Exercise 9.6 Page: 413

For each of the differential equations given in Exercises 1 to 12, find the general solution:

1:

$$\frac{dy}{dx}+2y=\sin x$$

Solution:

The given differential equation is $\frac{dy}{dx}+2y=\sin x$

This is in the form of $\frac{dy}{dx}+py=Q$ (where p = 2 and Q = sinx)

Now, I.F = $e^{\int pdx}=e^{\int 2dx}=2^{2x}$

The solution of the given differential equation is given by the relation,

$$Y(l.f)=\int(Q\times I.F)dx+C$$

$$\Rightarrow ye^{2x}=\int \sin x.e^{2x}dx+C \qquad .....(1)$$

Let $I=\int \sin x.e^{2x}$

$$\Rightarrow I=\int \sin x.\int e^{2x}dx-\int\left(\frac{d}{dx}(\sin x).\int e^{2x}dx\right)dx$$

$$\Rightarrow I=\sin x.\frac{e^{2x}}{2}-\int\left(\cos x.\frac{e^{2x}}{2}\right)dx$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x.\int e^{2x} - \int\left(\frac{d}{dx}(\cos x).\int e^{2x}dx\right)dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\cos x.\int\frac{e^{2x}}{2} - \int\left[(-\sin x).\frac{e^{2x}}{2}\right]dx\right]$$

$$\Rightarrow I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} = \frac{1}{4}\int\left(\sin x.e^{2x}\right)dx$$

$$\Rightarrow I = \frac{e^{2x}}{4}(2\sin x - \cos x) - \frac{1}{4}t$$

$$\Rightarrow \frac{5}{4}I = \frac{e^{2x}}{4}(2\sin x - \cos x)$$

$$\Rightarrow I = \frac{e^{2x}}{5}(2\sin x - \cos x)$$

Therefore, equation (1) becomes:

$$ye^{2x} = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

$$\Rightarrow y = \frac{1}{5}(2\sin - \cos x) + Ce^{-2x}$$

This is the required general solution of the given differential equation.

2:

$$\frac{dy}{dx} + 3y = e^{-2x}$$

Solution:

The given differential equation is $\frac{dy}{dx} + 3y = e^{-2x}$(where p = 3 and Q = e^{-2x}

Now I.F = $e^{\int pdx} = e^{\int 3dx} = 3^{3x}$

The solution of the given differential equation is given by the relation,

$$\Rightarrow ye^{3x} = \int\left(e^{-2x} \times e^{3x}\right) + C$$

$$\Rightarrow ye^{3x} = \int e^{x}dx + C$$

$$\Rightarrow ye^{3x} = e^{x} + C$$

$$\Rightarrow y = e^{-2x} + Ce^{-3x}$$

This is the required general solution of the given differential equation.

3:
$$\frac{dy}{dx}=\frac{y}{x}=x^2$$

Solution:
The given differential equation is:

(Where $p=\frac{1}{x}$ and $Q=x^2$)

Now, I.F = $e^{\int pdx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$
The solution of the given differential equation is given by the relation,

$$\Rightarrow y(x)=\int\left(x^2.x\right)dx+C$$
$$\Rightarrow xy=\int x^3dx+C$$
$$\Rightarrow xy=\frac{x^4}{4}+C$$
This is the required general solution of the given differential equation.

4:
$$\frac{dy}{dx}+\sec xy=\tan x\left(0\le x\le\frac{\pi}{2}\right)$$

Solution:
The given differential equation is:

(where p = sexx and Q = tanx)

Now I.F. $e^{\int pdx}=e^{\int \sec .xdx}=e^{\log(\sec x+\tan x)}$ = sec x + tan x
The general solution of the given differential equation is given by the relation,

$$\Rightarrow y(\sec x+\tan x)=\int \tan x(\sec x+\tan x)dx+C$$

$$\Rightarrow y(\sec x+\tan x)=\int \sec x\tan xdx+\int \tan^2 xdx+C$$

$$\Rightarrow y(\sec x+\tan x)=\sec x+\int\left(\sec^2 x-1\right)dx+C$$

$$\Rightarrow y\left(\sec x+\tan x\right)=\sec x+\tan x-x+C$$

5:

$$\int_0^{\frac{x}{2}} \cos 2xdx$$

Solution:

