Exercise 9.1 Page: 382

Determine order and degree (if defined) of differential equations given in Exercise 1 to 10.

1. $\frac{d^4y}{dx^4}+\sin(y''')=0$

Solution: $\frac{d^4y}{dx^4}+\sin(y''')=0$

$\Rightarrow y''''+\sin(y''')=0$

The highest order derivative present in the differential equation is y''''. Therefore, its order is four.

The given differential equation is not a polynomial equation in its derivatives. Hence, its degree is not defined.

2. y '+ 5y = 0

Solution: The given differential equation is:

y '+5y = 0

The highest order derivative present in the differential equation is y'. Therefore, its order is one.

It is a polynomial equation in y'. The highest power raised to y' is 1. Hence, its degree is one.

3: $\left(\frac{ds}{dt}\right)^4+3s\frac{d^2s}{dt^2}=0$

Solution: $\left(\frac{ds}{dt}\right)^4+3s\frac{d^2s}{dt^2}=0$

The highest order derivative present in the given differential equation is $\frac{d^2s}{dt^2}$. Therefore, its order is two.

It is a polynomial equation in $\frac{d^2s}{dt^2}$ and $\frac{ds}{dt}$. The power raised to $\frac{d^2s}{dt^2}$ is 1.

Hence, its degree is one.

4:

$$\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$$

Solution: $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is 2.

The given differential equation is not a polynomial equation in its derivatives. Hence, its degree is not defined.

5:

$$\left(\frac{d^2y}{dx^2}\right)^2 = \cos 3x + \sin 3x$$

Solution: $\left(\frac{d^2y}{dx^2}\right)^2 = \cos 3x + \sin 3x$

$$\Rightarrow \frac{d^2y}{dx^2} - \cos 3x + \sin 3x = 0$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

It is a polynomial equation $\frac{d^2y}{dx^2}$ in and the power raised to $\frac{d^2y}{dx^2}$ is 1.

Hence, its degree is one.

6:

$$(y''')^2 + \ y'')^3 + (y')^4 + y^5 = 0$$

Solution : $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

The highest order derivative present in the differential equation is y'''. Therefore, its order is three.

The given differential equation is a polynomial equation in y''', y'', and y'.

The highest power raised to y''' is 2. Hence, its degree is 2.

7: y''' + 2y'' + y' = 0

Solution: y''' + 2y'' + y ' = 0

The highest order derivative present in the differential equation is y'''. Therefore, its order is three.

It is a polynomial equation in y''', y'', and y'. The highest power raised to y'' is 1. Hence, its degree is 1.

8: y'+ y = e'

Solution: y' + y = e'

$\Rightarrow$ y' + y - e' = 0

The highest order derivative present in the differential equation is y'. Therefore, its order is one.

The given differential equation is a polynomial equation in y' and the highest power raised to y' is one. Hence, its degree is one.

9: $y' + (y')^2 + 2y = 0$

Solution: : $y' + (y')^2 + 2y = 0$

The highest order derivative present in the differential equation is y''. Therefore, its order is two.

The given differential equation is a polynomial equation in y '' and y' and the highest power raised to y'' is one.

Hence, its degree is one.

10: y'' + 2y' + sin y = 0

Solution: y'' + 2y' + sin y = 0

The highest order derivative present in the differential equation is y''. Therefore, its order is two.

This is a polynomial equation in y'' and y' and the highest power raised y'' to is one.

Hence, its degree is one.

11: The degree of the differential equation

$\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$ is

(A) 3

(B) 2

(C) 1

(D) Not defined

Solution $\left(\frac{d^2y}{dx^2}\right)^3+\left(\frac{dy}{dx}\right)^2+\sin\left(\frac{dy}{dx}\right)+1=0$

The given differential equation is not a polynomial equation in its derivatives. Therefore, its degree is not defined.

Hence, the correct answer is D.

12: The order of the differential equation

$2x^2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0$

(A) 2

(B) 1

(C) 0

(D) not defined

Solution: $2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$

The highest order derivative present in the given differential equation is $\frac{d^2y}{dx^2}$. Therefore, its order is two.

Hence, the correct answer is A.

Exercise 9.2 Page: 385

For each of the differential equations in Exercise 1 to 10. find the general solution:

1: $y = e^x + 1 \qquad : y'' - y' = 0$

Solution: $y = e^x + 1$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{dy}{dx} = \frac{d}{dx}(e^x + 1)$$

$$\Rightarrow y' = e^x \qquad ...(1)$$

Now, differentiating equation (1) with respect to x, we get:

$$\frac{d}{dx}(y') = \frac{d}{dx}(e^x)$$

$$\Rightarrow y^n = e^x$$

Substituting the values of and in the given differential equation, we get the L.H.S. as:

$$y'' - y' = e^x = 0 = \text{RHS}$$

Thus, the given function is the solution of the corresponding differential equation.

2: y = x^2 + 2x + C : y' – 2x – 2 = 0

Solution: y = x^2 + 2x + C

Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}(x^2 + 2x + C)$$

$$\Rightarrow y' = 2x + 2$$

Substituting the value of in the given differential equation, we get:

L.H.S.= y’ – 2x – 2 = 2x + 2 – 2x – 2 = 0 = R.H.S

Hence, the given function is the solution of the corresponding differential equation.

3: y = cosx + C : y' + sinx = 0

Solution: : y = cosx + C
Differentiating both sides of this equation with respect to x, we get:

$$y' = \frac{d}{dx}(\cos x + C)$$

$$\Rightarrow y' = \sin x$$

Substituting the value of in the given differential equation, we get:
L.H.S. = y' + sinx = -sinx + sinx = 0 = R.H.S.
Hence, the given function is the solution of the corresponding differential equation.

4: $y = \sqrt{1+x^2}$: $y' = \frac{xy}{1+x^2}$

Solution : $y = \sqrt{1+x^2}$
Differentiating both sides of the equation with respect to x, we get:

$$y' = \frac{d}{dx}(\sqrt{1+x^2})$$

$$y' = \frac{1}{2\sqrt{1+x^2}}.\frac{d}{dx}(1+x^2)$$

$$y' = \frac{2}{2\sqrt{1+x^2}}$$

$$y' = \frac{x}{2\sqrt{1+x^2}}$$

$$\Rightarrow y' = \frac{x}{1+x^2} \times \sqrt{1+x^2}$$

$$\Rightarrow y' = \frac{x}{1+x^2}.y$$

$$\Rightarrow y' = \frac{xy}{1+x^2}$$

$\therefore$ L.H.S. = R.H.S.
Hence, the given function is the solution of the corresponding differential equation

5: y = Ax $\therefore$ xy'=y(x $\neq$ 0)

Solution: y = Ax
Differentiating both sides with respect to x, we get:

$y' = \frac{d}{dx}(Ax)$

$\Rightarrow y' = A$

Substituting the value of in the given differential equation, we get:
L.H.S. = xy'= xA = Ax = y = R.H.S.
Hence, the given function is the solution of the corresponding differential equation.

6: $y = x\sin x$ $\quad : xy' = y + x\sqrt{x^2 - y^2}\,(x \neq 0 \text{ and } x > y) \text{or } x < -y)$

Solution : $y = x\sin x$

Differentiating both sides of this equation with respect to x, we get:

$y' = \frac{d}{dx}(x\sin x)$

$\Rightarrow y' = \sin x.\frac{d}{dx}(x).\frac{d}{dx}(\sin x)$

$\Rightarrow y' = \sin x + x\cos x$

Substituting the value of in the given differential equation, we get:

L.H.S.= $xy' = x(\sin x + x\cos x)$

= $x\sin x + x^2\cos x$

= $y + x^2.\sqrt{1-\sin^2 x}$

= $y + x^2\sqrt{1-\left(\frac{y}{x}\right)^2}$

= $y + x\sqrt{y^2 - x^2}$

= R.H.S

Hence, the given function is the solution of the corresponding differential equation.

7: $xy = \log y + C$ $\quad : y' = \frac{y^2}{1-xy}\,(xy \neq 1)$

Solution : $xy = \log y + C$

Differentiating both sides of this equation with respect to x, we get:

$\frac{d}{dx}(xy) = \frac{d}{dx}(\log y)$

$\Rightarrow y\frac{d}{dx}(x) + x.\frac{d}{dx} = \frac{1}{y}\frac{dy}{dx}$

$\Rightarrow y + xy' = \frac{1}{y}y'$

$\Rightarrow y^2 + xy\,y' = y'$

$\Rightarrow (xy-1)\,y' = -y^2$

$\Rightarrow y' = \dfrac{y^2}{1-xy}$

$\therefore$ L.H.S. = R.H.S.

Hence, the given function is the solution of the corresponding differential equation.

8: $y - \cos y = x \qquad : (y\sin y + \cos y + x)y' = 1$

Solution: $y - \cos y = x \qquad \ldots(1)$

Differentiating both sides of the equation with respect to x, we get:

$\dfrac{dy}{dx} - \dfrac{d}{dx}(\cos y) = \dfrac{d}{dx}(x)$

$\Rightarrow y'\sin y.y' = 1$

$\Rightarrow y'(1+\sin y) = 1$

$\Rightarrow y' = \dfrac{1}{1+\sin y}$

Substituting the value of in equation (1), we get:

L.H.S.= $(y\sin y + \cos y + x)y'$

= $(y\sin y + \cos y + y - \cos y) \times \dfrac{1}{1+\sin y}$

= $y(1+\sin y).\dfrac{1}{1+\sin y}$

= y

= R.H.S

Hence, the given function is the solution of the corresponding differential equation.

9: $x + y = \tan^{-1} y \qquad : y^2 y' + y^2 + 1 = 0$

Solution: $x + y = \tan^{-1} y$

Differentiating both sides of this equation with respect to x, we get:

$\dfrac{d}{dx}(x+y) = \dfrac{d}{dx}(\tan^{-1} y)$

$\Rightarrow 1 + y' = \left[\dfrac{1}{1+y^2}\right] y'$

$\Rightarrow y'\left[\dfrac{1}{1+y^2} - 1\right] = 1$

$$\Rightarrow y'\left[\frac{1-\left(1+y^2\right)}{1+y^2}\right]=1$$

$$\Rightarrow y'\left[\frac{-y^2}{1+y^2}\right]=1$$

$$\Rightarrow y'=\frac{(-1+y^2)}{y^2}$$

Substituting the value of in the given differential equation, we get:

L.H.S.= $y^2y'+y^2+1=y^2\left[\frac{-\left(1+y^2\right)}{y^2}\right]+y^2+1$

= -1-y^2+y^2+1

= 0

= R.H.S

Hence, the given function is the solution of the corresponding differential equation.

10: $y=\sqrt{a^2-x^2}x\in(-a,a)$ $\quad : x+y\frac{dy}{dx}=0\,(y\neq 0)$

Solution: $y=\sqrt{a^2-x^2}$

Differentiating both sides of this equation with respect to x, we get:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{a^2-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{2}.\frac{d}{dx}(a^2-x^2)$$

$$=\frac{1}{2\sqrt{a^2-x^2}}.(-2x)$$

$$=\frac{-x}{\sqrt{a^2-x^2}}$$

Substituting the value of $\frac{dy}{dx}$ in the given differential equation, we get:

L.H.S= $x+y\frac{dy}{dx}=x+\sqrt{a^2-x^2}\times\frac{-x}{\sqrt{a^2-x^2}}$

= x-x

= 0

= R.H.S

Hence, the given function is the solution of the corresponding differential equation.

11: The numbers of arbitrary constants in the general solution of a differential equation of fourth order are:

(A) 0

(B) 2

(C) 3
(D) 4

Solution: We know that the number of constants in the general solution of a differential equation of order n is equal to its order.
Therefore, the number of constants in the general equation of fourth order differential equation is four.
Hence, the correct answer is D.

12: The numbers of arbitrary constants in the particular solution of a differential equation of third order are:
(A) 3
(B) 2
(C) 1
(D) 0

Solution: In a particular solution of a differential equation, there are no arbitrary constants.
Hence, the correct answer is D.

Exercise 9.3 Page: 391

In each of the Exercises 1 to 5, form a differential equation representing the given family of curves by eliminating arbitrary constants a and b

1: $\frac{x}{a}+\frac{y}{b}=1$

Solution: $\frac{x}{a}+\frac{y}{b}=1$

Differentiating both sides of the given equation with respect to x, we get:

$\frac{1}{a}+\frac{1}{b}\frac{dy}{dx}=0$

$\Rightarrow \frac{1}{a}+\frac{1}{b}y'=0$

Again, differentiating both sides with respect to x, we get:

$0+\frac{1}{b}y''=0$

$\Rightarrow \frac{1}{b}y''=0$

$\Rightarrow y''=0$

Hence, the required differential equation of the given curve is $y''=0$.

2: $y^2=a(b^2-x^2)$

Solution:

Differentiating both sides with respect to x, we get:

$$2y\frac{dy}{dx}=a(-2x)$$

$$\Rightarrow 2yy'=-2ax$$

$$\Rightarrow yy'=-2ax \qquad \ldots(1)$$

Again, differentiating both sides with respect to x, we get:

$$y'.y'+yy''=-a$$

$$\Rightarrow (y')^2+yy''=-a \qquad \ldots(2)$$

Dividing equation (2) by equation (1), we get:

$$\frac{(y')^2+yy''}{yy''}=\frac{-a}{-ax}$$

$$\Rightarrow xyy''+x(y')^2-yy'=0$$

This is the required differential equation of the given curve.

3: $y=ae^{3x}+be^{-2x}$

Solution: $y=ae^{3x}+be^{-2x}\ldots(1)$

Differentiating both sides with respect to x, we get:

$$y'=3ae^{3x}-2be^{-2x} \qquad \ldots(2)$$

Again, differentiating both sides with respect to x, we get:

$$y''=9ae^{3x}+4be^{-2x} \qquad \ldots(3)$$

Multiplying equation (1) with (2) and then adding it to equation (2), we get:

$$(2ae^{3x}+2be^{-2x})+\left(3ae^{3x}-2be^{-2x}\right)=2y+y'$$

$$\Rightarrow 5ae^{3x}=2y+y'$$

$$\Rightarrow ae^{3x}=\frac{2y+y'}{5}$$

Now, multiplying equation (1) with equation (3) and subtracting equation (2) from it, we get:

$$(3ae^{3x}+3be^{-2x})-\left(3ae^{3x}-2be^{-2x}\right)=3y-y'$$

$$\Rightarrow 5ae^{3x}=3y+y'$$

$$\Rightarrow be^{-2x}=\frac{3y+y'}{5}$$

Substituting the values of in equation (3), we get:

$$y''=9\frac{2y+y'}{5}+4\frac{3y+y'}{5}$$

$$\Rightarrow y''=\frac{18y+y'}{5}+\frac{12y+y'}{5}$$

$$\Rightarrow y''=\frac{30y+5y'}{5}$$

$\Rightarrow y" = 6y + y'$

$\Rightarrow y'' - y' - 6y = 0$

This is the required differential equation of the given curve.

