Exercise 7.1 Page: 299

Find an anti-derivative (or integral) of the following function by the method of inspection.

1. sin 2x

Solution:

The anti-derivative of sin 2x is a function of x whose derivative is sin 2x. It is known that,

$$\frac{d}{dx}(\cos 2x) = -2\sin 2x$$

$$\Rightarrow \sin 2x = -\frac{1}{2}\frac{d}{dx}(\cos 2x)$$

$$\therefore \sin 2x = \frac{d}{dx}\left(-\frac{1}{2}\cos 2x\right)$$

Therefore, the anti-derivative of $\sin 2x$ is $-\frac{1}{2}\cos 2x$.

2. cos 3x

Solution:

The anti-derivative of cos 3x is a function of x whose derivative is cos 3x.
It is known that,

$$\frac{d}{dx}(\sin 3x) = 3\cos 3x$$

$$\Rightarrow \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$\therefore \cos 3x = \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Therefore, the anti-derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$.

3. e^{2x}

Solution:

The anti–derivative of e^{2x} is the function of x whose derivative is e^{2x}
It is known that,

$$\frac{d}{dx}(e^{2x}) = 2e^{2x}$$

$$\Rightarrow e^{2x} = \frac{1}{2}\frac{d}{dx}(e^{2x})$$

$$\therefore e^{2x} = \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Therefore, the anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

4. $(ax + b)^2$

Solution:

The anti-derivative of $(ax + b)^2$ is the function of x whose derivative is $(ax + b)^2$.

It is known that,

$$\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$$

$$\Rightarrow (ax+b)^2 = \frac{1}{3a}\frac{d}{dx}(ax+b)^3$$

$$\therefore (ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Therefore, the anti-derivative of $(ax+b)^2$ is $\frac{1}{3a}(ax+b)^3$.

5. $\sin 2x - 4e^{3x}$

Solution:

The anti-derivative of $\sin 2x - 4e^{3x}$ is the function of x whose derivative is $\sin 2x - 4e^{3x}$

It is known that,

$$\frac{d}{dx}\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right) = \sin 2x - 4e^{3x}$$

Therefore, the anti derivative of $\left(\sin 2x - 4e^{3x}\right)$ is $\left(-\frac{1}{2}\cos 2x - \frac{4}{3}e^{3x}\right)$.

Find the following integrals in Exercises 6 to 20:

6.

$$\int\left(4e^{3x} + 1\right)dx$$

Solution:

$$\int\left(4e^{3x} + 1\right)dx$$

$$= 4\int e^{3x}dx + \int 1dx$$

$$= 4\left(\frac{e^{3x}}{3}\right) + x + C$$

$$= \frac{4}{3}e^{3x} + x + C$$

where C is an arbitrary constant.

7.

$$\int x^2\left(1 - \frac{1}{x^2}\right)dx$$

Solution:

$$\int x^2\left(1 - \frac{1}{x^2}\right)dx$$

$$= \int (x^2 - 1)\,dx$$

$$= \int x^2 dx - \int 1 dx$$

$$= \frac{x^3}{3} - x + C$$

where C is an arbitrary constant.

8.

$$\int (ax^2 + bx + c)dx$$

Solution:

$$\int (ax^2 + bx + c)dx$$

$$= a\int x^2 dx + b\int x dx + c\int 1.dx$$

$$= a\left(\frac{x^3}{3}\right) + b\left(\frac{x^2}{2}\right) + cx + C$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$

where C is an arbitrary constant.

9.

$$\int (2x^2 + e^x)dx$$

Solution:

$$\int (2x^2 + e^x)dx$$

$= 2\int x^2 dx + \int e^x dx$

$= 2\left(\frac{x^3}{3}\right) + e^x + C$

$= \frac{2}{3}x^3 + e^x + C$

where C is an arbitrary constant.

10.

$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

Solution:

$\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

$= \int \left(x + \frac{1}{x} - 2\right) dx$

$= \int x dx + \int \frac{1}{x} dx - 2\int 1.dx$

$= \frac{x^2}{2} + \log|x| - 2x + C$

where C is an arbitrary constant.

11.

$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$

Solution:

$\int \frac{x^3 + 5x^2 - 4}{x^2} dx$

$= \int \left(x + 5 - 4x^{-2}\right) dx$

$= \int x dx + 5\int 1.dx - 4\int x^{-2} dx$

$= \frac{x^2}{2} + 5x - 4\left(\frac{x^{-1}}{-1}\right) + C$

$= \frac{x^2}{2} + 5x + \frac{4}{x} + C$

where C is an arbitrary constant.

12.

$$\int \frac{x^3+3x+4}{\sqrt{x}}dx$$

Solution:

$$\int \frac{x^3+3x+4}{\sqrt{x}}dx$$

$$=\int\left(x^{\frac{5}{2}}+3x^{\frac{1}{2}}+4x^{-\frac{1}{2}}\right)dx$$

$$\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{3\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+\frac{4\left(x^{\frac{1}{2}}\right)}{\frac{1}{2}}+C$$

$$=\frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$$

$$=\frac{2}{7}x^{\frac{7}{2}}+2x^{\frac{3}{2}}+8\sqrt{x}+C$$

where C is an arbitrary constant.

13.

$$\int \frac{x^3-x^2+x-1}{x-1}dx$$

Solution:

$$\int \frac{x^3-x^2+x-1}{x-1}dx$$

On factorising, we obtain

$$\int \frac{(x^2+1)(x-1)}{x-1}dx$$

$$=\int\left(x^2+1\right)dx$$

$$=\int x^2dx+\int 1dx$$

$$=\frac{x^3}{3}+x+C$$

where C is an arbitrary constant.

14.

$$\int(1-x)\sqrt{x}dx$$

Solution:

$\int(1-x)\sqrt{x}dx$

$=\int\left(\sqrt{x}-x^{\frac{3}{2}}\right)dx$

$=\int x^{\frac{1}{2}}dx-\int x^{\frac{3}{2}}dx$

$=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+C$

$=\frac{2}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}+C$

where C is an arbitrary constant.

15.

$\int\sqrt{x}\left(3x^2+2x+3\right)dx$

Solution:

$\int\sqrt{x}\left(3x^2+2x+3\right)dx$

$=3\int x^{\frac{5}{2}}dx+2\int x^{\frac{3}{2}}dx+3\int x^{\frac{1}{2}}dx$

$=3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}}\right)+2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)+3\frac{\left(x^{\frac{3}{2}}\right)}{\frac{3}{2}}+C$

$=\frac{6}{7}x^{\frac{7}{2}}+\frac{4}{5}x^{\frac{5}{2}}+2x^{\frac{3}{2}}+C$

where C is an arbitrary constant.

16.

$\int\left(2x-3\cos x+e^x\right)dx$

Solution:

$\int\left(2x-3\cos x+e^x\right)dx$

$=2\int xdx-3\int\cos xdx+\int e^xdx$

$=\frac{2x^2}{2}-3(\sin x)+e^x+C$

$=x^2-3\sin x+e^x+C$

where C is an arbitrary constant.

17.

$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$

Solution:

$\int\left(2x^2-3\sin x+5\sqrt{x}\right)dx$

$=2\int x^2dx-3\int \sin xdx+5\int x^{\frac{1}{2}}dx$

$=\frac{2x^3}{3}-3\left(-\cos x\right)+5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$

$=\frac{2}{3}x^3+3\cos x+\frac{10}{3}x^{\frac{3}{2}}+C$

where C is an arbitrary constant.

18.

$\int\sec x\left(\sec x+\tan x\right)dx$

Solution:

$\int\sec x\left(\sec x+\tan x\right)dx$

$=\int\left(\sec^2 x+\sec x\tan x\right)dx$

$=\int\sec^2 xdx+\int\sec x\tan xdx$

$=\tan x+\sec x+C$

where C is an arbitrary constant.

19.

$\int\frac{\sec^2 x}{\cos ec^2 x}dx$

Solution:

$\int\frac{\sec^2 x}{\cos ec^2 x}dx$

$=\int\frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}dx$

$=\int\frac{\sin^2 x}{\cos^2 x}dx$

$= \int \tan^2 x dx$

$= \int (\sec^2 x - 1) dx$

$= \int \sec^2 x dx - \int 1 dx$

$= \tan x - x + C$

where C is an arbitrary constant.

20.

$\int \frac{2 - 3\sin x}{\cos^2 x} dx$

Solution:

$\int \frac{2 - 3\sin x}{\cos^2 x} dx$

$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$

$= \int 2\sec^2 x dx - 3\int \tan x \sec x dx$

$= 2\tan x - 3\sec x + C$

where C is an arbitrary constant.

Chose the correct answer in Exercises 21 and 22.

21.

The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)$ equals

$(A) \frac{1}{3} x^{\frac{1}{3}} + 2x^{\frac{1}{2}} + C$ $(B) \frac{2}{3} x^{\frac{2}{3}} + \frac{1}{2} x^2 + C$

$(C) \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$ $(D) \frac{3}{2} x^{\frac{3}{2}} + \frac{1}{2} x^{\frac{1}{2}} + C$

Solution:

$\int \sqrt{x} + \frac{1}{\sqrt{x}} dx$

$= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$, where C is an arbitrary constant.

Hence, the correct Answer is C.

22.

If $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$ such that f(2) = 0, then f(x) is

$(A)\, x^4+\frac{1}{x^3}-\frac{129}{8}$ $\quad (B)\, x^3+\frac{1}{x^4}+\frac{129}{8}$

$(C)\, x^4+\frac{1}{x^3}+\frac{129}{8}$ $\quad (D)\, x^3+\frac{1}{x^4}-\frac{129}{8}$

Solution:

It is given that, $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$

Anti-derivative of $4x^3-\frac{3}{x^4}=f(x)$

$$\therefore f(x)=\int 4x^3-\frac{3}{x^4}dx$$

$$f(x)=4\int x^3dx-3\int\left(x^{-4}\right)dx$$

$$f(x)=4\left(\frac{x^4}{4}\right)-3\left(\frac{x^{-3}}{-3}\right)+C$$

$$f(x)=x^4+\frac{1}{x^3}+C$$

Also,

f(2) = 0

$$\therefore f(2)=(2)^4+\frac{1}{(2)^3}+C=0$$

$$\Rightarrow 16+\frac{1}{8}+C=0$$

$$\Rightarrow C=-\left(16+\frac{1}{8}\right)$$

$$\Rightarrow C=\frac{-129}{8}$$

$$\therefore f(x)=x^4+\frac{1}{x^3}-\frac{129}{8}$$

Hence, the correct Answer is A.

Exercise 7.2

Integrate the functions in Exercise 1 to 37

1.

Integrate $\frac{2x}{1+x^2}$

Solution:

Let $1+x^2 = t$

$\therefore$ 2x dx = dt

$$\Rightarrow \int \frac{2x}{1+x^2}dx = \int \frac{1}{t}dt$$

$$= \log|t| + C$$

$$= \log|1+x^2| + C$$

$$= \log(1+x^2) + C$$

where C is an arbitrary constant.

2.

Integrate $\frac{(\log x)^2}{x}$

Solution:

Let log x = t

$$\therefore \frac{1}{x}dx = dt$$

$$\Rightarrow \int \frac{(\log|x|)^2}{x}dx = \int t^2 dt$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(\log|x|)^3}{3} + C$$

where C is an arbitrary constant.

3.

Integrate $\frac{1}{x+x\log x}$

Solution:

The given function can be rewritten as

$$\frac{1}{x+x\log x} = \frac{1}{x(1+\log x)}$$

Let 1 + log x = t

$$\therefore \frac{1}{x}dx = dt$$

$$\Rightarrow \int \frac{1}{x(1+\log x)}dx = \int \frac{1}{t}dt$$
$$= \log|t| + C$$
$$= \log|1+\log x| + C$$
where C is an arbitrary constant.

4.
Integrate sin x . sin (cos x)

Solution:
Let cos x = t
∴ −sin x dx = dt
$$\Rightarrow \sin x.\sin(\cos x)dx \int\int - = \sin t dt$$
$$= -[-\cos t] + C$$
$$= \cos t + C$$
$$= \cos(\cos x) + C$$
where C is an arbitrary constant.

5.
Integrate sin (ax + b) cos (ax + b)

Solution:
The given function can be rewritten as
sin (ax + b) cos (ax + b) = $\frac{2\sin(ax+b)\cos(ax+b)}{2} = \frac{\sin 2(ax+b)}{2}$

Let 2 (ax + b) = t
∴ 2adx = dt
$$\Rightarrow \int \frac{\sin 2(ax+b)}{2}dx = \frac{1}{2}\int \frac{\sin t\, dt}{2a}$$
$$= \frac{1}{4a}[-\cos t] + C$$
$$= \frac{-1}{4a}\cos 2(ax+b) + C$$
where C is an arbitrary constant.

6.
Integrate $\sqrt{ax+b}$

Solution

Let ax + b = t

⇒ adx = dt

$\therefore dx = \frac{1}{a}dt$

$\Rightarrow \int (ax+b)^{\frac{1}{2}}dx = \frac{1}{a}\int t^{\frac{1}{2}}dt$

$= \frac{1}{a}\left(\frac{t^{\frac{1}{2}}}{\frac{3}{2}}\right)+C$

$= \frac{2}{3a}(ax+b)^{\frac{3}{2}}+C$

where C is an arbitrary constant.

7.

Integrate $x\sqrt{x+2}$

Solution:

Let x + 2 = t

dx = dt

$\Rightarrow \int x\sqrt{x+2}dx = \int (t-2)\sqrt{t}dt$

$= \int \left(t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)dt$

$= \int t^{\frac{3}{2}}dt - 2\int t^{\frac{1}{2}}dt$

$= \frac{t^{\frac{5}{2}}}{\frac{5}{2}} - 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+C$

$\therefore$

$= \frac{2}{5}t^{\frac{5}{2}} - \frac{4}{3}t^{\frac{3}{2}}+C$

$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{4}{3}(x+2)^{\frac{3}{2}}+C$

where C is an arbitrary constant.

8.

$x\sqrt{1+2x^2}$

Solution:

Let $1 + 2x^2 = t$

4x dx = dt

$$\Rightarrow \int x\sqrt{1+2x^2}\,dx = \int \frac{\sqrt{t}}{4}dt$$

$$= \frac{1}{4}\int t^{\frac{1}{2}}dt$$

$$= \frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{6}\left(1+2x^2\right)^{\frac{3}{2}} + C$$

where C is an arbitrary constant.

9.

Integrate $(4x+2)\sqrt{x^2+x+1}$

Solution:

Let $x^2 + x + 1 = t$

(2x +1)dx = dt

$$\int (4x+2)\sqrt{x^2+x+1}\,dx$$

$$= \int 2\sqrt{t}dt$$

$$= 2\int \sqrt{t}dt$$

$$= 2\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{4}{3}\left(x^2+x+1\right)^{\frac{3}{2}} + C$$

where C is an arbitrary constant.

10.

Integrate $\frac{1}{x-\sqrt{x}}$

Solution:

The given function can be rewritten as

$$\frac{1}{x-\sqrt{x}} = \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}$$

Let $\left(\sqrt{x}-1\right)$ = t

$\therefore \frac{1}{2\sqrt{x}}dx = dt$

$\Rightarrow \int \frac{1}{\sqrt{x}\left(\sqrt{x}-1\right)}dx = \int \frac{2}{t}dt$

$= 2\log|t| + C$

$= 2\log\left|\sqrt{x}-1\right| + C$

where C is an arbitrary constant.

11.

Integrate $\frac{x}{\sqrt{x+4}}, x>0$

Solution:

Let x + 4 = t

dx = dt

$\int \frac{x}{\sqrt{x+4}}dx = \int \frac{(t-4)}{\sqrt{t}}dt$

$= \int \left(\sqrt{t} - \frac{4}{\sqrt{t}}\right)dt$

$= \frac{t^{\frac{3}{2}}}{\frac{3}{2}} - 4\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$

$= \frac{2}{3}(t)^{\frac{3}{2}} - 8(t)^{\frac{1}{2}} + C$

$= \frac{2}{3}t.t^{\frac{1}{2}} - 8t^{\frac{1}{2}} + C$

$= \frac{2}{3}t^{\frac{1}{2}}(t-12) + C$

$= \frac{2}{3}(x+4)^{\frac{1}{2}}(x+4-12) + C$

$= \frac{2}{3}\sqrt{x+4}(x-8) + C$

where C is an arbitrary constant.

12.

Integrate $\left(x^3-1\right)^{\frac{1}{3}} x^5$

Solution:

Let $x^3 - 1 = t$

$\therefore 3x^2dx = dt$

$\Rightarrow \int \left(x^3-1\right)^{\frac{1}{3}} x^5 dx = \int \left(x^3-1\right)^{\frac{1}{3}} x^3.x^2dx$

$= \int t^{\frac{1}{3}}\left(t+1\right)\frac{dt}{3}$

$= \frac{1}{3}\int\left(t^{\frac{4}{3}}+t^{\frac{1}{3}}\right)dt$

$= \frac{1}{3}\left[\frac{t^{\frac{7}{3}}}{\frac{7}{3}}+\frac{t^{\frac{4}{3}}}{\frac{4}{3}}\right]+C$

$= \frac{1}{3}\left[\frac{3}{7}t^{\frac{7}{3}}+\frac{3}{4}t^{\frac{4}{3}}\right]+C$

$= \frac{1}{7}\left(x^3-1\right)^{\frac{7}{3}}+\frac{1}{4}\left(x^3-1\right)^{\frac{4}{3}}+C$

where C is an arbitrary constant.

13.

Integrate $\frac{x^2}{\left(2+3x^3\right)^3}$

Solution:

Let $2 + 3x^3 = t$

$9x^2\ dx = dt$

$\Rightarrow \int \frac{x^2}{\left(2+3x^3\right)^3}dx = \frac{1}{9}\int\frac{dt}{\left(t\right)^3}$

$= \frac{1}{9}\left[\frac{t^{-2}}{-2}\right]+C$

$= \frac{-1}{18}\left(\frac{1}{t^2}\right)+C$

$\therefore = \frac{-1}{18\left(2+3x^3\right)^2}+C$

where C is an arbitrary constant.

14.

Integrate $\frac{1}{x(\log x)^m}, x > 0$

Solution:

Let log x = t

$\frac{1}{x}dx = dt$

$\Rightarrow \int \frac{1}{x(\log x)^m}dx = \int \frac{dt}{(t)^m}$

$= \frac{t^{-m+1}}{-m+1} + C$

$= \frac{(\log x)^{1-m}}{(1-m)} + C$

where C is an arbitrary constant.

15.

Integrate $\frac{x}{9-4x^2}$

Solution:

Let $9 - 4x^2 = t$

−8x dx = dt

$\Rightarrow \int \frac{x}{9-4x^2}dx = \frac{-1}{8}\int \frac{1}{t}dt$

$= \frac{-1}{8}\log|t| + C$

$= \frac{-1}{8}\log|9-4x^2| + C$

where C is an arbitrary constant.

16.

Integrate e^{2x+3}

Solution:

Let 2x + 3 =

t 2dx = dt

$\Rightarrow \int e^{2x+3}dx = \frac{1}{2}\int e^t dt$

$=\frac{1}{2}\left(e^{t}\right)+C$

$=\frac{1}{2}e^{(2x+3)}+C$

where C is an arbitrary constant.

17.

Integrate $\frac{x}{e^{x^2}}$

Solution:

Let $x^2 = t$

$2x\,dx = dt$

$\Rightarrow \int \frac{x}{e^{x^2}}dx = \frac{1}{2}\int \frac{1}{e^t}dt$

$=\frac{1}{2}\int e^{-t}dt$

$=\frac{1}{2}\left(\frac{e^{-t}}{-1}\right)+C$

$=-\frac{1}{2}e^{-x^2}+C$

$\dot{=}\frac{-1}{2e^{x^2}}+C$

where C is an arbitrary constant.

18.

Integrate $\frac{e^{\tan^{-1}x}}{1+x^2}$

Solution:

Let $\tan^{-1}x = t$

$\therefore \frac{1}{1+x^2}dx = dt$

$\Rightarrow \int \frac{e^{\tan^{-1}x}}{1+x^2}dx = \int e^t dt$

$=e^t+C$

$=e^{\tan^{-1}x}+C$

where C is an arbitrary constant.

19.

Integrate $\frac{e^{2x}-1}{e^{2x}+1}$

Solution:

Dividing the given function's numerator and denominator by e^x, we obtain,

$$\frac{\frac{\left(e^{2x}-1\right)}{e^x}}{\frac{\left(e^{2x}+1\right)}{e^x}}=\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

Let $e^x+e^{-x}=t$

$$\left(e^x-e^{-x}\right)dx=dt$$

$$\Rightarrow\int\frac{e^{2x}-1}{e^{2x}+1}dx=\int\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$$

$$=\int\frac{dt}{t}$$

$$=\log|t|+C$$

$$=\log\left|e^x+e^{-x}\right|+C$$

where C is an arbitrary constant.

20.

Integrate $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

Solution:

Let $e^{2x}+e^{-2x}=t$

$$\Rightarrow 2e^{2x}-2e^{-2x}dx=dt$$

$$\Rightarrow 2\left(e^{2x}-e^{-2x}\right)dx=dt$$

$$\Rightarrow\int\left(\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\right)dx=\int\frac{dt}{2t}$$

$$=\frac{1}{2}\int\frac{1}{t}dt$$

$$=\frac{1}{2}\log|t|+C$$

$$=\frac{1}{2}\log\left|e^{2x}+e^{-2x}\right|+C$$

where C is an arbitrary constant.

21:

Integrate $\tan^2(2x-3)$

Solution:

$\tan^2(2x-3)=\sec^2(2x-3)-1$

Let 2x − 3 = t

2dx = dt

$\Rightarrow\int\tan^2(2x-3)dx=\int\left[\sec^2(2x-3)-1\right]dx$

$=\frac{1}{2}\int(\sec^2 t)dt-\int 1dx$

$=\frac{1}{2}\int\sec^2 tdt-\int 1dx$

$=\frac{1}{2}\tan t-x+C$

$=\frac{1}{2}\tan(2x-3)-x+C$

$\therefore$

where C is an arbitrary constant.

22.

Integrate $\sec^2(7-4x)$

Solution:

Let 7 − 4x = t

−4dx = dt

$\therefore\int\sec^2(7-4x)dx=\frac{-1}{4}\int\sec^2 tdt$

$=\frac{-1}{4}(\tan t)+C$

$=\frac{-1}{4}\tan(7-4x)+C$

where C is an arbitrary constant.

23.

Integrate $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$

Solution:

Let $\sin^{-1}$ x = t

$\frac{1}{\sqrt{1-x^2}}dx=dt$

$$\Rightarrow \int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int t\, dt$$

$$= \frac{t^2}{2} + C$$

$$= \frac{\left(\sin^{-1} x\right)^2}{2} + C$$

where C is an arbitrary constant.

24.

Integrate $\dfrac{2\cos x - 3\sin x}{6\cos x + 4\sin x}$

Solution:

The given function is,

$$\frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} = \frac{2\cos x - 3\sin x}{2\left(3\cos x + 2\sin x\right)}$$

Let $3\cos x + 2\sin x = t$

$$\left(-3\sin x + 2\cos x\right) dx = dt$$

$$\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx = \int \frac{dt}{2t}$$

$$= \frac{1}{2}\int \frac{1}{t} dt$$

$$= \frac{1}{2}\log|t| + C$$

$$= \frac{1}{2}\log|2\sin x + 3\cos x| + C$$

where C is an arbitrary constant.

25.

Integrate $\dfrac{1}{\cos^2 x\left(1-\tan x\right)^2}$

Solution:

The given function is

$$\frac{1}{\cos^2 x\left(1-\tan x\right)^2} = \frac{\sec^2}{\left(1-\tan x\right)^2}$$

Let $\left(1-\tan x\right) = t$

$$-\sec^2 x dx = dt$$

$$\Rightarrow \int \frac{\sec^2}{(1-\tan x)^2} dx = \int \frac{-dt}{t^2}$$

$$= -\int t^{-2} dt$$

$$= +\frac{1}{t} + C$$

$$= \frac{1}{(1-\tan x)} + C$$

where C is an arbitrary constant.

26.

Integrate $\frac{\cos\sqrt{x}}{\sqrt{x}}$

Solution:

Let $\sqrt{x} = t$

$$\frac{1}{2\sqrt{x}} dx = dt$$

$$\Rightarrow \int \frac{\cos\sqrt{x}}{\sqrt{x}} dx = 2\int \cos t dt$$

$$= 2\sin t + C$$

$$= 2\sin\sqrt{x} + C$$

where C is an arbitrary constant.

27.

Integrate $\sqrt{\sin 2x}\cos 2x$

Solution:

Let sin 2x = t

So, 2cos 2x dx = dt

$$\Rightarrow \int \sqrt{\sin 2x}\cos 2x dx = \frac{1}{2}\int \sqrt{t} dt$$

$$= \frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{1}{3} t^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(\sin 2x)^{\frac{3}{2}} + C$$

where C is an arbitrary constant.

28.

Integrate $\frac{\cos x}{\sqrt{1+\sin x}}$

Solution:

Let 1 + sin x = t

cos x dx = dt

$\Rightarrow \int \frac{\cos x}{\sqrt{1+\sin x}} dx = \int \frac{dt}{\sqrt{t}}$

$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= 2\sqrt{t} + C$

$= 2\sqrt{1+\sin x} + C$

where C is an arbitrary constant.

29.

Integrate $\cot x \log \sin x$

Solution:

Let log sin x =t

$\Rightarrow \frac{1}{\text{sinx}}.\cos x dx = dt$

$\therefore \cot x dx = dt$

$\Rightarrow \int \cot x \log \sin x dx = \int t dt$

$= \frac{t^2}{2} + C$

$= \frac{1}{2}(\log \sin x)^2 + C$

where C is an arbitrary constant.

30.

Integrate $\frac{\sin x}{1+\cos x}$

Solution:

Let 1 + cos x = t

−sin x dx = dt

$$\Rightarrow \int \frac{\sin x}{1+\cos x}dx = \int -\frac{dt}{t}$$

$$= -\log|t| + C$$

$$= -\log|1+\cos x| + C$$

where C is an arbitrary constant.

31.

Integrate $\dfrac{\sin x}{(1+\cos x)^2}$

Solution:

Let 1 + cos x = t

−sin x dx = dt

$$\Rightarrow \int \frac{\sin x}{(1+\cos x)^2}dx = \int -\frac{dt}{t^2}$$

$$= -\int t^{-2}dt$$

$$= \frac{1}{t} + C$$

$$= \frac{1}{1+\cos x} + C$$

where C is an arbitrary constant.

32.

Integrate $\dfrac{1}{1+\cot x}$

Solution:

Let $I = \int \frac{1}{1+\cot x}dx$

$$= \int \frac{1}{1+\frac{\cos x}{\sin x}}dx$$

$$= \int \frac{\sin x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)}dx$$

$$=\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$

$$=\frac{1}{2}(x)+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x+\cos x}dx$$

Let sin x + cos x = t ⇒ (cos x − sin x) dx = dt

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$

$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

$$=\frac{x}{2}-\frac{1}{2}\log|\sin x+\cos x|+C$$

where C is an arbitrary constant.

33.

Integrate $\frac{1}{1-\tan x}$

Solution:

Let $I=\int\frac{1}{1-\tan x}dx$

$$=\int\frac{1}{1-\frac{\sin x}{\cos x}}dx$$

$$=\int\frac{\cos x}{\cos x-\sin x}dx$$

$$=\frac{1}{2}\int\frac{2\cos x}{\cos x-\sin x}dx$$

$$=\frac{1}{2}\int\frac{(\cos x-\sin x)+(\cos x+\sin x)}{(\cos x-\sin x)}dx$$

$$=\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

$$=\frac{x}{2}+\frac{1}{2}\int\frac{\cos x+\sin x}{\cos x-\sin x}dx$$

Put cos x − sin x = t ⇒ (−sin x − cos x) dx = dt

$$\therefore I=\frac{x}{2}+\frac{1}{2}\int\frac{-(dt)}{t}$$

$$=\frac{x}{2}-\frac{1}{2}\log|t|+C$$

$$=\frac{x}{2}-\frac{1}{2}\log|\cos x-\sin x|+C$$

where C is an arbitrary constant.

