-\frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$=\frac{1}{\sin(a-b)}\left[\tan(x+a)-\tan(x+b)\right]$$

$$\int \frac{1}{\cos(x+a)\cos(x+b)}dx = \frac{1}{\sin(a-b)}\int[\tan(x+a)-\tan(x+b)]dx$$
$$= \frac{1}{\sin(a-b)}[-\log|\cos(x+a)|+\log|\cos(x+b)|]+C$$
$$= \frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

12:

Integrate $\frac{x^3}{\sqrt{1-x^8}}$

Solution:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t \Rightarrow 4x^3\,dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{1}{4}\sin^{-1}t + C$$
$$= \frac{1}{4}\sin^{-1}(x^4) + C$$

13:

Integrate $\frac{e^x}{(1+e^x)(2+e^x)}$

Solution:

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Let $e^x = t \Rightarrow e^x\,dx = dt$

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)}dx = \int \frac{dt}{(t+1)(t+2)}$$
$$= \int\left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right]dt$$
$$= \log|t+1| - \log|t+2| + C$$
$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$$= \log\left|\frac{1+e^x}{2+e^x}\right| + C$$

14:

Integrate $\frac{1}{(x^2+1)(x^2+4)}$

Solution:

$$\therefore \frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$$

$$\Rightarrow 1 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

A + C = 0

B + D = 0

4A + C = 0

4B + D = 1

On solving these equations, we obtain

$$A = 0,\ B = \frac{1}{3}, C = 0 \text{ and } D = -\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{1}{3(x^2+1)} - \frac{1}{3(x^2+4)}$$

$$\int \frac{1}{(x^2+1)(x^2+4)} dx = \frac{1}{3}\int \frac{1}{x^2+1} dx - \frac{1}{3}\int \frac{1}{x^2+4} dx$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$$

15:

Integrate $\cos^3 x e^{\log \sin x}$

Solution:

$\cos^3 x e^{\log \sin x}$ = $\cos^3$ x × sin x

Let cos x = t ⇒ −sin x dx = dt

$$\Rightarrow \int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x \sin x dx$$

$$= -\int t^3\, dx$$

$$= -\frac{t^4}{4} + C$$
$$= -\frac{\cos^4 x}{4} + C$$

16:

Integrate $e^{3\log x}\left(x^4+1\right)^{-1}$

Solution:

$$e^{3\log x}\left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

$$Let\ x^4+1 = t \Rightarrow 4x^3dx = dt$$

$$\Rightarrow \int e^{3\log x} = \left(x^4+1\right)^{-1}dx = \int \frac{x^3}{\left(x^4+1\right)}dx$$

$$= \frac{1}{4}\int \frac{dt}{t}$$
$$= \frac{1}{4}\log|t| + C$$
$$= \frac{1}{4}\log\left|x^4+1\right| + C$$
$$= \frac{1}{4}\log\left(x^4+1\right) + C$$

17:

Integrate $f'(ax+b)\left[f(ax+b)\right]^n$

Solution:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Let $f(ax+b) = t \Rightarrow a\,f'(ax+b)dx = dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$

$$= \frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1} + C$$

18:

Integrate $\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}$

Solution:

$$\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}=\frac{1}{\sqrt{\sin^3 x(\sin x\cos\alpha+\cos x\sin\alpha)}}$$

$$=\frac{1}{\sqrt{\sin^4 x\cos\alpha+\sin^3 x\cos x\sin\alpha}}$$

$$=\frac{1}{\sin^2 x\sqrt{\cos\alpha+\cot x\sin\alpha}}$$

$$=\frac{\cos ec^2 x}{\sqrt{\cos\alpha+\cot x\sin\alpha}}$$

Let $\cos\alpha+\cot x\sin\alpha=t\Rightarrow-\cos ec^2 x\sin\alpha dx=dt$

$$\therefore\int\frac{1}{\sqrt{\sin^3 x\sin(x+\alpha)}}dx=\int\frac{\cos ec^2 x}{\sqrt{\cos\alpha+\cot x\sin\alpha}}dx.$$