4: $y = e^{2x}(a + bx)$

Solution: $y = e^{2x}(a + bx)$ …(1)

Differentiating both sides with respect to x, we get:

$y' = 2e^{2x}(a + bx) + e^{2x}b$

$\Rightarrow y' = e^{2x}(2a + 2bx + b)$ …(2)

Multiplying equation (1) with equation (2) and then subtracting it from equation (2), we get:

$y' - 2y = e^{2x}(2a + 2bx + b) - e^{2x}(2a + 2bx)$

$\Rightarrow y' - 2 = be^{2x}$ …(3)

Differentiating both sides with respect to x, we get:

$y'k - 2y' = 2be^{2x}$ …(4)

Dividing equation (4) by equation (3), we get:

$$\frac{y"-2y'}{y'-2y} = 2$$

$\Rightarrow y" - 2y' = 2y' - 4y$

$\Rightarrow y" - 4y' + 4y = 0$

This is the required differential equation of the given curve.

5: $y = e^{3x}(a\cos x + b\sin x)$

Solution: $y = e^{3x}(a\cos x + b\sin x)$ …(1)

Differentiating both sides with respect to x, we get:

$y' = e^{x}(a\cos x + b\sin x) + e^{x}(-a\cos x + b\sin x)$

$\Rightarrow y' = e^{x}[(a+b)\cos x - (a-b)\sin x]$ …(2)

Again, differentiating with respect to x, we get:

$\Rightarrow y' = e^{x}[(a+b)\cos x - (a-b)\sin x] + e^{x}[(-a+b)\sin x - (a-b)\cos x)]$

$y" = e^{x}[2b\cos x - 2a\sin x]$

$y" = 2e^{x}(b\cos x - a\sin x)$

$$\Rightarrow \frac{y"}{2} = e^{x}(b\cos x - a\sin x) \quad \ldots(3)$$

Adding equations (1) and (3), we get:

$$y + \frac{y"}{2} = e^{x}[(a+b)\cos x - (a-b)\sin x]$$

$$\Rightarrow y + \frac{y"}{2} = y'$$

$\Rightarrow 2y + y" = y'$

$\Rightarrow y''-2y'+2y=0$

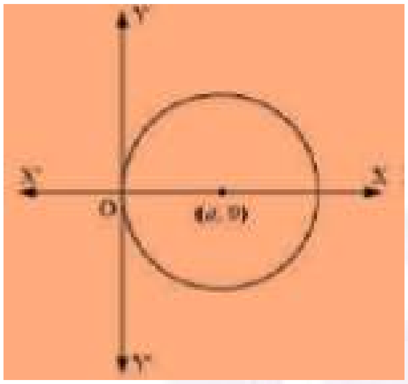
This is the required differential equation of the given curve.

6: Form the differential equation of the family of circles touching the y-axis at the origin.

Solution: The centre of the circle touching the y-axis at origin lies on the x-axis.
Let (a, 0) be the centre of the circle.
Since it touches the y-axis at origin, its radius is a.
Now, the equation of the circle with centre (a, 0) and radius (a) is

$(x-a)^2+y^2=a^2$

$\Rightarrow x^2+y^2=2ax \qquad \ldots(1)$



Differentiating equation (1) with respect to x, we get:

$2x+2yy'=2a$

$\Rightarrow x+yy'=a$

Now, on substituting the value of a in equation (1), we get:

$\Rightarrow x^2+y^2=2(x+yy')x$

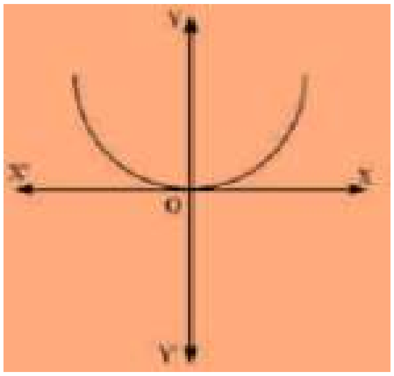
$\Rightarrow 2xyy'+x^2=y^2$

This is the required differential equation.

7: Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.

Solution: The equation of the parabola having the vertex at origin and the axis along the positive y-axis is:

$x^2=4ay \qquad \ldots(1)$



Differentiating equation (1) with respect to x, we get:

$2x = 4ay'$ ...(2)

Dividing equation (2) by equation (1), we get:

$$\frac{2x}{x^2} = \frac{4ay'}{4ay}$$

$$\Rightarrow \frac{2}{x} = \frac{y'}{y}$$

$$\Rightarrow xy' = 2y$$

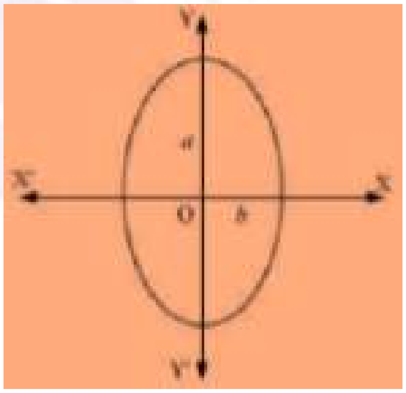
$$\Rightarrow xy' - 2y = 0$$

This is the required differential equation.

8: Form the differential equation of the family of ellipses having foci on y-axis and centre at origin.

Solution: The equation of the family of ellipses having foci on the y-axis and the centre at origin is as follows:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad \ldots(1)$$



Differentiating equation (1) with respect to x, we get:

$$\frac{2x}{b^2} + \frac{2yy'}{a^2} = 0$$

$$\Rightarrow \frac{x}{b^2} + \frac{yy'}{a^2} = 0 \quad \ldots(2)$$

Again, differentiating with respect to x, we get:

$$\frac{1}{b^2}+\frac{y'.y'+y.y''}{a^2}=0$$

$$\Rightarrow \frac{1}{b^2}+\frac{1}{a^2}(y'^2+yy'')=0$$

$$\Rightarrow \frac{1}{b^2}=-\frac{1}{a^2}(y'^2+yy'')=0$$

Substituting this value in equation (2), we get:

$$x\left[-\frac{1}{a^2}(y'^2+yy'')\right]+\frac{yy''}{a^2}=0$$

$$\Rightarrow -x(y')^2-xyy''+yy'=0$$

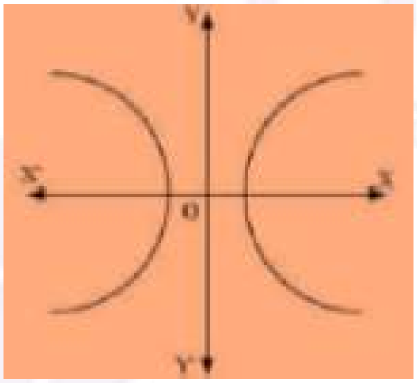
$$-xyy''-x(y)'^2-yy'=0$$

This is the required differential equation.

9: Form the differential equation of the family of hyperbolas having foci on x-axis and centre at origin.

Solution: The equation of the family of hyperbolas with the centre at origin and foci along the x axis is:

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$



Differentiating both sides of equation (1) with respect to x, we get:

$$\frac{2x}{b^2}+\frac{2yy'}{b^2}=0$$

$$\Rightarrow \frac{x}{a^2}+\frac{yy'}{b^2}=0 \qquad \ldots(2)$$

Again, differentiating both sides with respect to x, we get:

$$\frac{1}{a^2}-\frac{y'.y'+y.y''}{b^2}=0$$

Substituting the value of $\frac{1}{a^2}$ in equation (2), we get:

$$\frac{x}{b^2}\left[(y')^2+yy''\right]+\frac{yy'}{b^2}=0$$

$$\Rightarrow x(y')^2+xyy''-yy'=0$$

$\Rightarrow xyy'' - x(y')^2 - yy' = 0$

This is the required differential equation.

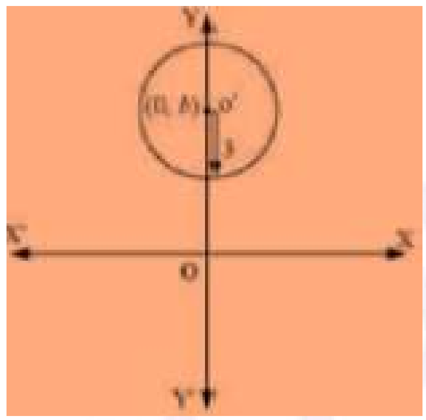
10: Form the differential equation of the family of circles having centre on y-axis and radius 3 units.

Solution: Let the centre of the circle on y-axis be (0, b).

The differential equation of the family of circles with centre at (0, b) and radius 3 is as follows:

$x^2 + (y-b)^2 = 3^2$

$\Rightarrow x^2 + (y-b)^2 = 9 \qquad \ldots(1)$



Differentiating equation (1) with respect to x, we get:

$2x + 2(y-b).y^2 = 0$

$\Rightarrow (y-b).y' = -x$

$\Rightarrow (y-b) = \frac{-x}{y'}$

Substituting the value of (y – b) in equation (1), we get:

$x^2 + \left(\frac{-x}{y'}\right) = 9$

$\Rightarrow x^2\left[1 + \frac{1}{(y')}\right] = 9$

$\Rightarrow x^2((y') + 1) = 9(y')^2$

$\Rightarrow (x^2 - 9)(y')^2 + x^2 = 0$

This is the required differential equation.

11: Which of the following differential equations has $y = c_1e^x + c_2e^{-x}$ as the general **solution?**

A. $\frac{d^2y}{dx^2} + y = 0$

B. $\frac{d^2y}{dx^2} - y = 0$

C. $\frac{d^2y}{dx^2}+1=0$

D. $\frac{d^2y}{dx^2}-y=0$

Solution: The given equation is:

$$y=c_1e^x+c_2e^x \qquad \ldots(1)$$

Differentiating with respect to x, we get:

$$\frac{dy}{dx}=c_1e^x-c_2e^{-x}$$

Again, differentiating with respect to x, we get:

$$\frac{d^2y}{dx^2}=c_1e^x+c_2e^{-x}$$

$$\Rightarrow \frac{d^2y}{dx^2}=y$$

$$\Rightarrow \frac{d^2y}{dx^2}-y=0$$

This is the required differential equation of the given equation of curve.

Hence, the correct answer is B.

12: Which of the following differential equation has y = x as one of its particular solution?

A. $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=x$

B. $\frac{d^2y}{dx^2}-x\frac{dy}{dx}+xy=x$

C. $\frac{d^2y}{dx^2}-x^2\frac{dy}{dx}+xy=x=0$

D. $\frac{d^2y}{dx^2}+x\frac{dy}{dx}+xy=0$

Solution: The given equation of curve is y = x.
Differentiating with respect to x, we get:

$$\frac{dy}{dx}=1 \qquad \ldots(1)$$

Again, differentiating with respect to x, we get:

$$\frac{d^2y}{dx^2}=0$$

Now, on substituting the values of y, $\frac{d^2y}{dx^2}$, and $\frac{dy}{dx}$ from equation (1) and (2) in each of the given alternatives, we find that only the differential equation given in alternative C is correct.

$$\frac{d^2y}{dx^2}-x\frac{dy}{dx}+xy=0-x^2.1+x.x$$

$= -x^2+x^2$

$= 0$

Hence, the correct answer is C.

Exercise 9.4 Page: 395

For each of the differential equations in Exercises 1 to 10, find the general solution:

1:
$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$

Solution:
The given differential equation is:
$$\frac{dy}{dx}=\frac{1-\cos x}{1+\cos x}$$
$$\Rightarrow \frac{dy}{dx}=\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}=\tan^2\frac{x}{2}$$
$$\Rightarrow \frac{dy}{dx}=\left(\sec^2\frac{x}{2}-1\right)$$
Separating the variables, we get:
$$dy=\left(\sec^2\frac{x}{2}-1\right)dx$$
Now, integrating both sides of this equation, we get:
$$\int dy=\int\left(\sec^2\frac{x}{2}-1\right)dx=\int\sec^2\frac{x}{2}dx-\int dx$$
$$\Rightarrow y=2\tan\frac{x}{2}-x+C$$
This is the required general solution of the given differential equation.

2:
$$\frac{dy}{dx}=\sqrt{4-y^2}\left(-2<y<2\right)$$

Solution:
The given differential equation is:
$$\frac{dy}{dx}=\sqrt{4-y^2}$$
Separating the variables, we get:
$$\Rightarrow \frac{dy}{\sqrt{4-y^2}}=dx$$
Now, integrating both sides of this equation, we get:
$$\int\frac{dy}{\sqrt{4-y^2}}=\int dx$$
$$\Rightarrow \sin^{-1}\frac{y}{2}=x+c$$
$$\Rightarrow \frac{y}{2}=\sin\left(x+C\right)$$
$$\Rightarrow y=2\sin\left(x+C\right)$$
This is the required general solution of the given differential equation.

3:
$$\frac{dy}{dx}=+y=1\left(y\neq 1\right)$$

Solution:
The given differential equation is:
$$\frac{dy}{dx}=+y=1\left(y\neq 1\right)$$
$$\Rightarrow dy+ydx=dx$$
$$\Rightarrow dy=\left(1-y\right)dx$$
Separating the variables, we get
$$\Rightarrow \frac{dy}{1-y}=dx$$
Now, integrating both sides, we get:

$$\int \frac{dy}{1-y} = \int dx$$

$$\Rightarrow \log(1-y) = x + \log C$$

$$\Rightarrow -\log C - \log(1-y) = x$$

$$\Rightarrow \log C(1-y) = -x$$

$$\Rightarrow C(1-y) = e^{-x}$$

$$\Rightarrow y = 1 - \frac{1}{C}e^{-x}$$

$$\Rightarrow y = 1 + Ae^{-x} \left(\text{Where } A = -\frac{1}{C} \right)$$

This is the required general solution of the given differential equation.

4:

$$\sec^2 c \tan y dx + \sec^2 y \tan x dy = 0$$

Solution:

The given differential equation is:

$$\sec^2 c \tan y dx + \sec^2 y \tan x dy = 0$$

$$\Rightarrow \frac{\sec^2 c \tan y dx + \sec^2 y \tan x dy = 0}{\tan x \tan y}$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx = -\frac{\sec^2 y}{\tan y} dy$$

Integrating both sides of this equation, we get:

$$\int \frac{\sec^2 x}{\tan x} dx = -\int \frac{\sec^2 y}{\tan y} dy \qquad \text{..... (1)}$$

Let tan x = t

$$\therefore \frac{d}{dx}\left(\tan x\right)=\frac{dt}{dx}$$

$$\Rightarrow \sec^2 x dx = dt$$

$$\frac{\sec^2 x}{\tan x}dx = \int \frac{1}{t}dt$$

$$= \log t$$

$$= \log\left(\tan x\right)$$

Similarly, $\int \frac{\sec^2 x}{\tan x}dy = \log\left(\tan\ y\right)$

Substituting these values in equation (1), we get:

log(tan x) = − log(tan y) + log C

$$\Rightarrow \log\left(\tan\ x\right)=\log\left(\frac{C}{\tan y}\right)$$

$$\Rightarrow \tan x = \frac{C}{\tan\ y}$$

$$\Rightarrow \tan x \tan y = C$$

This is the required general solution of the given differential equation.

