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In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1:

$\left(x^2 + xy\right)dy = \left(x^2 + y^2\right)dx$

Solution:

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \quad \text{.....(1)}$$

Let $F(x, y) = \dfrac{x^2 + y^2}{x^2 + xy}$

Now, $F(\lambda x, \lambda y) = \dfrac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)^2 + (\lambda x)(\lambda y)} = \dfrac{x^2 + y^2}{x^2 + xy} = \lambda^0 F(x, y)$

This shows that equation (1) is a homogeneous equation.
To solve it, we make the substitution as:
y = vx
Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x^2 + (vx)^2}{x^2 + x(vx)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1 + v^2}{1 + v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 + v^2}{1 + v} - v = \frac{(1 + v^2) - c(1 + v)}{1 + v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - v}{1 + v}$$

$$\Rightarrow \left(\frac{1 + v}{1 - v}\right) = dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2 - 1 + v}{1 - v}\right)dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2}{1 - v} - 1\right)dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$-2og(1-\nu)-\nu=\log x-\log k$$

$$\Rightarrow v=-2\log(1-\nu)-\log x+\log k$$

$$\Rightarrow \nu=\log\left[\frac{k}{x(1-\nu)^2}\right]$$

$$\Rightarrow \frac{y}{x}=\log\left[\frac{k}{x\left(1-\frac{y}{x}\right)^2}\right]$$

$$\Rightarrow \frac{y}{x}=\log\left[\frac{kx}{(x-y)^2}\right]$$

$$\Rightarrow \frac{kx}{(x-y)^2}=e^{\frac{-y}{x}}$$

$$\Rightarrow (x-y)^2=kxe^{\frac{-y}{x}}$$

This is the required solution of the given differential equation.

2:

$$y'=\frac{x+y}{x}$$

Solution:

The given differential equation is:

$$y'=\frac{x+y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y}{x} \qquad \text{..... (1)}$$

Let F(x, y) = $\frac{x+y}{x}$

Now, $(\lambda x,\lambda y)=\frac{\lambda x,\lambda y}{\lambda x}=\frac{x+y}{x}=\lambda^0 F(x,y)$

Thus, the given equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v$$

$$x\frac{dv}{dx} = 1$$

$$\Rightarrow dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$v = \log x + C$$

$$\Rightarrow \frac{y}{x} = \log x + C$$

$$\Rightarrow y = x\log x + Cx$$

This is the required solution of the given differential equation.

3:

$$(x - y)dy\,(-x + y)dx = 0$$

Solution:

$$(x - y)dy\,(-x + y)dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x + y}{x - y} \qquad \text{..... (1)}$$

Let F (x, y) $= \frac{x + y}{x - y}$

$$\therefore F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{x + y}{x - y} = \lambda^0 F(x, y)$$

Thus, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$\nu + x\frac{d\nu}{dx} = \frac{x + \nu x}{x - \nu x} = \frac{1+\nu}{1-\nu}$$

$$x\frac{dv}{dx} = \frac{1+\nu}{1-\nu} - \nu = \frac{1+\nu-\nu(1-\nu}{1-\nu}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1-\nu}$$

$$\Rightarrow \frac{1-\nu}{\left(1+\nu^2\right)}d\nu = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1+\nu^2} - \frac{1}{1-\nu^2}\right)dx = \frac{dx}{x}$$

Integrating both sides, we get

$$\tan^{-1}\nu - \frac{1}{2}\log\left(1+\nu^2\right) = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left[1+\left(\frac{y}{x}\right)^2\right] = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\left[\log\left(x^2+y^2\right) - \log x^2\right] = \log x + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log\left(x^2+y^2\right) + C$$

This is the required solution of the given differential equation.