34.

Integrate $\frac{\sqrt{\tan x}}{\sin x \cos x}$

Solution:

Let $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

$= \int \frac{\sqrt{\tan x} \times \cos x}{\sin x \cos x \times \cos x} dx$

$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x} dx$

$= \int \frac{\sec^2 x dx}{\sqrt{\tan x}}$

Let tan x = t $\Rightarrow$ $\sec^2 x$ dx = dt

$\therefore I = \int \frac{dt}{\sqrt{t}}$

$= 2\sqrt{t} + C$

$= 2\sqrt{\tan x} + C$

where C is an arbitrary constant.

35.

Integrate $\frac{(1+\log x)^2}{x}$

Solution:

Let 1 + log x = t

$\therefore \frac{1}{x} dx = dt$

$\Rightarrow \int \frac{(1+\log x)^2}{x} dx = \int t^2 dt$

$= \frac{t^3}{3} + C$

$= \frac{(1+\log x)^3}{3} + C$

where C is an arbitrary constant.

36.

Integrate $\frac{(x+1)(x+\log x)^2}{x}$

Solution:

The given function can be rewritten as

$$\frac{(x+1)(x+\log x)^2}{x}$$

$$=\left(1+\frac{1}{x}\right)(x+\log x)^2$$

Let $(x+\log x)=t$

$$\therefore\left(1+\frac{1}{x}\right)dx=dt$$

$$\Rightarrow\int\left(1+\frac{1}{x}\right)(x+\log x)^2\,dx=\int t^2dt$$

$$=\frac{t^3}{3}+C$$

$$=\frac{1}{3}(x+\log x)^3+C$$

where C is an arbitrary constant.

37.

Integrate $\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}$

Solution:

Let $x^4 = t$

$4x^3\, dx = dt$

$$\Rightarrow\int\frac{x^3\sin\left(\tan^{-1}x^4\right)}{1+x^8}dx=\frac{1}{4}\int\frac{\sin\left(\tan^{-1}t\right)}{1+t^2}dt \qquad ...(1)$$

Let $\tan^{-1}t=u$

$$\therefore\frac{1}{1+t^2}dt=du$$

From (1), we obtain

$$\int\frac{x^3\sin\left(\tan^{-1}x^4\right)dx}{1+x^8}=\frac{1}{4}\int\sin u\,du$$

$$\doteq\frac{1}{4}(-\cos u)+C$$

$$=-\frac{1}{4}\cos\left(\tan^{-1}t\right)+C$$

$$=\frac{-1}{4}\cos\left(\tan^{-1}x^4\right)+C$$

where C is an arbitrary constant.

Chose the correct answer in Exercises 38 and 39.

38.

$\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$ equals

(A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$

(C) $\left(10^x - x^{10}\right)^{-1} + C$ (D) $\log\left(10^x + x^{10}\right) + C$

Solution:

Let $x^{10} + 10^x = t$

$\therefore \left(10x^9 + 10^x \log_e 10\right) dx = \int \frac{dt}{t}$

$\Rightarrow \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10x} dx = \int \frac{dt}{t}$

$= \log t + C$

$= \log\left(10^x + x^{10}\right) + C$

Hence, the correct Answer is D.

39.

$\int \frac{dx}{\sin^2 x \cos^2 x}$ equals

(A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$

(C) $\tan x \cot x + C$ (D) $\tan x - \cot 2x + C$

Solution:

Let $I = \int \frac{dx}{\sin^2 x \cos^2 x}$

$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \sec^2 x dx + \int \cos ec^2 dx$

$= \tan x - \cot x + C$

Hence, the correct Answer is B.

Exercise 7.3

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Find the integrals of the functions in Exercises 1 to 22:

1.

$\sin^2(2x + 5)$

Solution:

The given function can be rewritten as

$$\sin^2(2x+5)=\frac{1-\cos 2(2x+5)}{2}=\frac{1-\cos(4x+10)}{2}$$

$$\Rightarrow \int \sin^2(2x+5)dx=\int\frac{1-\cos(4x+10)}{2}dx$$

$$=\frac{1}{2}\int 1dx-\frac{1}{2}\int\cos(4x+10)dx$$

$$=\frac{1}{2}x-\frac{1}{2}\left(\frac{\sin(4x+10)}{4}\right)+C`$$

$$=\frac{1}{2}x-\frac{1}{8}\sin(4x+10)+C$$

2.

sin3x.cos4x

Solution:

It is known that, $\sin A\cos B=\frac{1}{2}\{\sin(A+B)+\sin(A-B)\}$

$$\therefore \int\sin 3x\cos 4x dx=\frac{1}{2}\int\{\sin(3x+4x)+\sin(3x-4x)\}dx$$

$$=\frac{1}{2}\int\{\sin 7x+\sin(-x)\}dx$$

$$=\frac{1}{2}\int\{\sin 7x-\sin x\}dx$$

$$=\frac{1}{2}\int\sin 7x dx-\frac{1}{2}\int\sin x dx$$

$$=\frac{1}{2}\left(\frac{-\cos 7x}{7}\right)-\frac{1}{2}(-\cos x)+C$$

$$=\frac{-\cos 7x}{14}+\frac{\cos x}{2}+C$$

where C is an arbitrary constant.

3.

cos 2x cos 4x cos 6x

Solution:

It is known that, $\cos A\cos B=\frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$

$$\therefore \int \cos 2x(\cos 4x\cos 6x)\,dx=\int \cos 2x\left[\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right]dx$$

$$=\frac{1}{2}\int\{\cos 2x\cos 10x+\cos 2x\cos(-2x)\}\,dx$$

$$=\frac{1}{2}\int\{\cos 2x\cos 10x+\cos^2 2x\}\,dx$$

$$=\frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\right\}+\left(\frac{1+\cos 4x}{2}\right)\right]dx$$

$$=\frac{1}{4}\int(\cos 12x+\cos 8x+1+\cos 4x)\,dx$$

$$=\frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}+C\right]$$

where C is an arbitrary constant.

4.

Integrate $\sin^3(2x+1)$

Solution:

Let $I=\int \sin^3(2x+1)dx$

$$\Rightarrow \int \sin^3(2x+1)dx=\int \sin^2(2x+1).\sin(2x+1)\,dx$$

$$=\int\left(1-\cos^2(2x+1)\right)\sin(2x+1)\,dx$$

Let $\cos(2x+1)=t$

$$\Rightarrow -2\sin(2x+1)\,dx=dt$$

$$\Rightarrow \sin(2x+1)\,dx=\frac{-dt}{2}$$

$$\Rightarrow I=\frac{-1}{2}\int(1-t^2)\,dt$$

$$=\frac{-1}{2}\left\{t-\frac{t^3}{3}\right\}+C$$

$$=\frac{-1}{2}\left\{\cos(2x+1)-\frac{\cos^3(2x+1)}{3}\right\}+C$$

$$=\frac{-\cos(2x+1)}{2}+\frac{\cos^3(2x+1)}{6}+C$$

where C is an arbitrary constant.

5.

Integrate $\sin^3 x \cos^3 x$

Solution:

Let $I = \int \sin^3 x \cos^3 x.dx$

$= \int \cos^3 x.\sin^2 x.\sin x.dx$

$= \int \cos^3 x\left(1-\cos^2 x\right)\sin x.dx$

Let cosx = t

$\Rightarrow$ -sinx.dx $= dt$

$\Rightarrow I = -\int t^3\left(1-t^2\right)dt$

$= -\int\left(t^3 - t^5\right)dt$

$= -\left\{\frac{t^4}{4} - \frac{t^6}{6}\right\} + C$

$= -\left\{\frac{\cos^4 x}{4} - \frac{\cos^6 x}{6}\right\} + C$

$= \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$

where C is an arbitrary constant.

6.

Integrate $\sin x \sin 2x \sin 3x$

Solution:

It is known that, $\sin A \sin B = \frac{1}{2}\left\{\cos\left(A-B\right) - \cos\left(A+B\right)\right\}$

$\therefore \int \sin x \sin 2x \sin 3x dx = \int\left[\sin x.\frac{1}{2}\left\{\cos\left(2x-3x\right) - \cos\left(2x+3x\right)\right\}\right]dx$

$= \frac{1}{2}\int\left(\sin x \cos\left(-x\right) - \sin x \cos 5x\right)dx$

$= \frac{1}{2}\int\left(\sin \cos x - \sin x \cos 5x\right)dx$

$= \frac{1}{2}\int \frac{\sin 2x}{2}dx - \frac{1}{2}\int \sin x \cos 5x dx$

$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int \frac{1}{2}\{\sin(x+5x) + \sin(x-5x)\}dx$

$= \frac{-\cos 2x}{8} - \frac{1}{4}\int\left(\sin 6x + \sin\left(-4x\right)\right)dx$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{6} + \frac{\cos 4x}{4}\right] + C$$

$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3} + \frac{\cos 4x}{2}\right] + C$$

$$= \frac{1}{8}\left[\frac{\cos 6x}{3} - \frac{\cos 4x}{2} - \cos 2x\right] + C$$

where C is an arbitrary constant.

7.

Integrate $\sin 4x \sin 8x$

Solution:

It is known that, $\sin A \sin B = \frac{1}{2}\cos(A-B) - \cos(A+B)$

$$\therefore \int \sin 4x \sin 8x dx = \int \left\{\frac{1}{2}\cos(4x-8x) - \cos(4x+8x)\right\} dx$$

$$= \frac{1}{2}\int (\cos(-4x) - \cos 12x) dx$$

$$= \frac{1}{2}\int (\cos 4x - \cos 12x) dx$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 12x}{12}\right] + \text{C}$$

where C is an arbitrary constant.

8.

Integrate $\frac{1-\cos x}{1+\cos x}$

Solution:

Consider,

$$\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}} \qquad \left[2\sin^2 \frac{x}{2} = 1 - \cos x \text{ and } 2\cos^2 \frac{x}{2} = 1 + \cos x\right]$$

$$= \tan^2 \frac{x}{2}$$

$$= \left(\sec^2 \frac{x}{2} - 1\right)$$

$$\therefore \int \frac{1-\cos x}{1+\cos x}dx = \int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$= \left[\frac{\tan\frac{x}{2}}{\frac{1}{2}} - x\right] + C$$

$$= 2\tan\frac{x}{2} - x + C$$

where C is an arbitrary constant.

9.

Integrate $\dfrac{\cos x}{1+\cos x}$

Solution:

$$\frac{\cos x}{1+\cos x} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2} \text{ and } \cos x = 2\cos^2\frac{x}{2}-1\right]$$

$$= \frac{1}{2}\left[1-\tan^2\frac{x}{2}\right]$$

$$\therefore \int \frac{\cos x}{1+\cos x}dx = \frac{1}{2}\int\left(1-\tan^2\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\int\left(1-\sec^2\frac{x}{2}+1\right)dx$$

$$= \frac{1}{2}\int\left(2-\sec^2\frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left[2x - \frac{\tan\frac{x}{2}}{\frac{1}{2}}\right] + C$$

$$= x - \tan\frac{x}{2} + C$$

where C is an arbitrary constant.

10.

Integrate $\sin^4 x$

Solution:

Consider $\sin^4 x = \sin^2 x \sin^2 x$

$= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)$

$= \frac{1}{4}(1-\cos 2x)^2$

$= \frac{1}{4}\left[1+\cos^2 2x-2\cos 2x\right]$

$= \frac{1}{4}\left[1+\left(\frac{1+\cos 4x}{2}\right)-2\cos 2x\right]$

$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]$

$= \frac{1}{4}\left[\frac{3}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]$

$\therefore \int \sin^4 x dx = \frac{1}{4}\int\left[\frac{3}{2}+\frac{1}{2}\cos 4x-2\cos 2x\right]dx$

$= \frac{1}{4}\left[\frac{3}{2}x+\frac{1}{2}\left(\frac{\sin 4x}{4}\right)-\frac{2\sin 2x}{2}\right]+C$

$= \frac{1}{8}\left[3x+\frac{\sin 4x}{4}-2\sin 2x\right]+C$

$= \frac{3x}{8}-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+C$

where C is an arbitrary constant.

11.

Integrate $\cos^4 2x$

Solution:

$\cos^4 2x = \left(\cos^2 2x\right)^2$

$= \left(\frac{1+\cos 4x}{2}\right)^2$

$= \frac{1}{4}\left[1+\cos^2 4x+2\cos 4x\right]$

$= \frac{1}{4}\left[1+\left(\frac{1+\cos 8x}{2}\right)+2\cos 4x\right]$

$= \frac{1}{4}\left[1+\frac{1}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]$

$= \frac{1}{4}\left[\frac{3}{2}+\frac{\cos 8x}{2}+2\cos 4x\right]$

$\therefore \int \cos^4 2x dx = \int\left(\frac{3}{8}+\frac{\cos 8x}{8}+\frac{\cos 4x}{2}\right)dx$

$$=\frac{3}{8}x+\frac{\sin 8x}{64}+\frac{\sin 4x}{8}+C$$

where C is an arbitrary constant.

12.

Integrate $\frac{\sin^2 x}{1+\cos x}$

Solution:

$$\frac{\sin^2 x}{1+\cos x}=\frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}} \qquad \left[\sin x=2\sin\frac{x}{2}\cos\frac{x}{2};\cos x=2\cos^2\frac{x}{2}-1\right]$$

$$=\frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

$$=2\sin^2\frac{x}{2}$$

$$=1-\cos x$$

$$\therefore\int\frac{\sin^2 x}{1+\cos x}dx=\int(1-\cos x)dx$$

$$=x-\sin x+C$$

where C is an arbitrary constant.

13.

Integrate $\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$

Solution:

Consider,

$$\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}=\frac{-2\sin\frac{2x+2\alpha}{2}\sin\frac{2x-2a}{2}}{-2\sin\frac{x+\alpha}{2}\sin\frac{x-\alpha}{2}} \qquad \left[\cos C-\cos D=-2\sin\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$=\frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= \frac{\left[2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right)\right]\left[2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)\right]}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}$$

$$= 4\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)$$

$$= 2\left[\cos\left(\frac{x+\alpha}{2}+\frac{x-\alpha}{2}\right)+\cos\frac{x+\alpha}{2}-\frac{x-\alpha}{2}\right]$$

$$= 2\left[\cos(x)+\cos\alpha\right]$$

$$= 2\cos x + 2\cos\alpha$$

$$\therefore \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}dx = \int 2\cos x + 2\cos\alpha$$

$$= 2\left[\sin x + x\cos\alpha\right] + C$$

where C is an arbitrary constant.

14.

Integrate $\frac{\cos x - \sin x}{1+\sin 2x}$

Solution:

$$\frac{\cos x - \sin x}{1+\sin 2x} = \frac{\cos x - \sin x}{\left(\sin^2 x + \cos^2 x\right) + 2\sin x\cos x}$$

$\left[\sin^2 x + \cos^2 x = 1; \sin 2x = 2\sin x\cos x\right]$

$$= \frac{\cos x - \sin x}{(\sin x + \cos x)^2}$$

Let $\sin x + \cos x = t$

$$\therefore (\cos x - \sin x)dx = dt$$

$$\Rightarrow \int \frac{\cos x - \sin x}{1+\sin 2x}dx = \int \frac{\cos x - \sin x}{(\sin x + \cos x)^2}dx$$

$$= \int \frac{dt}{t^2}$$

$$= \int t^{-2}dt$$

$$= -t^{-1} + C$$

$$= -\frac{1}{t} + C$$

$$= \frac{-1}{\sin x + \cos x} + C$$

where C is an arbitrary constant.

15.
Integrate $\tan^3 2x \sec 2x$

Solution:

$$\tan^3 2x\sec 2x = \tan^2 2x\tan 2x\sec 2x$$
$$= \left(\sec^2 2x - 1\right)\tan 2x\sec 2x$$
$$= \sec^2 2x.\tan 2x\sec 2x - \tan 2x\sec 2x$$
$$\therefore \int \tan^3 2x\sec 2x\,dx = \int \sec^2 2x.\tan 2x\sec 2x\,dx - \int \tan 2x\sec 2x dx$$
$$= \int \sec^2 2x.\tan 2x\sec 2x\,dx - \frac{\sec 2x}{2} + C$$

$Let \sec 2x = t$

$\therefore 2\sec 2x\tan 2xdx = dt$

$$\therefore \int \tan^3 2x\sec 2xdx = \frac{1}{2}\int t^2 dt - \frac{\sec 2x}{2} + C$$
$$= \frac{t^3}{6} - \frac{\sec 2x}{2} + C$$
$$= \frac{\sec^3 2x}{6} - \frac{\sec 2x}{2} + C$$

where C is an arbitrary constant.

16.
Integrate $\tan^4 x$

Solution:

$\tan^4 x$

$= \tan^2 x.\tan^2 x$

$= \left(\sec^2 x - 1\right)\tan^2 x$

$= \sec^2 x\tan^2 x - \tan^2 x$

$= \sec^2 x\tan^2 x - \left(\sec^2 x - 1\right)$

$= \sec^2 x\tan^2 x - \sec^2 x + 1$

$$\therefore \int \tan^4 xdx = \int \sec^2 x\tan^2 xdx - \int \sec^2 xdx + \int 1dx$$
$$= \int \sec^2 x\tan^2 xdx - \tan x + x + C \qquad ...(1)$$

Consider $\int \sec^2 x\tan^2 xdx$

$Let \tan x = 1 \Rightarrow \sec^2 xdx = dt$

$$\Rightarrow \int \sec^2 x\tan^2 xdx = \int t^2 dt = \frac{t^3}{3} = \frac{\tan^3 x}{3}$$

From equation (1), we obtain

$$\int \tan^4 x dx = \frac{1}{3}\tan^3 x - \tan x + x + C$$

where C is an arbitrary constant.

17.

Integrate $\dfrac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}$

Solution:

$$\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} = \frac{\sin^3 x}{\sin^2 x \cos^2 x} + \frac{\cos^3 x}{\sin^2 x \cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x}$$

$$= \tan x \sec x + \cot x \cos ecx$$

$$\therefore \int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int (\tan x \sec x + \cot x \cos ecx) dx$$

$$= \sec x - \text{cosec}\, x + C$$

where C is an arbitrary constant.

18.

Integrate $\dfrac{\cos 2x + 2\sin^2 x}{\cos^2 x}$

Solution:

$$\frac{\cos 2x + 2\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos 2x + (1 - \cos 2x)}{\cos^2 x} \qquad \left[\cos 2x = 1 - 2\sin^2 x\right]$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx = \int \sec^2 x dx = \tan x + C$$

where C is an arbitrary constant.

19.

Integrate $\dfrac{1}{\sin x \cos^3 x}$

Solution:

$$\frac{1}{\sin x\cos^3 x}=\frac{\sin^2 x+\cos^2 x}{\sin x\cos^3 x}$$
$$=\frac{\sin x}{\cos^3 x}+\frac{1}{\sin x\cos x}$$
$$=\tan x\sec^2 x+\frac{\frac{1/\cos^2 x}{\sin x\cos x}}{\cos^2 x}$$
$$=\tan x\sec^2 x+\frac{\sec^2 x}{\tan x}$$
$$\therefore\int\frac{1}{\sin x\cos^3 x}dx=\int\tan x\sec^2 xdx+\int\frac{\sec^2 x}{\tan x}dx$$
$$Let\ \tan x=1\Rightarrow\sec^2 xdx=dt$$
$$\Rightarrow\int\frac{1}{\sin x\cos^3 x}dx=\int tdt+\int\frac{1}{t}dt$$
$$=\frac{t^2}{2}+\log|t|+C$$
$$=\frac{1}{2}\tan^2 x+\log|\tan x|+C$$

where C is an arbitrary constant.

20.

Integrate $\frac{\cos 2x}{(\cos x+\sin x)^2}$

Solution:

$$\frac{\cos 2x}{(\cos x+\sin x)^2}=\frac{\cos 2x}{\cos^2 x+\sin^2 x+2\sin x\cos x}=\frac{\cos 2x}{1+\sin 2x}$$
$$\therefore\int\frac{\cos 2x}{(\cos x+\sin x)^2}dx=\int\frac{\cos 2x}{(1+\sin 2x)}dx$$
$$Let\ 1+\sin 2x=t$$
$$\Rightarrow 2\cos 2xdx=dt$$
$$\therefore\int\frac{\cos 2x}{(\cos x+\sin x)^2}dx=\frac{1}{2}\int\frac{1}{t}dt$$
$$=\frac{1}{2}\log|t|+C$$
$$=\frac{1}{2}\log|1+\sin 2x|+C$$

$$= \frac{1}{2}\log\left|(\sin x + \cos x)^2\right| + C$$
$$= \log\left|\sin x + \cos x\right| + C$$

where C is an arbitrary constant.

21.

Integrate $\sin^{-1}(\cos x)$

Solution:

$\sin^{-1}(\cos x)$

$Let\ \cos x = t$

$Then,\ \sin x = \sqrt{1-t^2}$

$\Rightarrow (-\sin x)dx = dt$

$$dx = \frac{-dt}{\sin x}$$
$$dx = \frac{-dt}{\sqrt{1-t^2}}$$
$$\therefore \int \sin^{-1}(\cos x)dx = \int \sin^{-1} t\left(\frac{-dt}{\sqrt{1-t^2}}\right)$$
$$= -\int \frac{\sin^{-1} t}{\sqrt{1-t^2}}dt$$

$Let\ \sin^{-1} t = u$

$$\Rightarrow \frac{1}{\sqrt{1-t^2}}dt = du$$
$$\therefore \int \sin^{-1}(\cos x)dx = -\int 4du$$
$$= -\frac{u^2}{2} + C$$
$$= \frac{-\left(\sin^{1} t\right)^2}{2} + C$$
$$= \frac{-\left[\sin^{-1}(\cos x)\right]^2}{2} + C \quad ...(1)$$

It is known that,

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$
$$\therefore \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x) = \left(\frac{\pi}{2} - x\right)$$

Substituting in equation (1), we obtain

$$\int \sin^{-1}(\cos x)dx = \frac{-\left[\frac{\pi}{2}-x\right]^2}{2}+C$$
$$= -\frac{1}{2}\left(\frac{\pi^2}{2}+x^2-\pi x\right)+C$$
$$= -\frac{\pi^2}{8}-\frac{x^2}{2}+\frac{1}{2}\pi x+C$$
$$= \frac{\pi x}{2}-\frac{x^2}{2}+\left(C-\frac{\pi^2}{8}\right)$$
$$= \frac{\pi x}{2}-\frac{x^2}{2}+C$$

22.

Integrate $\frac{1}{\cos(x-a)\cos(x-b)}$

Solution:

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\right]$$
$$= \frac{1}{\sin(a-b)}\left[\frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}\right]$$
$$= \frac{1}{\sin(a-b)}\frac{[\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)]}{\cos(x-a)\cos(x-b)}$$
$$= \frac{1}{\sin(a-b)}[\tan(x-b)-\tan(x-a)]$$
$$\Rightarrow \int \frac{1}{\cos(x-a)\cos(x-b)}dx = \frac{1}{\sin(a-b)}\int[\tan(x-b)-\tan(x-a)]dx$$
$$= \frac{1}{\sin(a-b)}\left[-\log|\cos(x-b)|+\log|\cos(x-a)|\right]$$
$$= \frac{1}{\sin(a-b)}\left[\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|\right]+C$$

where C is an arbitrary constant.

Chose the correct answer in Exercises 23 and 24.

23.

$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ is equal to

(A) tan x + cot x + C
(B) tan x + cosec x + C
(C) -tan x + cot x + C
(D) tan x + sec x + C

Solution:

$$\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} - \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx$$
$$= \int \left(\sec^2 x - \operatorname{cosec}^2 x \right) dx$$
$$= \tan x + \cot x + C$$

Hence, the correct Answer is A.

24.

$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$ equals

(A) –cot (ex^x) + C
(B) tan (xe^x) + C
(C) tan (e^x) + C
(D) cot (e^x) + C

Solution:

$$\int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx$$

Let $ex^x = t$

$$\Rightarrow (e^x . x + e^x . 1) dx = dt$$
$$e^x (x+1) dx = dt$$
$$\therefore \int \frac{e^x (1+x)}{\cos^2 (e^x x)} dx = \int \frac{dt}{\cos^2 t}$$
$$= \int \sec^2 t dt$$
$$= \tan t + C$$
$$= \tan (e^x . x) + C$$

Hence, the correct Answer is B.

Exercise 7.4

1.

Integrate $\frac{3x^2}{x^6+1}$

Solution:

Let $x^3 = t$

$3x^2\,dx = dt$

$$\Rightarrow \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1}$$

$= \tan^{-1} t + C$

$= \tan^{-1}(x^3) + C$

where C is an arbitrary constant.

$\therefore$

2.

Integrate $\frac{1}{\sqrt{1+4x^2}}$

Solution:

Let $2x = t$

$2dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x^2}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{1}{2}\left[\log\left|t+\sqrt{t^2+1}\right|\right]+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$\therefore \qquad = \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+C$$

where C is an arbitrary constant.

3.

Integrate $\frac{1}{\sqrt{(2-x)^2+1}}$

Solution:

Let $2 - x = t$

$\Rightarrow -dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{(2-x)^2+1}}dx = -\int \frac{1}{\sqrt{t^2+1}}dt$$

$$= -\log\left|t+\sqrt{t^2+1}\right|+C \qquad \left[\int \frac{1}{\sqrt{x^2+a^2}}dt = \log\left|x+\sqrt{x^2+a^2}\right|\right]$$

$$= -\log\left|2-x+\sqrt{(2-x)^2+1}\right|+C$$

$$= \log\left|\frac{1}{(2-x)+\sqrt{x^2-4x+5}}\right|+C$$

where C is an arbitrary constant.

4.

Integrate $\frac{1}{\sqrt{9-25x^2}}$

Solution:

Let 5x = t

5dx = dt

$$\Rightarrow \int \frac{1}{\sqrt{9-25x^2}} dx = \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}} dt$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{t}{3}\right) + C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

where C is an arbitrary constant.

5.

Integrate $\frac{3x}{1+2x^4}$

Solution:

Let $\sqrt{2}x^2 = t$

$\therefore 2\sqrt{2}xdx = dt$

$$\Rightarrow \int \frac{3x}{1+2x^4} dx = \frac{3}{2\sqrt{2}}\int \frac{dt}{1+t^2}$$

$$= \frac{3}{2\sqrt{2}}\left[\tan^{-1} t\right] + C$$

$$= \frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}x^2\right) + C$$

where C is an arbitrary constant.

6.

Integrate $\frac{x^2}{1-x^6}$

Solution:

Let x^3 = t

$3x^2$ dx = dt

$$\Rightarrow \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2}$$
$$= \frac{1}{3}\left[\frac{1}{2}\log\left|\frac{1+t}{1-t}\right|\right] + C$$
$$= \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

where C is an arbitrary constant.

7.

Integrate $\frac{x-1}{\sqrt{x^2-1}}$

Solution:

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad ...(1)$$

For $\int \frac{x}{\sqrt{x^2-1}}dx$, let $x^2-1 = t \Rightarrow 2xdx = dt$

$$\therefore \int \frac{x}{\sqrt{x^2-1}}dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}$$
$$= \frac{1}{2}\int t^{-\frac{1}{2}}dt$$
$$= \frac{1}{2}\left[2t^{\frac{1}{2}}\right]$$
$$= \sqrt{t}$$
$$= \sqrt{x^2-1}$$

From (1), we obtain

$$\int \frac{x-1}{\sqrt{x^2-1}}dx = \int \frac{x}{\sqrt{x^2-1}}dx - \int \frac{1}{\sqrt{x^2-1}}dx \qquad \left[\int \frac{x}{\sqrt{x^2-a^2}}dt = \log\left|x+\sqrt{x^2-a^2}\right|\right]$$
$$= \sqrt{x^2-1} - \log\left|x+\sqrt{x^2-1}\right| + C$$

where C is an arbitrary constant.

8.