5:

$$\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$$

Solution:

The given differential equation is:

$$\left(e^x + e^{-x}\right)dy - \left(e^x - e^{-x}\right)dx = 0$$

$$\Rightarrow \left(e^x + e^{-x}\right)dy = \left(e^x - e^{-x}\right)dx$$

$$\Rightarrow dy = \left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx$$

Integrating both sides of this equation, we get:

$$\int dy = \int\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx + C$$

$$\Rightarrow y = \int\left[\frac{e^x - e^{-x}}{e^x + e^{-x}}\right]dx + C$$

Let (ex + e – x) = t.

Differentiating both sides with respect to x, we get:

$$\frac{d}{dx}\left(e^{x}+e^{-x}\right)=\frac{dt}{dx}$$

$$\Rightarrow e^{x}-e^{-x}=\frac{dt}{dx}$$

$$\Rightarrow \left(e^{x}-e^{-x}\right)dx=dt$$

Substituting this value in equation (1), we get:

$$y=\int\frac{1}{t}+C$$

$$\Rightarrow y=\log(t)+C$$

$$\Rightarrow y=\log\left(e^{x}+e^{-x}\right)+C$$

This is the required general solution of the given differential equation.

6:

$$\frac{dy}{dx}=\left(1+x^{2}\right)\left(1+y^{2}\right)$$

Solution:

The given differential equation is:

$$\frac{dy}{dx}=\left(1+x^{2}\right)\left(1+y^{2}\right)$$

$$\Rightarrow \frac{dy}{1+y^{2}}=\left(1+x^{2}\right)dx$$

Integrating both sides of this equation, we get:

$$\int\frac{dy}{1+y^{2}}=\int\left(1+x^{2}\right)dx$$

$$\Rightarrow \tan^{-1}y=\int dx+\int x^{2}dx$$

$$\Rightarrow \tan^{-1}y=x+\frac{x^{3}}{3}+C$$

This is the required general solution of the given differential equation.

7:

$$y\log y\,dx-x\,dy=0$$

Solution:

The given differential equation is:

$$y\log y dx - x dy = 0$$

$$\Rightarrow y\log y dx = x dy$$

$$\Rightarrow \frac{dy}{y\log y} = \frac{dx}{x}$$

Integrating both sides, we get:

$$\Rightarrow \int \frac{dy}{y\log y} = \int \frac{dx}{x} \quad \ldots\ldots (1)$$

Let log y = t

$$\therefore \frac{d}{dx}(\log y) = \frac{dt}{dy}$$

$$= \frac{1}{y} = \frac{dt}{dy}$$

$$\Rightarrow \frac{1}{y} dy = dt$$

Substituting this value in equation (1), we get:

$$\int \frac{dt}{t} = \int \frac{dx}{x}$$

$$\Rightarrow \log t = \log x + \log c$$

$$\Rightarrow \log(\log y) = \log Cx$$

$$\Rightarrow \log y = Cx$$

$$\Rightarrow y = e^{cx}$$

This is the required general solution of the given differential equation.

8:

$$x^5 \frac{dy}{dx} = -y^5$$

Solution:

The given differential equation is:

$$x^5 \frac{dy}{dx} = -y^5$$

$$\Rightarrow \frac{dy}{y^5} = -\frac{dx}{x^5}$$

$$\Rightarrow \frac{dx}{x^5} + \frac{dy}{y^5} = 0$$

Integrating both sides, we get:

$$\int \frac{dx}{x^5} + \int \frac{dy}{y^5} = k \qquad \text{(Where k is any constant)}$$

$$\Rightarrow \int x^{-5}dx + \int y^{-5}dy = k$$

$$\Rightarrow \frac{x^{-4}}{-4} + \frac{y^{-4}}{-4} = k$$

$$\Rightarrow x^{-4} + y^{-4} = -4k$$

$$\Rightarrow x^{-4} + y^{-4} = C \quad (c = -4k)$$

This is the required general solution of the given differential equation.

9:

$$\frac{dy}{dx} = \sin^{-1} x$$

Solution:

The given differential equation is:

$$\frac{dy}{dx} = \sin^{-1} x$$

$$\Rightarrow dy = \sin^{-1} x dx$$

Integrating both sides, we get:

$$\int dy = \int \sin^{-1} x dx$$

$$\Rightarrow y = \int (\sin^{-1} x.1) dx$$

$$\Rightarrow y = \sin^{-1} x.\int (1) dx - \int \left[\left(\frac{d}{dx}(\sin^{-1} x)\int (1) dx\right)\right] dx$$

$$\Rightarrow y = \sin^{-1} x.x - \int \left(\frac{1}{\sqrt{1-x^2}} x\right) dx$$

$$\Rightarrow y = \sin^{-1} x + \int \frac{-x}{\sqrt{1-x^2}} dx \qquad \ldots\ldots (1)$$

Let $1-x^2=t$

$\Rightarrow \frac{d}{dx}\left(1-x^2\right)=\frac{dt}{dx}$

$\Rightarrow -2x=\frac{dt}{dx}$

$\Rightarrow xdx=-\frac{1}{2}dt$

Substituting this value in equation (1), we get:

$y=x\sin^{-1}x+\int\frac{1}{2\sqrt{t}}dt$

$\Rightarrow y=x\sin^{-1}x+\frac{1}{2}\int(t)^{\frac{1}{2}dt}$

$\Rightarrow y=x\sin^{-1}x+\frac{1}{2}.\frac{t^{\frac{1}{2}}}{1}+C$

$\Rightarrow y=x\sin^{-1}x+\sqrt{t}+C$

$\Rightarrow y=x\sin^{-1}x+\sqrt{1-x^2}+C$

This is the required general solution of the given differential equation.

10:

$e^x\tan ydx+\left(1-e^x\right)\sec^2 ydy=0$

Solution:

The given differential equation is:

$e^x\tan ydx+\left(1-e^x\right)\sec^2 ydy=0$

$\left(1-e^x\right)\sec^2 ydy=-e^x\tan ydx$

Separating the variables, we get:

$\frac{\sec^2 y}{\tan y}dy=\frac{-e^x}{1-e^x}dx$

Integrating both sides, we get:

$\int\frac{\sec^2 y}{\tan y}dy=\int\frac{-e^x}{1-e^x}dx \qquad \ldots\ldots (1)$

Let tan y = u

$$\Rightarrow \frac{d}{dy}(\tan\ y)=\frac{du}{dy}$$

$$\Rightarrow \sec^2\ y=\frac{du}{dy}$$

$$\Rightarrow \sec^2\ ydy=du$$

$$\therefore \int\frac{\sec^2\ y}{\tan y}dy=\int\frac{du}{u}=\log u=\log(\tan y)$$

Now, $1-e^x=t$

$$\therefore \frac{d}{dx}(1-e^x)=\frac{dt}{dx}$$

$$\Rightarrow -e^x=\frac{du}{dx}$$

$$\Rightarrow -e^xdx=dt$$

$$\Rightarrow \int\frac{-e^x}{1-e^x}dx=\int\frac{dt}{t}=\log t=\log(1-e^x)$$

Substituting the values of $\int\frac{\sec^2\ y}{\tan y}dy$ and $\int\frac{-e^x}{1-e^x}dx$ in equation (1), we get

$$\Rightarrow \log(\tan y)=\log(1-e^x)+\log C$$

$$\Rightarrow \log(\tan y)=\log\left[C(1-e^x)\right]$$

$$\Rightarrow \tan y=C(1-e^x)$$

This is the required general solution of the given differential equation.

11:

$$(x^3+x^2+x+1)\frac{dy}{dx}=2x^2+x; y=1$$

Solution:

The given differential equation is:

$$(x^3+x^2+x+1)\frac{dy}{dx}=2x^2+x; y=1$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2 + x}{\left(x^3 + x^2 + x + 1\right)}$$

$$\Rightarrow dy = \frac{2x^2 + x}{(x+1)\left(x^2+1\right)}dx$$

Integrating both sides, we get:

$$\int dy = \int \frac{2x^2 + x}{(x+1)\left(x^2+1\right)}dx \qquad \text{.... (1)}$$

Let $\dfrac{2x^2 + x}{(x+1)\left(x^2+1\right)} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2+1}$ .... (2)

$$\Rightarrow \frac{2x^2 + x}{(x+1)\left(x^2+1\right)} = \frac{Ax^2 + A + (Bx + C)(x+1)}{(x+1)\left(x^2+1\right)}$$

$$\Rightarrow 2x^2 + x = Ax^2 + A + bx^2 + Bx + Cx + C$$

$$\Rightarrow 2x^2 + x = (A+B)x^2 + (B+C)x + (A+C)$$

Comparing the coefficients of x^2 and x, we get:

$A + B = 2$

$B + C = 1$

$A + C = 0$

Solving these equations, we get:

$$A = \frac{1}{2}, b = \frac{3}{2} \text{ and } C = \frac{-1}{2}$$

Substituting the values of A, B and C in equation (2), we get:

$$\frac{2x^2 + x}{(x+1)\left(x^2+1\right)} = \frac{1}{2}\frac{1}{(x+1)} + \frac{1}{2}\frac{(3x-1)}{\left(x^2+1\right)}$$

Therefore, equation (1) becomes:

$$\int dy = \frac{1}{2}\int\frac{1}{x+1}dx + \int\frac{3x-1}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{2}\int\frac{x}{x^2+1}dx - \frac{1}{2}\int\frac{x}{x^2+1}dx$$

$$\Rightarrow y = \frac{1}{2}\log(x+1)\frac{3}{4}\int\frac{2x}{x^2+1}dx - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{2}\log(x+1)\frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[2\log(x+1) + 3\log(x^2+1)\right] - \frac{1}{2}\tan^{-1}x + C$$

$$\Rightarrow y = \frac{1}{4}\left[(x^2+1)(x^2+1)\right] - \frac{1}{2}\tan^{-1}x + C \qquad \text{.....(3)}$$

Now, y = 1 when x = 0

$$\Rightarrow 1 = \frac{1}{4}\log(1) - \frac{1}{2}\tan^{-1}0 + C$$

$$\Rightarrow 1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + C$$

$$\Rightarrow C = 1$$

Substituting C = 1 in equation (3), we get:

$$y = \frac{1}{4}\left[\log(x^2+1)^2(x^2+1)^3\right] - \frac{1}{2}\tan^{-1}x + 1$$

12:

$$x(x^2-1)\frac{dy}{dx} = 1; y = 0 \text{ when } x = 2$$

Solution:

$$x(x^2-1)\frac{dy}{dx} = 1$$

$$\Rightarrow dy = \frac{dx}{x(x^2-1)}$$

$$\Rightarrow dy = \frac{1}{x(x-1)(x+1)}dx$$

Integrating both sides, we get:

$$\int dy = \int \frac{1}{x(x-1)(x+1)}dx \quad \text{..... (1)}$$

Let $\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$ ....(2)

$$\Rightarrow \frac{1}{x(x-1)(x+1)} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}$$

$$= \frac{(A+B+C)x^2 + (B-C)x - A}{x(x-1)(x+1)}$$

Comparing the coefficients of x^2, x and constant, we get:

A = -1

B – C = 0

A + B + C = 0

Solving these equations, we get B = $\frac{1}{2}$ and C = $\frac{1}{2}$

Substituting the values of A, B, and C in equation (2), we get:

$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1}{x(x-1)} + \frac{1}{2(x+1)}$$

Therefore, equation (1) becomes:

$$\int dy = -\int \frac{1}{x}dx + \frac{1}{2}\int \frac{1}{x-1}dx + \frac{1}{2}\int \frac{1}{x+1}dx$$

$$\Rightarrow y = -\log x + \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1) + \log k$$

Now,

$y = 0$ when $\ = 2$

$$0 = -\log 2 + \frac{1}{2}\log(2+1) + \frac{1}{2}\log(2-1) + \log C$$

$$0 = -\log 2 + \frac{\log 3}{2} + \log C$$

$$\log 1 = \log\left(\frac{\sqrt{3}C}{2}\right)$$

$$\frac{\sqrt{3}}{2}C = 1$$

$$C = \frac{2}{\sqrt{3}}$$

$$y = -\log x + \frac{1}{2}\log(x+1) + \frac{1}{2}\log(x-1) + \log\frac{2}{\sqrt{3}}$$

$$y = \log\left(\frac{2\sqrt{(x+1)(x-1)}}{\sqrt{3}x}\right)$$

$$y = \log\left(\frac{\sqrt{4(x^2-1)}}{\sqrt{3x^2}}\right)$$

$$y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

13:

$$\cos\left(\frac{dy}{dx}\right) = a\left(a \in R\right); 1 \text{ when } x = 0$$

Solution:

$$\cos\left(\frac{dy}{dx}\right) = a$$

$$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$$

$$\Rightarrow dy = \cos^{-1} a dx$$

Integrating both sides, we get:

$$\int dy = \cos^{-1} a \int dx dx$$

$$\Rightarrow y = \cos^{-1} a.x + C$$

$$\Rightarrow y = x\cos^{-1} a + C \qquad \ldots.(1)$$

Now, y =1 when x = 0

$$\Rightarrow 1 = 0.\cos^{-1} a + C$$

$$\Rightarrow C = 1$$

Substituting C = 1 in equation (1), we get

$y = x\cos^{-1} a + 1$

$\Rightarrow \dfrac{y-1}{x} = \cos^{-1} a$

$\Rightarrow \cos\left(\dfrac{y-1}{x}\right) = a$

14:

$\dfrac{dy}{dx} = y\tan x, y = 1$ when x = 0

Solution:

$\dfrac{dy}{dx} = y\tan x$

$\Rightarrow \dfrac{dy}{dx} = \tan\ xdx$

integrating both sides, we get:

$\int \dfrac{dy}{y} = -\int \tan\ xdx$

$\Rightarrow \log y = \log(\sec x) + \log C$

$\Rightarrow \log\ y = \log(C\sec x)$

$\Rightarrow y = C\sec x \qquad (1)$

Now y = 1 when x = 0

$\Rightarrow 1 = C \times \sec 0$

$\Rightarrow 1 = C \times 1$

$\Rightarrow C = 1$

Substituting C = 1 in equation (1), we get

y = sec x

15:

Find the equation of a curve passing through the point (0, 0) and whose differential equation is

$y' = e^x \sin x$

Solution:

The differential equation of the curve is:

$y' = e^x \sin x$

$$\Rightarrow \frac{dy}{dx} = e^x \sin x$$

$$\Rightarrow dy = e^x \sin x$$

Intergrating both sides, we get:

$$\int dy = \int e^x \sin x dx \qquad \text{….. (1)}$$

Let $I = \int e^x \sin x dx$

$$\Rightarrow I = \sin x \int e^x dx - \int \left[\frac{d}{dx}(\sin x).\int e^x dx \right] dx$$

$$\Rightarrow I = \sin x.e^x - \int \cos x.e^x dx$$

$$\Rightarrow I = \sin x.e^x - \left[\cos x.\int \left(\frac{d}{dx}(\cos x).\int e^x dx \right) dx \right]$$

$$\Rightarrow I = \sin x.e^x - \left[\cos x e^x - \int (-\sin x) e^x dx \right]$$

$$\Rightarrow I = e^x \sin x - e^x \cos x - 1$$

$$\Rightarrow 2I = e^x (\sin x - \cos x)$$

$$\Rightarrow I = \frac{e^x (\sin x - \cos x)}{2}$$

Substituting this value in equation (1), we get

$$y = \frac{e^x (\sin x - \cos x)}{2} + C \qquad \text{….. (2)}$$

Now, the curve passes through point (0, 0)

$$\therefore\ 0 = \frac{e^0 (\sin 0 - \cos 0)}{2} + C$$

$$\Rightarrow 0 = \frac{1(0-1)}{2} + c$$

$$\Rightarrow C = \frac{1}{2}$$

Substituting $C = \frac{1}{2}$ in equation (2), we get:

$$y = \frac{e^x(\sin x - \cos x)}{2} + \frac{1}{2}$$

$$\Rightarrow 2y = e^x(\sin x - \cos x) + 1$$

$$\Rightarrow 2y - 1 = e^x(\sin x - \cos x)$$

Hence, the required equation of the curve is $2y - 1 = e^x(\sin x - \cos x)$

16:

For the differential equation $xy = \frac{dy}{dx} = (x+2)(y+2)$ find the solution curve passing through the
point (1, –1).