4:

$$\left(x^2 - y^2\right)dx + 2xy\,dy = 0$$

Solution:

The given differential equation is:

$$\left(x^2 - y^2\right)dx + 2xy\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left(x^2 - y^2\right)}{2xy} \qquad \ldots\ldots (1)$$

Let (x, y) $= \frac{-\left(x^2 - y^2\right)}{2xy}$

$$\therefore\ F(\lambda x, \lambda y) = \left[\frac{(\lambda x)^2 - (\lambda y)^2}{2(\lambda x)(\lambda y)}\right] = \frac{-\left(x^2 - y^2\right)}{2xy} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$\nu + x\frac{d\nu}{dx} = -\left[\frac{x^2 - (vx)^2}{2x.(vx)}\right]$$

$$\nu + x\frac{d\nu}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x\frac{d\nu}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x\frac{d\nu}{dx} = -f\frac{\left(1 + v^2\right)}{2v}$$

$$\Rightarrow \frac{2v}{1 + v^2}dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left(1 + v^2\right) = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow 1+v^2=\frac{C}{x}$$

$$\Rightarrow \left[1+\frac{y^2}{x^2}\right]=\frac{C}{x}$$

$$\Rightarrow x^2+y^2=Cx$$

This is the required solution of the given differential equation.

5:

$$x^2\frac{dy}{dx}-x^2-2y^2+xy$$

Solution:

The given differential equation is:

$$x^2\frac{dy}{dx}-x^2-2y^2+xy$$

$$\frac{dy}{dx}=\frac{x^2-2y^2+xy}{x^2} \quad \text{.... (1)}$$

Let $F(x, y)=\dfrac{x^2-2y^2+xy}{x^2}$

$$\therefore\ F(\lambda x,\lambda y)=\frac{(\lambda x)^2-2(\lambda y)^2+(\lambda x)(\lambda y)}{(\lambda x)^2}=\frac{x^2-2y^2+xy}{x^2}=\lambda^0F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$\nu+x\frac{d\nu}{dx}=\frac{x^2-2(vx)^2+x.(vx)}{x^2}$$

$$\Rightarrow \nu+x\frac{d\nu}{dx}=1-2v^2+v$$

$$\Rightarrow x\frac{d\nu}{dx}=1-2v^2$$

$$\Rightarrow \frac{dv}{1-2v^2}=\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\frac{dv}{\frac{1}{2}-v^2}=\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2-v^2}\right]=\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\frac{1}{2\times\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right|=\log|x|+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}}+\frac{y}{x}}{\frac{1}{\sqrt{2}}-\frac{y}{x}}\right|=\log|x|+C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right|=\log|x|+C$$

This is the required solution for the given differential equation.

6:

$$xdy-ydy=\sqrt{x^2+y^2}dx$$

Solution:

$$xdy-ydy=\sqrt{x^2+y^2}dx$$

$$\Rightarrow xdy=\left[y+\sqrt{x^2+y^2}\right]dx$$

$$\frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x^2} \quad \ldots.(1)$$

Let $F(x,y)=\frac{y+\sqrt{x^2+y^2}}{x^2}$

$$\therefore\ F(\lambda x,\lambda y) = \frac{(\lambda x)+\sqrt{(\lambda x)^2(\lambda y)^2}}{\lambda x} = \frac{y+\sqrt{x^2+y^2}}{x} = \lambda^0 F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of v and $\frac{dy}{dx}$ in equation (1), we get:

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx+\sqrt{x^2+(vx)^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|v+\sqrt{1+v^2}\right| = \log|x| + \log C$$

$$\Rightarrow \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \log|Cx|$$

$$\Rightarrow \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right| = \log|Cx|$$

$$\Rightarrow y+\sqrt{x^2+y^2} = Cx^2$$

This is the required solution of the given differential equation.