Integrate $\frac{x^2}{\sqrt{x^6+a^6}}$

Solution:

Let $x^3 = t$

$\Rightarrow 3x^2\, dx = dt$

$$\therefore \int \frac{x^2}{\sqrt{x^6 + a^6}} dx = \frac{1}{3}\int \frac{dt}{\sqrt{t^2 + (a^3)^2}}$$

$$= \frac{1}{3}\log\left|t + \sqrt{t^6 + a^6}\right| + C$$

$$= \frac{1}{3}\log\left|x^3 + \sqrt{x^6 + a^6}\right| + C$$

where C is an arbitrary constant.

9.

Integrate $\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}$

Solution:

Let tan x = t

$\therefore \sec^2 x\, dx = dt$

$$\Rightarrow \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx = \int \frac{dt}{\sqrt{t^2 + 2^2}}$$

$$= \log\left|t + \sqrt{t^2 + 4}\right| + C$$

$$= \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$$

where C is an arbitrary constant.

10.

Integrate $\frac{1}{\sqrt{x^2 + 2x + 2}}$

Solution:

$$\int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx$$

Let x+1=t

$\therefore$ dx=dt

$$\Rightarrow \int \frac{1}{\sqrt{x^2 + 2x + 2}} dx = \int \frac{1}{\sqrt{t^2 + 1}} dt$$

$$= \log\left|t + \sqrt{t^2 + 1}\right| + C$$

$$= \log\left|(x+1) + \sqrt{(x+1)^2 + 1}\right| + C$$

$$=\log\left|(x+1)+\sqrt{x^2+2x+2}\right|+C$$

where C is an arbitrary constant.

11.

Integrate $\dfrac{1}{\sqrt{9x^2+6x+5}}$

Solution:

$$\int\frac{1}{\sqrt{9x^2+6x+5}}dx=\int\frac{1}{\sqrt{(3x+1)^2+2^2}}dx$$

$Let\ (3x+1)=t$

$\therefore 3dx=dt$

$$\int\frac{1}{\sqrt{(3x+1)^2+2^2}}dx=\frac{1}{3}\int\frac{1}{\sqrt{t^2+2^2}}dt$$

$$=\frac{1}{3}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]+C$$

$$=\frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

where C is an arbitrary constant.

12.

Integrate $\dfrac{1}{\sqrt{7-6x-x^2}}$

Solution:

$7-6x-x^2$ can be written as $7-\left(x^2+6x+9-9\right)$

Therefore,

$7-\left(x^2+6x+9-9\right)$

$=16-\left(x^2+6x+9\right)$

$=16-(x+3)^2$

$=(4)^2-(x+3)^2$

$$\therefore\int\frac{1}{\sqrt{7-6x-x^2}}dx=\int\frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$$

$Let\ x+3=1$

$\Rightarrow dx=dt$

$$\Rightarrow \int \frac{1}{\sqrt{(4)^2-(x+3)^2}}dx = \int \frac{1}{\sqrt{(4)^2-(t)^2}}dt$$

$$= \sin^{-1}\left(\frac{t}{4}\right)+C$$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

where C is an arbitrary constant.

13.

Integrate $\frac{1}{\sqrt{(x-1)(x-2)}}$

Solution:

(x – 1) (x – 2) can be written as x^2-3x+2.

Therefore,

x^2-3x+2

$$= x^2-3x+\frac{9}{4}-\frac{9}{4}+2$$

$$= \left(x-\frac{3}{2}\right)^2-\frac{1}{4}$$

$$= \left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\therefore \int \frac{1}{\sqrt{(x-1)(x-2)}}dx = \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx$$

$Let\ x-\frac{3}{2}=t$

$\therefore dx = dt$

$$\Rightarrow \int \frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}dx = \int \frac{1}{\sqrt{t^2-\left(\frac{1}{2}\right)^2}}dt$$

$$= \log\left|t+\sqrt{t^2-\left(\frac{1}{2}\right)^2}\right|+C$$

$$= \log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

where C is an arbitrary constant.

14.

Integrate $\frac{1}{\sqrt{8+3x-x^2}}$

Solution:

$8+3x-x^2$ can be written as $8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$

Therefore,

$$8-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)$$

$$=\frac{41}{4}-\left(x-\frac{3}{2}\right)^2$$

$$\Rightarrow \int \frac{1}{\sqrt{8+3x-x^2}}dx=\int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx$$

$$Let\ x-\frac{3}{2}=t$$

$$\therefore dx=dt$$

$$\Rightarrow \int \frac{1}{\sqrt{\frac{41}{4}-\left(x-\frac{3}{2}\right)^2}}dx=\int \frac{1}{\sqrt{\left(\frac{41}{4}\right)-t^2}}dt$$

$$=\sin^{-1}\left(\frac{t}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{41}}{2}}\right)+C$$

$$=\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$$

where C is an arbitrary constant.

15.

Integrate $\frac{1}{\sqrt{(x-a)(x-b)}}$

Solution:

$(x-a)(x-b)$ can be written as $x^2-(a+b)x+ab$.

Therefore,

$$x^2-(a+b)x+ab$$

$$=x^2-(a+b)x+\frac{(a+b)^2}{4}-\frac{(a+b)^2}{4}+ab$$

$$=\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}$$

$$\int\frac{1}{\sqrt{(x-a)(x-b)}}dx=\int\frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}}}dx$$

$$Let\ x-\left(\frac{a+b}{2}\right)=t$$

$$\therefore dx=dt$$

$$\int\frac{1}{\sqrt{\left[x-\left(\frac{a+b}{2}\right)\right]^2-\frac{(a-b)^2}{4}}}dx=\int\frac{1}{\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}}dt$$

$$=\log\left|t+\sqrt{t^2-\left(\frac{a-b}{2}\right)^2}\right|+C$$

$$=\log\left|\left\{x-\left(\frac{a+b}{2}\right)\right\}+\sqrt{(x-a)(x-b)}\right|+C$$

16:

Integrate $\frac{4x+1}{\sqrt{2x^2+x-3}}$

Solution:

Let $2x^2 + x - 3 = t$

$\therefore$ (4x + 1) dx = dt

$$\Rightarrow\int\frac{4x+1}{\sqrt{2x^2+x-3}}dx=\int\frac{1}{\sqrt{t}}dt$$

$$=2\sqrt{t}+C$$

$$=2\sqrt{2x^2+x-3}+C$$

where C is an arbitrary constant.

17.

Integrate $\frac{x+2}{\sqrt{x^2-1}}$

Solution:

Let x + 2 = $A\frac{d}{dx}\left(x^2-1\right)+B$ ...(1)

$\Rightarrow x+2=A(2x)+B$

Equating the coefficients of x and constant terms on both sides, we obtain

$2A=1\Rightarrow A=\frac{1}{2}$

$B=2$

From (1), we obtain

$(x+2)=\frac{1}{2}(2x)+2$

Then, $\int\frac{x+2}{\sqrt{x^2-1}}dx=\int\frac{\frac{1}{2}(2x)+2}{\sqrt{x^2-1}}dx$

$=\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx+\int\frac{2}{\sqrt{x^2-1}}dx$ ...(2)

In $\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx$ let $x^2-1=t\Rightarrow 2xdx=dt$

$\frac{1}{2}\int\frac{2x}{\sqrt{x^2-1}}dx=\frac{1}{2}\int\frac{dt}{\sqrt{t}}$

$=\frac{1}{2}\left[2\sqrt{t}\right]$

$=\sqrt{t}$

$=\sqrt{x^2-1}$

Then, $\int\frac{2}{\sqrt{x^2-1}}dx=2\int\frac{1}{\sqrt{x^2-1}}dx=2\log\left|x+\sqrt{x^2-1}\right|$

From equation (2), we obtain

$\int\frac{x+2}{\sqrt{x^2-1}}dx=\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+C$

where C is an arbitrary constant.

18.

Integrate $\frac{5x-2}{1+2x+3x^2}$

Solution:

Let $5x-2$ = $A\frac{d}{dx}\left(1+2x+3x^2\right)+B$

$\Rightarrow 5x-2=A(2+6x)+B$

Equating the coefficient of x and constant term on both sides, we obtain

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

$$2A + B = -2 \Rightarrow B = -\frac{11}{3}$$

$$\therefore 5x - 2 = \frac{5}{6}(2 + 6x) + \left(-\frac{11}{3}\right)$$

$$\Rightarrow \int \frac{5x-2}{1+2x+3x^2}dx = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}dx$$

$$= \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}dx$$

Let $I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$ and $I_2 = \int \frac{1}{1+2x+3x^2}dx$

$$\therefore \int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}I_1 - \frac{11}{3}I_2 \qquad ...(1)$$

$$I_1 = \int \frac{2+6x}{1+2x+3x^2}dx$$

Let $1 + 2x + 3x^2 = t$

$$\Rightarrow (2+6x)dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t}$$

$$I_1 = \log|t|$$

$$I_1 = \log|1+2x+3x^2| \qquad ...(2)$$

$$I_2 = \int \frac{1}{1+2x+3x^2}dx$$

$1 + 2x + 3x^2$ can be written as $1+3\left(x^2 + \frac{2}{3}x\right)$

Therefore,

$$1+3\left(x^2 + \frac{2}{3}x\right)$$

$$= 1 + 3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right)$$

$$= 1 + 3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3}$$

$$= \frac{2}{3} + 3\left(x + \frac{1}{3}\right)^2$$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}\right]$$

$$= 3\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$I_2 = \frac{1}{3}\int \frac{1}{\left[\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]}dx$$

$$= \frac{1}{3}\left[\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right)\right]$$

$$= \frac{1}{3}\left[\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(3)$$

Substituting equations (2) and (3) in equation (1), we obtain

$$\int \frac{5x-2}{1+2x+3x^2}dx = \frac{5}{6}\left[\log\left|1+2x+3x^2\right|\right] - \frac{11}{3}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)\right] + C$$

$$= \frac{5}{6}\log\left|1+2x+3x^2\right| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

where C is an arbitrary constant.

19.

Integrate $\frac{6x+7}{\sqrt{(x-5)(x-4)}}$

Solution:

$$\frac{6x+7}{\sqrt{(x-5)(x-4)}} = \frac{6x+7}{\sqrt{x^2-9x+20}}$$

Let $6x+7 = A\frac{d}{dx}\left(x^2-9x+20\right)+B$

$\Rightarrow 6x+7 = A(2x-9)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 6 \Rightarrow A = 3$

$-9A + B = 7 \Rightarrow B = 34$

$\therefore 6x + 7 = 3(2x - 9) + 34$

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}} = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}dx$$

$$=3\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx+34\int\frac{1}{\sqrt{x^2-9x+20}}dx$$

Let $I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$ and $I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$

$$\therefore\int\frac{6x+7}{\sqrt{x^2-9x+20}}=3I_1+34I_2 \qquad (1)$$

Then,

$$I_1=\int\frac{2x-9}{\sqrt{x^2-9x+20}}dx$$

$Let\ x^2-9x+20=t$

$$\Rightarrow(2x-9)dx=dt$$

$$\Rightarrow I_1=\frac{dt}{\sqrt{t}}$$

$$I_1=2\sqrt{t}$$

$$I_1=2\sqrt{x^2-9x+20} \qquad ...(2)$$

and $I_2=\int\frac{1}{\sqrt{x^2-9x+20}}dx$

$x^2-9x+20$ can be written as $x^2-9x+20+\frac{81}{4}-\frac{81}{4}$.

Therefore,

$$x^2-9x+20+\frac{81}{4}-\frac{81}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\frac{1}{4}$$

$$=\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2$$

$$\Rightarrow I_2=\int\frac{1}{\left(x-\frac{9}{2}\right)^2-\left(\frac{1}{2}\right)^2}dx$$

$$I_2=\log\left|\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right| \qquad ...(3)$$

Substituting equations (2) and (3) in (1), we obtain

$$\int\frac{6x+7}{\sqrt{x^2-9x+20}}dx=3\left[2\sqrt{x^2-9x+20}\right]+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

$$=6\sqrt{x^2-9x+20}+34\log\left[\left(x-\frac{9}{2}\right)+\sqrt{x^2-9x+20}\right]+C$$

where C is an arbitrary constant.

20.

Integrate $\frac{x+2}{\sqrt{4x-x^2}}$

Solution:

Let x + 2 = $A\frac{d}{dx}\left(4x-x^2\right)+B$

$\Rightarrow x+2=A(4-2x)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$-2A=1 \Rightarrow A=-\frac{1}{2}$

$4A+B=2 \Rightarrow B=4$

$\Rightarrow (x+2)=-\frac{1}{2}(4-2x)+4$

$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx=\int \frac{-\frac{1}{2}(4-2x)+4}{\sqrt{4x-x^2}}dx$

$=-\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}dx+4\int \frac{1}{\sqrt{4x-x^2}}dx$

$Let\ I_1=\int \frac{4-2x}{\sqrt{4x-x^2}}dx\ and\ I_2\int \frac{1}{\sqrt{4x-x^2}}dx$

$\therefore \int \frac{x+2}{\sqrt{4x-x^2}}dx=-\frac{1}{2}I_1+4I_2$ .... (1)

Then, $I_1=\int \frac{4-2x}{\sqrt{4x-x^2}}dx$

$Let\ 4x-x^2=t$

$\Rightarrow (4-2x)dx=dt$

$\Rightarrow I_1=\int \frac{dt}{\sqrt{t}}=2\sqrt{t}=2\sqrt{4x-x^2}$ ...(2)

$I_2=\int \frac{1}{\sqrt{4x-x^2}}dx$

$\Rightarrow 4x-x^2=-\left(-4x+x^2\right)$

$=\left(-4x+x^2+4-4\right)$

$=4-(x-2)^2$

$=(2)^2-(x-2)^2$

$\therefore I_2=\int \frac{1}{\sqrt{(2)^2-(x-2)^2}}dx=\sin^{-1}\left(\frac{x-2}{2}\right)$ ...(3)

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{4x-x^2}}dx = -\frac{1}{2}\left(2\sqrt{4x-x^2}\right)+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$
$$= -\sqrt{4x-x^2}+4\sin^{-1}\left(\frac{x-2}{2}\right)+C$$

Where C is an arbitrary constant

21.

Integrate $\frac{x+2}{\sqrt{x^2+2x+3}}$

Solution:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\int \frac{2(x+2)}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \frac{1}{2}\int \frac{2}{\sqrt{x^2+2x+3}}dx$$
$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$Let\ I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx\ and\ I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\therefore \int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}I_1 + I_2 \qquad ...(1)$$

Then, $I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}dx$

Let $x^2 + 2x + 3 = t$

$\Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad ...(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$\Rightarrow x^2+2x+3 = x^2+2x+1+2 = (x+1)^2+\left(\sqrt{2}\right)^2$$

$$\therefore I_2 = \int \frac{1}{\sqrt{(x+1)^2+\left(\sqrt{2}\right)^2}}dx = \log\left|(x+1)+\sqrt{x^2+2x+3}\right| \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}dx = \frac{1}{2}\left[2\sqrt{x^2+2x+3}\right]+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$
$$= \sqrt{x^2+2x+3}+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$$

Where C is an arbitrary constant

22:

Integrate $\frac{x+3}{x^2-2x-5}$

Solution:

Let $(x+3)=A\frac{d}{dx}(x^2-2x-5)+B$

$(x+3)=A(2x-2)+B$

Equating the coefficients of x and constant term on both sides, we obtain

$2A=1\Rightarrow A=\frac{1}{2}$

$-2A+B=3\Rightarrow B=4$

$\therefore (x+3)=\frac{1}{2}(2x-2)+4$

$\Rightarrow\int\frac{x+3}{x^2-2x-5}dx=\int\frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$

$=\frac{1}{2}\int\frac{2x-2}{x^2-2x-5}dx+4\int\frac{1}{x^2-2x-5}dx$

Let $I_1=\int\frac{2x-2}{x^2-2x-5}dx$ and $I_2=\int\frac{1}{x^2-2x-5}dx$

$\therefore\int\frac{x+3}{x^2-2x-5}dx=\frac{1}{2}I_1+4I_2$ ...(1)

Then, $I_1=\int\frac{2x-2}{x^2-2x-5}dx$

Let $x^2-2x-5=\text{t}$

$\Rightarrow(2x-2)dx=dt$

$\Rightarrow I_1=\int\frac{dt}{t}=\log|t|=\log|x^2-2x-5|$ ...(2)

$I_2=\int\frac{1}{x^2-2x-5}dx$

$=\int\frac{1}{(x^2-2x+1)-6}dx$

$=\int\frac{1}{(x-1)^2-(\sqrt{6})^2}dx$

$=\frac{1}{2\sqrt{6}}\log\left(\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right)$ ...(3)

Substituting (2) and (3) in (1), we obtain

$$\int \frac{x+3}{x^2-2x-5}dx = \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{4}{2\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

$$= \frac{1}{2}\log\left|x^2-2x-5\right| + \frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + C$$

Where C is an arbitrary constant.

23.

Integrate $\frac{5x+3}{\sqrt{x^2+4x+10}}$

Solution:

Let $5x+3 = A\frac{d}{dx}\left(x^2+4x+10\right)+B$

$\Rightarrow 5x+3 = A(2x+4)+B$

Equating the coefficients of x and constant term, we obtain

$2A = 5 \Rightarrow A = \frac{5}{2}$

$4A+B = 3 \Rightarrow B = -7$

$\therefore 5x+3 = \frac{5}{2}(2x+4)-7$

$$\Rightarrow \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \int \frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\int \frac{2x+4}{\sqrt{x^2+4x+10}}dx - 7\int \frac{1}{\sqrt{x^2+4x+10}}dx$$

Let $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$ and $I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$

$$\therefore \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx = \frac{5}{2}I_1 - 7I_2 \qquad \ldots.(1)$$

Then, $I_1 = \int \frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Let $x^2+4x+10 = t$

$\therefore (2x+4)dx = dt$

$$\Rightarrow I_1 = \int \frac{dt}{t} = 2\sqrt{t} = 2\sqrt{x^2+4x+10} \qquad \ldots(2)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+4x+10}}dx$$

$$=\int \frac{1}{\sqrt{(x^2+4x+4)+6}}dx$$

$$=\int \frac{1}{(x+2)^2+(\sqrt{6})^2}dx$$

$$=\log|(x+2)+\sqrt{x^2+4x+10}| \qquad \ldots (3)$$

Using equations (2) and (3) in (1), we obtain

$$\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx=\frac{5}{2}\left[2\sqrt{x^2+4x+10}\right]-7\log\left|(x+2)\sqrt{x^2+4x+10}\right|+C$$

$$=5\sqrt{x^2+4x+10}-7\log\left|(x+2)\sqrt{x^2+4x+10}\right|+C$$

Where C is an arbitrary constant.

24.

$\int \frac{dx}{x^2+2x+2}$ equals

(A) x $\tan^{-1}$ (x + 1) + C
(B) $\tan^{-1}$ (x + 1) + C
(C) (x + 1) $\tan^{-1}$ x + C
(D) $\tan^{-1}$x + C

Solution:

$$\int \frac{dx}{x^2+2x+2}=\int \frac{dx}{(x^2+2x+1)+1}$$

$$=\int \frac{1}{(x+1)^2+(1)^2}dx$$

$$=\left[\tan^{-1}(x+1)\right]+C$$

Hence, the correct Answer is B.

25.

$\int \frac{dx}{\sqrt{9x-4x^2}}$ equals

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$
(B) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$
(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$
(D) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

Solution:

$$\int \frac{dx}{\sqrt{9x-4x^2}}$$

$$=\int \frac{1}{\sqrt{-4\left(x^2-\frac{9}{4}x\right)}}dx$$

$$=\int \frac{1}{-4\left(x^2-\frac{9}{4}x+\frac{81}{64}-\frac{81}{64}\right)}dx$$

$$=\int \frac{1}{\sqrt{-4\left[\left(x-\frac{9}{8}\right)^2-\left(\frac{9}{8}\right)^2\right]}}dx$$

$$=\frac{1}{2}\left[\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right)\right]+C \qquad \left(\int \frac{dy}{\sqrt{a^2-y^2}}=\sin^{-1}\frac{y}{a}+C\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$$

Hence, the correct Answer is B.

Exercise 7.5

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1.

Integrate $\frac{x}{(x+1)(x+2)}$

Solution:

Let $\frac{x}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$

$\Rightarrow x = A(x+2) + B(x+1)$

Equating the coefficients of x and constant term, we obtain

A + B = 1

2A + B = 0

On solving, we obtain

A = −1 and B = 2

$\therefore \frac{x}{(x+1)(x+2)} = \frac{-1}{(x+1)} + \frac{2}{(x+2)}$

$\Rightarrow \int \frac{x}{(x+1)(x+2)} dx = \int \frac{-1}{(x+1)} + \frac{2}{(x+2)} dx$

$= -\log|x+1| + 2\log|x+2| + C$

$= \log(x+2)^2 - \log|x+1| + C$

$= \log\frac{(x+2)^2}{(x+1)} + C$

Where C is an arbitrary constant

$= -\log|x+1| + 2\log|x+2| + C$

$= \log(x+2)^2 - \log|x+1| + C$

$= \log\dfrac{(x+2)^2}{(x+1)} + C$

Where C is an arbitrary constant

2.

Integrate $\dfrac{1}{x^2-9}$

Solution:

Let $\dfrac{1}{(x+3)(x-3)} = \dfrac{A}{(x+3)} + \dfrac{B}{(x-3)}$

$1 = A(x-3) + B(x+3)$

Equating the coefficients of x and constant term, we obtain

A + B = 0

−3A + 3B = 1

On solving, we obtain

$A = -\dfrac{1}{6}$ and $B = \dfrac{1}{6}$

$\therefore \dfrac{1}{(x+3)(x-3)} = \dfrac{-1}{6(x+3)} + \dfrac{1}{6(x-3)}$

$\Rightarrow \int \dfrac{1}{(x^2-9)}dx = \int\left(\dfrac{-1}{6(x+3)} + \dfrac{1}{6(x-3)}\right)dx$

$= -\dfrac{1}{6}\log|x+3| + \dfrac{1}{6}\log|x-3| + C$

$= \dfrac{1}{6}\log\dfrac{|(x-3)|}{|(x+3)|} + C$

Where C is an arbitrary constant

3.

Integrate $\dfrac{3x-1}{(x-1)(x-2)(x-3)}$

Solution:

Let $\dfrac{3x-1}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$

$3x-1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad ...(1)$

Equating the coefficients of x^2, x and constant term, we obtain

A + B + C = 0

– 5A – 4B – 3C = 3

6A + 3B + 2C = – 1

Solving these equations, we obtain

A = 1, B = −5, and C = 4

$$\therefore \frac{3x-1}{(x-1)(x-2)(x-3)}=\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}$$

$$\Rightarrow \int \frac{3x-1}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{(x-1)}-\frac{5}{(x-2)}+\frac{4}{(x-3)}\right\}dx$$

$$=\log|x-1|-5\log|x-2|+4\log|x-3|+C$$

Where C is an arbitrary constant.

4.

Integrate $\frac{x}{(x-1)(x-2)(x-3)}$

Solution:

Let $\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$

$$x=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \quad ...(1)$$

Equating the coefficients of x^2, x and constant term, we obtain

A + B + C = 0

–5A – 4B – 3C = 1

6A + 4B + 2C = 0

Solving these equations, we obtain

$A=\frac{1}{2}, B=2$ and $C=\frac{3}{2}$

$$\therefore \frac{x}{(x-1)(x-2)(x-3)}=\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)(x-3)}dx=\int\left\{\frac{1}{2(x-1)}-\frac{2}{(x-2)}+\frac{3}{2(x-3)}\right\}dx$$

$$=\frac{1}{2}\log|x-1|-2\log|x-2|+\frac{3}{2}\log|x-3|+C$$

5.

Integrate $\frac{2x}{x^2+3x+2}$

Solution:

Let $\frac{2x}{x^2+3x+2}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$

$2x=A(x+2)+B(x+1) \quad ...(1)$

Equating the coefficients of x^2, x and constant term, we obtain

A + B = 2

2A + B = 0

Solving these equations, we obtain

A = −2 and B = 4

$\therefore \frac{2x}{(x+1)(x+2)}=\frac{-2}{(x+1)}+\frac{4}{(x+2)}$

$\Rightarrow \int \frac{2x}{(x+1)(x+2)}dx=\int\left\{\frac{4}{(x+2)}-\frac{2}{(x+1)}\right\}dx$

$=4\log|x+2|-2\log|x+1|+C$

Where C is an arbitrary constant.

6.

Integrate $\frac{1-x^2}{x(1-2x)}$

Solution:

It can be seen that the given integrand is not a proper fraction.

Therefore, on dividing $(1 - x^2)$ by $x(1 - 2x)$, we obtain

$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left(\frac{2-x}{x(1-2x)}\right)$

Let $\frac{2-x}{x(1-2x)}=\frac{A}{x}+\frac{B}{(1-2x)}$

$\Rightarrow (2-x)=A(1-2x)+Bx \quad ...(1)$

Equating the coefficients of x^2, x and constant term, we obtain

–2A + B = –1

And A = 2

Solving these equations, we obtain

A = 2 and B = 3

$\therefore \frac{2-x}{x(1-2x)}=\frac{2}{x}+\frac{3}{1-2x}$

Substituting in equation (1), we obtain

$\frac{1-x^2}{x(1-2x)}=\frac{1}{2}+\frac{1}{2}\left\{\frac{2}{x}+\frac{3}{(1-2x)}\right\}$

$$\int \frac{1-x^2}{x(1-2x)}dx = \int\left\{\frac{1}{2}+\frac{1}{2}\left(\frac{2}{x}+\frac{3}{(1-2x)}\right)\right\}dx$$

$$=\frac{x}{2}+\log|x|+\frac{3}{2(-2)}\log|1-2x|+C$$

$$=\frac{x}{2}+\log|x|-\frac{3}{4}\log|1-2x|+C$$

Where C is an arbitrary constant.

7.

Integrate $\frac{x}{(x^2+1)(x-1)}$

Solution:

Let $\frac{x}{(x^2+1)(x-1)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x-1)}$

$$x=(Ax+B)(x-1)+C(x^2+1)$$

$$x=Ax^2-Ax+Bx-B+Cx^2+C$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + C = 0

−A + B = 1

−B + C = 0

On solving these equations, we obtain

$$A=-\frac{1}{2}, B=\frac{1}{2}, \text{ and } C=\frac{1}{2}$$

From equation (1), we obtain

$$\therefore \frac{x}{(x^2+1)(x-1)}=\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{x^2+1}+\frac{\frac{1}{2}}{(x-1)}$$

$$\Rightarrow \int\frac{x}{(x^2+1)(x-1)}=-\frac{1}{2}\int\frac{x}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x^2+1}dx+\frac{1}{2}\int\frac{1}{x-1}dx$$

$$=-\frac{1}{4}\int\frac{2x}{x^2+1}dx+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

Consider $\int\frac{2x}{x^2+1}dx, let\ (x^2+1)=t \Rightarrow 2xdx=dt$

$$\Rightarrow \int\frac{2x}{x^2+1}dx=\int\frac{dt}{t}=\log|t|=\log|x^2+1|$$

$$\therefore \int\frac{x}{(x^2+1)(x-1)}=-\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+\frac{1}{2}\log|x-1|+C$$

$$=\frac{1}{2}\log|x-1|-\frac{1}{4}\log\left|x^2+1\right|+\frac{1}{2}\tan^{-1}x+C$$

Where C is an arbitrary constant.

8.

Integrate $\frac{x}{(x-1)^2(x+2)}$

Solution:

Let $\frac{x}{(x-1)^2(x+2)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+2)}$

$$x=A(x-1)(x+2)+B(x+2)+C(x-1)^2$$

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 0

A + B – 2C = 1

−2A + 2B + C = 0

On solving, we obtain

$A=\frac{2}{9}$ and $C=\frac{-2}{9}$

$B=\frac{1}{3}$

$$\therefore\frac{x}{(x-1)^2(x+2)}=\frac{2}{9(x-1)}+\frac{1}{3(x-1)^2}-\frac{2}{9(x+2)}$$

$$\Rightarrow\int\frac{x}{(x-1)^2(x+2)}dx=\frac{2}{9}\int\frac{1}{(x-1)}dx+\frac{1}{3}\int\frac{1}{(x-1)^2}dx-\frac{2}{9}\int\frac{1}{(x+2)}dx$$

$$=\frac{2}{9}\log|x-1|+\frac{1}{3}\left(\frac{-1}{x-1}\right)-\frac{2}{9}\log|x+2|+C$$

$$=\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+C$$

Where C is an arbitrary constant.