Solution:

The differential equation of the given curve is:

$$xy = \frac{dy}{dx} = (x+2)(y+2)$$

$$\Rightarrow \left(\frac{y}{y+2}\right)dy = \left(\frac{x+2}{x}\right)dx$$

Integrating both sides, we get:

$$\int\left(1 - \frac{2}{y+2}\right)dy = \int\left(1 + \frac{2}{x}\right)dx$$

$$\Rightarrow \int dy - 2\int \frac{1}{y+2}dy = \int dx + 2\int \frac{1}{x}dx$$

$$\Rightarrow y - 2\log(y+2) = x + 2\log x + C$$

$$\Rightarrow y - x - C = \log x^2 + \log(y+2)^2$$

$$\Rightarrow y - x - C = \log\left[x^2(y+2)^2\right] \qquad \ldots..(1)$$

Now, the curve passes through point (1, -1)

$$\Rightarrow -1 - 1 - C = \log\left[(1)^2(-1+2)^2\right]$$

$$\Rightarrow -2 - C = \log 1 = 0$$

$$\Rightarrow C = -2$$

Substituting C = -2 in equation (1), we get:

$$y - x + 2 = \log\left[x^2(y+2)^2\right]$$

This is the required solution of the given curve.

17:

Find the equation of a curve passing through the point (0, –2) given that at any point (x, y) on the curve, the product of the slope of its tangent and y-coordinate of the point is equal to the x-coordinate of the point.

Solution:

Let x and y be the x-coordinate and y-coordinate of the curve respectively.

We know that the slope of a tangent to the curve in the coordinate axis is given by the $\frac{dy}{dx}$

According to the given information, we get:

$$y\frac{dy}{dx} = x$$

$$\Rightarrow ydy = xdx$$

Integrating both sides, we get:

$$\int ydy = \int xdx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$$\Rightarrow y^2 - x^2 = 2C \qquad \text{..... (1)}$$

Now, the curve passes through point (0, –2).

$\therefore (-2)^2 - 02 = 2C$

$\Rightarrow 2C = 4$

Substituting 2C = 4 in equation (1), we get: $y^2 - x^2 = 4$

This is the required equation of the curve.

18:

At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (–4, –3). Find the equation of the curve given that it passes through (–2, 1).

Solution:

It is given that (x, y) is the point of contact of the curve and its tangent.

The slope (m_1) of the line segment joining (x, y) and (–4, –3) is $\frac{y+3}{x+4}$

We know that the slope of the tangent to the curve is given by the relation, $\frac{dy}{dx}$

∴ Slope (m_2) of tangent =

According to the given information: **19:**
$m_2 = 2m_1$

$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$

$$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$$

Integrating both sides, we get:

$$\int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$$

$$\Rightarrow \log(y+3) = 2\log(x+4) + \log C$$

$$\Rightarrow \log(y+3)\log C(x+4)^2$$

$$\Rightarrow y+3 = C(x+4)^2 \qquad \text{..... (1\_}$$

This is the general equation of the curve.
It is given that it passes through point (–2, 1).

$$\Rightarrow 1+3 = C(-2+4)^2$$

$$\Rightarrow 4 = 4C$$

$$\Rightarrow C = 1$$

Substituting C = 1 in equation (1), we get:
$y + 3 = (x + 4)^2$
This is the required equation of the curve.

19:
The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after t seconds.

Solution:
Let the rate of change of the volume of the balloon be k (where k is a constant).

$$\Rightarrow \frac{dy}{dx} = k$$

$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \qquad \left[\text{Volume of sphere} = \frac{4}{3}\pi r^3\right]$$

$$\Rightarrow \frac{4}{3}\pi.3r^2.\frac{dr}{dt} = k$$

$$\Rightarrow 4\pi r^2 dr = k\,dt$$

Integrating both sides, we get:

$$\Rightarrow 4\pi\frac{r^3}{3} = kt + C$$

$$\Rightarrow 4\pi r^3 = 3(kt + C) \qquad \text{..... (1)}$$

Now, at t = 0, r = 3

$$\Rightarrow 4\pi \times 33 = 3(k \times 0 + C)$$

$$\Rightarrow 108\pi = 3C$$

$$\Rightarrow C = 36\pi$$

At t = 3, r = 6:

$$\Rightarrow 4\pi \times 6^3 = 3(k \times 3 + C)$$

$$\Rightarrow 864\pi = 3(3k + 36\pi)$$

$$\Rightarrow 3k = -288\pi - 36\pi = 252\pi$$

$$\Rightarrow k = 84\pi$$

Substituting the values of k and C in equation (1), we get:

$$4\pi r^3 = 3[84\pi t + 36\pi]$$

$$\Rightarrow 4\pi r^3 = 4\pi(63t + 27)$$

$$\Rightarrow r^3 = 63t + 27$$

$$\Rightarrow r = (63t + 27)^{\frac{1}{3}}$$

Thus, the radius of the balloon after t seconds is $(63t + 27)^{\frac{1}{3}}$

20:

In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs 100 doubles itself in 10 years (loge 2 = 0.6931).

Solution:

Let p, t, and r represent the principal, time, and rate of interest respectively.
It is given that the principal increases continuously at the rate of r% per year.

$$\Rightarrow \frac{dp}{dt} = \left(\frac{r}{100}\right)p$$

$$\Rightarrow \frac{dp}{p} = \left(\frac{r}{100}\right)dt$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{r}{100}\int dt$$

$$\Rightarrow \log p = \frac{rt}{100} + k$$

$$\Rightarrow p = e^{\frac{n}{100}+k} \qquad \ldots..(1)$$

It is given that when t = 0, p = 100

$$\Rightarrow 100 = e^{k} \quad \ldots..(2)$$

Now, if t = 10, then p = 2 × 100 = 200

Therefore, equation (1) becomes:

$$200 = e^{\frac{n}{10}+k}$$

$$\Rightarrow 200 = e^{\frac{n}{10}+k}.e^{k}$$

$$\Rightarrow 200 = e^{\frac{n}{10}+k}.100 \qquad \text{From (2)}$$

$$\Rightarrow e^{\frac{r}{10}} = 2$$

$$\Rightarrow \frac{r}{10} = \log.2$$

$$\Rightarrow \frac{r}{10} = 0.6931$$

$$\Rightarrow r = 6.931$$

Hence, the value of r is 6.93%

21:

In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years $\left(e^{0.5} = 1.648\right)$.

Solution:

Let p and t be the principal and time respectively.

It is given that the principal increases continuously at the rate of 5% per year.

$$\Rightarrow \frac{d}{dt}\left(\frac{5}{100}\right)p$$

$$\Rightarrow \frac{dp}{dt} = \frac{p}{20}$$

$$\Rightarrow \frac{dp}{p} = \frac{dt}{20}$$

Integrating both sides, we get:

$$\int \frac{dp}{p} = \frac{1}{20}\int dt$$

$$\Rightarrow \log p = \frac{t}{20} + c$$

$$\Rightarrow p = e^{\frac{1}{20}+c} \qquad .....(1)$$

Now, when t = 0, p = 100

$$\Rightarrow 1000 = e^{c} \qquad ....(2)$$

At t = 10, equation (1) becomes:

$$p = e^{\frac{1}{20}+c}$$

$$\Rightarrow p = e^{0.5} \times e^{C}$$

$$\Rightarrow P = 1.648 \times 1000$$

$$\Rightarrow P = 1648$$

Hence, after 10 years the amount will worth Rs. 1648.

22:

In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

Solution:

Let y be the number of bacteria at any instant t.

It is given that the rate of growth of the bacteria is proportional to the number present.

$$\therefore \frac{dy}{dt} \infty y$$

$$\Rightarrow \frac{dy}{dt} = ky \text{ (Where k is a constant)}$$

$$\Rightarrow \frac{dy}{y} = kdt$$

Integrating both sides, we get:

$$\int \frac{dy}{y} = k \int dt$$

$$\Rightarrow \log y = kt + C \quad \text{..... (1)}$$

Let y_0 be the number of bacteria at t = 0.

$\Rightarrow \log y_0 = C$

Substituting the value of C in equation (1), we get:

$$\Rightarrow \log y = kt + \log y_0$$

$$\Rightarrow \log y - \log y_0 = kt$$

$$\Rightarrow \log\left(\frac{y}{y_0}\right) = kt$$

$$\Rightarrow kt = \log\left(\frac{y}{y_0}\right) \qquad \text{....(2)}$$

Also, it is given that the number of bacteria increases by 10% in 2 hours.

$$\Rightarrow y = \frac{110}{100} y_0$$

$$\Rightarrow \frac{y}{y_0} = \frac{11}{10} \quad \text{..... (3)}$$

Substituting this value in equation (2), we get:

$$k.2 = \log\left(\frac{11}{10}\right)$$

$$\Rightarrow k = \frac{1}{2}\log\left(\frac{11}{10}\right)$$

Therefore, equation (2) becomes:

$$\frac{1}{2}\log\left(\frac{11}{10}\right)t = \log\left(\frac{y}{y_0}\right)$$

$$\Rightarrow t = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} \qquad \text{.....(4)}$$

Now, let the time when the number of bacteria increases from 100000 to 200000 be $t_1 \Rightarrow y = y_0$ at $t = t_1$

From equation (4), we get

$$t_1 = \frac{2\log\left(\frac{y}{y_0}\right)}{\log\left(\frac{11}{10}\right)} = \frac{2\log 2}{\log\left(\frac{11}{10}\right)}$$

Hence, $\frac{2\log 2}{\log\left(\frac{11}{10}\right)}$ in hours the number of bacteria increases from 100000 to 200000.

23:

The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

A. $e^x + e^{-y} = C$

B. $e^x + e^y = C$

C. $e^{-x} + e^y = C$

D. $e^{-x} + e^{-y} = C$

Solution:

$$\frac{dy}{dx} = e^{x+y} = e^x . e^y$$

$$\Rightarrow \frac{dy}{e^y} = e^x dx$$

$$\Rightarrow e^{-y} dy = e^x dx$$

Intergrating both sides, we get:

$$\int e^{-y} dy = \int e^x dx$$

$$\Rightarrow -e^{-y} = e^x + k$$

$$\Rightarrow e^x + e^{-y} = -k$$

$$\Rightarrow e^x + e^{-y} = c \qquad (c = -k)$$

Hence, the correct answer is A.

Exercise 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1:

$$\left(x^2 + xy\right)dy \left(= x^2 + y^2\right)dx$$

Solution:

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \qquad \text{.....(1)}$$

Let $F(x, y) = \dfrac{x^2 + y^2}{x^2 + xy}$

Now, $F(\lambda x, \lambda y) = \dfrac{(\lambda x)^2 + (\lambda y)^2}{(\lambda x)^2 + (\lambda x)(\lambda y)} = \dfrac{x^2 + y^2}{x^2 + xy} = \lambda^0 F(x, y)$

This shows that equation (1) is a homogeneous equation.
To solve it, we make the substitution as:
y = vx
Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = v + x\frac{d\nu}{dx}$$

Substituting the values of v and in equation (1), we get:

$$v + x\frac{d\nu}{dx} = \frac{x^2 + (\nu x)^2}{x^2 + x(\nu x)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+\nu^2}{1+\nu}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+\nu^2}{1+\nu} - v = \frac{\left(1+\nu^2\right) - c(1+\nu)}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-\nu}{1+\nu}$$

$$\Rightarrow \left(\frac{1+\nu}{1-\nu}\right) = d\nu = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2-1+\nu}{1-\nu}\right)d\nu = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{2}{1-\nu} - 1\right)d\nu = \frac{dx}{x}$$

Integrating both sides, we get:

$$-2og(1-\nu)-\nu=\log x-\log k$$

$$\Rightarrow v=-2\log(1-\nu)-\log x+\log k$$

$$\Rightarrow \nu=\log\left[\frac{k}{x(1-\nu)^2}\right]$$

$$\Rightarrow \frac{y}{x}=\log\left[\frac{k}{x\left(1-\frac{y}{x}\right)^2}\right]$$

$$\Rightarrow \frac{y}{x}=\log\left[\frac{kx}{(x-y)^2}\right]$$

$$\Rightarrow \frac{kx}{(x-y)^2}=e^{\frac{-y}{x}}$$

$$\Rightarrow (x-y)^2=kxe^{\frac{-y}{x}}$$

This is the required solution of the given differential equation.

2:

$$y'=\frac{x+y}{x}$$

Solution:

The given differential equation is:

$$y'=\frac{x+y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y}{x} \qquad \text{….. (1)}$$

Let F(x, y) = $\frac{x+y}{x}$

Now, $(\lambda x,\lambda y)=\frac{\lambda x,\lambda y}{\lambda x}=\frac{x+y}{x}=\lambda^0 F(x,y)$

Thus, the given equation is a homogeneous equation.

To solve it, we make the substitution as:

y = vx

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{x + vx}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = 1 + v$$

$$x\frac{dv}{dx} = 1$$

$$\Rightarrow dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$v = \log x + C$$

$$\Rightarrow \frac{y}{x} = \log x + C$$

$$\Rightarrow y = x\log x + Cx$$

This is the required solution of the given differential equation.

3:

$$(x - y)dy\,(-x + y)dx = 0$$

Solution:

$$(x - y)dy\,(-x + y)dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x + y}{x - y} \qquad \text{..... (1)}$$

Let F (x, y) $= \dfrac{x + y}{x - y}$

$$\therefore\ F(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{x + y}{x - y} = \lambda^0 F(x, y)$$

Thus, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$\nu + x\frac{d\nu}{dx} = \frac{x + \nu x}{x - \nu x} = \frac{1+\nu}{1-\nu}$$

$$x\frac{dv}{dx} = \frac{1+\nu}{1-\nu} - \nu = \frac{1+\nu-\nu(1-\nu}{1-\nu}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1-\nu}$$

$$\Rightarrow \frac{1-\nu}{\left(1+\nu^2\right)}d\nu = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1+\nu^2} - \frac{1}{1-\nu^2}\right)dx = \frac{dx}{x}$$

Integrating both sides, we get

$$\tan^{-1}\nu - \frac{1}{2}\log\left(1+\nu^2\right) = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left[1+\left(\frac{y}{x}\right)^2\right] = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) = \log x + C$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\left[\log\left(x^2+y^2\right) - \log x^2\right] = \log x + c$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log\left(x^2+y^2\right) + C$$

This is the required solution of the given differential equation.