Let $I=\int_0^{\frac{x}{2}} \cos 2xdx$

$$\int \cos 2xdx=\left(\frac{\sin 2x}{2}\right)=F\left(x\right)$$

By second fundamental theorem of calculus, we obtain

$$I=F\left(\frac{\pi}{2}\right)-F=\left(0\right)$$

$$=\frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right)-\sin 0\right]$$

$$=\frac{1}{2}[\sin\pi-\sin 0]$$

$$=\frac{1}{2}[0-0]=0$$

6:

$$x\frac{dy}{dx}+2y=x^2\log x$$

Solution:

The given differential equation is:

$$x\frac{dy}{dx}+2y=x^2\log x$$

$$\Rightarrow \frac{dy}{dx}+\frac{2}{x}y=x\log x$$

This equation is in the form of a linear differential equation as:

(Where $p=\frac{2}{x}$ and Q = x log x)

Now, IF = $e^{\int p dx}=e^{\int \frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F})=\int(Q\times \text{I.F})dx+C$$

$$\Rightarrow y.x^2=\int\left(x\log x.x^2\right)dx+C$$

$$\Rightarrow x^2y=\int\left(x^3\log x\right)dx+C$$

$$\Rightarrow x^2y=\log x.\int x^3dx-\int\left[\frac{d}{dx}(\log x).\int x^3dx\right]dx+C$$

$$\Rightarrow x^2y=\log x.\frac{x^4}{4}-\int\left(\frac{1}{x}.\frac{x^4}{4}\right)dx+C$$

$$\Rightarrow x^2y=\frac{x^4\log x}{4}-\frac{1}{4}\int x^3dx+C$$

$$\Rightarrow x^2y=\frac{x^4\log x}{4}-\frac{1}{4}\frac{x^4}{4}+C$$

$$\Rightarrow x^2y=\frac{1}{16}x^4\left(4\log x-1\right)+C$$

$$\Rightarrow y=\frac{1}{16}x^4\left(4\log x-1\right)+Cx^{-2}$$

7:

$$x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$$

Solution:

The given differential equation is:

$$x\log x\frac{dy}{dx}+y=\frac{2}{x}\log x$$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x\log x}=\frac{2}{x^2}$$

This equation is the form of a linear differential equation as:

(Where $p = \frac{1}{x\log x}$ and $Q = \frac{2}{x^2}$)

Now, I.F = $e^{\int p dx} = e^{\int \frac{1}{x\log} dx} = e^{\log(\log x)} = \log x$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dx + C$$

$$\Rightarrow y\log x = \int \left(\frac{2}{x^2}\log x\right)dx + C \;.....(1)$$

Now, $\int \left(\frac{2}{x^2}\log x\right)dx = 2\int \left(\log x.\frac{1}{x^2}\right)dx$

$$= 2\left[\log x.\int \frac{1}{x^2}dx - \int\left\{\frac{d}{dx}(\log x).\int \frac{1}{x^2}dx\right\}dx\right]$$

$$= 2\left[\log x\left(-\frac{1}{x}\right) - \int\left(\frac{1}{x}\left(-\frac{1}{x}\right)\right)dx\right]$$

$$= 2\left[-\frac{\log x}{x} + \int \frac{1}{x^2}dx\right]$$

$$= 2\left[-\frac{\log x}{x} - \frac{1}{x}\right]$$

$$= -\frac{2}{x}(1 + \log x)$$

Substituting the value $\int \left(\frac{2}{x^2}\log x\right)dx$ of in equation (1), we get:

$$y\log x = -\frac{2}{x}(1 + \log x) + C$$

This is the required general solution of the given differential equation.

8:

$$(1 + x^2)dy + 2xy\,dx = \cot x dx\,(x \neq 0)$$

Solution:

$$(1 + x^2)dy + 2xy\,dx = \cot x dx$$

$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{\cot x}{1 + x^2}$$

This equation is a linear differential equation of the form:

$$\left(\text{Where } p=\frac{2x}{1+x^2} \text{ and } Q=\frac{\cot x}{1+x^2}\right)$$

Now, I.F. $e^{\int pdx}=e^{\int \frac{2x}{1+x^2}dx}=e^{\log(1+x^2)}=1+x^2$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F})=\int(Q\times \text{I.F})dx+C$$

$$\Rightarrow y(1+x^2)=\int\left[\frac{\cot x}{1+x^2}\times(1+x^2)\right]dx+C$$

$$\Rightarrow y(1+x^2)=\int \cot\ xdx+C$$

$$\Rightarrow y(1+x^2)=\log|\sin x|+C$$

9:

$$x\frac{dy}{dx}+y-x+xy\cot x=0(x\neq 0)$$

Solution:

$$x\frac{dy}{dx}+y-x+xy\cot x=0$$

$$\Rightarrow x\frac{dy}{dx}+y(1\cot x)y=1$$

$$\Rightarrow x\frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$$

This equation is a linear differential equation of the form:

$$(\text{Where } p=\frac{1}{x}+\cot x \text{ and } Q=1)$$

Now, I.F. $=e^{\int pdx}=e^{\int\left(\frac{1}{x}+\cot x\right)dx}e^{\log+\log(\sin x)}=e^{\log(x\sin x)}=x\sin x$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F})=\int(Q\times \text{I.F})dx+C$$

$$\Rightarrow y(x\sin x)=\int(1\times x\sin x)dx+C$$

$$\Rightarrow y(x\sin x)=\int(x\sin x)dx+C$$

$$\Rightarrow y(x\sin x)=x\int\sin x\,dx-\int\left[\frac{d}{dx}(x).\int\sin xdx\right]+C$$

$$\Rightarrow y(x\sin x) = x - (\cos x) - \int 1.(-\cos x)dx + C$$

$$\Rightarrow y(x\sin x) = -x\cos x + \sin x + C$$

$$\Rightarrow y = \frac{-x\cos x}{x\sin x} + \frac{\sin x}{x\sin x} + \frac{C}{x\sin x}$$

$$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}$$

10:

$$\left(x + y\frac{dy}{dx}\right) = 1$$

Solution:

$$\left(x + y\frac{dy}{dx}\right) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+y}$$

$$\Rightarrow \frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{dy}{dx} - x = y$$

This is a linear differential equation of the form:

(Where p = − 1 and Q = y)

Now, I.F = $= e^{\int pdx} = \int e^{-dy} = e^{-y}$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dy + C$$

$$\Rightarrow xe^{-y} = \int (y.e^{-y})dy + C$$

$$\Rightarrow xe^{-y} = y.\int e^{-y}dy - \int\left[\frac{d}{dy}(y).\int e^{-y}dy\right]dy + C$$

$$\Rightarrow xe^{-y} = y(-e^{-y}) - \int(-e^{-y})dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} + \int e^{-y}dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$$

$$\Rightarrow x = -y - 1 + Ce^{y}$$

$$\Rightarrow x + y + 1 = Ce^{y}$$

11:

$$ydx + \left(x - y^2\right)dy = 0$$

Solution:

$$ydx + \left(x - y^2\right)dy = 0$$

$$\Rightarrow ydx + \left(y^2 - x\right)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{y^2 - x}{y} = y - \frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{y} = y$$

This is a linear differential equation of the form:

(Where $p = \frac{1}{y}$ and Q = y)

Now, I.F. $= e^{\int pdx} = e^{\int \frac{1}{2}dy} = e^{-y} = e^{\log y} = y$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int \left(Q \times \text{I.F}\right)dy + C$$

$$\Rightarrow xy = \int (y.y)dy + C$$

$$\Rightarrow xy = \int y^2 dy + C$$

$$\Rightarrow xy = \frac{y^3}{3} + C$$

$$\Rightarrow x = \frac{y^3}{3} + \frac{C}{y}$$

12:

$$\left(x + 3y^3\right)\frac{dy}{dx} = y\left(y > 0\right)$$

Solution:

$$\left(x+3y^3\right)\frac{dy}{dx}=y$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x+3y^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+3y^2}{y}=\frac{x}{y}+3y$$

$$\Rightarrow \frac{dx}{dy}-\frac{x}{y}=3y$$

This is a linear differential equation of the form:

(Where $p=-\frac{1}{y}$ and $Q=3y$)

Now, I.F. $=e^{\int pdy}=e^{-\int\frac{dy}{y}}=e^{-\log y}=e^{\log\left(\frac{1}{y}\right)}=\frac{1}{y}$

The general solution of the given differential equation is given by the relation,

$$y\left(I.F\right)=\int\left(Q\times I.F\right)dy+C$$

$$\Rightarrow x\times\frac{1}{y}=\int\left(3y\times\frac{1}{y}\right)dy+C$$

$$\Rightarrow \frac{x}{y}=3y+C$$

$$\Rightarrow x=3y^2+Cy$$

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13:

$$\frac{dy}{dx}+2y\tan x=\sin x; y=0 \text{ when } x=\frac{\pi}{3}$$

Solution:

The given differential equation is $\frac{dy}{dx}+2y\tan x=\sin x$

This is a linear equation of the form:

(Where p = 2tanx and Q = sin x)