9.

Integrate $\frac{3x+5}{x^3-x^2-x+1}$

Solution:

$$\frac{3x+5}{x^3-x^2-x+1}=\frac{3x+5}{(x-1)^2(x+1)}$$

Let $\frac{3x+5}{(x-1)^2(x+1)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$

$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$

$3x+5=A(x^2-1)+B(x+1)+C(x^2+1-2x)$ ….(1)

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 0

B − 2C = 3

– A + B + C = 5

On solving, we obtain

B = 4

$A=-\frac{1}{2}$ and $C=\frac{1}{2}$

$\therefore \frac{3x+5}{(x-1)^2(x+1)}=\frac{-1}{2(x-1)}+\frac{4}{(x-1)^2}+\frac{1}{2(x+1)}$

$\Rightarrow \int\frac{3x+5}{(x-1)^2(x+1)}dx=-\frac{1}{2}\int\frac{1}{x-1}dx+4\int\frac{1}{(x-1)^2}dx+\frac{1}{2}\int\frac{1}{(x+1)}dx$

$=-\frac{1}{2}\log|x-1|+4\left(\frac{-1}{x-1}\right)+\frac{1}{2}\log|x+1|+C$

$=\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{(x-1)}+C$

Where C is an arbitrary constant.

10.

Integrate $\frac{2x-3}{(x^2-1)(2x+3)}$

Solution:

$\frac{2x-3}{(x^2-1)(2x+3)}=\frac{2x-3}{(x+1)(x-1)(2x+3)}$

Let $\frac{2x-3}{(x+1)(x-1)(2x+3)}=\frac{A}{(x+1)}+\frac{B}{(x-1)}+\frac{C}{(2x+3)}$

$\Rightarrow(2x-3)=A(x-1)(2x+3)+B(x+1)(2x+3)+C(x+1)(x-1)$

$\Rightarrow(2x-3)=A(2x^2+x-3)+B(2x^2+5x+3)+C(x^2-1)$

$\Rightarrow(2x-3)=(2A+2B+C)x^2+(A+5B)x+(-3A+3B-C)$

Equating the coefficients of x^2, x and constant, we obtain

2A + 2B + C = 0

A + 5B = 2

– 3A + 3B – C = – 3

On solving, we obtain

$B=-\frac{1}{10}, A=\frac{5}{2},$ and C=-$\frac{24}{5}$

$$\therefore \frac{2x-3}{(x+1)(x-1)(2x+3)}=\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$$

$$\Rightarrow \int \frac{2x-3}{(x^2-1)(2x+3)}dx=\frac{5}{2}\int \frac{1}{(x+1)}dx-\frac{1}{10}\int \frac{1}{x-1}dx-\frac{24}{5}\int \frac{1}{(2x+3)}dx$$

$$=\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{24}{5\times 2}\log|2x+3|$$

$$=\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+C$$

Where C is an arbitrary constant.

11.

Integrate $\frac{5x}{(x+1)(x^2-4)}$

Solution:

$$\frac{5x}{(x+1)(x^2-4)}=\frac{5x}{(x+1)(x+2)(x-2)}$$

Let $\frac{5x}{(x+1)(x+2)(x-2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}+\frac{C}{(x-2)}$

$$5x=A(x+2)(x-2)+B(x+1)(x-2)+C(x+1)(x+2) \quad ...(1)$$

Equating the coefficients of x^2, x and constant, we obtain

A + B + C = 0

– B + 3C = 5 and

– 4A – 2B + 2C = 0

On solving, we obtain

$$A=\frac{5}{3}, B=-\frac{5}{2}, \text{ and } C=\frac{5}{6}$$

$$\therefore \frac{5x}{(x+1)(x+2)(x-2)}=\frac{5}{3(x+1)}+-\frac{5}{2(x+2)}+\frac{5}{6(x-2)}$$

$$\Rightarrow \int \frac{5x}{(x+1)(x^2-4)}dx=\frac{5}{3}\int \frac{1}{(x+1)}dx-\frac{5}{2}\int \frac{1}{(x+2)}dx+\frac{5}{6}\int \frac{1}{(x-2)}dx$$

$$=\frac{5}{3}\log|x+1|-\frac{5}{2}\log|x+2|+\frac{5}{6}\log|x-2|+C$$

Where C is an arbitrary constant.

12.

Integrate $\frac{x^3+x+1}{x^2-1}$

Solution:

It can be seen that the given integrand is not a proper fraction.
Therefore, on dividing ($x^3 + x + 1$) by $x^2 - 1$, we obtain

$$\frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1}$$

Let $\frac{2x+1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$$2x+1 = A(x-1) + B(x+1) \qquad ...(1)$$

Equating the coefficients of x and constant, we obtain
A + B = 2
– A + B = 1
On solving, we obtain

$$A = \frac{1}{2} \text{ and } B = \frac{3}{2}$$

$$\therefore \frac{x^3+x+1}{x^2-1} = x + \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

$$\Rightarrow \int \frac{x^3+x+1}{x^2+1}dx = \int xdx + \frac{1}{2}\int \frac{1}{(x+1)}dx + \frac{3}{2}\int \frac{1}{(x-1)}dx$$

$$= \frac{x^2}{2} + \frac{1}{2}\log|x+1| + \frac{3}{2}\log|x-1| + C$$

Where C is an arbitrary constant.

13.

Integrate $\frac{2}{(1-x)(1+x^2)}$

Solution:

Let $\frac{2}{(1-x)(1+x^2)} = \frac{A}{(1-x)} + \frac{Bx+C}{(1+x^2)}$

$$2 = A(1+x^2) + (Bx+C)(1-x)$$

$$2 = A + Ax^2 + Bx - Bx^2 + C - Cx$$

Equating the coefficient of x^2, x, and constant term, we obtain
A − B = 0
B − C = 0
A + C = 2
On solving these equations, we obtain
A = 1, B = 1, and C = 1

$$\therefore \frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

$$\Rightarrow \int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{1}{1-x}dx + \int \frac{x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\log|x-1| + \frac{1}{2}\log|1+x^2| + \tan^{-1} x + C$$

Where C is an arbitrary constant.

14.

Integrate $\frac{3x-1}{(x+2)^2}$

Solution:

Let $\frac{3x-1}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$

$\Rightarrow 3x-1 = A(x+2)+B$

Equating the coefficient of x and constant term, we obtain

A = 3

2A + B = −1 ⇒ B = −7

$$\therefore \frac{3x-1}{(x+2)^2} = \frac{3}{(x+2)} - \frac{7}{(x+2)^2}$$

$$\Rightarrow \int \frac{3x-1}{(x+2)^2}dx = 3\int \frac{1}{(x+2)}dx - 7\int \frac{1}{(x+2)^2}dx$$

$$= 3\log|x+2| - 7\left(\frac{-1}{(x+2)}\right) + C$$

$$= 3\log|x+2| + \frac{7}{(x+2)} + C$$

Where C is an arbitrary constant.

15.

Integrate $\frac{1}{x^4-1}$

Solution:

$$\frac{1}{(x^4-1)} = \frac{1}{(x^2-1)(x^2+1)} = \frac{1}{(x+1)(x-1)(1+x^2)}$$

Let $\frac{1}{(x+1)(x-1)(1+x^2)} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{Cx+D}{(x^2+1)}$

$1 = A(x-1)(1+x^2) + B(x+1)(1+x^2) + (Cx+D)(x^2-1)$

$1 = A(x^3+x-x^2-1) + B(x^3+x+x^2+1) + Cx^3 + Dx^2 - Cx - D$

$1 = (A+B+C)x^3 + (-A+B+D)x^2 + (A+B-C)x + (-A+B-D)$

Equating the coefficient of x^3, x^2, x, and constant term, we obtain

$A+B+C=0$

$-A+B+D=0$

$A+B-C=0$

$-A+B-D=1$

$A=-\frac{1}{4}, B=\frac{1}{4}, C=0, \text{ and } D=-\frac{1}{2}$

$\therefore \frac{1}{(x^4-1)} = \frac{-1}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x^2+1)}$

$\Rightarrow \int \frac{1}{x^4-1}dx = -\frac{1}{4}\log|x-1| + \frac{1}{4}\log|x-1| - \frac{1}{2}\tan^{1} x + C$

$= \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{1} x + C$

Where C is an arbitrary constant.

16.

Integrate $\frac{1}{x(x^n+1)}$

[*Hint*: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

Solution:

$\frac{1}{x(x^n+1)}$

Multiplying numerator and denominator by x^{n-1}, we obtain

$\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x^{n-1}x(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}$

Let $x^n = t \Rightarrow n\, x^{n-1}\, dx = dt$

$\therefore \int \frac{1}{x(x^n+1)}dx = \int \frac{x^{n-1}}{x^n(x^n+1)}dx = \frac{1}{n}\int \frac{1}{t(t+1)}dt$

$Let\ \frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{(t+1)}$

$1 = A(1+t) + Bt \qquad ...(1)$

Equating the coefficients of t and constant, we obtain

A = 1 and B = −1

$$\therefore \frac{1}{t(t+1)}=\frac{1}{t}-\frac{1}{(1+t)}$$

$$\Rightarrow \int \frac{1}{x\left(x^n+1\right)} dx=\frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{(1+t)}\right\}dx$$

$$=\frac{1}{n}\left[\log|t|-\log|t+1|\right]+C$$

$$=\frac{1}{n}\left[\log|x^n|-\log|x^n+1|\right]+C$$

$$=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

Where C is an arbitrary constant.

17.

Integrate $\frac{\cos x}{(1-\sin x)(2-\sin x)}$

[Hint: Put sin x = t]

Solution:

$$\frac{\cos x}{(1-\sin x)(2-\sin x)}$$

Let $\sin x=t \Rightarrow \cos x dx=dt$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int \frac{dt}{(1-t)(2-t)}$$

Let $\frac{1}{(1-t)(2-t)}=\frac{A}{(1-t)}+\frac{B}{(2-t)}$

$1=A(2-t)+B(1-t) \qquad ...(1)$

Equating the coefficients of t and constant, we obtain

– A – B = 0 and

2A + B = 1

On solving, we obtain

A = 1 and B = -1

$$\therefore \frac{1}{(1-t)(2-t)}=\frac{1}{(1-t)}-\frac{1}{(2-t)}$$

$$\Rightarrow \int \frac{\cos x}{(1-\sin x)(2-\sin x)}dx=\int\left\{\frac{1}{1-t}-\frac{1}{(2-t)}\right\}dt$$

$$=-\log\lfloor 1-t\rfloor+\log|2-t|+C$$

$$=\log\left|\frac{2-t}{1-t}\right|+C$$

$$= \log\left|\frac{2-\sin x}{1-\sin x}\right| + C$$

Where C is an arbitrary constant.

18.

Integrate $\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

Solution:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \frac{4x^2+10}{(x^2+3)(x^2+4)}$$

Let $\frac{(4x^2+10)}{(x^2+3)(x^2+4)} = \frac{Ax+B}{(x^2+3)} + \frac{Cx+D}{(x^2+4)}$

$$4x^2+10 = (Ax+B)(x^2+4) + (Cx+D)(x^2+3)$$

$$4x^2+10 = Ax^3+4Ax+Bx^2+4B+Cx^3+3Cx+Dx^2+3D$$

$$4x^2+10 = (A+C)x^3+(B+D)x^2+(4A+3C)x+(4B+3D)$$

Equating the coefficients of x^3, x^2, x and constant term, we obtain

A + C = 0

B + D = 4

4A + 3C = 0

4B + 3D = 10

On solving these equations, we obtain

A = 0, B = −2, C = 0, and D = 6

$$\therefore \frac{(4x^2+10)}{(x^2+3)(x^2+4)} = \frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}$$

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 - \left(\frac{-2}{(x^2+3)} + \frac{6}{(x^2+4)}\right)$$

$$\Rightarrow \int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}dx = \int \left\{1 + \frac{2}{(x^2+3)} - \frac{6}{(x^2+4)}\right\}dx$$

$$= \int \left\{1 + \frac{2}{x^2+(\sqrt{3})^2} - \frac{6}{x^2+2^2}\right\}$$

$$= x + 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right) - 6\left(\frac{1}{2}\tan^{-1}\frac{x}{2}\right) + C$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} - 3\tan^{-1}\frac{x}{2} + C$$

19.

Integrate $\frac{2x}{(x^2+1)(x^2+3)}$

Solution:

$\frac{2x}{(x^2+1)(x^2+3)}$

Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int \frac{2x}{(x^2+1)(x^2+3)}dx = \int \frac{dt}{(t+1)(t+3)} \qquad ...(1)$$

Let $\frac{1}{(t+1)(t+3)} = \frac{A}{(t+1)} + \frac{B}{(t+3)}$

$$1 = A(t+3) + B(t+1) \qquad ...(2)$$

Equating the coefficients of t and constant, we obtain

A + B = 0 and 3A + B = 1

On solving, we obtain

$A = \frac{1}{2}$ and $B = -\frac{1}{2}$

$$\therefore \frac{1}{(t+1)(t+3)} = \frac{1}{2(t+1)} - \frac{1}{2(t+3)}$$

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+3)}dx = \int \left\{ \frac{1}{2(t+1)} - \frac{1}{2(t+3)} \right\} dt$$

$$= \frac{1}{2}\log|(t+1)| - \frac{1}{2}\log|t+3| + C$$

$$= \frac{1}{2}\log\left|\frac{t+1}{t+3}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right| + C$$

Where C is an arbitrary constant.

20.

Integrate $\frac{1}{x(x^4-1)}$

Solution:

$\frac{1}{x(x^4-1)}$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x\left(x^4-1\right)}=\frac{x^3}{x^4\left(x^4-1\right)}$$

$$\therefore\int\frac{1}{x\left(x^4-1\right)}dx=\int\frac{x^3}{x^4\left(x^4-1\right)}dx$$

Let $x^4 = t \Rightarrow 4x^3\,dx = dt$

$$\therefore\int\frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int\frac{dt}{t(t-1)}$$

Let $\frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{(t-1)}$

$1=A(t-1)+Bt \qquad ...(1)$

Equating the coefficients of t and constant, we obtain

A + B = 0 and –A = 1

A = −1 and B = 1

$$\Rightarrow\frac{1}{t(t-1)}=\frac{-1}{t}+\frac{1}{t-1}$$

$$\Rightarrow\int\frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int\left\{\frac{-1}{t}+\frac{1}{t-1}\right\}dt$$

$$=\frac{1}{4}\left[-\log|t|+\log|t-1|\right]+C$$

$$=\frac{1}{4}\log\left|\frac{t-1}{t}\right|+C$$

$$=\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$$

Where C is an arbitrary constant.

21.

Integrate $\frac{1}{\left(e^x-1\right)}$

[Hint: Put $e^x = t$]

Solution:

Let $e^x = t \Rightarrow e^x\,dx = dt$

$$\Rightarrow\int\frac{1}{\left(e^x-1\right)}dx=\int\frac{1}{t-1}\times\frac{dt}{t}=\int\frac{1}{t(t-1)}dt$$

$Let\ \frac{1}{t(t-1)}=\frac{A}{t}+\frac{B}{t-1}$

$1=A(t-1)+Bt \qquad ...(1)$

Equating the coefficients of t and constant, we obtain

A + B = 0 and –A = 1
A = −1 and B = 1

$$\therefore \frac{1}{t(t-1)} = \frac{-1}{t} + \frac{1}{t-1}$$

$$\Rightarrow \int \frac{1}{t(t-1)} dt = \log\left|\frac{t-1}{t}\right| + C$$

$$= \log\left|\frac{e^x - 1}{e^x}\right| + C$$

Where C is an arbitrary constant.

Chose the correct answer in Exercises 22 and 23.

22.

$$\int \frac{xdx}{(x-1)(x-2)} \text{ equals}$$

A. $\log\left|\frac{(x-1)^2}{x-2}\right| + C$

B. $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

C. $\log\left|\left(\frac{x-1}{x-2}\right)^2\right| + C$

D. $\log|(x-1)(x-2)| + C$

Solution:

Let $\frac{x}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$

$x = A(x-2) + B(x-1) \qquad ...(1)$

Equating the coefficients of x and constant, we obtain
A + B = 1 and –2A – B = 0
A = −1 and B = 2

$$\therefore \frac{x}{(x-1)(x-2)} = -\frac{1}{(x-1)} + \frac{2}{(x-2)}$$

$$\Rightarrow \int \frac{x}{(x-1)(x-2)} dx = \int \left\{\frac{-1}{(x-1)} + \frac{2}{(x-2)}\right\} dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log\left|\frac{(x-2)^2}{x-1}\right| + C$$

Hence, the correct Answer is B.

23.

$\int \frac{dx}{x\left(x^2+1\right)}$ equals

A. $\log|x|-\frac{1}{2}\log\left(x^2+1\right)+C$

B. $\log|x|+\frac{1}{2}\log\left(x^2+1\right)+C$

C. $-\log|x|+\frac{1}{2}\log\left(x^2+1\right)+C$

D. $\frac{1}{2}\log|x|+\log\left(x^2+1\right)+C$

Solution:

Let $\frac{1}{x\left(x^2+1\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$

$1=A\left(x^2+1\right)+\left(Bx+C\right)x$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

C = 0

A = 1

On solving these equations, we obtain

A = 1, B = −1, and C = 0

$\therefore \frac{1}{x\left(x^2+1\right)}=\frac{1}{x}+\frac{-x}{x^2+1}$

$\Rightarrow \int \frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{1}{x}-\frac{x}{x^2+1}\right\}dx$

$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$

Hence, the correct Answer is A.

Alternative Method:

$\Rightarrow \int \frac{1}{x\left(x^2+1\right)}dx=\int\left\{\frac{x}{x^2(x^2+1)}\right\}dx$

Let $x^2 = t$, therefore, 2x dx = dt

$\therefore \int \frac{x}{x^2(x^2+1)}dx=\frac{1}{2}\int\frac{dt}{t(t+1)}=\frac{1}{2}\int\frac{(t+1)-t}{t(t+1)}dt=\frac{1}{2}\int\frac{1}{t}-\frac{1}{t+1}dt$

$=\frac{1}{2}[\log t-\log(t+1)]+C$

$=\log|x|-\frac{1}{2}\log\left|x^2+1\right|+C$

Exercise 7.6 Page: 327

1.
Integrate x sin x

Solution:
Let I = $\int x\sin x dx$
Taking x as first function and sin x as second function and integrating by parts, we obtain,
I = $x\int \sin x dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin x\, dx\right\}dx$

$= x(-\cos x) - \int 1.(-\cos x)\,dx$

$= -x\cos x + \sin x + C$

Where C is an arbitrary constant.

2:
Integrate x sin 3x

Solution:
Let I = $\int x\sin 3x dx$
Taking x as first function and sin 3x as second function and integrating by parts, we obtain
$I = x\int \sin 3x dx - \int\left\{\left(\frac{d}{dx}x\right)\int \sin 3x dx\right\}$

$= x\left(\frac{-\cos 3x}{3}\right) - \int 1.\left(\frac{-\cos 3x}{3}\right)dx$

$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x dx$

$= \frac{-x\cos 3x}{3} + \frac{1}{9}\sin 3x + C$

Where C is an arbitrary constant.

3:
Integrate $x^2 e^x$

Solution:
Let $I = \int x^2 e^x dx$
Taking x^2 as first function and ex as second function and integrating by parts, we obtain
$I = x^2\int e^x dx - \int\left\{\left(\frac{d}{dx}x^2\right)\int e^x dx\right\}dx$

$= x^2e^x - \int 2xe^x dx$

$= x^2e^x - 2\int x \cdot e^x dx$

Again integrating by parts, we obtain

$= x^2e^x - 2\left[x \cdot \int e^x dx - \int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx\right]$

$= x^2e^x - 2\left[xe^x - \int e^x dx\right]$

$= x^2e^x - 2\left[xe^x - e^x\right]$

$= x^2e^x - 2xe^x + 2e^x + C$

$= e^x\left(x^2 - 2x + 2\right) + C$

Where C is an arbitrary constant.

4.

Integrate x log x

Solution:

Let $I = \int x \log x dx$

Taking log x as first function and x as second function and integrating by parts, we obtain

$I = \log x \int xdx - \int\left\{\left(\frac{d}{dx}\log x\right)\int xdx\right\}dx$

$= \log x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx$

$= \frac{x^2 \log x}{2} - \int \frac{x}{2}dx$

$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$

Where C is an arbitrary constant.

5.

Integrate *x* log 2x

Solution:

Let $I = \int x \log 2x dx$

Taking log 2x as first function and x as second function and integrating by parts, we obtain

$I = \log 2x \int x\,dx - \int\left\{\left(\frac{d}{dx}\log 2x\right)\int x\,dx\right\}dx$

$= \log 2x.\frac{x^2}{2} - \int \frac{2}{2x}.\frac{x^2}{2}dx$

$$=\frac{x^2\log 2x}{2}-\int\frac{x}{2}dx$$

$$=\frac{x^2\log 2x}{2}-\frac{x^2}{4}+C$$

Where C is an arbitrary constant.

6.

Integrate $x^2\log x$

Solution:

Let $I=\int x^2\log x dx$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I=\log x\int x^2dx-\int\left\{\left(\frac{d}{dx}\log x\right)\int x^2dx\right\}dx$$

$$=\log x.\left(\frac{x^3}{3}\right)-\int\frac{1}{x}.\frac{x^3}{3}dx$$

$$=\frac{x^3\log x}{3}-\int\frac{x^2}{3}dx$$

$$=\frac{x^3\log x}{3}-\frac{x^3}{9}+C$$

Where C is an arbitrary constant.

7.

Integrate $x\sin^{-1}x$

Solution:

Let $I=\int x\sin^{-1}x dx$

Taking $\sin^{-1}$ x as first function and x as second function and integrating by parts, we obtain

$$I=\sin^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\sin^{-1}x\right)\int x\,dx\right\}dx$$

$$=\sin^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{1}{\sqrt{1-x^2}}.\frac{x^2}{2}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\int\left\{\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right\}dx$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\int\sqrt{1-x^2}dx-\int\frac{1}{\sqrt{1-x^2}}dx\right\}$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{1}{2}\left\{\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x-\sin^{-1}x\right\}+C$$

$$=\frac{x^2\sin^{-1}x}{2}+\frac{x}{4}\sqrt{1-x^2}+\frac{1}{4}\sin^{-1}x-\frac{1}{2}\sin^{-1}x+C$$

$$\frac{1}{4}\left(2x^2-1\right)\sin^{-1}x+\frac{x}{4}\sqrt{1-x^2}+C$$

Where C is an arbitrary constant.

8.

Integrate $x\tan^{-1}x$

Solution:

Let $I=\int x\tan^{-1}x\,dx$

Taking $\tan^{-1}x$ as first function and x as second function and integrating by parts, we obtain

$$I=\tan^{-1}x\int x\,dx-\int\left\{\left(\frac{d}{dx}\tan^{-1}x\right)\int x\,dx\right\}dx$$

$$=\tan^{-1}x\left(\frac{x^2}{2}\right)-\int\frac{1}{1+x^2}.\frac{x^2}{2}dx$$

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\left(\frac{x^2+1}{1+x^2}-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2\tan^{-1}x}{2}-\frac{1}{2}\left(x-\tan^{-1}x\right)+C$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+C$$

Where C is an arbitrary constant.

9.

Integrate $x\cos^{-1}x$

Solution:

Let $I = \int x\cos^{-1} x dx$

Taking $\cos^{-1}$ x as first function and x as second function and integrating by parts, we obtain

$$I = \cos^{-1} x\int xdx - \int\left\{\left(\frac{d}{dx}\cos^{-1} x\right)\int xdx\right\}dx$$

$$= \cos^{-1} x\frac{x^2}{2} - \int\frac{-1}{\sqrt{1-x^2}}.\frac{x^2}{2}dx$$

$$= \frac{x^2\cos^{-1} x}{2} - \frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2\cos^{-1} x}{2} - \frac{1}{2}\int\left\{\sqrt{1-x^2} + \left(\frac{-1}{\sqrt{1-x^2}}\right)\right\}dx$$

$$= \frac{x^2\cos^{-1} x}{2} - \frac{1}{2}\int\sqrt{1-x^2}dx - \frac{1}{2}\int\left(\frac{-1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^2\cos^{-1} x}{2} - \frac{1}{2}\left(\frac{x}{2}\sqrt{1-x^2}\right) - \frac{1}{4}\cos^{-1} x + C$$

Where C is an arbitrary constant.

10.

Integrate $\left(\sin^{-1} x\right)^2$

Solution:

Let $I = \int\left(\sin^{-1} x\right)^2.1dx$

Taking $(\sin^{-1} x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = \int\left(\sin^{-1} x\right).\int 1dx - \int\left\{\frac{d}{dx}\left(\sin^{-1} x\right)^2.\int 1.dx\right\}dx$$

$$= \left(\sin^{-1} x\right)^2.x - \int\frac{2\sin^{-1} x}{\sqrt{1-x^2}}.xdx$$

$$= x\left(\sin^{-1} x\right)^2 + \int\sin^{-1} x.\left(\frac{-2x}{\sqrt{1-x^2}}\right)dx$$

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x\int\frac{-2x}{\sqrt{1-x^2}}dx - \int\left\{\left(\frac{d}{dx}\sin^{-1} x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$$

$$= x\left(\sin^{-1} x\right)^2 + \left[\sin^{-1} x.2\sqrt{1-x^2} - \int\frac{1}{\sqrt{1-x^2}}.2\sqrt{1-x^2}dx\right]$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\sin^{-1} x - \int 2dx$$

$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x + C$$

Where C is an arbitrary constant.

11.

Integrate $\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$

Solution:

Let $I=\int\frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx$

$I=\frac{-1}{2}\int\frac{-2x}{\sqrt{1-x^2}}.\cos^{-1}xdx$

Taking $\cos^{-1}$ x as first function and $\left(\frac{-2x}{\sqrt{1-x^2}}\right)$ as second function and integrating by parts, we obtain

$I=\frac{-1}{2}\left[\cos^{-1}x\int\frac{-2x}{\sqrt{1-x^2}}dx-\int\left\{\left(\frac{d}{dx}\cos^{-1}x\right)\int\frac{-2x}{\sqrt{1-x^2}}dx\right\}dx\right]$

$=\frac{-1}{2}\left[\cos^{-1}x.2\sqrt{1-x^2}-\int\frac{-1}{\sqrt{1-x^2}}.2\sqrt{1-x^2}dx\right]$

$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+\int 2dx\right]$

$=\frac{-1}{2}\left[2\sqrt{1-x^2}\cos^{-1}x+2x\right]+C$

$=-\left[\sqrt{1-x^2}\cos^{-1}x+x\right]+C$

Where C is an arbitrary constant.

12.

Integrate $x\sec^2x$

Solution:

Let $I=\int x\sec^2xdx$

Taking x as first function and $\sec^2$ x as second function and integrating by parts, we obtain

$I=x\int\sec^2xdx-\int\left\{\left\{\frac{d}{dx}x\right\}\int\sec^2xdx\right\}dx$

$=x\tan x-\int 1.\tan xdx$

$=x\tan x+\log|\cos x|+C$

Where C is an arbitrary constant.

13.

Integrate $\tan^{-1} x$

Solution:

Let $I = \int 1.\tan^{-1} x dx$

Taking $\tan^{-1}$ x as first function and 1 as second function and integrating by parts, we obtain

$$I = \tan^{-1} x \int 1 dx - \int \left\{ \left(\frac{d}{dx} \tan^{-1} x \right) \int 1.dx \right\} dx$$

$$= \tan^{-1} x.x - \int \frac{1}{1+x^2}.xdx$$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$= x\tan^{-1} x - \frac{1}{2}\log\left|1+x^2\right| + C$$

$$= x\tan^{-1} x - \frac{1}{2}\log\left(1+x^2\right) + C$$

Where C is an arbitrary constant.

14.