$$=\frac{-1}{\sin\alpha}\int\frac{dt}{\sqrt{t}}$$

$$=\frac{-1}{\sin\alpha}\left[2\sqrt{t}\right]+C$$

$$=\frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha+\cot x\sin\alpha}\right]+C$$

$$=\frac{-2}{\sin\alpha}\sqrt{\cos\alpha+\frac{\cos x\sin\alpha}{\sin x}}+C$$

$$=\frac{-2}{\sin\alpha}\sqrt{\frac{\sin x\cos\alpha+\cos x\sin\alpha}{\sin x}}+C$$

$$=\frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}}+C$$

19:

Integrate $\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}, x\in[0,1]$

Solution:

Let $I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx$

It is known that, $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\frac{\pi}{2}$

$$\Rightarrow I=\int\frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$=\frac{2}{\pi}\int\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

$$=\frac{2}{\pi}.\frac{\pi}{2}\int 1.dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx$$

$$=x-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx \qquad ...(1)$$

$$Let\ I_1=\int\cos^{-1}\sqrt{x}dx$$

Also. let $\sqrt{x}=t\Rightarrow dx=2t\,dt$

$$\Rightarrow I_1=2\int\cos^{-1}t.t\,dt$$

$$=2\left[\cos^{-1}t.\frac{t^2}{2}-\int\frac{-1}{\sqrt{1-t^2}}.\frac{t^2}{2}dt\right]$$

$$=t^2\cos^{-1}t+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\frac{1-t^2-1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\sqrt{1-t^2}dt+\int\frac{1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\frac{1}{2}\sqrt{1-t^2}-\frac{1}{2}\sin^{-1}t+\sin^{-1}t$$

$$=t^2\cos^{-1}t-\frac{1}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t$$

From equation (1), we obtain

$$I=x-\frac{4}{\pi}\left[t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]$$

$$=x-\frac{4}{\pi}\left[x\cos^{-1}\sqrt{x}-\frac{\sqrt{x}}{2}\sqrt{1-x}+\frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-\frac{4}{\pi}\left[x\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)-\frac{\sqrt{x-x^2}}{2}+\frac{\pi}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-2x+\frac{4x}{\pi}\sin^{-1}\sqrt{x}+\frac{2}{\pi}\sqrt{x-x^2}-\frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$=-x+\frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right]+\frac{2}{\pi}\sqrt{x-x^2}+C$$

$$=\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2}{\pi}\sqrt{x-x^2}-x+C$$

20:

Integrate $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Solution:

$$I=\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$$

Let $x=\cos^2\theta \Rightarrow dx=-2\sin\theta\cos\theta\, d\theta$

$$I=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\left(-2\sin\theta\cos\theta\right)d\theta$$

$$=-\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta\, d\theta$$

$$=-\int\tan\frac{\theta}{2}.2\sin\theta\cos\theta\, d\theta$$

$$=-2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta\, d\theta$$

$$=-4\int\sin^2\frac{\theta}{2}\cos\theta\, d\theta$$

$$=-4\int\sin^2\frac{\theta}{2}.\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$=-4\int\left(2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$$

$$=-8\int\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$=-2\int\sin^2\theta\, d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$=-2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta+4\int\frac{1-\cos\theta}{2}d\theta$$

$$=-2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]+4\left[\frac{\theta}{2}-\frac{\sin\theta}{2}\right]+C$$

$$=-\theta+\frac{\sin 2\theta}{2}+2\theta-2\sin\theta+C$$

$$=\theta+\frac{\sin 2\theta}{2}+2\sin\theta+C$$

$$=\theta+\frac{2\sin\theta\cos\theta}{2}-2\sin\theta+C$$

$$= \theta + \sqrt{1-\cos^2\theta}.\cos\theta - 2\sqrt{1-\cos^2\theta} + C$$
$$= \cos^{-1}\sqrt{x} + \sqrt{1-x}.\sqrt{x} - 2\sqrt{1-x} + C$$
$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + C$$
$$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$$