7:

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx = \left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdx$$

Solution:

The given differential equation is:

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdx$$

$$\frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x} \quad \ldots\ldots (1)$$

$$\text{Let } F(x,y)=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\therefore F(\lambda x,\lambda y)=\frac{\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)+\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda y}{\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda x}$$

$$=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$=\lambda^0 F.(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get;

$$v+x\frac{dv}{dx}=\frac{(x\cos v+vx\sin v).vx}{(vx\sin v-x\cos v)x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v - v^2 \sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \left[\frac{v\sin v - \cos v}{v\cos v}\right]dv = \frac{2dx}{x}$$

$$\Rightarrow \left(\tan v - \frac{1}{v}\right)dv = \frac{2dx}{x}$$

Integrating both sides, we get:

log(sec v) – log v = 2log x + log C

$$\Rightarrow \log\left(\frac{\sec v}{v}\right) = \log\left(Cx^2\right)$$

$$\Rightarrow \left(\frac{\sec v}{v}\right) = Cx^2$$

$$\Rightarrow \sec v = Cx^2 v$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = C.x^2.\frac{y}{x}$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = Cxy$$

$$\Rightarrow \cos\left(\frac{y}{x}\right) = \frac{1}{Cxy} = \frac{1}{C}.\frac{1}{xy}$$

$$\Rightarrow xy\cos\left(\frac{y}{x}\right) = k \quad \left(k = \frac{1}{C}\right)$$

This is the required solution of the given differential equation.

8:

$$x\frac{dy}{dx} - y + \sin\left(\frac{y}{x}\right) = 0$$

Solution:

$$x\frac{dy}{dx} - y + \sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} \quad \ldots\ldots (1)$$

Let $F(x, y) = \dfrac{y - x\sin\left(\frac{y}{x}\right)}{x}$

$$\therefore\ F(\lambda x, \lambda y)\frac{dy}{dx} = \frac{\lambda y - \lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
$y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx - x\sin v}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow -\frac{dv}{\sin v} = \frac{dx}{x}$$

$$\Rightarrow \operatorname{cosec} v\, dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|\operatorname{cosec} v - \cot v\right| = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow \operatorname{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = \frac{C}{x}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{y}{x}\right)} - \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{c}{x}$$

$$\Rightarrow x\left[1-\cos\left(\frac{y}{x}\right)\right] = C\sin\left(\frac{y}{x}\right)$$

This is the required solution of the given differential equation.

9:

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

Solution:

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

$$\Rightarrow ydx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \quad \text{..... (1)}$$

Let $F(x,y) = \dfrac{y}{2x - x\log\left(\frac{y}{x}\right)}$

$$\therefore\ F(\lambda x,\lambda y) = \frac{\lambda y}{2(\lambda x) - (\lambda x)\log\left(\frac{\lambda y}{\lambda x}\right)} = \frac{y}{2x - \log\left(\frac{y}{x}\right)} = \lambda^0.F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
$y = vx$

$$\Rightarrow \frac{d}{dx} = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\dfrac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\log v - x}{2 - \log v}$$

$$\Rightarrow \frac{2 - \log v}{v(\log v - 1)}dv = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1 + (1 - \log v)}{v(\log v - 1)}\right]dv = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{v(\log v - 1)} - \frac{1}{v}\right]dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{1}{v(\log v - 1)}dv - \int \frac{1}{v}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{dv}{v(\log v - 1)} - \log v = \log x + \log C \qquad \text{..... (2)}$$

$$\Rightarrow \text{Let } \log v - 1 = t$$

$$\Rightarrow \frac{d}{dv}(\log v - 1) = \frac{dt}{dv}$$

$$\Rightarrow \frac{1}{v} = \frac{dt}{dv}$$

$$\Rightarrow \frac{dv}{v} = dt$$

Therefore, equation (1) becomes:

$$\Rightarrow \int \frac{dt}{t} - \log v = \log x + \log C$$

$$\Rightarrow \log t - \log\left(\frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \log\left[\log\left(\frac{y}{x}\right) - 1\right] - \log\left(\frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \log\left[\frac{\log\left(\frac{y}{x}\right) - 1}{\frac{y}{x}}\right] = \log(Cx)$$

$$\Rightarrow \frac{x}{y}\left[\log\left(\frac{y}{x}\right) - 1\right] = Cx$$

$$\Rightarrow \log\left(\frac{y}{x}\right) - 1 = Cy$$

This is the required solution of the given differential equation.