4:

$$\left(x^2 - y^2\right)dx + 2xy\,dy = 0$$

Solution:

The given differential equation is:

$$\left(x^2 - y^2\right)dx + 2xy\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left(x^2 - y^2\right)}{2xy} \qquad \text{….. (1)}$$

Let (x, y) $= \dfrac{-\left(x^2 - y^2\right)}{2xy}$

$$\therefore\ F(\lambda x, \lambda y) = \left[\frac{(\lambda x)^2 - (\lambda y)^2}{2(\lambda x)(\lambda y)}\right] = \frac{-\left(x^2 - y^2\right)}{2xy} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$v + x\frac{dv}{dx} = -\left[\frac{x^2 - (vx)^2}{2x.(vx)}\right]$$

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -f\frac{\left(1 + v^2\right)}{2v}$$

$$\Rightarrow \frac{2v}{1 + v^2}dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left(1 + v^2\right) = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow 1+v^2=\frac{C}{x}$$

$$\Rightarrow \left[1+\frac{y^2}{x^2}\right]=\frac{C}{x}$$

$$\Rightarrow x^2+y^2=Cx$$

This is the required solution of the given differential equation.

5:

$$x^2\frac{dy}{dx}-x^2-2y^2+xy$$

Solution:

The given differential equation is:

$$x^2\frac{dy}{dx}-x^2-2y^2+xy$$

$$\frac{dy}{dx}=\frac{x^2-2y^2+xy}{x^2} \quad \text{.... (1)}$$

Let $F(x, y) = \dfrac{x^2-2y^2+xy}{x^2}$

$$\therefore\ F(\lambda x,\lambda y)=\frac{(\lambda x)^2-2(\lambda y)^2+(\lambda x)(\lambda y)}{(\lambda x)^2}=\frac{x^2-2y^2+xy}{x^2}=\lambda^0F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$\nu+x\frac{d\nu}{dx}=\frac{x^2-2(vx)^2+x.(vx)}{x^2}$$

$$\Rightarrow \nu+x\frac{d\nu}{dx}=1-2v^2+v$$

$$\Rightarrow x\frac{d\nu}{dx}=1-2v^2$$

$$\Rightarrow \frac{dv}{1-2v^2} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\frac{dv}{\frac{1}{2}-v^2} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{dv}{\left(\frac{1}{\sqrt{2}}\right)^2 - v^2}\right] = \frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\frac{1}{2\times\frac{1}{\sqrt{2}}}\log\left|\frac{\frac{1}{\sqrt{2}}+v}{\frac{1}{\sqrt{2}}-v}\right| = \log|x| + C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{\frac{1}{\sqrt{2}}+\frac{y}{x}}{\frac{1}{\sqrt{2}}-\frac{y}{x}}\right| = \log|x| + C$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right| = \log|x| + C$$

This is the required solution for the given differential equation.

6:

$$xdy - ydy = \sqrt{x^2+y^2}dx$$

Solution:

$$xdy - ydy = \sqrt{x^2+y^2}dx$$

$$\Rightarrow xdy = \left[y + \sqrt{x^2+y^2}\right]dx$$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x^2} \quad \ldots.(1)$$

Let $F(x, y) = \dfrac{y+\sqrt{x^2+y^2}}{x^2}$

$$\therefore\ F(\lambda x,\lambda y)=\frac{(\lambda x)+\sqrt{(\lambda x)^2(\lambda y)^2}}{\lambda x}=\frac{y+\sqrt{x^2+y^2}}{x}=\lambda^0 F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of v and in equation (1), we get:

$$\Rightarrow \nu+x\frac{d\nu}{dx}=\frac{vx+\sqrt{x^2+(vx)^2}}{x}$$

$$\Rightarrow \nu+x\frac{d\nu}{dx}=v+\sqrt{1+v^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|v+\sqrt{1+v^2}\right|=\log|x|+\log C$$

$$\Rightarrow \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right|=\log|Cx|$$

$$\Rightarrow \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right|=\log|Cx|$$

$$\Rightarrow y+\sqrt{x^2+y^2}=Cx^2$$

This is the required solution of the given differential equation.

7:

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdx$$

Solution:

The given differential equation is:

$$\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}ydx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}xdx$$

$$\frac{dy}{dx}=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x} \quad \ldots\ldots (1)$$

$$\text{Let } F(x,y)=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$\therefore F(\lambda x,\lambda y)=\frac{\left\{\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)+\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda y}{\left\{\lambda y\sin\left(\frac{\lambda y}{\lambda x}\right)-\lambda x\cos\left(\frac{\lambda y}{\lambda x}\right)\right\}\lambda x}$$

$$=\frac{\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y}{\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x}$$

$$=\lambda^0 F.(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get;

$$v+x\frac{dv}{dx}=\frac{(x\cos v+vx\sin v).vx}{(vx\sin v-x\cos v)x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v - v^2 \sin v + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \left[\frac{v\sin v - \cos v}{v\cos v}\right]dv = \frac{2dx}{x}$$

$$\Rightarrow \left(\tan v - \frac{1}{v}\right)dv = \frac{2dx}{x}$$

Integrating both sides, we get:

log(sec v) – log v = 2log x + log C

$$\Rightarrow \log\left(\frac{\sec v}{v}\right) = \log\left(Cx^2\right)$$

$$\Rightarrow \left(\frac{\sec v}{v}\right) = Cx^2$$

$$\Rightarrow \sec v = Cx^2 v$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = C.x^2.\frac{y}{x}$$

$$\Rightarrow \sec\left(\frac{y}{x}\right) = Cxy$$

$$\Rightarrow \cos\left(\frac{y}{x}\right) = \frac{1}{Cxy} = \frac{1}{C}.\frac{1}{xy}$$

$$\Rightarrow xy\cos\left(\frac{y}{x}\right) = k \quad \left(k = \frac{1}{C}\right)$$

This is the required solution of the given differential equation.

8:

$$x\frac{dy}{dx} - y + \sin\left(\frac{y}{x}\right) = 0$$

Solution:

$$x\frac{dy}{dx}-y+\sin\left(\frac{y}{x}\right)=0$$

$$\Rightarrow x\frac{dy}{dx}=y-x\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y-x\sin\left(\frac{y}{x}\right)}{x} \quad \ldots\ldots (1)$$

Let $F(x, y)=\dfrac{y-x\sin\left(\frac{y}{x}\right)}{x}$

$$\therefore\ F(\lambda x,\lambda y)\frac{dy}{dx}=\frac{\lambda y-\lambda x\sin\left(\frac{\lambda y}{\lambda x}\right)}{\lambda x}=\frac{y-x\sin\left(\frac{y}{x}\right)}{x}=\lambda^0F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{vx-x\sin v}{x}$$

$$\Rightarrow v+x\frac{dv}{dx}=v-\sin v$$

$$\Rightarrow -\frac{dv}{\sin v}=\frac{dx}{x}$$

$$\Rightarrow \operatorname{cosec} v\,dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\log\left|\operatorname{cosec} v-\cot v\right|=-\log x+\log C=\log\frac{C}{x}$$

$$\Rightarrow \operatorname{cosec}\left(\frac{y}{x}\right)-\cot\left(\frac{y}{x}\right)=\frac{C}{x}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{y}{x}\right)} - \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{c}{x}$$

$$\Rightarrow x\left[1-\cos\left(\frac{y}{x}\right)\right] = C\sin\left(\frac{y}{x}\right)$$

This is the required solution of the given differential equation.

9:

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

Solution:

$$ydx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$$

$$\Rightarrow ydx = \left[2x - x\log\left(\frac{y}{x}\right)\right]dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)} \qquad \text{..... (1)}$$

Let $F(x,y) = \dfrac{y}{2x - x\log\left(\frac{y}{x}\right)}$

$$\therefore\ F(\lambda x, \lambda y) = \frac{\lambda y}{2(\lambda x) - (\lambda x)\log\left(\frac{\lambda y}{\lambda x}\right)} = \frac{y}{2x - \log\left(\frac{y}{x}\right)} = \lambda^0 . F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
$y = vx$

$$\Rightarrow \frac{d}{dx} = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{vx}{2x - x\log v}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{2 - \log v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{2 - \log v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - 2v + v\log v}{2 - \log v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\log v - x}{2 - \log v}$$

$$\Rightarrow \frac{2 - \log v}{v(\log v - 1)}dv = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1 + (1 - \log v)}{v(\log v - 1)}\right]dv = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{v(\log v - 1)} - \frac{1}{v}\right]dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$\int \frac{1}{v(\log v - 1)}dv - \int \frac{1}{v}dv = \int \frac{1}{x}dx$$

$$\Rightarrow \int \frac{dv}{v(\log v - 1)} - \log v = \log x + \log C \qquad \text{….. (2)}$$

$$\Rightarrow \text{Let } \log v - 1 = t$$

$$\Rightarrow \frac{d}{dv}(\log v - 1) = \frac{dt}{dv}$$

$$\Rightarrow \frac{1}{v} = \frac{dt}{dv}$$

$$\Rightarrow \frac{dv}{v} = dt$$

Therefore, equation (1) becomes:

$$\Rightarrow \int \frac{dt}{t} - \log v = \log x + \log C$$

$$\Rightarrow \log t - \log\left(\frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \log\left[\log\left(\frac{y}{x}\right) - 1\right] - \log\left(\frac{y}{x}\right) = \log(Cx)$$

$$\Rightarrow \log\left[\frac{\log\left(\frac{y}{x}\right) - 1}{\frac{y}{x}}\right] = \log(Cx)$$

$$\Rightarrow \frac{x}{y}\left[\log\left(\frac{y}{x}\right) - 1\right] = Cx$$

$$\Rightarrow \log\left(\frac{y}{x}\right) - 1 = Cy$$

This is the required solution of the given differential equation.

10:

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

Solution:

$$\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\Rightarrow \left(1 + e^{\frac{x}{y}}\right)dx = -e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{-e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)}{1 + e^{\frac{x}{y}}} \qquad \ldots\ldots (1)$$

Let $F(x, y) = \dfrac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$

$$\therefore\ F(\lambda x, \lambda y) = \frac{-e^{\frac{\lambda x}{y\lambda}}\left(1-\frac{\lambda x}{\lambda y}\right)}{1+e^{\frac{\lambda x}{y\lambda}}} = \frac{-e^{\frac{\lambda x}{y\lambda}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{\lambda}}} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
$x = vy$

$$\Rightarrow \frac{d}{dy}(x) = \frac{d}{dy}(vy)$$

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the values of x and $\dfrac{dx}{dy}$ in equation (1), we get:

$$v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v}{1+e^v} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{-e^v + ve^v - v - ve^v}{1+e^v}$$

$$\Rightarrow y\frac{dv}{dy} = \left[\frac{v+e^v}{1+e^v}\right]$$

$$\Rightarrow \left[\frac{1+e^v}{v+e^v}\right]dv = -\frac{dy}{y}$$

Integrating both sides, we get:

$$\Rightarrow \log\left(v+e^{v}\right) = -\log y + \log C = \log\left(\frac{C}{y}\right)$$

$$\Rightarrow \left[\frac{x}{y}+e^{\frac{x}{y}}\right] = \frac{c}{y}$$

$$\Rightarrow x + xy^{\frac{x}{y}} = c$$

This is the required solution of the given differential equation.

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11:

$$(x\ +y)dy\,(+x\ -y)dx\ =0; y=1 \text{ when } x=1$$

Solution:

$$(x\ +y)dy\,(+x\ -y)dx\ =0$$

$$\Rightarrow (x+y)dy-(x-y)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(x-y)}{x+y} \quad \ldots\ldots (1)$$

Let $F(x, y) = \dfrac{-(x-y)}{x+y}$

$$\therefore\ F(\lambda x, \lambda y) = \frac{-(\lambda x-\lambda y)}{\lambda x+\lambda y} = \frac{-(x-y)}{x+y} = \lambda^{0}F(x,y)$$

Therefore, the given differential equation is a homogeneous equation.

To solve it, we make the substitution as:

$y = vx$

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\dfrac{dy}{dx}$ in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{-(x - vx)}{x + vx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v - 1 - v(v+1)}{v+1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v - 1 - v^2 - v}{v+1} = \frac{-(1+v^2)}{v+1}$$

$$\Rightarrow \frac{(v+1)}{1+v^2}dv = -\frac{dx}{x}$$

$$\Rightarrow \left[\frac{v}{1+v^2} + \frac{1}{1+v^2}\right]dv = -\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\log(1+v^2) + \tan^{-1} v = -\log x + k$$

$$\Rightarrow \log(1+v^2) + 2\tan^{-1} v = -2\log x + 2k$$

$$\Rightarrow \log\left[(1+v^2)x^2\right] + 2\tan^{-1} v = 2k$$

$$\Rightarrow \log\left[\left(1+\frac{y^2}{x^2}\right)x^2\right] + 2\tan^{-1}\frac{y}{x} = 2k$$

$$\Rightarrow \log(x^2 + y^2) + 2\tan^{-1}\frac{y}{x} = 2k \qquad \ldots\ldots (2)$$

Now, $y = 1$ at $x = 1$

$$\Rightarrow \log 2 + 2\tan^{-1} 1 = 2k$$

$$\Rightarrow \log 2 + 2 \times \frac{\pi}{4} = 2k$$

$$\Rightarrow \frac{\pi}{2} + \log 2 = 2k$$

Substituting the value of 2k in equation (2), we get:

$$\log(x^2 + y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2$$

This is the required solution of the given differential equation.

12:

$x^2 dy\left(+xy \ +y^2\right)dx \ =0; y=1 \text{ where } x=1$

Solution:

$$x^2dy+\left(xy+y^2\right)dx=0$$

$$\Rightarrow x^2dy=-\left(xy+y^2\right)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\left(xy+y^2\right)}{x^2} \qquad .....(1)$$

Let $F(x, y) = \dfrac{-\left(xy+y^2\right)}{x^2}$

$$\therefore \ F\left(\lambda x,\lambda y\right)=\frac{\left[\lambda x.\lambda y+\left(\lambda y\right)^2\right]}{\left(\lambda x\right)^2}=\frac{-\left(xy+y^2\right)}{x^2}=\lambda^0 F\left(x,y\right)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}\left(y\right)=\frac{d}{dx}\left(vx\right)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$v+x\frac{dy}{dx}=\frac{-\left[x.vx+\left(vx\right)^2\right]}{x^2}=-v-v^2$$

$$\Rightarrow x\frac{dy}{dx}=-v^2-2v=-v(v+2)$$

$$\Rightarrow \frac{dv}{v\left(v+2\right)}=\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{\left(v+2\right)-v}{v\left(v+2\right)}\right]dv=-\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left[\frac{1}{2}-\frac{1}{v+2}\right]dv=-\frac{dx}{x}$$

Integrating both sides, we get:

$$\frac{1}{2}\left[\log v-\log\left(v+2\right)\right]=-\log x+\log C$$

$$\Rightarrow\frac{1}{2}\log\left(\frac{v}{v+2}\right)=\log\frac{C}{x}$$

$$\Rightarrow\frac{v}{v+2}=\left(\frac{C}{x}\right)^{2}$$

$$\Rightarrow\frac{\frac{y}{x}}{\frac{y}{x}+2}=\left(\frac{C}{x}\right)^{2}$$

$$\Rightarrow\frac{y}{y+2x}=\frac{c^{2}}{x^{2}}$$

$$\Rightarrow\frac{x^{2}y}{y+2x}=C^{2}\quad \ldots\ldots (2)$$

Now, $y = 1$ at $x = 1$

$$\Rightarrow\frac{1}{1+2}=C^{2}$$

$$\Rightarrow C^{2}=\frac{1}{3}$$

Substituting $C^{2}=\frac{1}{3}$ in equation (2), we get;

$$\frac{x^{2}y}{y+2x}=\frac{1}{3}$$

$$\Rightarrow y+2x=3x^{2}y$$

This is the required solution of the given differential equation.