21:

Integrate $\frac{2+\sin 2x}{1+\cos 2x}e^x$

Solution:

$$I = \int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$$
$$= \int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$$
$$= \int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$$
$$= \int\left(\sec^2 x + \tan x\right)e^x$$
$$Let\ f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$$
$$\therefore I = \int\left(f(x) + f'(x)\right)e^x dx$$
$$= e^x f(x) + C$$
$$= e^x \tan x + C$$

22:

Integrate $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Solution:

Let $\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$ ...(1)

$$\Rightarrow x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x^2+2x+1)$$
$$\Rightarrow x^2+x+1 = A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)$$
$$\Rightarrow x^2+x+1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 1

3A + B + 2C = 1

2A + 2B + C = 1

On solving these equations, we obtain
A = −2, B = 1, and C = 3
From equation (1), we obtain

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

23:

Integrate $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Solution:

$\text{I} = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Let $x=\cos\theta \Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}-(\sin\theta d\theta)$$

$$=-\int\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$$

$$=-\int\tan^{-1}\tan\frac{\theta}{2}.\sin\theta d\theta$$

$$=-\frac{1}{2}\int\theta.\sin\theta d\theta$$

$$=-\frac{1}{2}\left[\theta.(-\cos\theta)-\int1.(-\cos\theta)d\theta\right]$$

$$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]$$

$$=+\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x.x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{x}{2}\cos^{-1}-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

24:

Integrate $\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$

Solution:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$Let\ 1+\frac{1}{x^2}=t\Rightarrow\frac{-2}{x^3}dx=dt$$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t\,dt$$

$$=-\frac{1}{2}\int t^{\frac{1}{2}}.\log t\,dt$$

Integrating by parts, we obtain

$$I=-\frac{1}{2}\left[\log t.\int t^{\frac{1}{2}}.dt-\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

25:

$$\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Solution:

$$I=\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$=\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$=\int_{\frac{\pi}{2}}^{\pi}\left(\frac{\cos ec^2\frac{x}{2}}{2}-\cot\frac{x}{2}\right)dx$$

$$Let\ \ f(x)=-\cot\frac{x}{2}$$

$$\Rightarrow f'(x)=-\left(-\frac{1}{2}\cos ec^2\frac{x}{2}\right)=\frac{1}{2}\cos ec^2\frac{x}{2}$$

$$\therefore I=\int_{\frac{\pi}{2}}^{\pi} e^x\left(f(x)+f'(x)\right)dx$$

$$=\left[e^x.f(x)dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\left[e^x.\cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$=-\left[e^{\pi}.\cot\frac{\pi}{2}-e^{\frac{\pi}{2}}.\cot\frac{\pi}{4}\right]$$

$$=-\left[e^{\pi}.0-e^{\frac{\pi}{2}}.1\right]$$

$$=e^{\frac{\pi}{2}}$$

26:

$$\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\frac{(\sin x\cos x)}{\cos^4 x}}{\frac{\left(\cos^4 x+\sin^4 x\right)}{\cos^4 x}}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\tan x\sec^2 x}{1+\tan^4 x}dx$

Let $\tan^2 x=t\Rightarrow 2\tan x\sec^2 xdx=dt$

When x = 0, t = 0 and when $x=\frac{\pi}{4},t=1$

$\therefore I=\frac{1}{2}\int_0^1\frac{dt}{1+t^2}$

$=\frac{1}{2}\left[\tan^{-1}t\right]_0^1$

$=\frac{1}{2}\left[\tan^{-1}1\tan^{-1}0\right]$

$=\frac{1}{2}\left[\frac{\pi}{4}\right]$

$=\frac{\pi}{8}$

27:

$\int_0^{\frac{\pi}{2}}\frac{\cos^2 xdx}{\cos^2 x+4\sin^2 x}$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\left(1-\cos^2 x\right)}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4}{4-3\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}1dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4\sec^2 x-3}dx$

$$\Rightarrow I = \frac{-1}{3}[x]_0^{\frac{\pi}{2}} + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1+\tan^2 x)-3}dx$$

$$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx \qquad ...(1)$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx$

Let 2 tanx = t $\Rightarrow$ 2sec^{2}xdx = dt

When x = 0, t = 0 and when x = $\frac{\pi}{2}$, t = ∞

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx = \int_0^{\infty} \frac{dt}{1+t^2}$$

$$= \left[\tan^{-1} t\right]_0^{\infty}$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{\pi}{2}$$

Therefore, from (1), we obtain

$$I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin\cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x\cos^2 x - 2\sin x\cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)dx}{\sqrt{1-(\sin x - \cos x)^2}}$$

Let $(\sin x - \cos x) = t = (\sin x + \cos x)dx = dt$

when $x=\frac{\pi}{6}, t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when x = $\frac{\pi}{3}, t=\left(\frac{\sqrt{3}-1}{2}\right)$

$$I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I=\int_{-\left(\frac{1-\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}}=\frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function.

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

It is known that if f(x) is an even function, then

$$\Rightarrow I=2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$=\left[2\sin^{-1} t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

29:

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Solution:

Let $I=\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$I=\int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)} \times \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)} dx$$

$$=\int_0^1 \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{1+x-x} dx$$

$$=\int_0^1 \sqrt{1+x}dx+\int_0^1 \sqrt{x}dx$$

$$=\left[\frac{2}{3}(1+x)^{\frac{2}{3}}\right]_0^1 \left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{3}\left[(2)^{\frac{2}{3}}-1\right]+\frac{2}{3}[1]$$

$$=\frac{2}{3}(2)^{\frac{2}{3}}$$

$=\frac{2.2\sqrt{2}}{3}$

$=\frac{4\sqrt{2}}{3}$

30:

$$\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$$

Solution:

Let I = $\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

Also let sin x – cos x = t $\Rightarrow$ (cos x + sinx) dx = dt

When x = 0, t = - 1 and when x = $\frac{\pi}{4}$, t = 0

$\Rightarrow(\sin x-\cos x)^2=t^2$

$\Rightarrow\sin^2+\cos^2-2\sin x\cos x=t^2$

$\Rightarrow 1-\sin 2x=t^2$

$\Rightarrow\sin 2x=1-t^2$

$\therefore I=\int_{-1}^{0}\frac{dt}{9+16(1-t^2)}$

$=\int_{-1}^{0}\frac{dt}{9+16-16t^2}$

$=\int_{-1}^{0}\frac{dt}{25-16t^2}=\int_{-1}^{0}\frac{dt}{(5)^2-(4t)^2}$

$=\frac{1}{4}\left[\frac{1}{2(5)}\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$

$=\frac{1}{40}\left[\log(1)-\log\left|\frac{1}{9}\right|\right]$

$=\frac{1}{40}\log 9$

31:

$$\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)dx=\int_0^{\frac{\pi}{2}}2\sin x\cos x\tan^{-1}(\sin x)dx$

Also, let sin x = t $\Rightarrow$ cos x dx = dt

When x = 0, t = 0 and when x = $\frac{\pi}{2}$, t = 1

$$\Rightarrow I = 2\int_0^1 t\tan^{-1}(t)\,dt \qquad \ldots (1)$$

Consider $\int t.\tan^{-1} t dt = \tan^{-1} t.\int t dt - \int\left\{\frac{d}{dt}\left(\tan^{-1} t\right)\int t dt\right\}dt$

$$= \tan^{-1} t.\frac{t^2}{2} - \int \frac{1}{1+t^2}.\frac{t^2}{2}dt$$

$$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}\int\frac{t^2+1-1}{1+t^2}dt$$

$$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}\int 1.dt + \frac{1}{2}\int\frac{1}{1+t^2}dt$$

$$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}.t + \frac{1}{2}\tan^{-1} t$$

$$\Rightarrow \int_0^1 t.\tan^{-1} t dt = \left[\frac{t^2\tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2}\tan^{-1} t\right]_0^1$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 1 + \frac{\pi}{4}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 1\right] = \frac{\pi}{4} - \frac{1}{2}$$