Integrate $x(\log x)^2 dx$

Solution:

$$I = \int x(\log x)^2 dx$$

Taking $(\log x)^2$ as first function and 1 as second function and integrating by parts, we obtain

$$I = (\log)^2 \int xdx - \int \left[\left\{ \left(\frac{d}{dx} \log x \right)^2 \right\} \int xdx \right] dx$$

$$= \frac{x^2}{2}(\log x)^2 - \left[\int 2\log x.\frac{1}{x}.\frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2}(\log x)^2 - \int x\log xdx$$

Again integrating by parts, we obtain

$$I = \frac{x^2}{2}(\log x)^2 - \left[\log x \int xdx - \int \left\{ \left(\frac{d}{dx}\log x \right) \int xdx \right\} dx \right]$$

$$= \frac{x^2}{2}(\log x)^2 - \left[\frac{x^2}{2}\log x - \int \frac{1}{x}\cdot\frac{x^2}{2} dx \right]$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{1}{2}\int xdx$$

$$= \frac{x^2}{2}(\log x)^2 - \frac{x^2}{2}\log x + \frac{x^2}{4} + C$$

Where C is an arbitrary constant.

15.

Integrate $(x^2+1)\log x$

Solution:

Let $I = \int (x^2+1)\log x dx = \int x^2 \log x dx + \int \log x dx$

Let I = I_1 + I_2 … (1)

Where, $I_1 = \int x^2 \log x dx$ and $I_2 = \int \log x dx$

$$I_1 = \int x^2 \log x dx$$

Taking log x as first function and x^2 as second function and integrating by parts, we obtain

$$I_1 = \log x \int x^2\, dx - \int \left\{ \left(\frac{d}{dx}\log x \right) \int x^2\, dx \right\} dx$$

$$= \log x . \frac{x^3}{3} - \int \frac{1}{x} . \frac{x^3}{3} dx$$

$$= \frac{x^3}{3}\log x - \frac{1}{3}\left(\int x^2 dx \right)$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 \qquad ...(2)$$

$$I_2 = \int \log x dx$$

Taking log x as first function and 1 as second function and integrating by parts, we obtain

$$I_2 = \log x \int 1.dx - \int \left\{ \left(\frac{d}{dx}\log x \right) \int 1.dx \right\}$$

$$= \log x . x - \int \frac{1}{x} . x dx$$

$$= x\log x - \int 1 dx$$

$$= x\log x - x + C_2 \qquad ...(3)$$

Using equations (2) and (3) in (1), we obtain

$$I = \frac{x^3}{3}\log x - \frac{x^3}{9} + C_1 + x\log x - x + C_2$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + (C_1 + C_2)$$

$$= \left(\frac{x^3}{3} + x \right)\log x - \frac{x^3}{9} - x + C$$

Where C is an arbitrary constant.

16.

Integrate $e^x(\sin x+\cos x)$

Solution:

Let $I=\int e^x(\sin x+\cos x)dx$

Let f (x) = sin x

$f'(x)=\cos x$

$I=\int e^x\{f(x)+f'(x)\}dx$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$\therefore I=e^x\sin x+C$

Where C is an arbitrary constant.

17.

Integrate $\frac{xe^x}{(1+x)^2}$

Solution:

Let $I=\int\frac{xe^x}{(1+x)^2}dx=\int e^x\left\{\frac{x}{(1+x)^2}\right\}dx$

$=\int e^x\left\{\frac{1+x-1}{(1+x)^2}\right\}dx$

$=\int e^x\left\{\frac{1}{1+x}-\frac{1}{(1+x)^2}\right\}dx$

Let $f(x)=\frac{1}{1+x}$ $\qquad f'(x)=\frac{-1}{(1+x)^2}$

$\Rightarrow\int\frac{xe^x}{(1+x)^2}dx=\int e^x\{f(x)+f'(x)\}dx$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$\therefore\int\frac{xe^x}{(1+x)^2}dx=\frac{e^x}{1+x}+C$

Where C is an arbitrary constant.

18.

Integrate $e^x\left(\frac{1+\sin x}{1+\cos x}\right)$

Solution:

$$e^x\left(\frac{1+\sin x}{1+\cos x}\right)$$

$$=e^x\left(\frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$$

$$=\frac{e^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}{2\cos^2\frac{x}{2}}$$

$$=\frac{1}{2}e^x.\left(\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)^2$$

$$=\frac{1}{2}e^x\left[\tan\frac{x}{2}+1\right]^2$$

$$=\frac{1}{2}e^x\left(1+\tan\frac{x}{2}\right)^2$$

$$=\frac{1}{2}e^x\left[1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$=\frac{1}{2}e^x\left[\sec^2\frac{x}{2}+2\tan\frac{x}{2}\right]$$

$$\frac{e^x(1+\sin x)dx}{(1+\cos x)}=e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right] \qquad ...(1)$$

Let $\tan\frac{x}{2}=f(x) \qquad so \qquad f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

From equation (1), we obtain

$$\int\frac{e^x(1+\sin x)}{(1+\cos x)}dx=e^x\tan\frac{x}{2}+C$$

Where C is an arbitrary constant.

19:

Integrate $e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$

Solution:

Let $I=\int e^x\left[\frac{1}{x}-\frac{1}{x^2}\right]dx$

Also, let $\frac{1}{x}=f(x)$ $\qquad f'(x)=\frac{-1}{x^2}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$\therefore I=\frac{e^x}{x}+C$

Where C is an arbitrary constant.

20:

Integrate $\frac{(x-3)e^x}{(x-1)^3}$

Solution:

$$\int e^x\left\{\frac{x-3}{(x-1)^3}\right\}dx=\int e^x\left\{\frac{x-1-2}{(x-1)^3}\right\}dx$$

$$=\int e^x\left\{\frac{1}{(x-1)^2}-\frac{2}{(x-1)^3}\right\}dx$$

Let $f(x)=\frac{1}{(x-1)^2}$ $\qquad f'(x)=\frac{-2}{(x-1)^3}$

It is known that, $\int e^x\{f(x)+f'(x)\}dx=e^x f(x)+C$

$$\therefore \int e^x\left\{\frac{(x-3)}{(x-1)^2}\right\}dx=\frac{e^x}{(x-1)^2}+C$$

Where C is an arbitrary constant.

21:

Integrate $e^{2x}\sin x$

Solution:

Let $I=\int e^{2x}\sin x dx \qquad ...(1)$

Integrating by parts, we obtain

$$I=\sin x\int e^{2x}dx-\int\left\{\left(\frac{d}{dx}\sin x\right)\int e^{2x}dx\right\}dx$$

$$\Rightarrow I=\sin x.\frac{e^{2x}}{2}-\int\cos x.\frac{e^{2x}}{2}dx$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\int e^{2x}\cos x dx$$

Again integrating by parts, we obtain

$$I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x\int e^{2x}dx - \int\left\{\left(\frac{d}{dx}\cos x\right)\int e^{2x}dx\right\}dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x.\frac{e^{2x}}{2} - \int(-\sin x)\frac{e^{2x}}{2}dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\frac{e^{2x}\cos x}{2} + \frac{1}{2}\int e^{2x}\sin x dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}I \qquad [\text{From } (1)]$$

$$\Rightarrow I + \frac{1}{4}I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}$$

$$\Rightarrow \frac{5}{4}I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}$$

$$\Rightarrow I = \frac{4}{5}\left[\frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4}\right] + C$$

$$\Rightarrow I = \frac{e^{2x}}{5}\left[2\sin x - \cos x\right] + C$$

Where C is an arbitrary constant.

22:

Integrate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Solution:

Let $x = \tan\theta \qquad dx = \sec^2\theta d\theta$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\int\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int 2\theta.\sec^2\theta d\theta = 2\int\theta.\sec^2\theta d\theta$$

Integrating by parts, we obtain

$$2\left[\theta.\int\sec^2\theta d\theta - \int\left\{\left(\frac{d}{d\theta}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]$$

$$= 2\left[\theta.\tan\theta - \int\tan\theta d\theta\right]$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right] + C$$

$$= 2\left[x\tan^{-1}x + \log\left|\frac{1}{\sqrt{1+x^2}}\right|\right] + C$$

$$= 2x\tan^{-1}x + 2\left[-\frac{1}{2}\log\left(1+x^2\right)\right] + C$$

$$= 2x\tan^{-1}x - \log\left(1+x^2\right) + C$$

Where C is an arbitrary constant.

Chose the correct answer in Exercises 23 and 24.

23.

$\int x^2 e^{x^3} dx$ equals

(A) $\frac{1}{3}e^{x^3} + C$ (B) $\frac{1}{3}e^{x^2} + C$

(C) $\frac{1}{2}e^{x^3} + C$ (D) $\frac{1}{3}e^{x^2} + C$

Solution:

Let $I = \int x^2 e^{x^3} dx$

Also, let $x^3 = t$ so $3x^2 dx = dt$

$$\Rightarrow I = \frac{1}{3}\int e^t dt$$

$$= \frac{1}{3}\left(e^t\right) + C$$

$$= \frac{1}{3}e^{x^3} + C$$

Hence, the correct Answer is A.

24.

$\int e^x \sec x\left(1+\tan x\right)dx$ equals

(A) $e^x \cos x + C$ (B) $e^x \sec x + C$

(C) $e^x \sin x + C$ (D) $e^x \tan x + C$

Solution:

$$\int e^x \sec x\left(1+\tan x\right)dx$$

Let $I = \int e^x \sec x\left(1+\tan x\right)dx = \int e^x\left(\sec x + \sec x\tan x\right)dx$

Also, let $\sec x = f(x)$ $\sec x\tan x = f'(x)$

It is known that, $\int e^x\left\{f(x) + f'(x)\right\}dx = e^x f(x) + C$

$$\therefore I = e^x \sec x + C$$

Hence, the correct Answer is B.

Exercise 7.7

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1: Integrate

$$\sqrt{4-x^2}$$

Solution:

Let $I=\int\sqrt{4-x^2}dx=\int\sqrt{(2)^2-(x)^2}dx$

It is known that,

$$\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$$

$$\therefore I=\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}+C$$

$$=\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$$

Where C is an arbitrary constant.

2:

Integrate $\sqrt{1-4x^2}$

Solution:

Let $I=\int\sqrt{1-4x^2}dx=\int\sqrt{(1)^2-(2x)^2}dx$

Let $2x=t\Rightarrow 2dx=dt$

$$\therefore I=\frac{1}{2}\int\sqrt{(1)^2-(t)^2}$$

It is known that,

$$\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$$

$$\Rightarrow I=\frac{1}{2}\left[\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]+C$$

$$=\frac{t}{4}\sqrt{1-t^2}+\frac{1}{4}\sin^{-1}t+C$$

$$=\frac{2x}{4}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$$

$$=\frac{x}{2}\sqrt{1-4x^2}+\frac{1}{4}\sin^{-1}2x+C$$

Where C is an arbitrary constant.

3: Integrate

$$\sqrt{x^2+4x+6}$$

Solution:

Let $I=\int\sqrt{x^2+4x+6}dx$

$=\int\sqrt{x^2+4x+4+2}dx$

$=\int\sqrt{\left(x^2+4x+4\right)+2}dx$

$=\int\sqrt{(x+2)^2+\left(\sqrt{2}\right)^2}\,dx$

It is known that,

$\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+C$

$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\frac{2}{2}\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

$=\frac{(x+2)}{2}\sqrt{x^2+4x+6}+\log\left|(x+2)+\sqrt{x^2+4x+6}\right|+C$

Where C is an arbitrary constant.

4:

Integrate $\sqrt{x^2+4x+1}$

Solution:

Let $I=\int\sqrt{x^2+4x+1}dx$

$=\int\sqrt{\left(x^2+4x+4\right)-3}dx$

$=\int\sqrt{(x+2)^2-\left(\sqrt{3}\right)^2}\,dx$

It is known that,

$\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|+C$

$\therefore I=\frac{(x+2)}{2}\sqrt{x^2+4x+1}-\frac{3}{2}\log\left|(x+2)+\sqrt{x^2+4x+1}\right|+C$

Where C is an arbitrary constant.

5:

Integrate $\sqrt{1-4x-x^2}$

Solution:

Let $I = \int \sqrt{1-4x-x^2}\,dx$

$= \int \sqrt{1-\left(x^2+4x+4-4\right)}dx$

$= \int \sqrt{1+4-(x+2)^2}\,dx$

$= \int \sqrt{\left(\sqrt{5}\right)^2-(x+2)^2}\,dx$

It is known that,

$\sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

$\therefore I = \frac{(x+2)}{2}\sqrt{1-4x-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right)+C$

Where C is an arbitrary constant.

6:

Integrate $\sqrt{x^2+4x-5}$

Solution:

Let $I = \int \sqrt{x^2+4x-5}dx$

$= \int \sqrt{\left(x^2+4x+4\right)-9}dx$

$= \int \sqrt{(x+2)^2-(3)^2}\,dx$

It is known that, $\int \sqrt{x^2-a^2}\ dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I = \frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\log\left|(x+2)+\sqrt{x^2+4x-5}\right|+C$

Where C is an arbitrary constant.

7:

Integrate $\sqrt{1+3x-x^2}$

Solution:

Let $I = \int \sqrt{1+3x-x^2}\,dx$

$= \int \sqrt{1-\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)}dx$

$= \int \sqrt{\left(1+\frac{9}{4}\right)-\left(x-\frac{3}{2}\right)^2}\ dx$

$$= \int \sqrt{\left(\frac{\sqrt{13}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}\, dx$$

It is known that,

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$\therefore I = \frac{x - \frac{3}{2}}{2}\sqrt{1 + 3x - x^2} + \frac{13}{4 \times 2}\sin^{-1}\left(\frac{x - \frac{3}{2}}{\frac{\sqrt{13}}{2}}\right) + C$$

$$= \frac{2x-3}{4}\sqrt{1+3x-x^2} + \frac{13}{8}\sin^{-1}\left(\frac{2x-3}{\sqrt{13}}\right) + C$$

Where C is an arbitrary constant.

8:

Integrate $\sqrt{x^2 + 3x}$

Solution:

Let $I = \int \sqrt{x^2 + 3x}\, dx$

$$= \int \sqrt{x^2 + 3x + \frac{9}{4} - \frac{9}{4}} dx$$

$$= \int \sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{3}{2}\right)^2}\, dx$$

It is known that,

$$\sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$$

$$\therefore I = \frac{\left(x + \frac{3}{2}\right)}{2}\sqrt{x^2 + 3x} - \frac{\frac{9}{4}}{2}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$$

$$= \frac{(2x+3)}{4}\sqrt{x^2 + 3x} - \frac{9}{8}\log\left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x}\right| + C$$

Where C is an arbitrary constant.

9:

Integrate $\sqrt{1 + \frac{x^2}{9}}$

Solution:

Let $I=\int\sqrt{1+\frac{x^2}{9}}dx=\frac{1}{3}\int\sqrt{9+x^2}dx=\frac{1}{3}\int\sqrt{(3)^2+x^2}dx$

It is known that,

$\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

$\therefore I=\frac{1}{3}\left[\frac{x}{2}\sqrt{x^2+9}+\frac{9}{2}\log\left|x+\sqrt{x^2+9}\right|\right]+C$

$=\frac{x}{6}\sqrt{x^2+9}+\frac{3}{2}\log\left|x+\sqrt{x^2+9}\right|+C$

Where C is an arbitrary constant.

10:

$\int\sqrt{1+x^2}$ is equal to

A. $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

B. $\frac{2}{3}\left(1+x^2\right)^{\frac{2}{3}}+C$

C. $\frac{2}{3}x\left(1+x^2\right)^{\frac{2}{3}}+C$

D. $\frac{x^3}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log\left|x+\sqrt{1+x^2}\right|+C$

Solution:

It is known that,

$\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log|x+\sqrt{x^2+a^2}|+C$

$\therefore\int\sqrt{1+x^2}dx=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left|x+\sqrt{1+x^2}\right|+C$

Hence, the correct Answer is A.

11:

$\int\sqrt{x^2-8x+7}dx$ is equal to

A. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

B. $\frac{1}{2}(x+4)\sqrt{x^2-8x+7}+9\log\left|x+4+\sqrt{x^2-8x+7}\right|+C$

C. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

D. $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|x-4+\sqrt{x^2-8x+7}\right|+C$

Solution:

Let $I=\int\sqrt{x^2-8x+7}dx$

$=\int\sqrt{\left(x^2-8x+16\right)-9}dx$

$=\int\sqrt{(x-4)^2-(3)^2}dx$

It is known that, $\sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

$\therefore I=\frac{(x-4)}{2}\sqrt{x^2-8x+7}-\frac{9}{2}\log\left|(x-4)+\int\sqrt{x^2-8x+7}\right|+C$

Hence, the correct Answer is D.

Exercise 7.8 Page: 334

1:

$\int_a^b x dx$

Solution:

It is known that,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f\left(a+(n-1)h\right)\right] \text{ where } h=\frac{b-a}{n}$$

$$Here, a=a, b=b, \text{ and } f(x)=x$$

$$\therefore \int_a^b x dx = (b-a)\lim_{n\to\infty}\frac{1}{n}\left[a+(a+h)\ldots(a+2h)\ldots a+(n-1)h\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[\left(a+\underbrace{a}_{n\,times}+a+\ldots+a\right)+\left(h+2h+3h+\ldots+(n-1)h\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left(1+2+3+\ldots+(n-1)\right)\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+h\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[na+\frac{n(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\frac{n}{n}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)h}{2}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{(n-1)(b-a)}{2n}\right]$$

$$=(b-a)\lim_{n\to\infty}\left[a+\frac{\left(1-\frac{1}{n}\right)(b-a)}{2}\right]$$

$$=(b-a)\left[a+\frac{(b-a)}{2}\right]$$

$$=(b-a)\left[\frac{2a+b-a}{2}\right]$$

$$=\frac{(b-a)(b+a)}{2}$$

$$=\frac{1}{2}\left(b^2-a^2\right)$$

2:

$$\int (x+1)\,dx$$

Solution:

Let I = $\int_0^b (x+1)\,dx$

It is known that,

$$\int_a^b f(x)\,dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f\left(a+(n-1)h\right)\right],\text{ where } h=\frac{b-a}{n}$$

$Here, a=0, b=5,$ and $f(x)=(x+1)$

$$\Rightarrow h=\frac{5-0}{n}=\frac{5}{n}$$

$$\therefore \int_0^5 (x+1)dx=(5-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f\left(\frac{5}{n}\right)+\ldots f\left((n-1)\frac{5}{n}\right)\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\left(\frac{5}{n}+1\right)+\ldots\left\{1+\left(\frac{5(n-1)}{n}\right)\right\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+1\ldots1}_{n\text{ times}}\right)+\left[\frac{5}{n}+2.\frac{5}{n}+3.\frac{5}{n}+\ldots(n-1)\frac{5}{n}\right]\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}\{1+2+3...(n-1)\}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5}{n}.\frac{(n-1)n}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{5(n-1)}{2}\right]$$

$$=5\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{5}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=5\left[1+\frac{5}{2}\right]$$

$$=5\left[\frac{7}{2}\right]$$

$$=\frac{35}{2}$$

3:

$$\int_2^3 x^2dx$$

Solution:

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f(a+2h)...+f\{a+(n-1)h\}\right],\text{ where } h=\frac{b-a}{n}$$

Here, $a=2, b=3$, and $f(x)=x^2$

$$\Rightarrow h=\frac{3-2}{n}=\frac{1}{n}$$

$$\therefore\int_2^3 x^2dx=(3-2)\lim_{n\to\infty}\frac{1}{n}\left[f(2)+f\left(2+\frac{1}{n}\right)+f\left(2+\frac{2}{n}\right)...f\left\{2+(n-1)\frac{1}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[(2)^2+\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+...\left(2+\frac{(n-1)}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[2^2+\left\{2^2+\left(\frac{1}{n}\right)^2+2.2.\frac{1}{n}\right\}+...+\left\{(2)^2+\frac{(n-1)^2}{n^2}+2.2.\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\left(2^2+\underset{n\ times}{...}+2^2\right)+\left\{\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+...+\left(\frac{n-1}{n}\right)^2\right\}+2.2.\left\{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+...+\frac{(n-1)}{n}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{1^2+2^2+3^2...+(n-1)^2\right\}+\frac{4}{n}\{1+2+...+(n-1)\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{1}{n^2}\left\{\frac{n(n-1)(2n-1)}{6}\right\}+\frac{4}{n}\left\{\frac{n(n-1)}{2}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[4n+\frac{n\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{6}+\frac{4n-4}{2}\right]$$

$$=\lim_{n\to\infty}\left[4+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+2-\frac{2}{n}\right]$$

$$=4+\frac{2}{6}+2$$

$$=\frac{19}{3}$$

4:

$$\int_1^4\left(x^2-x\right)dx$$

Solution:

Let $I=\int_1^4\left(x^2-x\right)dx$

$$=\int_1^4 x^2dx-\int_1^4 xdx$$

Let $I=I_1-I_2$, where $I_1=\int_1^4 x^2dx$ and $I_2=\int_1^4 xdx$ ...(1)

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+f\left(a+(n-1)h\right)\right],\text{ where } h=\frac{b-a}{n}$$

For, $I_1=\int_1^4 x^2dx,$

a = 1, b = 4, and f (x) = x^2

$$\therefore h=\frac{4-1}{n}=\frac{3}{n}$$

$$I_1=\int_1^4 x^2dx=(4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots+f\left(1+(n-1)h\right)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left(1+\frac{3}{n}\right)^2+\left(1+2.\frac{3}{n}\right)^2+\ldots\left(1+\frac{(n-1)3}{n}\right)^2\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left\{1^2+\left(\frac{3}{n}\right)^2+2.\frac{3}{n}\right\}+\ldots+\left\{1^2+\left(\frac{(n-1)3}{n}\right)^2+\frac{2.(n-1).3}{2}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[\left(1^2+\underset{n\,times}{\ldots}+1^2\right)+\left(\frac{3}{n}\right)^2\left\{1^2+2^2+\ldots+(n-1)^2\right\}+2.\frac{3}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$

$$=3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$

$$=3[1+3+3]$$

$$=3[7]$$

$$I_1=21 \qquad \ldots(2)$$

For $I_2=\int_1^4 xdx,$

$a=1, b=4,$ and $f(x)=x$

$$\Rightarrow h=\frac{4-1}{n}=\frac{3}{n}$$

$$\therefore I_2=(4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots f(a+(n-1)h)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+(1+h)+\ldots+(1+(n-1)h)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+\ldots+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1+1+\ldots+1}_{n\text{ times}}\right)+\frac{3}{n}(1+2+\ldots+(n-1))\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{3}{2}\left(1-\frac{1}{n}\right)\right]$$

$$=3\left[1+\frac{3}{2}\right]$$

$$=3\left[\frac{5}{2}\right]$$

$$I_2=\frac{15}{2} \qquad \ldots(3)$$

From equations (2) and (3), we obtain

$$I=I_1-I_2=21-\frac{15}{2}=\frac{27}{2}$$

5:

$$\int_{-1}^{1}e^x dx$$

Solution:

Let $I=\int_{-1}^{1}e^{x}dx \qquad ...(1)$

It is known that,

$Here, a=-1, b=1, \text{ and } f(x)=e^{x}$

$\therefore h=\frac{1+1}{n}=\frac{2}{n}$

$\therefore I=(1+1)\lim_{n\to\infty}\frac{1}{n}\left[f(-1)+f\left(-1+\frac{2}{n}\right)+f\left(-1+2.\frac{2}{n}\right)+...+f\left(-1+\frac{(n-1)2}{n}\right)\right]$

$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}+e^{\left(-1+\frac{2}{n}\right)}+e^{\left(-1+2.\frac{2}{n}\right)}+...e^{\left(-1+(n-1)\frac{2}{n}\right)}\right]$

$=2\lim_{n\to\infty}\frac{1}{n}\left[e^{-1}\left\{1+e^{\frac{2}{n}}+e^{\frac{4}{n}}+e^{\frac{6}{n}}+e^{(n-1)\frac{2}{n}}\right\}\right]$

$=2\lim_{n\to\infty}\frac{e^{-1}}{n}\left[\frac{e^{\frac{2n}{n}-1}}{e^{\frac{2}{n}-1}}\right]$

$=e^{-1}\times 2\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^{2}-1}{e^{\frac{2}{n}-1}}\right]$

$=\frac{e^{-1}\times 2\left(e^{2}-1\right)}{\lim_{\frac{2}{n}\to 0}\left(\frac{e^{\frac{2}{n}-1}}{\frac{2}{n}}\right)\times 2}$

$=e^{-1}\left[\frac{2\left(e^{2}-1\right)}{2}\right] \qquad \left[\lim_{h\to 0}\left(\frac{e^{h}-1}{h}\right)=1\right]$

$=\frac{e^{2}-1}{e}$

$=\left(e-\frac{1}{e}\right)$

6:

$\int_{0}^{4}\left(x+e^{2x}\right)dx$

Solution:

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\cdots+f\left(a+(n-1)h\right)\right] \text{ where } h=\frac{b-a}{n}$$

$$Here, a=0, b=4, \text{ and } f(x)=x+e^{2x}$$

$$\therefore h=\frac{4-0}{n}=\frac{4}{n}$$

$$\Rightarrow \int_0^4 \left(x+e^{2x}\right)dx=(4-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(h)+f(2h)+\ldots+f\left((n-1)h\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left(0+e^{0}\right)+\left(h+e^{2h}\right)+\left(2h+e^{2.2h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[1+\left(h+e^{2h}\right)+\left(2h+e^{4h}\right)+\ldots+\left\{(n-1)h+e^{2(n-1)h}\right\}\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\left\{h+2h+3h+\ldots+(n-1)h\right\}+\left(1+e^{2h}+e^{4h}+\ldots+e^{2(n-1)h}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[h\left\{1+2+\ldots(n-1)\right\}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{\left(h(n-1)n\right)}{2}+\left(\frac{e^{2hn}-1}{e^{2h}-1}\right)\right]$$

$$=4\lim_{n\to\infty}\frac{1}{n}\left[\frac{4}{n}.\frac{(n-1)n}{2}+\left(\frac{e^{8}-1}{e^{\frac{8}{n}}-1}\right)\right]$$

$$=4(2)+4\lim_{n\to\infty}\frac{\left(e^{8}-1\right)}{\left(\frac{e^{\frac{8}{n}}-1}{\frac{8}{n}}\right)8}$$

$$=8+\frac{4.\left(e^{8}-1\right)}{8} \qquad \left(\lim_{x\to 0}\frac{e^{x}-1}{x}=1\right)$$

$$=8+\frac{e^{8}-1}{2}$$

$$=\frac{15+e^{8}}{2}$$

Exercise 7.9

1:

$\int_{-1}^{1}(x+1)dx$

Solution:

Let $I=\int_{-1}^{1}(x+1)dx$

$\int(x+1)dx=\frac{x^2}{2}+x=F(x)$

By second fundamental theorem of calculus, we obtain

$I=F(1)-F(-1)$

$=\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)$

$=\frac{1}{2}+1-\frac{1}{2}+1$

$=2$

2:

$\int_{2}^{3}\frac{1}{x}dx$

Solution:

Let $I=\int_{2}^{3}\frac{1}{x}dx$

$\int\frac{1}{x}dx=\log|x|=F(x)$

By second fundamental theorem of calculus, we obtain

$I=F(3)-F(2)$

$=\log|3|-\log|2|=\log\frac{3}{2}$

3:

$\int_{1}^{2}(4x^3-5x^2+6x+9)dx$

Solution:

Let $I=\int_{1}^{2}(4x^3-5x^2+6x+9)dx$

$\int(4x^3-5x^2+6x+9)dx=4\left(\frac{x^4}{4}\right)-5\left(\frac{x^3}{3}\right)+6\left(\frac{x^2}{2}\right)+9(x)$