10:

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

Solution:

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\Rightarrow \left(1 + e^{\frac{x}{y}}\right)dx = -e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{-e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}} \qquad \ldots\ldots (1)$$

Let $F(x, y) = \dfrac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$

$$\therefore\ F(\lambda x, \lambda y) = \frac{-e^{\frac{\lambda x}{y\lambda}}\left(1-\frac{\lambda x}{\lambda y}\right)}{1+e^{\frac{\lambda x}{y\lambda}}} = \frac{-e^{\frac{\lambda x}{y\lambda}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{\lambda}}} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
$x = vy$

$$\Rightarrow \frac{d}{dy}(x) = \frac{d}{dy}(vy)$$

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the values of x and $\frac{dx}{dy}$ in equation (1), we get:

$$v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v}{1+e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \left[\frac{v+e^v}{1+e^v}\right]$$

$$\Rightarrow \left[\frac{1+e^v}{v+e^v}\right]dv = -\frac{dy}{y}$$

Integrating both sides, we get:

$$\Rightarrow \log\left(v+e^{v}\right)=-\log y+\log C=\log\left(\frac{C}{y}\right)$$

$$\Rightarrow \left[\frac{x}{y}+e^{\frac{x}{y}}\right]=\frac{c}{y}$$

$$\Rightarrow x+xy^{\frac{x}{y}}=c$$

This is the required solution of the given differential equation.

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11:

$(x+y)dy\,(+x-y)dx=0; y=1$ when $x=1$

Solution:

$(x+y)dy\,(+x-y)dx=0$

$$\Rightarrow (x+y)dy-(x-y)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-y)}{x+y} \quad \text{..... (1)}$$

Let $F(x, y)=\dfrac{-(x-y)}{x+y}$

$$\therefore\ F(\lambda x,\lambda y)=\frac{-(\lambda x-\lambda y)}{\lambda x+\lambda y}=\frac{-(x-y)}{x+y}=\lambda^{0}F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{-(x - vx)}{x + vx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v - 1}{v + 1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v = \frac{v - 1 - v(v + 1)}{v + 1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - 1 - v^2 - v}{v + 1} = \frac{-(1 + v^2)}{v + 1}$$

$$\Rightarrow \frac{(v + 1)}{1 + v^2}dv = -\frac{dx}{x}$$

$$\Rightarrow \left[\frac{v}{1 + v^2} + \frac{1}{1 + v^2}\right]dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\log(1 + v^2) + \tan^{-1} v = -\log x + k$$

$$\Rightarrow \log(1 + v^2) + 2\tan^{-1} v = -2\log x + 2k$$

$$\Rightarrow \log\left[(1 + v^2)x^2\right] + 2\tan^{-1} v = 2k$$

$$\Rightarrow \log\left[\left(1 + \frac{y^2}{x^2}\right)x^2\right] + 2\tan^{-1}\frac{y}{x} = 2k$$

$$\Rightarrow \log(x^2 + y^2) + 2\tan^{-1}\frac{y}{x} = 2k \qquad \text{..... (2)}$$

Now, $y = 1$ at $x = 1$

$$\Rightarrow \log 2 + 2\tan^{-1} 1 = 2k$$

$$\Rightarrow \log 2 + 2 \times \frac{\pi}{4} = 2k$$

$$\Rightarrow \frac{\pi}{2} + \log 2 = 2k$$

Substituting the value of 2k in equation (2), we get:

$$\log(x^2 + y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2$$

This is the required solution of the given differential equation.