13:

$$\left[x\sin^{2}\left(\frac{x}{y}-y\right)\right]dx+xdy=0;\ y\frac{\pi}{4}\text{ when } x=1$$

Solution:

$$\left[x\sin^{2}\left(\frac{x}{y}-y\right)\right]dx+xdy=0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x} \qquad \ldots. (1)$$

Let F (x, y) $= \dfrac{-\left[x\sin^2\left(\frac{y}{x}\right) - y\right]}{x}$

$$\therefore\ F(\lambda x, \lambda y) = = \frac{\left[\lambda x\sin^2\left(\frac{\lambda x}{\lambda y} - \lambda y\right)\right]}{x} = \frac{-\left[x\sin^2\left(\frac{y}{x} - y\right)\right]}{x} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve this differential equation, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{-\left[x\sin^2 v - vx\right]}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\left[\sin^2 v - v\right] = v - \sin^2 v$$

$$\Rightarrow x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \frac{dv}{\sin^2 v} = -\frac{dx}{dx}$$

$$\Rightarrow \cos ec^2 dv = -\frac{dx}{dx}$$

Integrating both sides, we get;

$$-\cot v = -\log|x| - C$$

$$\Rightarrow \cot v = \log|x| - C$$

$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|x| + \log C$$

$$\Rightarrow \cot\left(\frac{y}{x}\right) = \log|Cx| \qquad \text{.... (2)}$$

Now, $y = \frac{\pi}{4}$ at $x = 1$

$$\Rightarrow \cot\left(\frac{\pi}{4}\right) = \log|C|$$

$$\Rightarrow 1 = \log C$$

$$\Rightarrow C = e^1 = e$$

Substituting C = e in equation (2), we get:

$$\cot\left(\frac{y}{x}\right) = \log|ex|$$

This is the required solution of the given differential equation.

14:

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0; y = 0 \text{ when } x = 1$$

Solution:

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) \qquad \text{..... (1)}$$

Let F (x, y) $= \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right)$

$$\therefore\ F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} - \operatorname{cosec}\left(\frac{\lambda y}{\lambda x}\right)$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{y}{x} - \operatorname{cosec}\left(\frac{y}{x}\right) = F(x, y)\ \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the value of y and in equation (1), we get:

$$v+x\frac{dv}{dx}=-\operatorname{cosec} v$$
$$\Rightarrow -\frac{dv}{\operatorname{cosec} v}=-\frac{dx}{x}$$
$$\Rightarrow -\sin v dx=\frac{dx}{x}$$

Integrating both sides, we get:

$$\cos v=\log x+\log C=\log|Cx|$$
$$\Rightarrow \cos\left(\frac{y}{x}\right)=\log|Cx| \qquad \text{..... (2)}$$

This is the required solution of the given differential equation.

Now, y = 0 at x = 1.

$$\Rightarrow \cos(0)=\log C$$
$$\Rightarrow 1=\log C$$
$$\Rightarrow C=e^1=e$$

Substituting C = e in equation (2), we get:

$$\cos\left(\frac{y}{x}\right)=\log|(ex)|$$

This is the required solution of the given differential equation.

15:

$$2xy+y^2-2x^2\frac{dy}{dx}=0; y=2 \text{ when} x=1$$

Solution:

$$2xy+y^2-2x^2\frac{dy}{dx}=0$$
$$\Rightarrow 2x^2\frac{dy}{dx}=2xy+y^2$$
$$\Rightarrow \frac{dy}{dx}=\frac{2xy+y^2}{2x^2} \qquad \text{..... (1)}$$

Let $F(x, y)=\frac{2xy+y^2}{2x^2}$

$$\therefore\ F(\lambda x, \lambda y) = \frac{2(\lambda x)(\lambda y) + (\lambda y)^2}{2(\lambda x)^2} = \frac{2xy + y^2}{2x^2} = \lambda^0 F(x, y)$$

Therefore, the given differential equation is a homogeneous equation.
To solve it, we make the substitution as:
y = vx

$$\Rightarrow \frac{d}{dx}(y) = \frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the value of y and in equation (1), we get:

$$v + x\frac{dv}{dx} = \frac{2x(vx) + (vx)^2}{2x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2v + v^2}{2x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \frac{v^2}{2}$$

$$\Rightarrow \frac{2}{v^2}dv = \frac{dx}{x}$$

Integrating both sides, we get:

$$2.\frac{v^{-2+1}}{-2+1} = \log|x| + C$$

$$\Rightarrow -\frac{2}{v} = \log|x| + C$$

$$\Rightarrow -\frac{2}{\frac{y}{x}} = \log|x| + C$$

$$\Rightarrow -\frac{2x}{y} = \log|x| + C \qquad \text{.....(2)}$$

Now, y = 2 at x = 1

$$\Rightarrow -1 = \log(1) + C$$

$$\Rightarrow C = -1$$

Substituting C = -1 in equation (2) we get:

$$-\frac{2x}{y} = \log|x| + 1$$

$$\Rightarrow \frac{2x}{y} = \log|x|$$

$$\Rightarrow y = \frac{2x}{1-\log|x|}, (x \neq 0, x \neq e)$$

This is the required solution of the given differential equation.

16:

A homogeneous differential equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the

Substitution
A. y = vx
B. v = yx
C. x = vy
D. x = v

Solution:

For solving the homogeneous equation of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$, we need to make the

substitution as x = vy.
Hence, the correct answer is C.

17:
Which of the following is a homogeneous differential equation?

A. $(4x+6y+5)dy - (3y+2x+4)dx = 0$

B. $(xy)dx - (x^3+y^3)dy = 0$

C. $(x^3+2y^2)dx + 2xydy = 0$

D. $y^2dx + (x^2-xy^2-y^2)dy = 0$

Solution:
Function F (x, y) is said to be the homogenous function of degree n, if $F(\lambda x, \lambda v) = \lambda' F(x, y)$ for any non-zero constant (λ).
Consider the equation given in alternative D:

$$y^2dx + (x^2-xy^2-y^2)dy = 0$$

$$\Rightarrow \frac{dx}{dy} = \frac{-y^2}{x^2-xy^2-y^2} = \frac{y^2}{y^2+xy-x^2}$$

Let $F(x, y) = \frac{y^2}{y^2 + xy - x^2}$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{(\lambda y)^2}{(\lambda y)^2 + (\lambda x)(\lambda y) - (\lambda x)^2}$$

$$= \frac{\lambda^2 y^2}{\lambda^2 (y^2 + xy - x^2)}$$

$$= \lambda^0 \left(\frac{y^2}{y^2 + xy - x^2} \right)$$

$$= \lambda^0 F(x, y)$$

Hence, the differential equation given in alternative D is a homogenous equation.

Exercise 9.6 Page: 413

For each of the differential equations given in Exercises 1 to 12, find the general solution:

1:

$$\frac{dy}{dx}+2y=\sin x$$

Solution:

The given differential equation is $\frac{dy}{dx}+2y=\sin x$

This is in the form of $\frac{dy}{dx}+py=Q$ (where p = 2 and Q = sinx)

Now, I.F = $e^{\int pdx}=e^{\int 2dx}=2^{2x}$

The solution of the given differential equation is given by the relation,

$$Y(I.f)=\int(Q\times I.F)dx+C$$

$$\Rightarrow ye^{2x}=\int \sin x.e^{2x}dx+C \qquad .....(1)$$

Let $I=\int \sin x.e^{2x}$

$$\Rightarrow I=\int \sin x.\int e^{2x}dx-\int\left(\frac{d}{dx}(\sin x).\int e^{2x}dx\right)dx$$

$$\Rightarrow I=\sin x.\frac{e^{2x}}{2}-\int\left(\cos x.\frac{e^{2x}}{2}\right)dx$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x.\int e^{2x}-\int\left(\frac{d}{dx}(\cos x).\int e^{2x}dx\right)dx\right]$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{1}{2}\left[\cos x.\int\frac{e^{2x}}{2}-\int\left[(-\sin x).\frac{e^{2x}}{2}\right]dx\right]$$

$$\Rightarrow I=\frac{e^{2x}\sin x}{2}-\frac{e^{2x}\cos x}{4}=\frac{1}{4}\int\left(\sin x.e^{2x}\right)dx$$

$$\Rightarrow I=\frac{e^{2x}}{4}(2\sin x-\cos x)-\frac{1}{4}t$$

$$\Rightarrow \frac{5}{4}I=\frac{e^{2x}}{4}(2\sin x-\cos x)$$

$$\Rightarrow I=\frac{e^{2x}}{5}(2\sin x-\cos x)$$

Therefore, equation (1) becomes:

$$ye^{2x}=\frac{e^{2x}}{5}(2\sin x-\cos x)+C$$

$$\Rightarrow y=\frac{1}{5}(2\sin-\cos x)+Ce^{-2x}$$

This is the required general solution of the given differential equation.

2:

$$\frac{dy}{dx}+3y=e^{-2x}$$

Solution:

The given differential equation is $\frac{dy}{dx}+3y=e^{-2x}$(where p = 3 and Q = e^{-2x}

Now I.F = $e^{\int pdx}=e^{\int 3dx}=3^{3x}$

The solution of the given differential equation is given by the relation,

$$\Rightarrow ye^{3x}=\int\left(e^{-2x}\times e^{3x}\right)+C$$

$$\Rightarrow ye^{3x}=\int e^{x}dx+C$$

$$\Rightarrow ye^{3x}=e^{x}+C$$

$$\Rightarrow y=e^{-2x}+Ce^{-3x}$$

This is the required general solution of the given differential equation.

3:

$$\frac{dy}{dx}=\frac{y}{x}=x^2$$

Solution:

The given differential equation is:

(Where $p=\frac{1}{x}$ and $Q=x^2$)

Now, I.F = $e^{\int pdx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$

The solution of the given differential equation is given by the relation,

$$\Rightarrow y(x)=\int\left(x^2.x\right)dx+C$$

$$\Rightarrow xy=\int x^3dx+C$$

$$\Rightarrow xy=\frac{x^4}{4}+C$$

This is the required general solution of the given differential equation.

4:

$$\frac{dy}{dx}+\sec xy=\tan x\left(0\le x\le\frac{\pi}{2}\right)$$

Solution:

The given differential equation is:

(where p = sexx and Q = tanx)

Now I.F. $e^{\int pdx}=e^{\int \sec .xdx}=e^{\log(\sec x+\tan x)}$ = sec x + tan x

The general solution of the given differential equation is given by the relation,

$$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \int \sec x \tan x dx + \int \tan^2 x dx + C$$

$$\Rightarrow y(\sec x + \tan x) = \sec x + \int \left(\sec^2 x - 1\right)dx + C$$

$$\Rightarrow y\left(\sec x + \tan x\right) = \sec x + \tan x - x + C$$

5:

$$\int_0^{\frac{x}{2}} \cos 2x dx$$

Solution:

Let $I = \int_0^{\frac{x}{2}} \cos 2x dx$

$$\int \cos 2x dx = \left(\frac{\sin 2x}{2}\right) = F\left(x\right)$$

By second fundamental theorem of calculus, we obtain

$$I = F\left(\frac{\pi}{2}\right) - F = \left(0\right)$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{1}{2}\left[\sin \pi - \sin 0\right]$$

$$= \frac{1}{2}\left[0 - 0\right] = 0$$

6:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

Solution:

The given differential equation is:

$$x\frac{dy}{dx} + 2y = x^2 \log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = x \log x$$

This equation is in the form of a linear differential equation as:

(Where $p = \frac{2}{x}$ and Q = x log x)

Now, IF = $e^{\int pdx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dx + C$$

$$\Rightarrow y.x^2 = \int (x\log x.x^2)dx + C$$

$$\Rightarrow x^2y = \int (x^3\log x)dx + C$$

$$\Rightarrow x^2y = \log x.\int x^3dx - \int\left[\frac{d}{dx}(\log x).\int x^3dx\right]dx + C$$

$$\Rightarrow x^2y = \log x.\frac{x^4}{4} - \int\left(\frac{1}{x}.\frac{x^4}{4}\right)dx + C$$

$$\Rightarrow x^2y = \frac{x^4\log x}{4} - \frac{1}{4}\int x^3dx + C$$

$$\Rightarrow x^2y = \frac{x^4\log x}{4} - \frac{1}{4}\frac{x^4}{4} + C$$

$$\Rightarrow x^2y = \frac{1}{16}x^4(4\log x - 1) + C$$

$$\Rightarrow y = \frac{1}{16}x^4(4\log x - 1) + Cx^{-2}$$

7:

$$x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$$

Solution:

The given differential equation is:

$$x\log x\frac{dy}{dx} + y = \frac{2}{x}\log x$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x\log x} = \frac{2}{x^2}$$

This equation is the form of a linear differential equation as:

(Where $p=\dfrac{1}{x\log x}$ and $Q=\dfrac{2}{x^2}$)

Now, I.F = $e^{\int p dx}=e^{\int \frac{1}{x\log}dx}=e^{\log(\log x)}=\log x$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F})=\int(Q\times \text{I.F})dx+C$$

$$\Rightarrow y\log x=\int\left(\frac{2}{x^2}\log x\right)dx+C \ \ldots..(1)$$

Now, $\int\left(\dfrac{2}{x^2}\log x\right)dx = 2\int\left(\log x.\dfrac{1}{x^2}\right)dx$

$$=2\left[\log x.\int\frac{1}{x^2}dx-\int\left\{\frac{d}{dx}(\log x).\int\frac{1}{x^2}dx\right\}dx\right]$$

$$=2\left[\log x\left(-\frac{1}{x}\right)-\int\left(\frac{1}{x}\left(-\frac{1}{x}\right)\right)dx\right]$$

$$=2\left[-\frac{\log x}{x}+\int\frac{1}{x^2}dx\right]$$

$$=2\left[-\frac{\log x}{x}-\frac{1}{x}\right]$$

$$=-\frac{2}{x}(1+\log x)$$

Substituting the value $\int\left(\dfrac{2}{x^2}\log x\right)dx$ of in equation (1), we get:

$$y\log x=-\frac{2}{x}(1+\log x)+C$$

This is the required general solution of the given differential equation.