Now, I.F $=e^{\int pdx}=e^{\int 2\tan xdx}=e^{2\log|\sec x|}=e^{\log\left(\sec^2 x\right)}=\sec^2 x$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F})\,dy + C$$
$$\Rightarrow y(\sec^2 x) = \int (\sin x.\sec^2 x)\,dx + C$$
$$\Rightarrow y\sec^2 x = \int (\sec x.\tan x)\,dx + C$$
$$\Rightarrow y\sec^2 x = \sec x + C \qquad \text{..... (1)}$$

Now, $y = 0$ at $x = \frac{\pi}{3}$

Therefore,

$$0 \times \sec^2 \frac{\pi}{3} = \sec\frac{\pi}{3} + C$$
$$\Rightarrow 0 = 2 + C$$
$$\Rightarrow C = -2$$

Substituting $C = -2$ in equation (1), we get:

$$y\sec^2 x = \sec x - 2$$
$$\Rightarrow y = \cos x - 2\cos^2 x$$

Hence, the required solution of the given differential equation is $y = \cos x - 2\cos^2 x$

14:

$$\left(1 + x^2\right)\frac{dy}{dx} + 2xy = \frac{1}{1 + x^2}; y = 0 \text{ when } x = 1$$

Solution:

$$\left(1 + x^2\right)\frac{dy}{dx} + 2xy = \frac{1}{1 + x^2}$$
$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{1}{\left(1 + x^2\right)^2}$$

This is a linear differential equation of the form:

(Where $p = \frac{2x}{1 + x^2}$ and $Q = \frac{1}{\left(1 + x^2\right)^2}$)

Now I.F. $= e^{\int p dx} = e^{\int \frac{2x dx}{1+x^2}} = e^{\log\left(1+x^2\right)} = 1 + x^2$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F})\,dx + C$$

$$\Rightarrow y\left(1+x^2\right)=\int\left[\frac{1}{\left(1+x^2\right)^2}.\left(1+x^2\right)\right]dx+C$$

$$\Rightarrow y\left(1+x^2\right)=\int\frac{1}{1+x^2}dx+C$$

$$\Rightarrow y\left(1+x^2\right)=\tan^{-1}x+C\ ......(1)$$

Now, y = 0 at x = 1.
Therefore,

$$\Rightarrow C=-\frac{\pi}{4}$$

Substituting $C=-\frac{\pi}{4}$ in equation (1), we get:

$$y\left(1+x^2\right)=\tan^{-1}x-\frac{\pi}{4}$$

This is the required general solution of the given differential equation.

15:

$$\frac{dy}{dx}-3y\cot x=\sin 2x;\ y=2\text{ when }x=\frac{\pi}{2}$$

Solution:

The given differential equation is $\frac{dy}{dx}-3y\cot x=\sin 2x$

This is a linear differential equation of the form:

(Where p = − 3 cot x and Q = sin 2x)

Now, $\text{I.F.}=e^{\int pdx}=e^{-3\int\cot xdx}=e^{-3\log|\sin x|}=e^{\log\left|\frac{1}{\sin^{-1}x}\right|}=\frac{1}{\sin^3 x}$

The general solution of the given differential equation is given by the relation,

$$y\left(\text{I.F}\right)=\int\left(Q\times\text{I.F}\right)dx+C$$

$$\Rightarrow y.\frac{1}{\sin^3 x}=\int\left[\sin 2x.\frac{1}{\sin^3 x}\right]dx+C$$

$$\Rightarrow y\operatorname{cosec}^3 x=2\int\left(\cot x\operatorname{cosec}x\right)dx+C$$

$$\Rightarrow y\operatorname{cosec}^3 x=2\operatorname{cosec}x+C$$

$$\Rightarrow y = -\frac{2}{\operatorname{cosec}^2 x} + \frac{3}{\operatorname{cosec}^2 x}$$

$$\Rightarrow y = -2\sin^2 x + C\sin^3 x$$

Now, $y = 2 \text{ at } x = \frac{\pi}{2}$

Therefore, we get:

$2 = -2 + C$

$\Rightarrow C = 4$

Substituting C = 4 in equation (1), we get:

$$y = -2\sin^2 x + 4\sin^3 x$$

$$\Rightarrow y = 4\sin^3 x - 2\sin^2 x$$

This is the required particular solution of the given differential equation.

16:

Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Solution:

Let F (x, y) be the curve passing through the origin.

At point (x, y), the slope of the curve will be $\frac{dy}{dx}$.