12:

$x^2 dy + (xy + y^2)dx = 0; y = 1 \text{ where } x = 1$

Solution:

$$x^2dy + (xy + y^2)dx = 0$$

$$\Rightarrow x^2dy = -(xy + y^2)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \qquad .....(1)$$

Let $F(x, y) = \dfrac{-(xy + y^2)}{x^2}$

$$\therefore F(\lambda x, \lambda y) = \frac{\left[\lambda x.\lambda y + (\lambda y)^2\right]}{(\lambda x)^2} = \frac{-(xy + y^2)}{x^2} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$v + x\frac{dy}{dx} = \frac{-\left[x.vx + (vx)^2\right]}{x^2} = -v - v^2$$

$$\Rightarrow x\frac{dy}{dx} = -v^2 - 2v = -v(v + 2)$$

$$\Rightarrow \frac{dv}{v(v+2)} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{(v+2) - v}{v(v+2)}\right]dv = -\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{2} - \frac{1}{v+2}\right]dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\left[\log v-\log(v+2)\right]=-\log x+\log C$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{v}{v+2}\right)=\log\frac{C}{x}$$

$$\Rightarrow \frac{v}{v+2}=\left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{\frac{y}{x}}{\frac{y}{x}+2}=\left(\frac{C}{x}\right)^2$$

$$\Rightarrow \frac{y}{y+2x}=\frac{c^2}{x^2}$$

$$\Rightarrow \frac{x^2y}{y+2x}=C^2 \quad \text{..... (2)}$$

Now, y = 1 at x = 1

$$\Rightarrow \frac{1}{1+2}=C^2$$

$$\Rightarrow C^2=\frac{1}{3}$$

Substituting $C^2=\frac{1}{3}$ in equation (2), we get;

$$\frac{x^2y}{y+2x}=\frac{1}{3}$$

$$\Rightarrow y+2x=3x^2y$$

This is the required solution of the given differential equation.

13:

$$\left[x\sin^2\left(\frac{x}{y}-y\right)\right]dx+xdy=0;\ y\frac{\pi}{4}\text{when}\ x=1$$

Solution:

$$\left[x\sin^2\left(\frac{x}{y}-y\right)\right]dx+xdy=0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x} \quad \ldots. (1)$$

Let F (x, y) $= \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}$

$$\therefore\ F(\lambda x, \lambda y) = = \frac{\left[\lambda x\sin^2\left(\frac{\lambda x}{\lambda y} - \lambda y\right)\right]}{x} = \frac{-\left[x\sin^2\left(\frac{y}{x} - y\right)\right]}{x} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve this differential equation, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{-\left[x\sin^2 v - vx\right]}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\left[\sin^2 v - v\right] = v - \sin^2 v$$

$$\Rightarrow x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \frac{dv}{\sin^2 v} = -\frac{dx}{dx}$$

$$\Rightarrow \operatorname{cosec}^2 dv = -\frac{dx}{dx}$$

Integrating both sides, we get;

$$-\cot v = -\log|x| - C$$

$$\Rightarrow \cot v = \log|x| - C$$

$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|x| + \log C$$

$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|Cx| \qquad \text{.... (2)}$$

Now, $y = \frac{\pi}{4}$ at $x = 1$

$$\Rightarrow \cot\left(\frac{\pi}{4}\right) = \log|C|$$

$$\Rightarrow 1 = \log C$$

$$\Rightarrow C = e^1 = e$$

Substituting C = e in equation (2), we get:

$$\cot\left(\frac{y}{x}\right) = \log|ex|$$

This is the required solution of the given differential equation.

14:

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0; y = 0 \text{ when } x = 1$$

Solution:

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \qquad \text{..... (1)}$$

Let F (x, y) $= \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$

$$\therefore\ F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \operatorname{cosec}\left(\frac{\lambda y}{\lambda x}\right)$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) = F(x, y)\ \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = -\cos ec v$$

$$\Rightarrow -\frac{dv}{\cos ec v} = -\frac{dx}{x}$$

$$\Rightarrow -\sin v dx = \frac{dx}{x}$$

Integrating both sides, we get:

$$\cos v = \log x + \log C = \log|Cx|$$

$$\Rightarrow \cos\left(\frac{y}{x}\right) = \log|Cx| \qquad \text{..... (2)}$$

This is the required solution of the given differential equation.