8:

$$\left(1+x^2\right)dy+2xy\,dx=\cot x dx\left(x\neq 0\right)$$

Solution:

$$\left(1+x^2\right)dy+2xy\,dx=\cot x dx$$

$$\Rightarrow \frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{\cot x}{1+x^2}$$

This equation is a linear differential equation of the form:

$$\left(\text{Where } p=\frac{2x}{1+x^2} \text{ and } Q=\frac{\cot x}{1+x^2}\right)$$

Now, I.F. $e^{\int pdx}=e^{\int \frac{2x}{1+x^2}dx}=e^{\log\left(1+x^2\right)}=1+x^2$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F})=\int(Q\times \text{I.F})dx+C$$

$$\Rightarrow y\left(1+x^2\right)=\int\left[\frac{\cot x}{1+x^2}\times\left(1+x^2\right)\right]dx+C$$

$$\Rightarrow y\left(1+x^2\right)=\int \cot\ xdx+C$$

$$\Rightarrow y\left(1+x^2\right)=\log|\sin x|+C$$

9:

$$x\frac{dy}{dx}+y-x+xy\cot x=0(x\neq 0)$$

Solution:

$$x\frac{dy}{dx}+y-x+xy\cot x=0$$

$$\Rightarrow x\frac{dy}{dx}+y(1\cot x)y=1$$

$$\Rightarrow x\frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$$

This equation is a linear differential equation of the form:

$$(\text{Where } p=\frac{1}{x}+\cot x \text{ and } Q=1)$$

Now, I.F. $=e^{\int pdx}=e^{\int\left(\frac{1}{x}+\cot x\right)dx}\ e^{\log+\log(\sin x)}=e^{\log(x\sin x)}=x\sin x$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F})=\int(Q\times \text{I.F})dx+C$$

$$\Rightarrow y(x\sin x)=\int(1\times x\sin x)dx+C$$

$$\Rightarrow y(x\sin x)=\int(x\sin x)dx+C$$

$$\Rightarrow y(x\sin x)=x\int \sin x\,dx-\int\left[\frac{d}{dx}(x).\int \sin xdx\right]+C$$

$$\Rightarrow y(x\sin x) = x - (\cos x) - \int 1.(-\cos x)dx + C$$

$$\Rightarrow y(x\sin x) = -x\cos x + \sin x + C$$

$$\Rightarrow y = \frac{-x\cos x}{x\sin x} + \frac{\sin x}{x\sin x} + \frac{C}{x\sin x}$$

$$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}$$

10:

$$\left(x + y\frac{dy}{dx}\right) = 1$$

Solution:

$$\left(x + y\frac{dy}{dx}\right) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+y}$$

$$\Rightarrow \frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{dy}{dx} - x = y$$

This is a linear differential equation of the form:

(Where p = − 1 and Q = y)

Now, I.F = $= e^{\int pdx} = \int e^{-dy} = e^{-y}$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dy + C$$

$$\Rightarrow xe^{-y} = \int (y.e^{-y})dy + C$$

$$\Rightarrow xe^{-y} = y.\int e^{-y}dy - \int\left[\frac{d}{dy}(y).\int e^{-y}dy\right]dy + C$$

$$\Rightarrow xe^{-y} = y(-e^{-y}) - \int(-e^{-y})dy + C$$

$$\Rightarrow xe^{-y} = -ye^{-y} + \int e^{-y}dy + C$$

$\Rightarrow xe^{-y} = -ye^{-y} - e^{-y} + C$

$\Rightarrow x = -y - 1 + Ce^{y}$

$\Rightarrow x + y + 1 = Ce^{y}$

11:

$ydx + \left(x - y^{2}\right)dy = 0$

Solution:

$ydx + \left(x - y^{2}\right)dy = 0$

$\Rightarrow ydx + \left(y^{2} - x\right)dy$

$$\Rightarrow \frac{dx}{dy} = \frac{y^{2} - x}{y} = y - \frac{x}{y}$$

$$\Rightarrow \frac{dy}{dx} + \frac{x}{y} = y$$

This is a linear differential equation of the form:

(Where $p = \frac{1}{y}$ and $Q = y$)

Now, I.F. $= e^{\int pdx} = e^{\int \frac{1}{2}dy} = e^{-y} = e^{\log y} = y$

The general solution of the given differential equation is given by the relation,

$y\left(I.F\right) = \int \left(Q \times I.F\right)dy + C$

$\Rightarrow xy = \int \left(y.y\right)dy + C$

$\Rightarrow xy = \int y^{2}dy + C$

$$\Rightarrow xy = \frac{y^{3}}{3} + C$$

$$\Rightarrow x = \frac{y^{3}}{3} + \frac{C}{y}$$

12:

$$\left(x + 3y^{3}\right)\frac{dy}{dx} = y\left(y > 0\right)$$

Solution:

$$\left(x+3y^3\right)\frac{dy}{dx}=y$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x+3y^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+3y^2}{y}=\frac{x}{y}+3y$$

$$\Rightarrow \frac{dx}{dy}-\frac{x}{y}=3y$$

This is a linear differential equation of the form:

(Where $p=-\frac{1}{y}$ and $Q=3y$)

Now, I.F. $=e^{\int p dy}=e^{-\int \frac{dy}{y}}=e^{-\log y}=e^{\log\left(\frac{1}{y}\right)}=\frac{1}{y}$

The general solution of the given differential equation is given by the relation,

$$y\left(I.F\right)=\int\left(Q\times I.F\right)dy+C$$

$$\Rightarrow x\times\frac{1}{y}=\int\left(3y\times\frac{1}{y}\right)dy+C$$

$$\Rightarrow \frac{x}{y}=3y+C$$

$$\Rightarrow x=3y^2+Cy$$

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13:

$$\frac{dy}{dx}+2y\tan x=\sin x; y=0 \text{ when } x=\frac{\pi}{3}$$

Solution:

The given differential equation is $\frac{dy}{dx}+2y\tan x=\sin x$

This is a linear equation of the form:

(Where p = 2tanx and Q = sin x)

Now, I.F $=e^{\int p dx}=e^{\int 2\tan x dx}=e^{2\log|\sec x|}=e^{\log\left(\sec^2 x\right)}=\sec^2 x$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dy + C$$
$$\Rightarrow y(\sec^2 x) = \int (\sin x.\sec^2 x)dx + C$$
$$\Rightarrow y\sec^2 x = \int (\sec x.\tan x)dx + C$$
$$\Rightarrow y\sec^2 x = \sec x + C \qquad \text{.....} (1)$$

Now, y = 0 at $x = \frac{\pi}{3}$

Therefore,

$$0 \times \sec^2 \frac{\pi}{3} = \sec\frac{\pi}{3} + C$$
$$\Rightarrow 0 = 2 + C$$
$$\Rightarrow C = -2$$

Substituting C = –2 in equation (1), we get:

$$y\sec^2 x = \sec x - 2$$
$$\Rightarrow y = \cos x - 2\cos^2 x$$

Hence, the required solution of the given differential equation is $y = \cos x - 2\cos^2 x$

14:

$$\left(1+x^2\right)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}; y = 0 \text{ when } x = 1$$

Solution:

$$\left(1+x^2\right)\frac{dy}{dx} + 2xy = \frac{1}{1+x^2}$$
$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{\left(1+x^2\right)^2}$$

This is a linear differential equation of the form:

(Where $p = \frac{2x}{1+x^2}$ and $Q = \frac{1}{\left(1+x^2\right)^2}$)

Now I.F. $= e^{\int p dx} = e^{\int \frac{2x dx}{1+x^2}} = e^{\log\left(1+x^2\right)} = 1 + x^2$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F})dx + C$$

$$\Rightarrow y\left(1+x^2\right)=\int\left[\frac{1}{\left(1+x^2\right)^2}.\left(1+x^2\right)\right]dx+C$$

$$\Rightarrow y\left(1+x^2\right)=\int\frac{1}{1+x^2}dx+C$$

$$\Rightarrow y\left(1+x^2\right)=\tan^{-1}x+C\ \ldots\ldots(1)$$

Now, y = 0 at x = 1.

Therefore,

$$\Rightarrow C=-\frac{\pi}{4}$$

Substituting $C=-\frac{\pi}{4}$ in equation (1), we get:

$$y\left(1+x^2\right)=\tan^{-1}x-\frac{\pi}{4}$$

This is the required general solution of the given differential equation.

15:

$$\frac{dy}{dx}-3y\cot x=\sin 2x;\ y=2\text{ when }x=\frac{\pi}{2}$$

Solution:

The given differential equation is $\frac{dy}{dx}-3y\cot x=\sin 2x$

This is a linear differential equation of the form:

(Where p = − 3 cot x and Q = sin 2x)

Now, $\text{I.F.}=e^{\int pdx}=e^{-3\int \cot xdx}=e^{-3\log|\sin x|}=e^{\log\left|\frac{1}{\sin^{-1}x}\right|}=\frac{1}{\sin^3 x}$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F})=\int(Q\times \text{I.F})dx+C$$

$$\Rightarrow y.\frac{1}{\sin^3 x}=\int\left[\sin 2x.\frac{1}{\sin^3 x}\right]dx+C$$

$$\Rightarrow y\operatorname{cosec}^3 x=2\int(\cot x\operatorname{cosec} x)dx+C$$

$$\Rightarrow y\operatorname{cosec}^3 x=2\operatorname{cosec} x+C$$

$$\Rightarrow y = -\frac{2}{\operatorname{cosec}^2 x} + \frac{3}{\operatorname{cosec}^2 x}$$

$$\Rightarrow y = -2\sin^2 x + C\sin^3 x$$

Now, $y = 2$ at $x = \frac{\pi}{2}$

Therefore, we get:

$2 = -2 + C$

$\Rightarrow C = 4$

Substituting C = 4 in equation (1), we get:

$$y = -2\sin^2 x + 4\sin^3 x$$

$$\Rightarrow y = 4\sin^3 x - 2\sin^2 x$$

This is the required particular solution of the given differential equation.

16:

Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

Solution:

Let F (x, y) be the curve passing through the origin.

At point (x, y), the slope of the curve will be .

According to the given information:

$$\frac{dy}{dx} = x + y$$

$$\Rightarrow \frac{dy}{dx} = -y = x$$

This is a linear differential equation of the form:

(Where p = −1 and Q = x)

Now, I.F = $= e^{\int p dx} = e^{\int (-1) dx} = e^{-x}$

The general solution of the given differential equation is given by the relation,

$$y(\text{I.F}) = \int (Q \times \text{I.F}) dx + C$$

Now, $\int xe^{-x} dx = x\int e^{-x} dx - \int \left[\frac{d}{dx}(x) . \int e^{-x} dx \right] dx$

$= -xe^{-x} - \int e^{-x} dx$

$= -xe^{-x} + \left(-e^{-x}\right)$

$= -e^{-x}\left(x+1\right)$

Substituting in equation (1), we get:

$ye^{-x} = -e^{-x}\left(x+1\right) + C$

$\Rightarrow y = -\left(x+1\right) + Ce^{x}$

$\Rightarrow x + y + 1 = Ce^{x}$ .....(2)

The curve passes through the origin.
Therefore, equation (2) becomes:
1 = C
Substituting C = 1 in equation (2), we get:

$\Rightarrow x + y + 1 = e^{x}$

Hence, the required equation of curve passing through the origin is $x + y + 1 = e^{x}$

17:
Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

Solution:
Let F (x, y) be the curve and let (x, y) be a point on the curve. The slope of the tangent to the

curve at (x, y) is

According to the given information:

$$\frac{dy}{dx} + 5 = x + y$$

$$\Rightarrow \frac{dy}{dx} - y = x + 5$$

This is a linear differential equation of the form:

(where p = −1 and Q = x – 5)

Now, I.F. $= e^{\int p dx} = e^{\int (-1) dx} = e^{-x}$

The general equation of the curve is given by the relation,

$$y\left(I.F\right) = \int \left(Q \times I.F\right) dx + C$$

$$\Rightarrow y.e^{-x} = \int \left(x-5\right) e^{-x} dx + C \quad ......(1)$$

Now, $\int \left(x-5\right) e^{-x} dx = \left(x-5\right)\int e^{-x} dx - \int \left[\frac{d}{dx}\left(x-5\right).\int e^{-x} dx\right] dx$

$$= (x-5)\left(-e^{-x}\right) - \int\left(-e^{-x}\right)dx$$

$$= (5-x)e^{-x} + \left(e^{-x}\right)$$

$$= (4-x)e^{-x}$$

Therefore, equation (1) becomes:

The curve passes through point (0, 2).
Therefore, equation (2) becomes:
0 + 2 – 4 = Ce0
⇒ – 2 = C
⇒ C = – 2
Substituting C = –2 in equation (2), we get:

$$x + y - 4 = -2e^{x}$$

$$\Rightarrow y = 4 - x - 2e^{x}$$

This is the required equation of the curve.

18:

The integrating factor of the differential equation $x\dfrac{dy}{dx} - y = 2x^2$ **is**

A. e^{-x}

B. e^{-y}

C. $\dfrac{1}{x}$

D. x

Solution:

The given differential equation is:

$$x\frac{dy}{dx} - y = 2x^2$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x$$

This is a linear differential equation of the form:

(Where $p = -\dfrac{1}{x}$ and Q = 2x)

The integrating factor (I.F) is given by the relation,

$$e^{\int pdx}$$

$$\therefore \text{I.F.} = e^{\int \frac{1}{x}dx} = e^{-\log x} = e^{\log\left(x^{-1}\right)} = x^{-1} = \frac{1}{x}$$

Hence, the correct answer is C.

19:
The integrating factor of the differential equation.

$$\left(1-y^2\right)\frac{dx}{dy}+yx=ay\left(-1>y<1\right)$$

A. $\frac{1}{y^2-1}$

B. $\frac{1}{\sqrt{y^2-1}}$

C. $\frac{1}{1-y^2}$

D. $\frac{1}{\sqrt{1-y^2}}$

Solution:
The given differential equation is:

$$\left(1-y^2\right)\frac{dx}{dy}+yx=ay$$

$$\Rightarrow\frac{dy}{dx}+\frac{yx}{1-y^2}=\frac{ay}{1-y^2}$$

This is a linear differential equation of the form:

(where $p=-\frac{y}{1-y^2}$ and $Q=\frac{ay}{1-y^2}$)

The integrating factor (I.F) is given by the relation,

$e^{\int pdx}$

$$\therefore I.F=e^{\int pdx}=e^{\int\frac{1}{1-y^2}dx}=e^{\frac{1}{2}\log x\left(1-y^2\right)}=e^{\log\left[\frac{1}{\sqrt{1-y^2}}\right]}=\frac{1}{1-y^2}$$

Hence, the correct answer is D.

Miscellaneous Exercise

1:

For each of the differential equations given below, indicate its order and degree (if defined).

(i) $\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$

(ii) $\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$

(iii) $\frac{d^4y}{dx^2}-\sin\left(\frac{d^3y}{dx^3}\right)=0$

Solution:

(i)The differential equation is given as:

$$\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y=\log x$$

$$\Rightarrow\frac{d^2y}{dx^2}+5x\left(\frac{dy}{dx}\right)^2-6y-\log x$$

The highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. Thus, its order is two.

The highest power raised to $\frac{d^2y}{dx^2}$ is one. Hence, its degree is one.