From equation (1), we obtain

$$I = 2\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{2} - 1$$

32:

$$\int_0^\pi \frac{x\tan x}{\sec x + \tan x}dx$$

Solution:

Let $\int_0^\pi \frac{x\tan x}{\sec x + \tan x}dx \qquad \ldots(1)$

$$I = \int_0^\pi \left\{\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\right\}dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^\pi \left\{\frac{-(\pi - x)\tan x}{-(\sec x + \tan x)}\right\}dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi - x)\tan x}{\sec x + \tan x}dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\pi}\frac{\pi\tan x}{\sec x+\tan x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x+1-1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}1.dx-\pi\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi[x]_0^{\pi}-\pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$\Rightarrow 2I=\pi^2-\pi\int_0^{\pi}\left(\sec^2 x-\tan x\sec x\right)dx$$

$$\Rightarrow 2I=\pi^2-\pi\left[\tan x-\sec x\right]_0^{\pi}$$

$$\Rightarrow 2I=\pi^2-\pi\left[\tan\pi-\sec\pi-\tan 0+\sec 0\right]$$

$$\Rightarrow 2I=\pi^2-\pi\left[0-(-1)-0+1\right]$$

$$\Rightarrow 2I=\pi^2-2\pi$$

$$\Rightarrow 2I=\pi(\pi-2)$$

$$\Rightarrow I=\frac{\pi}{2}(\pi-2)$$

33:

$$\int_1^4\left[|x-1|+|x-2|+|x-3|\right]dx$$

Solution:

Let $I=\int_1^4\left[|x-1|+|x-2|+|x-3|\right]dx$

$$\Rightarrow I=\int_1^4|x-1|\,dx+\int_1^4|x-2|\,dx+\int_1^4|x-3|\,dx$$

$$I=I_1+I_2+I_3 \qquad ...(1)$$

where, $I_1=\int_1^4|x-1|\,dx,\ I_2=\int_1^4|x-2|\,dx, and\ I_3=\int_1^4|x-3|\,dx$

$$I_1=\int_1^4|x-1|\,dx$$

$(x-1)\geq 0\ \ for\ 1\leq x\leq 4$

$$\therefore I_1=\int_1^4(x-1)\,dx$$

$$\Rightarrow I_1=\left[\frac{x^2}{2}-x\right]_1^4$$

$$\Rightarrow I_1 = \left[8-4-\frac{1}{2}+1\right] = \frac{9}{2} \qquad ...(2)$$

$$I_2 = \int_1^4 |x-2|\,dx$$

$$x-2 \geq 0 \text{ for } 2 \leq x \leq 4 \text{ and } x-2 \leq 0 \text{ for } 1 \leq x \leq 2$$

$$\therefore I_2 = \int_1^2 (2-x)\,dx + \int_2^4 (x-2)\,dx$$

$$\Rightarrow I_2 = \left[2x-\frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}-2x\right]_2^4$$

$$\Rightarrow I_2 = \left[4-2-2+\frac{1}{2}\right] + [8-8-2+4]$$

$$\Rightarrow I_2 = \frac{1}{2}+2 = \frac{5}{2} \qquad ...(3)$$

$$I_3 = \int_1^4 |x-3|\,dx$$

$$x-3 \geq 0 \text{ for } 3 \leq x \leq 4 \text{ and } x-3 \leq 0 \text{ for } 1 \leq x \leq 3$$

$$\therefore I_3 = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx$$

$$\Rightarrow I_3 = \left[3x-\frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2}-3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9-\frac{9}{2}-3+\frac{1}{2}\right] + \left[8-12-\frac{9}{2}+9\right]$$

$$\Rightarrow I_3 = [6-4] + \left[\frac{1}{2}\right] = \frac{5}{2} \qquad ...(4)$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2}+\frac{5}{2}+\frac{5}{2} = \frac{19}{2}$$