$=x^4-\frac{5x^3}{3}+3x^2+9x=F(x)$

By second fundamental theorem of calculus, we obtain

I = F (2) – F (1)

$$I=\left\{2^4-\frac{5.(2)^3}{3}+3(2)^2+9(2)\right\}-\left\{(1)^4-\frac{5(1)^3}{3}+3(1)^2+9(1)\right\}$$

$$=\left(16-\frac{40}{3}+12+18\right)-\left(1-\frac{5}{3}+3+9\right)$$

$$=16-\frac{40}{3}+12+18-1+\frac{5}{3}-3-9$$

$$=33-\frac{35}{3}$$

$$=\frac{99-35}{3}$$

$$=\frac{64}{3}$$

4:

$$\int_0^{\frac{x}{4}}\sin 2x dx$$

Solution:

Let $I=\int_0^{\frac{x}{4}}\sin 2x dx$

$$\int \sin 2x dx=\left(\frac{-\cos 2x}{2}\right)=F(x)$$

By second fundamental theorem of calculus, we obtain

$$I=F\left(\frac{\pi}{4}\right)-F(0)$$

$$=-\frac{1}{2}\left[\cos 2\left(\frac{\pi}{4}\right)-\cos 0\right]$$

$$=-\frac{1}{2}[0-1]$$

$$=\frac{1}{2}$$

5:

$$\int_0^{\frac{\pi}{2}}\cos 2x dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\cos 2x dx$

$$\int \cos 2x dx = \left(\frac{\sin 2x}{2}\right) = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{2}\right) - F(0)$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{1}{2}[\sin \pi - \sin 0]$$

$$= \frac{1}{2}[0 - 0] = 0$$

6:

$$\int_4^5 e^x dx$$

Solution:

Let $I = \int_4^5 e^x dx$

$$\int e^x dx = e^x = F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (5) – F (4)

$$= e^5 - e^4$$

$$= e^4(e - 1)$$

7:

$$\int_0^{\frac{\pi}{4}} \tan x dx$$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} \tan x dx$

$$\int \tan x dx = -\log|\cos x| = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= -\log\left|\cos\frac{\pi}{4}\right| + \log|\cos 0|$$

$$= -\log\left|\frac{1}{\sqrt{2}}\right| + \log|1|$$

$= -\log(2)^{-\frac{1}{2}}$

$= \frac{1}{2}\log 2$

8:

$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \operatorname{cosec} x\, dx$

$\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F\left(\frac{\pi}{4}\right) - F\left(\frac{\pi}{6}\right)$

$= \log\left|\operatorname{cosec}\frac{\pi}{4} - \cot\frac{\pi}{4}\right| - \log\left|\operatorname{cosec}\frac{\pi}{6} - \cot\frac{\pi}{6}\right|$

$= \log\left|\sqrt{2} - 1\right| - \log\left|2 - \sqrt{3}\right|$

$= \log\left(\frac{\sqrt{2}-1}{2-\sqrt{3}}\right)$

9:

$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Solution:

Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x = F(x)$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$= \sin^{-1}(1) - \sin^{-1}(0)$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

10:

$$\int_0^1 \frac{dx}{1+x^2}$$

Solution:

Let $I = \int_0^1 \frac{dx}{1+x^2}$

$$\int \frac{dx}{1+x^2} = \tan^{-1} = F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4}$$

11:

$$\int_2^3 \frac{dx}{x^2-1}$$

Solution 11:

Let $I = \int_2^3 \frac{dx}{x^2-1}$

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| = F(x)$$

By second fundamental theorem of calculus, we obtain

I =F (3) -F (2)

$$= \frac{1}{2}\left[\log\left|\frac{3-1}{3+1}\right| - \log\left|\frac{2-1}{2+1}\right|\right]$$

$$= \frac{1}{2}\left[\log\left|\frac{2}{4}\right| - \log\left|\frac{1}{3}\right|\right]$$

$$= \frac{1}{2}\left[\log\frac{1}{2} - \log\frac{1}{3}\right]$$

$$= \frac{1}{2}\left[\log\frac{3}{2}\right]$$

12:

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\cos^2 x dx$

$$\int \cos^2 x dx=\int\left(\frac{1+\cos 2x}{2}\right)dx=\frac{x}{2}+\frac{\sin 2x}{4}=\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)=F(x)$$

By second fundamental theorem of calculus, we obtain

$$I=\left[F\left(\frac{\pi}{2}\right)-F(0)\right]$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-\left(0+\frac{\sin 0}{2}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}+0-0-0\right]$$

$$=\frac{\pi}{4}$$

13:

$$\int_2^3 \frac{xdx}{x^2+1}$$

Solution:

Let $I=\int_2^3 \frac{x}{x^2+1}dx$

$$\int \frac{x}{x^2+1}dx=\frac{1}{2}\int\frac{2x}{x^2+1}dx=\frac{1}{2}\log\left(1+x^2\right)=F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (3) – F (2)

$$=\frac{1}{2}\left[\log\left(1+(3)^2\right)-\log\left(1+(2)^2\right)\right]$$

$$=\frac{1}{2}\left[\log(10)-\log(5)\right]$$

$$=\frac{1}{2}\log\left(\frac{10}{5}\right)=\frac{1}{2}\log 2$$

14:

$$\int_0^1 \frac{2x+3}{5x^2+1}dx$$

Solution:

Let $I=\int_0^1 \frac{2x+3}{5x^2+1}dx$

$$\int \frac{2x+3}{5x^2+1}dx = \frac{1}{5}\int \frac{5(2x+3)}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x+15}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5x^2+1}dx$$

$$= \frac{1}{5}\int \frac{10x}{5x^2+1}dx + 3\int \frac{1}{5\left(x^2+\frac{1}{5}\right)}dx$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{5}.\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x}{\frac{1}{\sqrt{5}}}$$

$$= \frac{1}{5}\log\left(5x^2+1\right) + \frac{3}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\right)x$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$= \left\{\frac{1}{5}\log(5+1) + \frac{3}{\sqrt{5}}\tan^{-1}(\sqrt{5})\right\} - \left\{\frac{1}{5}\log(1) + \frac{3}{\sqrt{5}}\tan^{-1}(0)\right\}$$

$$= \frac{1}{5}\log 6 + \frac{3}{\sqrt{5}}\tan^{-1}\sqrt{5}$$

15:

$$\int_0^1 xe^{x^2}dx$$

Solution:

Let $I = \int_0^1 xe^{x^2}dx$

Put $x^2 = t \Rightarrow 2xdx = dt$

As $x \to 0, t \to 0$ *and as* $x \to 1, t \to 1,$

$$\therefore I = \frac{1}{2}\int_0^1 e^t dt$$

$$\frac{1}{2}\int e^t dt = \frac{1}{2}e^t = F(t)$$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$= \frac{1}{2}e - \frac{1}{2}e^0$$

$$= \frac{1}{2}(e-1)$$

16:

$$\int_0^1 \frac{5x^2}{x^2+4x+3}$$

Solution:

Let $I = \int_1^2 \frac{5x^2}{x^2+4x+3}$ dx

Dividing $5x^2$ by x^2+4x+3, we obtain

$$I = \int_1^2 \left\{5 - \frac{20x+15}{x^2+4x+3}\right\}dx$$

$$= \int_1^2 5dx - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$$

$$= [5x]_1^2 - \int_1^2 \frac{20x+15}{x^2+4x+3}dx$$

$I = 5 - I_1$, where $I_1 = \int_1^2 \frac{20x+15}{x^2+4x+3}dx$ … (1)

Consider

Let $20x+15 = A\frac{d}{dx}\left(x^2+4x+3\right)+B$

$$= 2Ax + (4A+B)$$

Equating the coefficients of x and constant term, we obtain

A = 10 and B = −25

Let $x^2+4x+3 = t$

$\Rightarrow (2x+4)dx = dt$

$$\Rightarrow I_1 = 10\int \frac{dt}{t} - 25\int \frac{dx}{(x+2)^2 - 1^2}$$

$$= 10\log t - 25\left[\frac{1}{2}\log\left(\frac{x+2-1}{x+2-1}\right)\right]$$

$$= \left[10\log\left(x^2+4x+3\right)\right]_1^2 - 25\left[\frac{1}{2}\log\left(\frac{x+1}{x+3}\right)\right]_1^2$$

$$= [10\log 15 - 10\log 8] - 25\left[\frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\log\frac{2}{4}\right]$$

$$= [10\log(5\times3) - 10\log(4\times2)] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$$

$$= [10\log 5 + 10\log 3 - 10\log 4 - 10\log 2] - \frac{25}{2}[\log 3 - \log 5 - \log 2 + \log 4]$$

$$= \left[10 + \frac{25}{2}\right]\log 5 + \left[-10 - \frac{25}{2}\right]\log 4 + \left[10 - \frac{25}{2}\right]\log 3 + \left[-10 + \frac{25}{2}\right]\log 2$$

$$= \frac{45}{2}\log 5 = \frac{45}{2}\log 4 - \frac{5}{2}\log 3 + \frac{5}{2}\log 2$$

$$= \frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}$$

Substituting the value of I_1 in (1), we obtain

$$I = 5 - \left[\frac{45}{2}\log\frac{5}{4} - \frac{5}{2}\log\frac{3}{2}\right]$$

$$= 5 - \frac{5}{2}\left[9\log\frac{5}{4} - \log\frac{3}{2}\right]$$

17:

$$\int_0^{\frac{\pi}{4}}\left(2\sec^2 x + x^3 + 2\right)dx$$

Solution:

Let $I = \int_0^{\frac{\pi}{4}}\left(2\sec^2 x + x^3 + 2\right)dx$

$$\int\left(2\sec^2 x + x^3 + 2\right)dx = 2\tan x + \frac{x^4}{4} + 2x = F(x)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{4}\right) - F(0)$$

$$= \left\{\left(2\tan\frac{\pi}{4} + \frac{1}{4}\left(\frac{\pi}{4}\right)^4 + 2\left(\frac{\pi}{4}\right)\right) - (2\tan 0 + 0 + 0)\right\}$$

$$= 2\tan\frac{\pi}{4} + \frac{\pi^4}{4^5} + \frac{\pi}{2}$$

$$= 2 + \frac{\pi}{2} + \frac{\pi^4}{1024}$$

18:

$$\int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$$

Solution:

Let $I = \int_0^{\pi}\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)dx$

$$= -\int_0^{\pi}\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)dx$$

$$= -\int_0^{\pi}\cos x\,dx$$

$-\int_0^{\pi} \cos x\,dx = -\sin x = F(x)$

By second fundamental theorem of calculus, we obtain

$I = F(\pi) - F(0)$

$= -\sin \pi + \sin 0$

$= 0$

19:

$\int_0^2 \frac{6x+3}{x^2+4}dx$

Solution:

Let $I = \int_0^2 \frac{6x+3}{x^2+4}dx$

$\int \frac{6x+3}{x^2+4}dx = 3\int \frac{2x+1}{x^2+4}dx$

$= 3\int \frac{2x}{x^2+4}dx + 3\int \frac{1}{x^2+4}dx$

$= 3\log(x^2+4) + \frac{3}{2}\tan^{-1}\frac{x}{2} = F(x)$

By second fundamental theorem of calculus, we obtain

I = F(2) – F(0)

$= \left\{3\log(2^2+4) + \frac{3}{2}\tan^{-1}\left(\frac{2}{2}\right)\right\} - \left\{3\log(0+4) + \frac{3}{2}\tan^{-1}\left(\frac{0}{2}\right)\right\}$

$= 3\log 8 + \frac{3}{2}\tan^{-1}1 - 3\log 4 - \frac{3}{2}\tan^{-1}0$

$= 3\log 8 + \frac{3}{2}\left(\frac{\pi}{4}\right) - 3\log 4 - 0$

$= 3\log\left(\frac{8}{4}\right) + \frac{3\pi}{8}$

$= 3\log 2 + \frac{3\pi}{8}$

20:

$\int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$

Solution:

Let $I = \int_0^1 \left(xe^x + \sin\frac{\pi x}{4}\right)dx$

$$\int_0^1\left(xe^x+\sin\frac{\pi x}{4}\right)dx = x\int e^x dx-\int\left\{\left(\frac{d}{dx}x\right)\int e^x dx\right\}dx+\left\{\frac{-\cos\frac{\pi x}{4}}{\frac{\pi}{4}}\right\}$$

$$= xe^x-\int e^x dx-\frac{4}{\pi}\cos\pi\frac{x}{4}$$

$$= xe^x-e^x-\frac{4}{\pi}\cos\pi\frac{x}{4}$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

I = F (1) – F (0)

$$=\left(1.e^1-e^1-\frac{4}{\pi}\cos\frac{\pi}{4}\right)-\left(0.e^0-e^0-\frac{4}{\pi}\cos 0\right)$$

$$=e-e-\frac{4}{\pi}\left(\frac{1}{\sqrt{2}}\right)+1+\frac{4}{\pi}$$

$$=1+\frac{4}{\pi}-\frac{2\sqrt{2}}{\pi}$$

Chose the correct answer in Exercises 21 and 22.

21:

$$\int_1^{\sqrt{3}}\frac{dx}{1+x^2}$$

A. $\frac{\pi}{3}$

B. $\frac{2\pi}{3}$

C. $\frac{\pi}{6}$

D. $\frac{\pi}{12}$ equals

Solution:

$$\int\frac{dx}{1+x^2}=\tan^{-1}x=F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_1^{\sqrt{3}}\frac{dx}{1+x^2}=F\left(\sqrt{3}\right)-F(1)$$

$$=\tan^{-1}\sqrt{3}-\tan^{-1}1$$

$$=\frac{\pi}{3}-\frac{\pi}{4}$$

$$=\frac{\pi}{12}$$

Hence, the correct Answer is D.

22:

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2}$$

A. $\frac{\pi}{6}$

B. $\frac{\pi}{12}$

C. $\frac{\pi}{24}$

D. $\frac{\pi}{4}$ equals

Solution:

$$\int \frac{dx}{4+9x^2} = \int \frac{dx}{(2)^2+(3x)^2}$$

Put 3x=t $\Rightarrow$ 3dx=dt

$$\therefore \int \frac{dx}{(2)^2+(3x)^2} = \frac{1}{3}\int \frac{dt}{(2)^2+t^2}$$

$$= \frac{1}{3}\left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)$$

$$= F(x)$$

By second fundamental theorem of calculus, we obtain

$$\int_0^{\frac{2}{3}} \frac{dx}{4+9x^2} = F\left(\frac{2}{3}\right) - F(0)$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3}{2}.\frac{2}{3}\right) - \frac{1}{6}\tan^{-1}0$$

$$= \frac{1}{6}\tan^{-1}1 - 0$$

$$= \frac{1}{6}\times\frac{\pi}{4}$$

$$= \frac{\pi}{24}$$

Hence, the correct Answer is C.

Exercise 7.ρπ

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1:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Solution:

$$\int_0^1 \frac{x}{x^2+1}dx$$

Let $x^2+1=t \Rightarrow 2xdx=dt$

When x = 0, t = 1 and when x = 1, t = 2

$$\therefore \int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\left[\log|t|\right]_1^2$$

$$= \frac{1}{2}[\log 2 - \log 1]$$

$$= \frac{1}{2}\log 2$$

2:

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$$

Solution:

Let

$$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi d\phi$$

Also, let $\sin\phi = t \Rightarrow \cos\phi d\phi = dt$

When $\phi = 0, t = 0$ and when $\phi = \frac{\pi}{2}, t = 1$

$$\therefore I = \int_0^1 \sqrt{t}(1-t^2)^2 dt$$

$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$

$$= \int_0^1 \left[t^{\frac{1}{2}} + t^{\frac{9}{2}} - 2t^{\frac{5}{2}}\right]dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + \frac{t^{\frac{11}{2}}}{\frac{11}{2}} - \frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7}$$

$$= \frac{154+42-132}{231}$$

$$=\frac{64}{231}$$

3:

$$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Solution:

Let $I=\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let x = tanθ ⇒ dx = $\sec^2$ θ dθ

When x = 0, θ = 0 and when x = 1, $\theta=\frac{\pi}{4}$

$$I=\int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta d\theta$$

$$=\int_0^{\frac{\pi}{4}} \sin^{-1}(\sin 2\theta)\sec^2\theta d\theta$$

$$=\int_0^{\frac{\pi}{4}} 2\theta.\sec^2\theta d\theta$$

$$=2\int_0^{\frac{\pi}{4}} \theta.\sec^2\theta d\theta$$

Taking θ as first function and $\sec^2$ θ as second function and integrating by parts, we obtain

$$I=2\left[\theta\int\sec^2\theta d\theta-\int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\theta\tan\theta-\int\tan\theta d\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\theta\tan\theta+\log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\frac{\pi}{4}\tan\frac{\pi}{4}+\log\left|\cos\frac{\pi}{4}\right|-\log|\cos 0|\right]$$

$$=2\left[\frac{\pi}{4}+\log\left(\frac{1}{\sqrt{2}}\right)-\log 1\right]$$

$$=2\left[\frac{\pi}{4}-\frac{1}{2}\log 2\right]$$

$$=\frac{\pi}{2}-\log 2$$

4:

$\int_0^2 x\sqrt{x+2}\left(\text{Put x+2=t}^2\right)$

Solution:

$\int_0^2 x\sqrt{x+2}dx$

Let x + 2 = t^2 ⇒ dx = 2t dt

When x = 0, $t=\sqrt{2}$ and when x = 2, t = 2

$$\therefore \int_0^2 x\sqrt{x+2}dx = \int_{\sqrt{2}}^2 (t^2-2)\sqrt{t^2}\,2t\,dt$$
$$= 2\int_{\sqrt{2}}^2 (t^2-2)t^2 dt = 2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2 = \frac{16(2+\sqrt{2})}{15}$$
$$= \frac{16\sqrt{2}(\sqrt{2}+1)}{15}$$

5:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Solution:

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx$

Let cos x = t ⇒ −sinx dx = dt

When x = 0, t = 1 and when $x=\frac{\pi}{2}, t=0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}dx = -\int_1^0 \frac{dt}{1+t^2}$$
$$= -\left[\tan^{-1} t\right]_1^0$$
$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$
$$= -\left[-\frac{\pi}{4}\right]$$
$$= \frac{\pi}{4}$$

6:

$\int_0^2 \frac{dx}{x+4-x^2}$

Solution:

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{-\left(x^2-x-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-4\right)}$$

$$= \int_0^2 \frac{dx}{-\left[\left(x-\frac{1}{2}\right)^2-\frac{17}{4}\right]}$$

$$= \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}$$

Let x-$\frac{1}{2}=t$ so dx=dt

when x=0, t=-$\frac{1}{2}$ and when x=2, t=$\frac{3}{2}$

$$\therefore \int_0^2 \frac{dx}{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} = \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{dt}{\left(\frac{\sqrt{17}}{2}\right)^2-t^2}$$

$$= \left[\frac{1}{2\left(\frac{\sqrt{17}}{2}\right)} \log \frac{\frac{\sqrt{17}}{2}+t}{\frac{\sqrt{17}}{2}-t}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\frac{\sqrt{17}}{2}+\frac{3}{2}}{\frac{\sqrt{17}}{2}-\frac{3}{2}} - \frac{\log \frac{\sqrt{17}}{2}-\frac{1}{2}}{\log \frac{\sqrt{17}}{2}+\frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{17}}\left[\log \frac{\sqrt{17}+3}{\sqrt{17}-3} - \log \frac{\sqrt{17}-1}{\sqrt{17}+1}\right]$$

$$= \frac{1}{\sqrt{17}} \log \frac{\sqrt{17}+3}{\sqrt{17}-3} \times \frac{\sqrt{17}+1}{\sqrt{17}-1}$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{17+3+4\sqrt{17}}{17+3-4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{20+4\sqrt{17}}{20-4\sqrt{17}}\right]$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{5+\sqrt{17}}{5-\sqrt{17}}\right)$$

$$= \frac{1}{\sqrt{17}} \log\left[\frac{(5+\sqrt{17})(5+\sqrt{17})}{25-17}\right]$$

$$= \frac{1}{\sqrt{17}} \log\left[\frac{25+17+10\sqrt{17}}{8}\right]$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{42+10\sqrt{17}}{8}\right)$$

$$= \frac{1}{\sqrt{17}} \log\left(\frac{21+5\sqrt{17}}{4}\right)$$

7:

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5}$$

Solution:

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \int_{-1}^{1} \frac{dx}{(x^2+2x+1)+4} = \int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2}$$

Let x + 1 = t ⇒ dx = dt

When x = −1, t = 0 and when x = 1, t = 2

$$\int_{-1}^{1} \frac{dx}{(x+1)^2+(2)^2} = \int_{0}^{2} \frac{dx}{t^2+2^2}$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{t}{2}\right]_0^2$$

$$= \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

8:

$$\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$$

Solution:

$$\int_{1}^{2} \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx$$

Let 2x = t ⇒ 2dx = dt

When x = 1, t = 2 and when x = 2, t = 4

$$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x^2}\right) e^{2x} dx = \frac{1}{2}\int_2^4 \left(\frac{2}{t} - \frac{2}{t^2}\right) e^t dt$$

Let $\frac{1}{t} = f(t)$

Then, $f'(t) = -\frac{1}{t^2}$

$$= \int_2^4 \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t dt = \int_2^4 (f(t) + f'(t)) e^t dt$$

$$= [e^t f(t)]_2^4$$

$$= \left[e^t \cdot \frac{1}{t}\right]_2^4$$

$$= \left[\frac{e^t}{t}\right]_2^4$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^2\left(e^2 - 2\right)}{4}$$

Chose the correct answer in Exercises 21 and 22.

9:

The value of the integral $\int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$ is

A. 6
B. 0
C. 3
D. 4

Solution:

Let $I = \int_{\frac{1}{3}}^{1} \frac{\left(x - x^3\right)^{\frac{1}{3}}}{x^4} dx$

Also, let x = sin $\theta \Rightarrow$ dx = cos θ d θ

When $x = \frac{1}{3}, \theta = \sin^{-1}\left(\frac{1}{3}\right)$ and when x=1, $\theta = \frac{\pi}{2}$

$$\Rightarrow I = \int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}} \frac{(\sin\theta - \sin^3\theta)^{\frac{1}{3}}}{\sin^4\theta} \cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}\left(1-\sin^2\theta\right)^{\frac{1}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^4\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\sin\theta)^{\frac{1}{3}}(\cos\theta)^{\frac{2}{3}}}{\sin^2\theta\sin^2\theta}\cos\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}\frac{(\cos\theta)^{\frac{5}{3}}}{(\sin\theta)^{\frac{5}{3}}}\cos ec^2\theta d\theta$$

$$=\int_{\sin^{-1}\left(\frac{1}{3}\right)}^{\frac{\pi}{2}}(\cot\theta)^{\frac{5}{3}}\cos ec^2\theta d\theta$$

Let $\cot\theta = t \Rightarrow -\text{cosec}^2\theta\, d\theta = dt$

When $\theta=\sin^{-1}\left(\frac{1}{3}\right), t=2\sqrt{2}$ and when $\theta=\frac{\pi}{2}, t=0$

$$\therefore I=-\int_{2\sqrt{2}}^{0}(t)^{\frac{5}{3}}dt$$

$$=-\left[\frac{3}{8}(t)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=-\frac{3}{8}\left[-\left(2\sqrt{2}\right)^{\frac{8}{3}}\right]_{2\sqrt{2}}^{0}$$

$$=\frac{3}{8}\left[\left(\sqrt{8}\right)^{\frac{8}{3}}\right]$$

$$=\frac{3}{8}\left[(8)^{\frac{4}{3}}\right]$$

$$=\frac{3}{8}[16]$$

$$=3\times 2$$

$$=6$$

Hence, the correct Answer is A.

10:

If $f(x)=\int_0^x t\sin t dt$, then $f'(x)$ is

A. cos x + x sin x

B. x sin x

C. x cos x

D. sin x + x cos x

Solution:

$f(x)=\int_0^x t\sin t dt$

Integrating by parts, we obtain

$f(x)=t\int_0^x \sin t dt-\int_0^x\left\{\left(\frac{d}{dt}t\right)\int \sin t dt\right\}dt$

$=\left[t(-\cos t)\right]_0^x-\int_0^x(-\cot t)dt$

$=\left[-t\cos t+\sin t\right]_0^x$

$=-x\cos x+\sin x$

$\Rightarrow f'(x)=-\left[\left\{x(-\sin x)\right\}+\cos x\right]+\cos x$

$=x\sin x-\cos x+\cos x$

$=x\sin x$

Hence, the correct Answer is B.

Exercise 7.ρ1

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1:

$$\int_0^{\frac{\pi}{2}} \cos^2 x dx$$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \cos^2 x dx \qquad ...(1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \cos^2\left(\frac{\pi}{2} - x\right)dx \qquad \left(\int_0^0 f(x)dx = \int_0^0 f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \sin^2 x dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^2 x\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1.dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

2:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

Solution:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

Let $I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \qquad ...(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \qquad .... (2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

3:

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} xdx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{2}} \frac{\sin^{\frac{3}{2}} xdx}{\sin^{\frac{3}{2}} x+\cos^{\frac{3}{2}} x}dx \qquad ...(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)+\cos^{\frac{3}{2}}\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^{\frac{3}{2}}+\cos^{\frac{3}{2}}x}{\sin^{\frac{3}{2}}x+\cos^{\frac{3}{2}}x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I=\frac{\pi}{2}$$

$$\Rightarrow I=\frac{\pi}{4}$$

4:

$$\int_0^{\frac{\pi}{2}}\frac{\cos^5 xdx}{\sin^5 x+\cos^5 x}dx$$

Solution:

$$\text{Let } I=\int_0^{\frac{\pi}{2}}\frac{\cos^5 xdx}{\sin^5 x+\cos^5 x}dx \qquad ...(1)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^5\left(\frac{\pi}{2}-x\right)}{\sin^5\left(\frac{\pi}{2}-x\right)+\cos^5\left(\frac{\pi}{2}-x\right)}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\frac{\pi}{2}}\frac{\sin^5 x+\cos^5 x}{\sin^5 x+\cos^5 x}dx$$

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}1.dx$$

$$\Rightarrow 2I=[x]_0^{\frac{\pi}{2}}$$

$\Rightarrow 2I = \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$

5:

$\int_{-5}^{5} |x+2| dx$

Solution:

Let $I = \int_{-5}^{5} |x+2| dx$

It can be seen that $(x + 2) \le 0$ on $[-5, -2]$ and $(x + 2) \ge 0$ on $[-2, 5]$.

$\therefore I = \int_{-5}^{-2} -(x+2)\,dx + \int_{-2}^{5} (x+2)\,dx$

$I = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$

$= -\left[\frac{(-2)^2}{2} + 2(-2) - \frac{(-5)^2}{2} - 2(-5)\right] + \left[\frac{(5)^2}{2} + 2(5) - \frac{(-2)^2}{2} - 2(-2)\right]$

$= -\left[2 - 4 - \frac{25}{2} + 10\right] + \left[\frac{25}{2} + 10 - 2 + 4\right]$

$= -2 + 4 + \frac{25}{2} - 10 + \frac{25}{2} + 10 - 2 + 4$

$= 29$

6:

$\int_{2}^{8} |x-5| dx$

Solution:

Let $I = \int_{2}^{8} |x-5| dx$

It can be seen that $(x - 5) \le 0$ on $[2, 5]$ and $(x - 5) \ge 0$ on $[5, 8]$.