According to the given information:

$$\frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{dy}{dx} = -y = x$$

This is a linear differential equation of the form:

(Where p = −1 and Q = x)

Now, I.F = $= e^{\int p dx} = e^{\int (-1) dx} = e^{-x}$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F}) dx + C$$

Now, $\int xe^{-x} dx = x\int e^{-x} dx - \int \left[\frac{d}{dx}(x).\int e^{-x} dx\right] dx$

$= -xe^{-x} - \int e^{-x} dx$

$= -xe^{-x} + \left(-e^{-x}\right)$

$= -e^{-x}\left(x+1\right)$

Substituting in equation (1), we get:

$ye^{-x} = -e^{-x}\left(x+1\right) + C$

$\Rightarrow y = -\left(x+1\right) + Ce^{x}$

$\Rightarrow x + y + 1 = Ce^{x}$ …..(2)

The curve passes through the origin.
Therefore, equation (2) becomes:
1 = C
Substituting C = 1 in equation (2), we get:

$\Rightarrow x + y + 1 = e^{x}$

Hence, the required equation of curve passing through the origin is $x + y + 1 = e^{x}$

17:
Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Solution:
Let F (x, y) be the curve and let (x, y) be a point on the curve. The slope of the tangent to the curve at (x, y) is $\frac{dy}{dx}$

According to the given information:

$\frac{dy}{dx} + 5 = x + y$

$\Rightarrow \frac{dy}{dx} - y = x + 5$

This is a linear differential equation of the form:

(where p = −1 and Q = x – 5)

Now, I.F. $= e^{\int p\,dx} = e^{\int(-1)dx} = e^{-x}$

The general equation of the curve is given by the relation,

$y\left(I.F\right) = \int\left(Q \times I.F\right)dx + C$

$\Rightarrow y.e^{-x} = \int\left(x-5\right)e^{-x}dx + C$ ……(1)

Now, $\int\left(x-5\right)e^{-x}dx = \left(x-5\right)\int e^{-x}dx - \int\left[\frac{d}{dx}\left(x-5\right).\int e^{-x}dx\right]dx$

$=(x-5)(-e^{-x})-\int(-e^{-x})dx$

$=(5-x)e^{-x}+(e^{-x})$

$=(4-x)e^{-x}$

Therefore, equation (1) becomes:

The curve passes through point (0, 2).
Therefore, equation (2) becomes:
0 + 2 – 4 = Ce0
⇒ – 2 = C
⇒ C = – 2
Substituting C = –2 in equation (2), we get:

$x+y-4=-2e^{x}$

$\Rightarrow y=4-x-2e^{x}$

This is the required equation of the curve.

18:

The integrating factor of the differential equation $x\frac{dy}{dx}-y=2x^2$ is

A. e^{-x}

B. e^{-y}

C. $\frac{1}{x}$

D. x

Solution:

The given differential equation is:

$x\frac{dy}{dx}-y=2x^2$

$\Rightarrow \frac{dy}{dx}-\frac{y}{x}=2x$

This is a linear differential equation of the form:

(Where $p=-\frac{1}{x}$ and Q = 2x)

The integrating factor (I.F) is given by the relation,

$e^{\int pdx}$

$\therefore$ I.F. $=e^{\int \frac{1}{x}dx}=e^{-\log x}=e^{\log(x^{-1})}=x^{-1}=\frac{1}{x}$

Hence, the correct answer is C.

19:

The integrating factor of the differential equation.

$$\left(1-y^2\right)\frac{dx}{dy}+yx=ay\left(-1>y<1\right)$$

A. $\frac{1}{y^2-1}$

B. $\frac{1}{\sqrt{y^2-1}}$

C. $\frac{1}{1-y^2}$

D. $\frac{1}{\sqrt{1-y^2}}$

Solution:

The given differential equation is:

$$\left(1-y^2\right)\frac{dx}{dy}+yx=ay$$

$$\Rightarrow\frac{dy}{dx}+\frac{yx}{1-y^2}=\frac{ay}{1-y^2}$$

This is a linear differential equation of the form:

(where $p=-\frac{y}{1-y^2}$ and $Q=\frac{ay}{1-y^2}$)

The integrating factor (I.F) is given by the relation,

$e^{\int pdx}$

$$\therefore I.F=e^{\int pdx}=e^{\int\frac{1}{1-y^2}dx}=e^{\frac{1}{2}\log x\left(1-y^2\right)}=e^{\log\left[\frac{1}{\sqrt{1-y^2}}\right]}=\frac{1}{1-y^2}$$

Hence, the correct answer is D.