34:

Prove $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Solution:

Let $I = \int_1^3 \frac{dx}{x^2(x+1)}$

Also, let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + C = 0

A + B = 0

B = 1

On solving these equations, we obtain

A = −1, C = 1, and B = 1

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence, the given result is proved.

35:

Prove $\int_0^4 xe^x dx = 1$

Solution:

Let $I = \int_0^4 xe^x dx$

Integrating by parts, we obtain

$$I = x\int_0^4 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence, the given result is proved.

36:

Prove $\int_{-1}^{1} x^{17}\cos^4 x dx = 0$

Solution:

Let $I = \int_{-1}^{1} x^{17}\cos^4 x dx$

Also, let $f(x) = x^{17}\cos^4 x$

$\Rightarrow f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

Therefore, f (x) is an odd function.

It is known that if f(x) is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

$\therefore I = \int_{-1}^{1} x^{17}\cos^4 x dx = 0$

Hence, the given result is proved.

37:

Prove $\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x dx$

$I = \int_0^{\frac{\pi}{2}} \sin^2 x.\sin x dx$

$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx$

$= \int_0^{\frac{\pi}{2}} \sin x dx - \int_0^{\frac{\pi}{2}} \cos^2 x.\sin x dx$

$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$

$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$

Hence, the given result is proved.

38:

Prove $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} 2\tan^3 x dx$

$$I=\int_0^{\frac{\pi}{4}}2\tan^2 x\tan x dx=2\int_0^{\frac{\pi}{4}}\left(\sec^2 x-1\right)\tan x dx$$

$$=2\int_0^{\frac{\pi}{4}}\sec^2 x\tan x dx-2\int_0^{\frac{\pi}{4}}\tan x dx$$

$$=2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}}+2\left[\log\cos x\right]_0^{\frac{\pi}{4}}$$

$$=1+2\left[\log\cos\frac{\pi}{4}-\log\cos 0\right]$$

$$=1+2\left[\log\frac{1}{\sqrt{2}}-\log 1\right]$$

$$=1-\log 2-\log 1=1-\log 2$$

Hence, the given result is proved.

39:

Prove $\int_0^1\sin^{-1}x dx=\frac{\pi}{2}-1$

Solution:

Let $\int_0^1\sin^{-1}x dx$

$$\Rightarrow I=\int_0^1\sin^{-1}x.1.dx$$

Integrating by parts, we obtain

$$I=\left[\sin^{-1}x.x\right]_0^1-\int_0^1\frac{1}{\sqrt{1-x^2}}.xdx$$

$$=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\int_0^1\frac{(-2x)}{\sqrt{1-x^2}}dx$$

Let $1-x^2=t\Rightarrow -2x\,dx=dt$

When x = 0, t = 1 and when x = 1, t = 0

$$I=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\int_0^1\frac{dt}{\sqrt{t}}$$

$$=\left[x\sin^{-1}x\right]_0^1+\frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$=\sin^{-1}(1)+\left[-\sqrt{1}\right]$$

$$=\frac{\pi}{2}-1$$

Hence, the given result is proved.

40:

Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Solution:

Let I = $\int_0^1 e^{2-3x}dx$

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right]$$

where, $h = \frac{b-a}{n}$

Here, $a=0,\ b=1$, and $f(x) = e^{2-3x}$

$$\Rightarrow h = \frac{1-0}{n} = \frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx = (1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+...+f(0+(n-1)h)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2 + e^{2-3x} + ... + e^{2-3(n-1)h}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+...+e^{-3(n-1)h}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$= e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$= \frac{-e^2\left(e^{-3}-1\right)}{3}(1) \qquad \left[\lim_{n\to\infty}\frac{x}{e^x-1}\right]$$

$=\frac{-e^{-1}+e^{2}}{3}$

$=\frac{1}{3}\left(e^{2}-\frac{1}{e}\right)$

Chose the correct answer in Exercises 41 to 44.