$I = \int_{2}^{5} -(x-5)\,dx + \int_{2}^{8} (x-5)\,dx \qquad \left(\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)\right)$

$= -\left[\frac{x^2}{2} - 5x\right]_{2}^{5} + \left[\frac{x^2}{2} - 5x\right]_{5}^{8}$

$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$

$= 9$

7:

$\int_0^1 x(1-x)^n\, dx$

Solution:

Let $I=\int_0^1 x(1-x)^n\, dx$

$\therefore I=\int_0^1 (1-x)(1-(1-x))^n\, dx$

$=\int_0^1 (1-x)(x)^n\, dx$

$=\int_0^1 \left(x^n-x^{n+1}\right)dx$

$=\left[\frac{x^{n+1}}{n+1}-\frac{x^{n+2}}{n+2}\right]_0^1 \qquad \left(\int_1^a f(x)dx=\int_0^a f(a-x)dx\right)$

$=\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$

$=\frac{(n+2)-(n+1)}{(n+1)(n+2)}$

$=\frac{1}{(n+1)(n+2)}$

8:

$\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

Solution:

Let $I=\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx \qquad ...(1)$

$\therefore I=\int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$

$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right\}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan x}{1+\tan x}\right\}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log\frac{2}{(1+\tan x)}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}} \log 2dx-\int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$

$$\Rightarrow I=\int_0^{\frac{\pi}{4}}\log 2dx-I \qquad [From(1)]$$

$$\Rightarrow 2I=[x\log 2]_0^{\frac{\pi}{4}}$$

$$\Rightarrow 2I=\frac{\pi}{4}\log 2$$

$$\Rightarrow I=\frac{\pi}{8}\log 2$$

9:

$$\int_0^2 x\sqrt{2-x}dx$$

Solution 9:

Let $I=\int_0^2 x\sqrt{2-x}dx$

$$I=\int_0^2(2-x)\sqrt{x}dx \qquad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$=\int_0^2\left\{2x^{\frac{1}{2}}-x^{\frac{3}{2}}\right\}dx$$

$$=\left[2\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^2$$

$$=\left[\frac{4}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\right]_0^2$$

$$=\frac{4}{3}(2)^{\frac{3}{2}}-\frac{2}{5}(2)^{\frac{5}{2}}$$

$$=\frac{4\times 2\sqrt{2}}{3}-\frac{2}{5}\times 4\sqrt{2}$$

$$=\frac{8\sqrt{2}}{3}-\frac{8\sqrt{2}}{5}$$

$$=\frac{40\sqrt{2}-24\sqrt{2}}{15}$$

$$=\frac{16\sqrt{2}}{15}$$

10:

$$\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}(2\log\sin x-\log\sin 2x)dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{2\log\sin x-\log(2\sin x\cos x)\}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{2\log\sin x-\log\sin x-\log\cos x-\log 2\}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{\log\sin x-\log\cos x-\log 2\}dx \quad ...(1)$

It is known that, $\left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\{\log\cos x-\log\sin x-\log 2\}dx \quad ...(2)$

Adding (1) and (2), we obtain

$2I=\int_0^{\frac{\pi}{2}}(-\log 2-\log 2)dx$

$\Rightarrow 2I=-2\log 2\int_0^{\frac{\pi}{2}}1.dx$

$\Rightarrow I=-\log 2\left[\frac{\pi}{2}\right]$

$\Rightarrow I=\frac{\pi}{2}(-\log 2)$

$\Rightarrow I=\frac{\pi}{2}\left[\log\frac{1}{2}\right]$

$\Rightarrow I=\frac{\pi}{2}\log\frac{1}{2}$

11:

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 xdx$

Solution:

Let $I=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\sin^2 xdx$

As $\sin^2(-x)=(\sin(-x))^2=(-\sin x)^2=\sin^2 x$, therefore, $\sin^2 x$ is an even function.

It is known that if f(x) is an even function, then $\int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx$

$I=2\int_0^{\frac{\pi}{2}}\sin^2 xdx$

$=2\int_0^{\frac{\pi}{2}}\frac{1-\cos 2x}{2}dx$

$$= \int_0^{\frac{\pi}{2}} (1-\cos 2x)\, dx$$

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

12:

$$\int_0^{\pi} \frac{x dx}{1+\sin x}$$

Solution:

Let $I = \int_0^{\pi} \frac{x dx}{1+\sin x}$ ...(1)

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin(\pi - x)} dx$ $\left(\int_0^{\pi} f(x)\, dx = \int_0^{\pi} f(a-x)\, dx\right)$

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{1+\sin x} dx$ ...(2)

Adding (1) and (2), we obtain

$$2I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \{\sec^2 x - \tan x \sec x\} dx$$

$$\Rightarrow 2I = \pi [2]$$

$$\Rightarrow I = \pi$$

13:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$$

Solution:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7 x dx$ ...(1)

As $\sin^7 (-x) = (\sin (-x))^7 = (-\sin x)^7 = -\sin^7 x$, therefore, $\sin^2 x$ is an odd function.

It is known that, if f(x) is an odd function, then $\int_{-a}^{a} f(x)\, dx = 0$

$$\therefore I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^7 x dx=0$$

14:

$$\int_0^{2\pi}\cos^5 x dx$$

Solution:

Let $I=\int_0^{2\pi}\cos^5 x dx \quad ...(1)$

$\cos^5(2\pi-x)=\cos^5 x$

It is known that,

$$\int_0^{2a} f(x)dx=2\int_0^a f(x)dx, \text{ if } f(2a-x)=f(x)$$
$$=0 \text{ if } f(2a-x)=-f(x)$$

$$\therefore I=2\int_0^{\pi}\cos^5 x\,dx$$

$$\Rightarrow I=2(0)=0 \quad [\cos^5(\pi-x)=-\cos^5 x]$$

15:

$$\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx \quad ...(1)$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx \quad \left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$$

$$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos x-\sin x}{1+\sin x\cos x}dx \quad ...(2)$$

Adding (1) and (2), we obtain

$$\Rightarrow 2I=\int_0^{\frac{\pi}{2}}\frac{0}{1+\sin x\cos x}dx$$

$$\Rightarrow I=0$$

16:

$$\int_0^{\pi}\log(1+\cos x)dx$$

Solution:

Let $I=\int_0^{\pi}\log(1+\cos x)\,dx \qquad ...(1)$

$\Rightarrow I=\int_0^{\pi}\log(1+\cos(\pi-x))\,dx \qquad \left(\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right)$

$\Rightarrow I=\int_0^{\pi}\log(1-\cos x)\,dx \qquad ...(2)$

Adding (1) and (2), we obtain

$2I=\int_0^{\pi}\{\log(1-\cos x)+\log(1-\cos x)\}\,dx$

$\Rightarrow 2I=\int_0^{\pi}\log(1-\cos^2 x)\,dx$

$\Rightarrow 2I=\int_0^{\pi}\log\sin^2 xdx$

$\Rightarrow 2I=2\int_0^{\pi}\log\sin xdx$

$\Rightarrow I=\int_0^{\pi}\log\sin xdx \qquad ...(3)$

sin (π − x) = sin x

$\therefore I=2\int_0^{\frac{\pi}{2}}\log\sin x\,dx \qquad ...(4)$

$\Rightarrow I=2\int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2}-x\right)dx=2\int_0^{\frac{\pi}{2}}\log\cos x\,dx \qquad ...(5)$

Adding (4) and (5), we obtain

$2I=2\int_0^{\frac{\pi}{2}}(\log\sin x+\log\cos x)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log\sin x+\log\cos x+\log 2-\log 2)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}(\log 2\sin x\cos x-\log 2)\,dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\log\sin 2xdx-\int_0^{\frac{\pi}{2}}\log 2dx$

Let 2x = t ⇒ 2dx = dt

When x = 0, t = 0 and when x = π/2, t = π

$\therefore I=\frac{1}{2}\int_0^{\pi}\log\sin t\,dt-\frac{\pi}{2}\log 2$

$\Rightarrow I=\frac{I}{2}-\frac{\pi}{2}\log 2$

$\Rightarrow \frac{I}{2}=-\frac{\pi}{2}\log 2$

$\Rightarrow I=-\pi\log 2$

17:

$$\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$$

Solution:

Let $I=\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx \qquad ...(1)$

It is known that, $\left(\int_0^a f(x)dx=\int_0^a f(a-x)dx\right)$

$I=\int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x}+\sqrt{x}}dx \qquad ...(2)$

Adding (1) and (2), we obtain

$2I=\int_0^a \frac{\sqrt{x}+\sqrt{a-x}}{\sqrt{x}+\sqrt{a-x}}dx$

$\Rightarrow 2I=\int_0^a 1.dx$

$\Rightarrow 2I=[x]_0^a$

$\Rightarrow 2I=a$

$\Rightarrow I=\frac{a}{2}$

18:

$$\int_0^4 |x-1|dx$$

Solution:

$I=\int_0^4 |x-1|dx$

It can be seen that, (x − 1) ≤ 0 when 0 ≤ x ≤ 1 and (x − 1) ≥ 0 when 1 ≤ x ≤ 4

$I=\int_0^1 |x-1|dx+\int_1^4 |x-1|dx \qquad \left(\int_a^b f(x)=\int_a^c f(x)+\int_c^b f(x)\right)$

$I=\int_0^1 -(x-1)dx+\int_0^4 (x-1)dx$

$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^4$

$=1-\frac{1}{2}+\frac{(4)^2}{2}-4-\frac{1}{2}+1$

$=1-\frac{1}{2}+8-4-\frac{1}{2}+1$

$=5$

19:

Show that $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$, if *f* and *g* are defined as f(x) = f(a – x) and g (x) + g (a – x) = 4

Solution:

Let $\int_0^a f(x)g(x)dx \quad ...(1)$

$\Rightarrow \int_0^a f(a-x)g(a-x)dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$

$\Rightarrow \int_0^a f(x)g(a-x)dx \quad ...(2)$

Adding (1) and (2), we obtain

$2I = \int_0^a \{f(x)g(x) + f(x)g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x)\{g(x)+g(a-x)\}dx$

$\Rightarrow 2I = \int_0^a f(x)\times 4dx \qquad [g(x)+g(a-x)=4]$

$\Rightarrow I = 2\int_0^a f(x)dx$

Chose the correct answer in Exercises 20 and 21.

20:

The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$ is

A. 0
B. 2
C. π
D. 1

Solution:

Let $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(x^3 + x\cos x + \tan^5 x + 1\right)dx$

$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^3dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos xdx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan^5 xdx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.dx$

It is known that if f (x) is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

if f (x) is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

and $I = 0+0+0+2\int_0^{\frac{\pi}{2}} 1dx$

$= 2[x]_0^{\frac{\pi}{2}}$

$= \frac{2\pi}{2}$

$= \pi$

Hence, the correct Answer is C.

21:

The value of $\int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

A. 2

B. $\frac{3}{4}$

C. 0

D. –2

Solution:

Let I = $\int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ …(1)

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \left[\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos\left(\frac{\pi}{2}-x\right)}\right]dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2I = \int_0^{\frac{\pi}{2}} \left\{\log\left(\frac{4+3\sin x}{4+3\cos x}\right) + \log\left(\frac{4+3\cos x}{4+3\sin x}\right)\right\}dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left(\frac{4+3\sin x}{4+3\cos x} \times \frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log 1 dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 0 dx$$

$$\Rightarrow I = 0$$

Hence, the correct Answer is C.

Miscellaneous Exercise on Chapter 7 Page: 352

1:

Integrate $\frac{1}{x-x^3}$

Solution:

$$\frac{1}{x-x^3}=\frac{1}{x\left(1-x^2\right)}=\frac{1}{x(1-x)(1+x)}$$

Let $\frac{1}{x(1-x)(1+x)}=\frac{A}{x}+\frac{B}{(1-x)}+\frac{C}{1+x}$ ...(1)

$$\Rightarrow 1=A\left(1-x^2\right)+Bx(1+x)+Cx(1-x)$$

$$\Rightarrow 1=A-Ax^2+Bx+Bx^2+Cx-Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

−A + B − C = 0

B + C = 0

A = 1

On solving these equations, we obtain

$$A=1,\ B=\frac{1}{2},\text{ and } C=-\frac{1}{2}$$

From equation (1), we obtain

$$\frac{1}{x(1-x)(1+x)}=\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

$$\Rightarrow \int\frac{1}{x(1-x)(1+x)}dx=\int\frac{1}{x}dx+\frac{1}{2}\int\frac{1}{(1-x)}dx-\frac{1}{2}\int\frac{1}{(1+x)}dx$$

$$=\log|x|-\frac{1}{2}\log|(1-x)|-\frac{1}{2}\log|(1+x)|$$

$$=\log|x|-\log\left|(1-x)^{\frac{1}{2}}\right|-\log\left|(1+x)^{\frac{1}{2}}\right|$$

$$=\log\left|\frac{x}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}}\right|+C$$

$$=\log\left|\left(\frac{x^2}{1-x^2}\right)^{\frac{1}{2}}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right|+C$$

2:

Integrate $\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}$

Solution:

$$\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$

$$=\frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x-b)}$$

$$=\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{a-b}$$

$$\Rightarrow\int\frac{1}{\sqrt{x+a}+\sqrt{(x+b)}}dx=\frac{1}{a-b}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\right]+C$$

3:

Integrate $\frac{1}{x\sqrt{ax-x^2}}$ $\left[\text{Hint: } x=\frac{a}{t}\right]$

Solution:

$$\frac{1}{x\sqrt{ax-x^2}}$$

Let $x=\frac{a}{t}\Rightarrow dx=-\frac{a}{t^2}dh$

$$\Rightarrow\int\frac{1}{x\sqrt{ax-x^2}}dx=\int\frac{1}{\frac{a}{t}\sqrt{a\cdot\frac{a}{t}-\left(\frac{a}{t}\right)^2}}\left(-\frac{a}{t^2}dt\right)$$

$$=-\int\frac{1}{at}\cdot\frac{1}{\sqrt{\frac{1}{t}-\frac{1}{t^2}}}dt$$

$$=-\frac{1}{a}\int\frac{1}{\sqrt{t-1}}dt$$

$$=-\frac{1}{a}\left[2\sqrt{t-1}\right]+C$$

$$=-\frac{1}{a}\left[2\sqrt{\frac{a}{x}-1}\right]+C$$

$$= -\frac{2}{a}\left(\frac{\sqrt{a-x}}{\sqrt{x}}\right)+C$$

$$= -\frac{2}{a}\left(\sqrt{\frac{a-x}{x}}\right)+C$$

4:

Integrate $\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$

Solution:

$$\frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}$$

Multiplying and dividing by x^{-3}, we obtain

$$\frac{x^{-3}}{x^2x^{-3}(x^4+1)^{\frac{3}{4}}} = \frac{x^{-3}(x^4+1)^{\frac{-3}{4}}}{x^2x^{-3}}$$

$$= \frac{\left(x^4+1\right)^{\frac{-3}{4}}}{x^5.\left(x^4\right)^{-\frac{3}{4}}}$$

$$= \frac{1}{x^5}\left(\frac{x^4+1}{x^4}\right)^{-\frac{3}{4}}$$

$$= \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}$$

Let $\frac{1}{x^4}=t \Rightarrow -\frac{4}{x^5}dx = dt \Rightarrow \frac{1}{x^5}dx = -\frac{dt}{4}$

$$\therefore \int \frac{1}{x^2\left(x^4+1\right)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$$

$$= -\frac{1}{4}\int (1+t)^{-\frac{3}{4}}dt$$

$$= -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right]+C$$

$$= -\frac{1}{4}\frac{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}{\frac{1}{4}} + C$$

$$= -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$$

5:

Integrate $\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$

$$\left[\text{Hint:}\frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} = \frac{1}{x^{\frac{1}{3}}\left(1+x^{\frac{1}{6}}\right)} \text{Put } x = t^6\right]$$

Solution:

Let $x = t^6 \Rightarrow dx = 6t^5 dt$

$$\int \frac{1}{x^{1/2} + x^{1/3}} dx = \int \frac{6t^5}{t^3 + t^2} dt$$

$$= \int \frac{6t^5}{t^2(1+t)} dt$$

$$= 6\int \frac{t^3}{(1+t)} dt$$

On dividing, we obtain

$$\int \frac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}} dx = 6\int \left\{(t^2 - t + 1) - \frac{1}{1+t}\right\} dt$$

$$= 6\left[\left(\frac{t^3}{3}\right) - \left(\frac{t^2}{2}\right) + t - \log|1+t|\right]$$

$$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1+x^{\frac{1}{6}}\right) + C$$

$$= 2\sqrt{x} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log\left(1-x^{\frac{1}{6}}\right) + C$$

6:

Integrate $\frac{5x}{(x+1)(x^2+9)}$

Solution:

Let $\frac{5x}{(x+1)(x^2+9)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+9)}$ ...(1)

$\Rightarrow 5x=A(x^2+9)+(Bx+C)(x+1)$

$\Rightarrow 5x=Ax^2+9A+Bx^2+Bx+Cx+C$

Equating the coefficients of x^2, x, and constant term, we obtain

A + B = 0

B + C = 5

9A + C = 0

On solving these equations, we obtain

$A=-\frac{1}{2}, B=\frac{1}{2}, and\ C=\frac{9}{2}$

From equation (1), we obtain

$$\frac{5x}{(x+1)(x^2+9)}=\frac{-1}{2(x+1)}+\frac{\frac{x}{2}+\frac{9}{2}}{(x^2+9)}$$

$$\int\frac{5x}{(x+1)(x^2+9)}dx=\int\left\{\frac{-1}{2(x+1)}+\frac{(x+9)}{2(x^2+9)}\right\}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{2}\int\frac{x}{x^2+9}dx+\frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\int\frac{2x}{x^2+9}dx+\frac{9}{2}\int\frac{1}{x^2+9}dx$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+9|+\frac{9}{2}.\frac{1}{3}\tan^{-1}\frac{x}{3}$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+9)+\frac{3}{2}\tan^{-1}\frac{x}{3}+C$$

7:

Integrate $\frac{\sin x}{\sin(x-a)}$

Solution:

$\frac{\sin x}{\sin(x-a)}$

Let x − a = t ⇒ dx = dt

$$\int \frac{\sin x}{\sin(x-a)}dx = \int \frac{\sin(t+a)}{\sin t}dt$$
$$= \int \frac{\sin t\cos a + \cos t\sin a}{\sin t}dt$$
$$= \int (\cos a + \cot t\sin a)\,dt$$
$$= t\cos a + \sin a\log|\sin t| + C_1$$
$$= (x-a)\cos a + \sin a\log|\sin(x-a)| + C_1$$
$$= x\cos a + \sin a\log|\sin(x-a)| - a\cos a + C_1$$
$$= \sin a\log|\sin(x-a)| + x\cos a + C$$

8:

Integrate $\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$

Solution:

$$\frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}} = \frac{e^{4\log x}\left(e^{\log x} - 1\right)}{e^{2\log x}\left(e^{\log x} - 1\right)}$$
$$= e^{2\log x}$$
$$= e^{\log x^2}$$
$$= x^2$$
$$\therefore \int \frac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}dx = \int x^2 dx = \frac{x^3}{3} + C$$

9:

Integrate $\frac{\cos x}{\sqrt{4-\sin^2 x}}$

Solution:

$$\frac{\cos x}{\sqrt{4-\sin^2 x}}$$

Let sin x = t ⇒ cos x dx = dt

$$\Rightarrow \int \frac{\cos x}{\sqrt{4-\sin^2 x}}dx = \int \frac{dt}{\sqrt{(2)^2 - (t)^2}}$$
$$= \sin^{-1}\left(\frac{t}{2}\right) + C$$
$$= \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

10:

Integrate $\frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}$

Solution:

$$\frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x} = \frac{\left(\sin^4 x + \cos^4 x\right)\left(\sin^4 x - \cos^4 x\right)}{1-2\sin^2 x + \cos^2 x}$$

$$= \frac{\left(\sin^4 x + \cos^4 x\right)\left(\sin^2 x + \cos^2 x\right)\left(\sin^2 x - \cos^2 x\right)}{1-2\sin^2 x + \cos^2 x}$$

$$= \frac{\left(\sin^4 x + \cos^4 x\right)\left(\sin^2 x - \cos^2 x\right)}{1-2\sin^2 x + \cos^2 x}$$

$$= \frac{-\left(\sin^4 x + \cos^4 x\right)\left(\cos^2 x - \sin^2 x\right)}{\left(\sin^4 x + \cos^4 x\right)}$$

$$= -\cos 2x$$

$$\therefore \int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}dx = \int -\cos 2x dx = -\frac{\sin 2x}{2} + C$$

11:

Integrate $\frac{1}{\cos(x+a)\cos(x+b)}$

Solution:

$$\frac{1}{\cos(x+a)\cos(x+b)}$$

Multiplying and dividing by sin (a – b), we obtain.

$$\frac{1}{\sin(a-b)}\left[\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin\left[(x+a)-(x+b)\right]}{\cos(x+a)\cos(x+b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x+a).\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)}\right]$$

$$= \frac{1}{\sin(a-b)}\left[\tan(x+a) - \tan(x+b)\right]$$

$$\int \frac{1}{\cos(x+a)\cos(x+b)}dx = \frac{1}{\sin(a-b)}\int\left[\tan(x+a)-\tan(x+b)\right]dx$$
$$= \frac{1}{\sin(a-b)}[-\log|\cos(x+a)|+\log|\cos(x+b)|]+C$$
$$= \frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$$

12:

Integrate $\frac{x^3}{\sqrt{1-x^8}}$

Solution:

$$\frac{x^3}{\sqrt{1-x^8}}$$

Let $x^4 = t \Rightarrow 4x^3\ dx = dt$

$$\Rightarrow \int \frac{x^3}{\sqrt{1-x^8}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{1}{4}\sin^{-1} t + C$$
$$= \frac{1}{4}\sin^{-1}\left(x^4\right) + C$$

13:

Integrate $\frac{e^x}{(1+e^x)(2+e^x)}$

Solution:

$$\frac{e^x}{(1+e^x)(2+e^x)}$$

Let $e^x = t \Rightarrow e^x\ dx = dt$

$$\Rightarrow \int \frac{e^x}{(1+e^x)(2+e^x)}dx = \int \frac{dt}{(t+1)(t+2)}$$
$$= \int\left[\frac{1}{(t+1)} - \frac{1}{(t+2)}\right]dt$$
$$= \log|t+1| - \log|t+2| + C$$
$$= \log\left|\frac{t+1}{t+2}\right| + C$$

$$=\log\left|\frac{1+e^x}{2+e^x}\right|+C$$

14:

Integrate $\frac{1}{(x^2+1)(x^2+4)}$

Solution:

$$\therefore \frac{1}{(x^2+1)(x^2+4)}=\frac{Ax+B}{(x^2+1)}+\frac{Cx+D}{(x^2+4)}$$

$$\Rightarrow 1=(Ax+B)(x^2+4)+(Cx+D)(x^2+1)$$

$$\Rightarrow 1=Ax^3+4Ax+Bx^2+4B+Cx^3+Cx+Dx^2+D$$

Equating the coefficients of x^3, x^2, x, and constant term, we obtain

A + C = 0

B + D = 0

4A + C = 0

4B + D = 1

On solving these equations, we obtain

$$A=0,\ B=\frac{1}{3}, C=0 \text{ and } D=-\frac{1}{3}$$

From equation (1), we obtain

$$\frac{1}{(x^2+1)(x^2+4)}=\frac{1}{3(x^2+1)}-\frac{1}{3(x^2+4)}$$

$$\int\frac{1}{(x^2+1)(x^2+4)}dx=\frac{1}{3}\int\frac{1}{x^2+1}dx-\frac{1}{3}\int\frac{1}{x^2+4}dx$$

$$=\frac{1}{3}\tan^{-1}x-\frac{1}{3}.\frac{1}{2}\tan^{-1}\frac{x}{2}+C$$

$$=\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+C$$

15:

Integrate $\cos^3 xe^{\log\sin x}$

Solution:

$\cos^3 xe^{\log\sin x}$ = $\cos^3$ x × sin x

Let cos x = t ⇒ −sin x dx = dt

$$\Rightarrow\int\cos^3 xe^{\log\sin x}dx=\int\cos^3 x\sin x dx$$

$$=-\int t^3\,dx$$

$$= -\frac{t^4}{4} + C$$

$$= -\frac{\cos^4 x}{4} + C$$

16:

Integrate $e^{3\log x}\left(x^4+1\right)^{-1}$

Solution:

$$e^{3\log x}\left(x^4+1\right)^{-1} = e^{\log x^3}\left(x^4+1\right)^{-1} = \frac{x^3}{\left(x^4+1\right)}$$

$$Let\ x^4+1 = t \Rightarrow 4x^3 dx = dt$$

$$\Rightarrow \int e^{3\log x} = \left(x^4+1\right)^{-1} dx = \int \frac{x^3}{\left(x^4+1\right)} dx$$

$$= \frac{1}{4}\int \frac{dt}{t}$$

$$= \frac{1}{4}\log|t| + C$$

$$= \frac{1}{4}\log\left|x^4+1\right| + C$$

$$= \frac{1}{4}\log\left(x^4+1\right) + C$$

17:

Integrate $f'(ax+b)\left[f(ax+b)\right]^n$

Solution:

$$f'(ax+b)\left[f(ax+b)\right]^n$$

Let $f(ax+b) = t \Rightarrow a\, f'(ax+b)dx = dt$

$$\Rightarrow \int f'(ax+b)\left[f(ax+b)\right]^n dx = \frac{1}{a}\int t^n dt$$

$$= \frac{1}{a}\left[\frac{t^{n+1}}{n+1}\right]$$

$$= \frac{1}{a(n+1)}\left(f(ax+b)\right)^{n+1} + C$$

18:

Integrate $\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Solution:

$$\frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} = \frac{1}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x \sin\alpha)}}$$

$$= \frac{1}{\sqrt{\sin^4 x \cos\alpha + \sin^3 x \cos x \sin\alpha}}$$

$$= \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

$$= \frac{\cos ec^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

Let $\cos\alpha + \cot x \sin\alpha = t \Rightarrow -\cos ec^2 x \sin\alpha dx = dt$

$$\therefore \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}} dx = \int \frac{\cos ec^2 x}{\sqrt{\cos\alpha + \cot x \sin\alpha}} dx.$$

$$= \frac{-1}{\sin\alpha}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{t}\right] + C$$

$$= \frac{-1}{\sin\alpha}\left[2\sqrt{\cos\alpha + \cot x \sin\alpha}\right] + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\cos\alpha + \frac{\cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}} + C$$

$$= \frac{-2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$$

19:

Integrate $\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}, x \in [0,1]$

Solution:

Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$

It is known that, $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$$\Rightarrow I=\int\frac{\left(\frac{\pi}{2}-\cos^{-1}\sqrt{x}\right)-\cos^{-1}\sqrt{x}}{\frac{\pi}{2}}dx$$

$$=\frac{2}{\pi}\int\left(\frac{\pi}{2}-2\cos^{-1}\sqrt{x}\right)dx$$

$$=\frac{2}{\pi}.\frac{\pi}{2}\int 1.dx-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx$$

$$=x-\frac{4}{\pi}\int\cos^{-1}\sqrt{x}dx \qquad ...(1)$$

$$Let\ I_1=\int\cos^{-1}\sqrt{x}dx$$

Also. let $\sqrt{\mathrm{x}}=t\Rightarrow dx=2t\,dt$

$$\Rightarrow I_1=2\int\cos^{-1}t.t\,dt$$

$$=2\left[\cos^{-1}t.\frac{t^2}{2}-\int\frac{-1}{\sqrt{1-t^2}}.\frac{t^2}{2}dt\right]$$

$$=t^2\cos^{-1}t+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\frac{1-t^2-1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\int\sqrt{1-t^2}dt+\int\frac{1}{\sqrt{1-t^2}}dt$$

$$=t^2\cos^{-1}t-\frac{1}{2}\sqrt{1-t^2}-\frac{1}{2}\sin^{-1}t+\sin^{-1}t$$

$$=t^2\cos^{-1}t-\frac{1}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t$$

From equation (1), we obtain

$$I=x-\frac{4}{\pi}\left[t^2\cos^{-1}t-\frac{t}{2}\sqrt{1-t^2}+\frac{1}{2}\sin^{-1}t\right]$$

$$=x-\frac{4}{\pi}\left[x\cos^{-1}\sqrt{x}-\frac{\sqrt{x}}{2}\sqrt{1-x}+\frac{1}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-\frac{4}{\pi}\left[x\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)-\frac{\sqrt{x-x^2}}{2}+\frac{\pi}{2}\sin^{-1}\sqrt{x}\right]$$

$$=x-2x+\frac{4x}{\pi}\sin^{-1}\sqrt{x}+\frac{2}{\pi}\sqrt{x-x^2}-\frac{2}{\pi}\sin^{-1}\sqrt{x}$$

$$=-x+\frac{2}{\pi}\left[(2x-1)\sin^{-1}\sqrt{x}\right]+\frac{2}{\pi}\sqrt{x-x^2}+C$$

$$=\frac{2(2x-1)}{\pi}\sin^{-1}\sqrt{x}+\frac{2}{\pi}\sqrt{x-x^2}-x+C$$

20:

Integrate $\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}$

Solution:

$$I=\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$$

Let $x=\cos^2\theta \Rightarrow dx=-2\sin\theta\cos\theta\, d\theta$

$$I=\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}(-2\sin\theta\cos\theta)d\theta$$

$$=-\int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin 2\theta\, d\theta$$

$$=-\int\tan\frac{\theta}{2}.2\sin\theta\cos\theta\, d\theta$$

$$=-2\int\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\theta d\theta$$

$$=-4\int\sin^2\frac{\theta}{2}\cos\theta d\theta$$

$$=-4\int\sin^2\frac{\theta}{2}.\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$=-4\int\left(2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right)d\theta$$

$$=-8\int\sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$=-2\int\sin^2\theta\, d\theta+4\int\sin^2\frac{\theta}{2}d\theta$$

$$=-2\int\left(\frac{1-\cos 2\theta}{2}\right)d\theta+4\int\frac{1-\cos\theta}{2}d\theta$$

$$=-2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]+4\left[\frac{\theta}{2}-\frac{\sin\theta}{2}\right]+C$$

$$=-\theta+\frac{\sin 2\theta}{2}+2\theta-2\sin\theta+C$$

$$=\theta+\frac{\sin 2\theta}{2}+2\sin\theta+C$$

$$=\theta+\frac{2\sin\theta\cos\theta}{2}-2\sin\theta+C$$

$= \theta + \sqrt{1-\cos^2\theta}.\cos\theta - 2\sqrt{1-\cos^2\theta} + C$

$= \cos^{-1}\sqrt{x} + \sqrt{1-x}.\sqrt{x} - 2\sqrt{1-x} + C$

$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + C$

$= -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x-x^2} + C$

21:

Integrate $\frac{2+\sin 2x}{1+\cos 2x}e^x$

Solution:

$I = \int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x$

$= \int\left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x$

$= \int\left(\frac{1+\sin x\cos x}{\cos^2 x}\right)e^x$

$= \int\left(\sec^2 x + \tan x\right)e^x$

$Let\ \ f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$

$\therefore I = \int\left(f(x) + f'(x)\right)e^x dx$

$= e^x f(x) + C$

$= e^x \tan x + C$

22:

Integrate $\frac{x^2+x+1}{(x+1)^2(x+2)}$

Solution:

Let $\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$ ...(1)

$\Rightarrow x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x^2+2x+1)$

$\Rightarrow x^2+x+1 = A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)$

$\Rightarrow x^2+x+1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$

Equating the coefficients of x^2, x and constant term, we obtain

A + C = 1

3A + B + 2C = 1

2A + 2B + C = 1

On solving these equations, we obtain
A = −2, B = 1, and C = 3
From equation (1), we obtain

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{-2}{(x+1)}+\frac{3}{(x+2)}+\frac{1}{(x+1)^2}$$

$$\int\frac{x^2+x+1}{(x+1)^2(x+2)}dx=-2\int\frac{1}{x+1}dx+3\int\frac{1}{(x+2)}dx+\int\frac{1}{(x+1)^2}dx$$

$$=-2\log|x+1|+3\log|x+2|-\frac{1}{(x+1)}+C$$

23:

Integrate $\tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Solution:

$\text{I} = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$

Let $x=\cos\theta \Rightarrow dx=-\sin\theta d\theta$

$$I=\int\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}-(\sin\theta d\theta)$$

$$=-\int\tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\sin\theta d\theta$$

$$=-\int\tan^{-1}\tan\frac{\theta}{2}.\sin\theta d\theta$$

$$=-\frac{1}{2}\int\theta.\sin\theta d\theta$$

$$=-\frac{1}{2}\left[\theta.(-\cos\theta)-\int 1.(-\cos\theta)d\theta\right]$$

$$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]$$

$$=+\frac{1}{2}\theta\cos\theta-\frac{1}{2}\sin\theta$$

$$=\frac{1}{2}\cos^{-1}x.x-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{x}{2}\cos^{-1}-\frac{1}{2}\sqrt{1-x^2}+C$$

$$=\frac{1}{2}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)+C$$

24:

Integrate $\dfrac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}$

Solution:

$$\frac{\sqrt{x^2+1}\left[\log\left(x^2+1\right)-2\log x\right]}{x^4}=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(x^2+1\right)-\log x^2\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\left[\log\left(\frac{x^2+1}{x^2}\right)\right]$$

$$=\frac{\sqrt{x^2+1}}{x^4}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{\frac{x^2+1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$=\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)$$

$$Let\ 1+\frac{1}{x^2}=t\Rightarrow\frac{-2}{x^3}dx=dt$$

$$\therefore I=\int\frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)dx$$

$$=-\frac{1}{2}\int\sqrt{t}\log t dt$$

$$=-\frac{1}{2}\int t^{\frac{1}{2}}.\log t dt$$

Integrating by parts, we obtain

$$I=-\frac{1}{2}\left[\log t.\int t^{\frac{1}{2}}.dt-\left\{\left(\frac{d}{dt}\log t\right)\int t^{\frac{1}{2}}dt\right\}dt\right]$$

$$=-\frac{1}{2}\left[\log t.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\int\frac{1}{t}.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{2}{3}\int t^{\frac{1}{2}}dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{\frac{3}{2}}\log t-\frac{4}{9}t^{\frac{3}{2}}\right]$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\log t+\frac{2}{9}t^{\frac{3}{2}}$$

$$=-\frac{1}{3}t^{\frac{3}{2}}\left[\log t-\frac{2}{3}\right]$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}\left[\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right]+C$$

25:

$$\int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

Solution:

$$I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$$

$$= \int_{\frac{\pi}{2}}^{\pi} e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$$

$$= \int_{\frac{\pi}{2}}^{\pi}\left(\frac{\operatorname{cosec}^2\frac{x}{2}}{2}-\cot\frac{x}{2}\right)dx$$

$$Let\ f(x) = -\cot\frac{x}{2}$$

$$\Rightarrow f'(x) = -\left(-\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\pi} e^x\left(f(x)+f'(x)\right)dx$$

$$= \left[e^x.f(x)dx\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^x.\cot\frac{x}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= -\left[e^{\pi}.\cot\frac{\pi}{2} - e^{\frac{\pi}{2}}.\cot\frac{\pi}{4}\right]$$

$$= -\left[e^{\pi}.0 - e^{\frac{\pi}{2}}.1\right]$$

$$= e^{\frac{\pi}{2}}$$

26:

$$\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$$

Solution:

Let $I=\int_0^{\frac{\pi}{4}}\frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\frac{(\sin x\cos x)}{\cos^4 x}}{\frac{\left(\cos^4 x+\sin^4 x\right)}{\cos^4 x}}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\tan x\sec^2 x}{1+\tan^4 x}dx$

Let $\tan^2 x=t\Rightarrow 2\tan x\sec^2 x dx=dt$

When x = 0, t = 0 and when $x=\frac{\pi}{4},t=1$

$\therefore I=\frac{1}{2}\int_0^1\frac{dt}{1+t^2}$

$=\frac{1}{2}\left[\tan^{-1}t\right]_0^1$

$=\frac{1}{2}\left[\tan^{-1}1\tan^{-1}0\right]$

$=\frac{1}{2}\left[\frac{\pi}{4}\right]$

$=\frac{\pi}{8}$

27:

$\int_0^{\frac{\pi}{2}}\frac{\cos^2 xdx}{\cos^2 x+4\sin^2 x}$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4\left(1-\cos^2 x\right)}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{\cos^2 x+4-4\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}\frac{4-3\cos^2 x}{4-3\cos^2 x}dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4}{4-3\cos^2 x}dx$

$\Rightarrow I=\frac{-1}{3}\int_0^{\frac{\pi}{2}}1dx+\frac{1}{3}\int_0^{\frac{\pi}{2}}\frac{4\sec^2 x}{4\sec^2 x-3}dx$

$$\Rightarrow I = \frac{-1}{3}[x]_0^{\frac{\pi}{2}} + \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{4\sec^2 x}{4(1+\tan^2 x)-3}dx$$

$$\Rightarrow I = -\frac{\pi}{6} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx \qquad ...(1)$$

Consider, $\int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx$

Let 2 tanx = t $\Rightarrow$ 2sec^{2}xdx = dt

When x = 0, t = 0 and when x = $\frac{\pi}{2}$, t = ∞

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{2\sec^2 x}{1+4\tan^2 x}dx = \int_0^{\infty} \frac{dt}{1+t^2}$$

$$= \left[\tan^{-1} t\right]_0^{\infty}$$

$$= \left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{\pi}{2}$$

Therefore, from (1), we obtain

$$I = -\frac{\pi}{6} + \frac{2}{3}\left[\frac{\pi}{2}\right] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

28:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$$

Solution:

Let $I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{-(-\sin 2x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x + \cos x}{\sqrt{-(-1+1-2\sin\cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)}{\sqrt{1-(\sin^2 x\cos^2 x - 2\sin x\cos x)}} dx$$

$$\Rightarrow I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\sin x + \cos x)dx}{\sqrt{1-(\sin x - \cos x)^2}}$$

Let $(\sin x - \cos x) = t = (\sin x + \cos x)dx = dt$

when $x=\frac{\pi}{6}, t=\left(\frac{1-\sqrt{3}}{2}\right)$ and when x = $\frac{\pi}{3}, t=\left(\frac{\sqrt{3}-1}{2}\right)$

$$I=\int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I=\int_{-\left(\frac{1-\sqrt{3}}{2}\right)}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

As $\frac{1}{\sqrt{1-(-t)^2}}=\frac{1}{\sqrt{1-t^2}}$, therefore, $\frac{1}{\sqrt{1-t^2}}$ is an even function.

$$\int_{-a}^{a} f(x)dx=2\int_0^a f(x)dx$$

It is known that if f(x) is an even function, then

$$\Rightarrow I=2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1-t^2}}$$

$$=\left[2\sin^{-1}t\right]_0^{\frac{\sqrt{3}-1}{2}}$$

$$=2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right)$$

29:

$$\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$$

Solution:

Let $I=\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}}$

$$I=\int_0^1 \frac{1}{\left(\sqrt{1+x}-\sqrt{x}\right)}\times\frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{\left(\sqrt{1+x}+\sqrt{x}\right)}dx$$

$$=\int_0^1 \frac{\left(\sqrt{1+x}+\sqrt{x}\right)}{1+x-x}dx$$

$$=\int_0^1 \sqrt{1+x}dx+\int_0^1\sqrt{x}dx$$

$$=\left[\frac{2}{3}(1+x)^{\frac{2}{3}}\right]_0^1\left[\frac{2}{3}(x)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{2}{3}\left[(2)^{\frac{2}{3}}-1\right]+\frac{2}{3}[1]$$

$$=\frac{2}{3}(2)^{\frac{2}{3}}$$

$=\frac{2.2\sqrt{2}}{3}$

$=\frac{4\sqrt{2}}{3}$

30:

$\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

Solution:

Let I = $\int_0^{\frac{\pi}{4}}\frac{\sin x+\cos x}{9+16\sin 2x}dx$

Also let sin x – cos x = t $\Rightarrow$ (cos x + sinx) dx = dt

When x = 0, t = - 1 and when x = $\frac{\pi}{4}$, t = 0

$\Rightarrow(\sin x-\cos x)^2=t^2$

$\Rightarrow\sin^2+\cos^2-2\sin x\cos x=t^2$

$\Rightarrow1-\sin 2x=t^2$

$\Rightarrow\sin 2x=1-t^2$

$\therefore I=\int_{-1}^{0}\frac{dt}{9+16(1-t^2)}$

$=\int_{-1}^{0}\frac{dt}{9+16-16t^2}$

$=\int_{-1}^{0}\frac{dt}{25-16t^2}=\int_{-1}^{0}\frac{dt}{(5)^2-(4t)^2}$

$=\frac{1}{4}\left[\frac{1}{2(5)}\log\left|\frac{5+4t}{5-4t}\right|\right]_{-1}^{0}$

$=\frac{1}{40}\left[\log(1)-\log\left|\frac{1}{9}\right|\right]$

$=\frac{1}{40}\log 9$

31:

$\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)dx$

Solution:

Let $I=\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)dx=\int_0^{\frac{\pi}{2}}2\sin x\cos x\tan^{-1}(\sin x)dx$

Also, let sin x = t $\Rightarrow$ cos x dx = dt

When x = 0, t = 0 and when x = $\frac{\pi}{2}$, t = 1

$$\Rightarrow I = 2\int_0^1 t\tan^{-1}(t)\,dt \qquad \ldots(1)$$

Consider $\int t.\tan^{-1} t dt = \tan^{-1} t.\int t dt - \int\left\{\frac{d}{dt}\left(\tan^{-1} t\right)\int t dt\right\}dt$

$$= \tan^{-1} t.\frac{t^2}{2} - \int\frac{1}{1+t^2}.\frac{t^2}{2}dt$$

$$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}\int\frac{t^2+1-1}{1+t^2}dt$$

$$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}\int 1.dt + \frac{1}{2}\int\frac{1}{1+t^2}dt$$

$$= \frac{t^2\tan^{-1} t}{2} - \frac{1}{2}.t + \frac{1}{2}\tan^{-1} t$$

$$\Rightarrow \int_0^1 t.\tan^{-1} t dt = \left[\frac{t^2\tan^{-1} t}{2} - \frac{t}{2} + \frac{1}{2}\tan^{-1} t\right]_0^1$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 1 + \frac{\pi}{4}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 1\right] = \frac{\pi}{4} - \frac{1}{2}$$

From equation (1), we obtain

$$I = 2\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{2} - 1$$

32:

$$\int_0^{\pi}\frac{x\tan x}{\sec x + \tan x}dx$$

Solution:

Let $\int_0^{\pi}\frac{x\tan x}{\sec x + \tan x}dx \qquad \ldots(1)$

$$I = \int_0^{\pi}\left\{\frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}\right\}dx \qquad \left(\int_0^a f(x)dx = \int_0^a f(a-x)dx\right)$$

$$\Rightarrow I = \int_0^{\pi}\left\{\frac{-(\pi - x)\tan x}{-(\sec x + \tan x)}\right\}dx$$

$$\Rightarrow I = \int_0^{\pi}\frac{(\pi - x)\tan x}{\sec x + \tan x}dx \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2I=\int_0^{\pi}\frac{\pi\tan x}{\sec x+\tan x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}+\frac{\sin x}{\cos x}}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sin x+1-1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}1.dx-\pi\int_0^{\pi}\frac{1}{1+\sin x}dx$$

$$\Rightarrow 2I=\pi[x]_0^{\pi}-\pi\int_0^{\pi}\frac{1-\sin x}{\cos^2 x}dx$$

$$\Rightarrow 2I=\pi^2-\pi\int_0^{\pi}\left(\sec^2 x-\tan x\sec x\right)dx$$

$$\Rightarrow 2I=\pi^2-\pi\left[\tan x-\sec x\right]_0^{\pi}$$

$$\Rightarrow 2I=\pi^2-\pi\left[\tan\pi-\sec\pi-\tan 0+\sec 0\right]$$

$$\Rightarrow 2I=\pi^2-\pi\left[0-(-1)-0+1\right]$$

$$\Rightarrow 2I=\pi^2-2\pi$$

$$\Rightarrow 2I=\pi(\pi-2)$$

$$\Rightarrow I=\frac{\pi}{2}(\pi-2)$$

33:

$$\int_1^4\left[|x-1|+|x-2|+|x-3|\right]dx$$

Solution:

Let $I=\int_1^4\left[|x-1|+|x-2|+|x-3|\right]dx$

$$\Rightarrow I=\int_1^4|x-1|\,dx+\int_1^4|x-2|\,dx+\int_1^4|x-3|\,dx$$

$I=I_1+I_2+I_3$ ...(1)

where, $I_1=\int_1^4|x-1|\,dx,\ I_2=\int_1^4|x-2|\,dx, and\ I_3=\int_1^4|x-3|\,dx$

$I_1=\int_1^4|x-1|\,dx$

$(x-1)\geq 0\ \ for\ 1\leq x\leq 4$

$$\therefore I_1=\int_1^4(x-1)\,dx$$

$$\Rightarrow I_1=\left[\frac{x^2}{2}-x\right]_1^4$$

$$\Rightarrow I_1 = \left[8 - 4 - \frac{1}{2} + 1\right] = \frac{9}{2} \qquad ...(2)$$

$$I_2 = \int_1^4 |x-2|\,dx$$

$$x - 2 \geq 0 \;\; for \;\; 2 \leq x \leq 4 \;\; and \;\; x - 2 \leq 0 \;\; for \;\; 1 \leq x \leq 2$$

$$\therefore I_2 = \int_1^2 (2-x)\,dx + \int_2^4 (x-2)\,dx$$

$$\Rightarrow I_2 = \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$$

$$\Rightarrow I_2 = \left[4 - 2 - 2 + \frac{1}{2}\right] + [8 - 8 - 2 + 4]$$

$$\Rightarrow I_2 = \frac{1}{2} + 2 = \frac{5}{2} \qquad ...(3)$$

$$I_3 = \int_1^4 |x-3|\,dx$$

$$x - 3 \geq 0 \;\; for \;\; 3 \leq x \leq 4 \;\; and \;\; x - 3 \leq 0 \;\; for \;\; 1 \leq x \leq 3$$

$$\therefore I_3 = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx$$

$$\Rightarrow I_3 = \left[3x - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$$

$$\Rightarrow I_3 = \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$$

$$\Rightarrow I_3 = [6 - 4] + \left[\frac{1}{2}\right] = \frac{5}{2} \qquad ...(4)$$

From equations (1), (2), (3), and (4), we obtain

$$I = \frac{9}{2} + \frac{5}{2} + \frac{5}{2} = \frac{19}{2}$$

34:

Prove $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$

Solution:

Let $I = \int_1^3 \frac{dx}{x^2(x+1)}$

Also, let $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$

$$\Rightarrow 1 = Ax(x+1) + B(x+1) + C(x^2)$$

$$\Rightarrow 1 = Ax^2 + Ax + Bx + B + Cx^2$$

Equating the coefficients of x^2, x, and constant term, we obtain

A + C = 0

A + B = 0

B = 1

On solving these equations, we obtain

A = −1, C = 1, and B = 1

$$\therefore \frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)}$$

$$\Rightarrow I = \int_1^3 \left\{ -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{(x+1)} \right\} dx$$

$$= \left[-\log x - \frac{1}{x} + \log(x+1) \right]_1^3$$

$$= \left[\log\left(\frac{x+1}{x}\right) - \frac{1}{x} \right]_1^3$$

$$= \log\left(\frac{4}{3}\right) - \frac{1}{3} - \log\left(\frac{2}{1}\right) + 1$$

$$= \log 4 - \log 3 - \log 2 + \frac{2}{3}$$

$$= \log 2 - \log 3 + \frac{2}{3}$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Hence, the given result is proved.

35:

Prove $\int_0^4 xe^x dx = 1$

Solution:

Let $I = \int_0^4 xe^x dx$

Integrating by parts, we obtain

$$I = x\int_0^4 e^x dx - \int_0^1 \left\{ \left(\frac{d}{dx}(x) \right) \int e^x dx \right\} dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x dx$$

$$= \left[xe^x \right]_0^1 - \left[e^x \right]_0^1$$

$$= e - e + 1$$

$$= 1$$

Hence, the given result is proved.

36:

Prove $\int_{-1}^{1} x^{17}\cos^4 x dx = 0$

Solution:

Let $I = \int_{-1}^{1} x^{17}\cos^4 x dx$

Also, let $f(x) = x^{17}\cos^4 x$

$\Rightarrow f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$

Therefore, f (x) is an odd function.

It is known that if f(x) is an odd function, then $\int_{-a}^{a} f(x) dx = 0$

$\therefore I = \int_{-1}^{1} x^{17}\cos^4 x dx = 0$

Hence, the given result is proved.

37:

Prove $\int_0^{\frac{\pi}{2}} \sin^3 x dx = \frac{2}{3}$

Solution:

Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x dx$

$I = \int_0^{\frac{\pi}{2}} \sin^2 x.\sin x dx$

$= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\sin x dx$

$= \int_0^{\frac{\pi}{2}} \sin x dx - \int_0^{\frac{\pi}{2}} \cos^2 x.\sin x dx$

$= [-\cos x]_0^{\frac{\pi}{2}} + \left[\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}}$

$= 1 + \frac{1}{3}[-1] = 1 - \frac{1}{3} = \frac{2}{3}$

Hence, the given result is proved.

38:

Prove $\int_0^{\frac{\pi}{4}} 2\tan^3 x dx = 1 - \log 2$

Solution:

Let $I = \int_0^{\frac{\pi}{4}} 2\tan^3 x dx$

$$I=\int_0^{\frac{\pi}{4}}2\tan^2 x\tan x dx=2\int_0^{\frac{\pi}{4}}\left(\sec^2 x-1\right)\tan x dx$$

$$=2\int_0^{\frac{\pi}{4}}\sec^2 x\tan x dx-2\int_0^{\frac{\pi}{4}}\tan x dx$$

$$=2\left[\frac{\tan^2 x}{2}\right]_0^{\frac{\pi}{4}}+2\left[\log\cos x\right]_0^{\frac{\pi}{4}}$$

$$=1+2\left[\log\cos\frac{\pi}{4}-\log\cos 0\right]$$

$$=1+2\left[\log\frac{1}{\sqrt{2}}-\log 1\right]$$

$$=1-\log 2-\log 1=1-\log 2$$

Hence, the given result is proved.

39:

Prove $\int_0^1\sin^{-1} x dx=\frac{\pi}{2}-1$

Solution:

Let $\int_0^1\sin^{-1} x dx$

$$\Rightarrow I=\int_0^1\sin^{-1} x.1.dx$$

Integrating by parts, we obtain

$$I=\left[\sin^{-1} x.x\right]_0^1-\int_0^1\frac{1}{\sqrt{1-x^2}}.x dx$$

$$=\left[x\sin^{-1} x\right]_0^1+\frac{1}{2}\int_0^1\frac{(-2x)}{\sqrt{1-x^2}}dx$$

Let $1 - x^2 = t \Rightarrow -2x\, dx = dt$

When x = 0, t = 1 and when x = 1, t = 0

$$I=\left[x\sin^{-1} x\right]_0^1+\frac{1}{2}\int_0^1\frac{dt}{\sqrt{t}}$$

$$=\left[x\sin^{-1} x\right]_0^1+\frac{1}{2}\left[2\sqrt{t}\right]_1^0$$

$$=\sin^{-1}(1)+\left[-\sqrt{1}\right]$$

$$=\frac{\pi}{2}-1$$

Hence, the given result is proved.

40:

Evaluate $\int_0^1 e^{2-3x}dx$ as a limit of a sum.

Solution:

Let I = $\int_0^1 e^{2-3x}dx$

It is known that,

$$\int_a^b f(x)dx=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+...+f(a+(n-1)h)\right]$$

where, $h=\frac{b-a}{n}$

Here, $a=0,\ b=1$, and $f(x)=e^{2-3x}$

$$\Rightarrow h=\frac{1-0}{n}=\frac{1}{n}$$

$$\therefore \int_0^1 e^{2-3x}dx=(1-0)\lim_{n\to\infty}\frac{1}{n}\left[f(0)+f(0+h)+...+f(0+(n-1)h)\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2+e^{2-3x}+...+e^{2-3(n-1)h}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{1+e^{-3h}+e^{-6h}+e^{-9h}+...+e^{-3(n-1)h}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-\left(e^{-3h}\right)^n}{1-\left(e^{-3h}\right)}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[e^2\left\{\frac{1-e^{-\frac{3}{n}\times n}}{1-e^{-\frac{3}{n}}}\right\}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left[\frac{e^2\left(1-e^{-3}\right)}{1-e^{-\frac{3}{n}}}\right]$$

$$=e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{e^{-\frac{3}{n}}-1}\right]$$

$$=e^2\left(e^{-3}-1\right)\lim_{n\to\infty}\left(-\frac{1}{3}\right)\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$=\frac{e^2\left(e^{-3}-1\right)}{3}\lim_{n\to\infty}\left[\frac{-\frac{3}{n}}{e^{-\frac{3}{n}}-1}\right]$$

$$=\frac{-e^2\left(e^{-3}-1\right)}{3}(1) \qquad \left[\lim_{n\to\infty}\frac{x}{e^x-1}\right]$$

$= \frac{-e^{-1}+e^{2}}{3}$

$= \frac{1}{3}\left(e^{2}-\frac{1}{e}\right)$

Chose the correct answer in Exercises 41 to 44.

41:

$\int \frac{dx}{e^{x}+e^{-x}}$ is equal to

A. $\tan^{-1}\left(e^{x}\right)+C$

B. $\tan^{-1}\left(e^{-x}\right)+C$

C. $\log\left(e^{x}-e^{-x}\right)+C$

D. $\log\left(e^{x}+e^{-x}\right)+C$

Solution:

Let $I=\int \frac{dx}{e^{x}+e^{-x}}dx=\int \frac{e^{x}}{e^{2x}+1}dx$

Also, let $e^{x}=t \Rightarrow e^{x}dx=dt$

$\therefore I=\int \frac{dt}{1+t^{2}}$

$=\tan^{-1}t+C$

$=\tan^{-1}\left(e^{x}\right)+C$

Hence, the correct Answer is A.

42:

$\int \frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$ is

A. $\frac{-1}{\sin x+\cos x}+C$

B. $\log|\sin x+\cos x|+C$

C. $\log|\sin x-\cos x|+C$

D. $\frac{1}{(\sin x+\cos x)^{2}}+C$ equal to

Solution:

Let $I=\int \frac{\cos 2x}{(\sin x+\cos x)^{2}}dx$

$$I=\int \frac{\cos^2 x-\sin^2 x}{(\sin x+\cos x)^2}dx$$
$$=\int \frac{(\cos x+\sin x)(\cos x-\sin x)}{(\sin x+\cos x)^2}dx$$
$$=\int \frac{\cos x-\sin x}{\cos x+\sin x}dx$$

Let $\cos x+\sin x=t \Rightarrow (\cos x-\sin x)dx=dt$

$$\therefore I=\int \frac{dt}{t}$$
$$=\log|t|+C$$
$$=\log|\cos x+\sin x|+C$$

Hence, the correct Answer is B.

43:

If f (a + b – x) = f (x), then $\int_a^b xf(x)dx$ is equal to

A. $\frac{a+b}{2}\int_a^b f(b-x)dx$

B. $\frac{a+b}{2}\int_a^b f(b+x)dx$

C. $\frac{b-a}{2}\int_a^b f(x)dx$

D. $\frac{a+b}{2}\int_a^b f(x)dx$

Solution:

Let I = $\int_a^b xf(x)dx$ … (1)

$$I=\int_a^b (a+b-x)f(a+b-x)dx \qquad \left(\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right)$$
$$\Rightarrow I=\int_a^b (a+b-x)f(x)dx$$
$$\Rightarrow I=(a+b)\int_a^b f(x)dx-I \qquad [\text{using}(1)]$$
$$\Rightarrow I+I=(a+b)\int_a^b f(x)dx$$
$$\Rightarrow 2I=(a+b)\int_a^b f(x)dx$$
$$\Rightarrow I=\left(\frac{a+b}{2}\right)\int_a^b f(x)dx$$

Hence, the correct Answer is D.

44: The value of $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ is

A. 1
B. 0
C. − 1
D. $\frac{\pi}{4}$

Solution:

Let $I=\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$

$\Rightarrow I=\int_0^1 \tan^{-1}\left(\frac{x-(1-x}{1+x(1-x)}\right)dx$

$\Rightarrow I=\int_0^1\left[\tan^{-1}x-\tan^{-1}(1-x)\right]dx$ ...(1)

$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}(1-1+x)\right]dx$

$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx$

$\Rightarrow I=\int_0^1\left[\tan^{-1}(1-x)-\tan^{-1}(x)\right]dx$ ...(2)

Adding (1) and (2), we obtain

$\Rightarrow 2I=\int_0^1\left(\tan^{-1}x-\tan^{-1}(1-x)-\tan^{-1}(1-x)-\tan^{-1}x\right)dx$

$\Rightarrow 2I=0$

$\Rightarrow I=0$

Hence, the correct Answer is B.