Now, y = 0 at x = 1.

$$\Rightarrow \cos(0) = \log C$$

$$\Rightarrow 1 = \log C$$

$$\Rightarrow C = e^1 = e$$

Substituting C = e in equation (2), we get:

$$\cos\left(\frac{y}{x}\right) = \log|(ex)|$$

This is the required solution of the given differential equation.

15:

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0; y = 2 \text{ when} x = 1$$

Solution:

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0$$

$$\Rightarrow 2x^2\frac{dy}{dx} = 2xy + y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2} \qquad \text{..... (1)}$$

Let $F(x, y) = \frac{2xy + y^2}{2x^2}$

$$\therefore\ F(\lambda x,\lambda y)=\frac{2(\lambda x)(\lambda y)+(\lambda y)^2}{2(\lambda x)^2}=\frac{2xy+y^2}{2x^2}=\lambda^0F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow\frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{2x(vx)+(vx)^2}{2x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{2v+v^2}{2x^2}$$

$$\Rightarrow v+x\frac{dv}{dx}=v+\frac{v^2}{2}$$

$$\Rightarrow\frac{2}{v^2}dv=\frac{dx}{x}$$

Integrating both sides, we get:

$$2.\frac{v^{-2+1}}{-2+1}=\log|x|+C$$

$$\Rightarrow-\frac{2}{v}=\log|x|+C$$

$$\Rightarrow-\frac{2}{\frac{y}{x}}=\log|x|+C$$

$$\Rightarrow-\frac{2x}{y}=\log|x|+C \qquad .....(2)$$

Now, y = 2 at x = 1

$$\Rightarrow-1=\log(1)+C$$

$$\Rightarrow C=-1$$

Substituting C = -1 in equation (2) we get:

$$-\frac{2x}{y}=\log|x|+1$$

$$\Rightarrow \frac{2x}{y} = \log|x|$$

$$\Rightarrow y = \frac{2x}{1 - \log|x|}, (x \neq 0, x \neq e)$$

This is the required solution of the given differential equation.

16:

A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the

Substitution
A. y = vx
B. v = yx
C. x = vy
D. x = v

Solution:

For solving the homogeneous equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$, we need to make the

substitution as x = vy.
Hence, the correct answer is C.

17:
Which of the following is a homogeneous differential equation?

A. $(4x + 6y + 5)dy - (3y + 2x + 4)dx = 0$

B. $(xy)dx - (x^3 + y^3)dy = 0$

C. $(x^3 + 2y^2)dx + 2xydy = 0$

D. $y^2dx + (x^2 - xy^2 - y^2)dy = 0$

Solution:
Function F (x, y) is said to be the homogenous function of degree n, if $F(\lambda x, \lambda v) = \lambda' F(x, y)$ for any non-zero constant (λ).
Consider the equation given in alternative D:

$$y^2dx + (x^2 - xy^2 - y^2)dy = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{-y^2}{x^2 - xy^2 - y^2} = \frac{y^2}{y^2 + xy - x^2}$$

Let $F(x, y) = \dfrac{y^2}{y^2 + xy - x^2}$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{(\lambda y)^2}{(\lambda y)^2 + (\lambda x)(\lambda y) - (\lambda x)^2}$$

$$= \frac{\lambda^2 y^2}{\lambda^2 (y^2 + xy - x^2)}$$

$$= \lambda^0 \left(\frac{y^2}{y^2 + xy - x^2} \right)$$

$$= \lambda^0 F(x, y)$$

Hence, the differential equation given in alternative D is a homogenous equation.