41:

$\int\frac{dx}{e^{x}+e^{-x}}$ is equal to

A. $\tan^{-1}\left(e^{x}\right)+C$

B. $\tan^{-1}\left(e^{-x}\right)+C$

C. $\log\left(e^{x}-e^{-x}\right)+C$

D. $\log\left(e^{x}+e^{-x}\right)+C$

Solution:

Let $I=\int\frac{dx}{e^{x}+e^{-x}}dx=\int\frac{e^{x}}{e^{2x}+1}dx$

Also, let $e^{x}=t\Rightarrow e^{x}dx=dt$

$\therefore I=\int\frac{dt}{1+t^{2}}$

$=\tan^{-1}t+C$

$=\tan^{-1}\left(e^{x}\right)+C$

Hence, the correct Answer is A.

42:

$\int\frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$ is

A. $\frac{-1}{\sin x+\cos x}+C$

B. $\log|\sin x+\cos x|+C$

C. $\log|\sin x-\cos x|+C$

D. $\frac{1}{(\sin x+\cos x)^{2}}+C$ equal to

Solution:

Let $I=\int\frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$

$$I=\int\frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)^2}dx$$
$$=\int\frac{(\cos x+\sin x)(\cos x-\sin x)}{(\sin x+\cos x)^2}dx$$
$$=\int\frac{\cos x-\sin x}{\cos x+\sin x}dx$$

Let $\cos x+\sin x=t\Rightarrow(\cos x-\sin x)dx=dt$

$$\therefore I=\int\frac{dt}{t}$$
$$=\log|t|+C$$
$$=\log|\cos x+\sin x|+C$$

Hence, the correct Answer is B.

43:

If f (a + b – x) = f (x), then $\int_a^b xf(x)dx$ is equal to

A. $\frac{a+b}{2}\int_a^b f(b-x)dx$

B. $\frac{a+b}{2}\int_a^b f(b+x)dx$

C. $\frac{b-a}{2}\int_a^b f(x)dx$

D. $\frac{a+b}{2}\int_a^b f(x)dx$

Solution:

Let I = $\int_a^b xf(x)dx$ … (1)

$$I=\int_a^b(a+b-x)f(a+b-x)dx \qquad \left(\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$$
$$\Rightarrow I=\int_a^b(a+b-x)f(x)dx$$
$$\Rightarrow I=(a+b)\int_a^b f(x)dx-I \qquad [\text{using}(1)]$$
$$\Rightarrow I+I=(a+b)\int_a^b f(x)dx$$
$$\Rightarrow 2I=(a+b)\int_a^b f(x)dx$$
$$\Rightarrow I=\left(\frac{a+b}{2}\right)\int_a^b f(x)dx$$

Hence, the correct Answer is D.

44: The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A. 1
B. 0
C. − 1
D. $\frac{\pi}{4}$

Solution:

Let $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$\Rightarrow I = \int_0^1 \tan^{-1}\left(\frac{x-(1-x}{1+x(1-x)}\right)dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1} x - \tan^{-1}(1-x)\right]dx \quad ...(1)$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(1-1+x)\right]dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx$

$\Rightarrow I = \int_0^1 \left[\tan^{-1}(1-x) - \tan^{-1}(x)\right]dx \quad ...(2)$

Adding (1) and (2), we obtain

$\Rightarrow 2I = \int_0^1 \left(\tan^{-1} x - \tan^{-1}(1-x) - \tan^{-1}(1-x) - \tan^{-1} x\right)dx$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

Hence, the correct Answer is B.