(ii)The differential equation is given as:

$$\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y=\sin x$$

$$\Rightarrow\left(\frac{dy}{dx}\right)^3-4\left(\frac{dy}{dx}\right)^2+7y-\sin x=0$$

The highest order derivative present in the differential equation is $\frac{dy}{dx}$. Thus, its order is one.

The highest power raised to $\frac{dy}{dx}$ is three. Hence, its degree is three.

(iii)The differential equation is given as:

$$\frac{d^4y}{dx^2}-\sin\left(\frac{d^3y}{dx^3}\right)=0$$

The highest order derivative present in the differential equation is $\frac{d^4y}{dx^4}$. Thus, its order is four.

However, the given differential equation is not a polynomial equation. Hence, its degree is not defined.

2:

For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.

(i) $y = ae^x + be^{-x} + x^2$: $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0$

(ii) $y - e^x\left(a\cos x + b\sin x\right)$: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

(iii) $y = x\sin 3x$: $\frac{d^2y}{dx^2} + 9y - 6\cos 3x = 0$

(iv) $x^2 = 2y^2\log y$: $\left(x^2 + y^2\right)\frac{dy}{dx} - xy = 0$

Solution:

(i) $y = ae^x + be^{-x} + x^2$

Differentiating both sides with respect to x, we get:

$$\frac{d^2y}{dx^2} = a\frac{d}{dx}\left(e^x\right) + b\frac{d}{dx}\left(e^{-x}\right) + \frac{d}{dx}\left(x^2\right)$$

$$\Rightarrow \frac{dy}{dx} = ae^x - be^{-x} + 2x$$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^2y}{dx^2} = ae^x + be^{-x} + 2$$

Now, on substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the differential equation, we get:

L.H.S

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2$$

$$= x\left(ae^{x} + be^{-x} + 2\right) + 2\left(ae^{x} - be^{-x} + 2x\right) - x\left(ae^{x} + be^{-x} + x^{2}\right) + x^{2} - 2$$

$$= \left(axe^{x} + bxe^{-x} + 2x\right) + \left(2ae^{x} - 2be^{-x} + 4x\right) - \left(axe^{x} + bxe^{-x} + x^{3}\right) + x^{2} - 2$$

$$= 2ae^{x} - 2be^{-x} + x^{2} + 6x - 2$$

$$\neq 0$$

L.H.S $\neq$ R.H.S

Hence, the given function is not a solution of the corresponding differential equation.

(ii) $y = e^{x}\left(a\cos x + b\sin x\right) = ae^{x}\cos x + be^{x}\sin x$

Differentiating both sides with respect to x, we get:

$$\frac{dy}{dx} = a.\frac{d}{dx}\left(e^{x}\cos x\right) + b.\frac{d}{dx}\left(e^{x}\sin x\right)$$

$$\Rightarrow \frac{dy}{dx} = a\left(e^{x}\cos x - e^{x}\sin x\right) + b.\left(e^{x}\sin x + e^{x}\cos x\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(a + b\right)e^{x}\cos x + \left(b - a\right)\sin x$$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^{2}y}{dx^{2}} = \left(a + b\right)\frac{d}{dx}\left(e^{x}\cos x\right) + \left(b - a\right)\frac{d}{dx}\left(e^{x}\sin x\right)$$

$$\Rightarrow \frac{d^{2}y}{dx^{2}} = \left(a + b\right).\left(e^{x}\cos x - e^{x}\sin x\right) + \left(b - a\right)\left(e^{x}\sin x + e^{x}\cos x\right)$$

$$\Rightarrow \frac{d^{2}y}{dx^{2}} = e^{x}\left[\left(a + b\right)\left(\cos x - \sin x\right) + \left(b - a\right)\left(\sin x + \cos x\right)\right]$$

$$\Rightarrow \frac{d^{2}y}{dx^{2}} = e^{x}\left[a\cos x - a\sin x + b\cos x - b\sin x + b\sin x + b\cos x - a\sin x - a\cos x\right]$$

$$\Rightarrow \frac{d^{2}y}{dx^{2}} = \left[2e^{x}\left(b\cos x - a\sin x\right)\right]$$

Now, on substituting the values of $\frac{d^{2}y}{dx^{2}}$ and $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$= 2e^x(b\cos x - a\sin x) - 2e^x\left[(a+b)\cos x + (b-a)\sin x\right] + 2e^x(a\cos x + b\sin x)$$

$$= e^x\begin{bmatrix}(2b\cos x - 2a\sin x) - (2a\cos x - 2bosx) \\ -(2b\sin x - 2a\sin x) + (2a\cos x - 2b\sin x)\end{bmatrix}$$

$$= e^x\left[(2b - 2a - 2b)\cos x\right] + e^x\left[(-2a - 2b + 2a + 2b)\sin x\right]$$

$$= 0$$

Hence, the given function is a solution of the corresponding differential equation.

(iii) y = x sin 3x

Differentiating both sides with respect to x, we get:

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x\sin 3x) = \sin 3x + x.\cos 3x.3$$

$$\Rightarrow \frac{dy}{dx} = \sin 3x + 3x\cos 3x$$

Again, differentiating both sides with respect to x, we get:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\sin 3x) + 3\frac{d}{dx}(x\cos 3x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 3\cos 3x + 3\left[\cos 3x + x(-\sin 3x).3\right]$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6\cos 3x - 9x\sin 3x$$

Substituting the value in the L.H.S. of the given differential equation, we get:

$$\frac{d^2y}{dx^2} + 9y - 6\cos 3x$$

$$= (6.\cos 3x - 9x\sin x) + 9x\sin 3x - 6\cos 3x$$

= 0

Hence, the given function is a solution of the corresponding differential equation.

(iv) $x^2 = 2y^2 \log y$

Differentiating both sides with respect to x, we get:

$$2x = 2\frac{d}{dx} = \left[y^2 \log y\right]$$
$$\Rightarrow x = \left[2y.\log y.\frac{dy}{dx} + y^2.\frac{1}{y}.\frac{dy}{dx}\right]$$
$$\Rightarrow x = \frac{dy}{dx}(2y\log y + y)$$
$$\Rightarrow \frac{dy}{dx} = \frac{x}{y(1+2\log y)}$$

Substituting the value of $\frac{dy}{dx}$ in the L.H.S. of the given differential equation, we get:

$$\left(x^2 + y^2\right)\frac{dy}{dx} - xy$$
$$= \left(2y^2 \log y + y^2\right).\frac{x}{y(1+2\log y)} - xy$$
$$= y^2(1+2\log y).\frac{x}{y(1+2\log y)} - xy$$
$$= xy - xy$$
$$= 0$$

Hence, the given function is a solution of the corresponding differential equation.

3:

Form the differential equation representing the family of curves given by $(x-a)^2 + 2y^2 = a^2$ where a is an arbitrary constant.

Solution:

$$(x-a)^2 + 2y^2 = a^2$$
$$\Rightarrow x^2 + a^2 - 2ax + 2y^2 = a^2$$
$$\Rightarrow 2y^2 = 2ax - x^2 \quad \text{….. (1)}$$

Differentiating with respect to x, we get:

$$2y\frac{dy}{dx} = \frac{2a-2x}{2}$$
$$\Rightarrow \frac{dy}{dx} = \frac{a-x}{2y}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2ax-2x^3}{4xy} \qquad .....(2)$$

From equation (1), we get:

$$2ax=2y^2+x^2$$

On substituting this value in equation (3), we get:

$$\frac{dy}{dx}=\frac{2y^2+x^2-2x^3}{4xy}$$

$$\Rightarrow \frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$$

Hence, the differential equation of the family of curves is given as $\frac{dy}{dx}=\frac{2y^2-x^2}{4xy}$

4:

Prove that $x^2-y^2=c\left(x^2+y^2\right)^2$ is the general solution of differential equation $\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$ where c is a parameter.

Solution:

$$\left(x^3-3xy^2\right)dx=\left(y^3-3x^2y\right)dy$$

$$\Rightarrow \frac{dy}{dx}=\frac{x^3-3xy^2}{y^3-3x^2y} \qquad ....(1)$$

This is a homogeneous equation. To simplify it, we need to make the substitution as:
$y = vx$

$$\Rightarrow \frac{d}{dx}(y)=\frac{d}{dx}(vx)$$

$$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dv}{dx}$ in equation (1), we get:

$$v+x\frac{dv}{dx}=\frac{x^3-3x(vx)^2}{(vx)^3-3x^2(vx)}$$

$$\Rightarrow v+x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2}{v^3-3v}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-3v^2-v\left(v^3-3v\right)}{v^3-3v}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{1-v^4}{v^3-3v}$$

$$\Rightarrow \left(\frac{v^3-3v}{1-v^4}\right)dv=\frac{dx}{x}$$

Integrating both sides, we get:

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv=\log x+\log C' \quad \ldots(2)$$

Now, $\int\left(\frac{v^3-3v}{1-v^4}\right)dv=\int\frac{v^3-3v}{1-v^4}-3\int\frac{vdv}{1-v^4}$

Now, $\int\left(\frac{v^3-3v}{1-v^4}\right)dv=I_1-3I_2$ Where $I_1=\int\frac{v^3dv}{1-v^4}$ and $I_2\int\frac{vdv}{1-v^4}$ ....(3)

Let $1-v^4=t$

$$\therefore \frac{d}{dv}\left(1-v^4\right)=\frac{dt}{dv}$$

$$\Rightarrow -4v^3=\frac{dt}{dv}$$

$$\Rightarrow v^3dv=-\frac{dt}{4}$$

Now, $I_1=\int\frac{dt}{4t}=-\frac{1}{4}\log t=-\frac{1}{4}\log\left(1-v^4\right)$

And, $I_2=\int\frac{vdv}{1-v^4}=\int\frac{vdv}{1-\left(v^2\right)^2}$

Let $v^2=p$

$$\therefore \frac{d}{dv}\left(v^2\right)=\frac{dp}{dv}$$

$$\Rightarrow 2v=\frac{dp}{dv}$$

$$\Rightarrow vdv = \frac{dp}{2}$$

$$\Rightarrow I_2 = \frac{1}{2}\int \frac{dp}{1-p^2} = \frac{1}{2\times 2}\log\left|\frac{1+p}{1-p}\right| = \frac{1}{4}\log\left|\frac{1+v^2}{1-v^2}\right|$$

Substituting the values of I1 and I2 in equation (3), we get:

$$\int\left(\frac{v^3-3v}{1-v^4}\right)dv = -\frac{1}{4}\log\left(1-v^4\right) - \frac{3}{4}\log\left|\frac{1-v^2}{1+v^2}\right|$$

Therefore, equation (2) becomes:

$$\frac{1}{4}\log\left(1-v^2\right) - \frac{3}{4}\log\left|\frac{1-v^2}{1+v^2}\right| = \log x + \log C'$$

$$\Rightarrow -\frac{1}{4}\log\left[\left(1-v^4\right)\left(\frac{1+v^2}{1-v^2}\right)^3\right] = \log C'x$$

$$\Rightarrow \frac{\left(1+v^2\right)^4}{\left(1-v^2\right)^2} = \left(C'x\right)^{-4}$$

$$\Rightarrow \frac{\left(1+\frac{y^2}{x^2}\right)^4}{\left(1-\frac{y^2}{x^2}\right)^2} = \frac{1}{C'^4x^4}$$

$$\Rightarrow \frac{\left(x^2+y^2\right)^4}{x^4\left(x^2-y^2\right)^2} = \frac{1}{C'^4x^4}$$

$$\Rightarrow \left(x^2+y^2\right)^2 = C'^4\left(x^2+y^2\right)^4$$

$$\Rightarrow \left(x^2-y^2\right) = C'^2\left(x^2+y^2\right)^2$$

$$\Rightarrow x^2-y^2 = C\left(x^2+y^2\right)^2, \text{where } C = C'^2$$

Hence, the given result is proved.

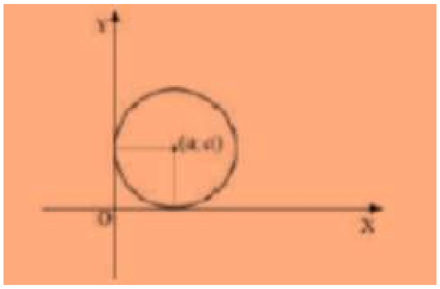
5:

Form the differential equation of the family of circles in the first quadrant which touch the coordinate axes.

Solution:

The equation of a circle in the first quadrant with centre (a, a) and radius (a) which touches the coordinate axes is:

$$(x-a)^2+(y-a)^2=a^2 \quad \ldots\ldots (1)$$



Differentiating equation (1) with respect to x, we get:

$$2(x-a)+2(y-a)\frac{dy}{dx}=0$$

$$\Rightarrow (x-a)+(y-a)y'=0$$

$$\Rightarrow x-a+yy'-ay'=0$$

$$\Rightarrow x+yy'-a(1+y')=0$$

$$\Rightarrow a=\frac{x+yy'}{1+y'}$$

Substituting the value of a in equation (1), we get:

$$\left[x-\left(\frac{x+yy'}{1+y'}\right)\right]^2+\left[y-\left(\frac{x+yy'}{1+y'}\right)\right]^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

$$\Rightarrow \left[\frac{(x-y)y'}{(1+y')}\right]^2+\left[\frac{y-x}{1+y'}\right]^2=\left[\frac{x+yy'}{1+y'}\right]^2$$

$$\Rightarrow (x-y)^2.y'^2+(x-y)^2=(x+yy')^2$$

$$\Rightarrow (x-y)^2\left[1+(y')^2\right]=(x+yy')^2$$

Hence, the required differential equation of the family of circles is

$$(x-y)^2\left[1+(y')^2\right]=(x+yy')^2$$

6:

Find the general solution of the differential equation $\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$

Solution:

$$\frac{dy}{dx}+\sqrt{\frac{1-y^2}{1-x^2}}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}}=\frac{-dx}{\sqrt{1-x^2}}$$

Integrating both sides, we get:

$$\sin^{-1} y=-\sin^{-1} x+C$$

$$\Rightarrow \sin^{-1} x+\sin^{-1} y=C$$

7:

Show that the general solution of the differential equation $\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$ is given by (x + y + 1) = A (1 – x – y – 2xy), where A is parameter.

Solution:

$$\frac{dy}{dx}+\frac{y^2+y+1}{x^2+x+1}=0$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\left(y^2+y+1\right)}{x^2+x+1}$$

$$\Rightarrow \frac{dy}{y^2+y+1}=\frac{-dx}{x^2+x+1}$$

$$\Rightarrow \frac{dy}{y^2+y+1}+\frac{dx}{x^2+x+1}=0$$

Integrating both sides, we get:

$$\int\frac{dy}{y^2+y+1}+\int\frac{dx}{x^2+x+1}=C$$

$$\Rightarrow \int \frac{dy}{y^2+y+1} + \int \frac{dx}{x^2+x+1} = C - \frac{\left(y^2+y+1\right)}{x^2+x+1}$$

$$\Rightarrow \int \frac{dy}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = C$$

$$\Rightarrow \frac{2}{\sqrt{3}} \tan^{-1}\left[\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + \frac{2}{\sqrt{3}} \tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] = C$$

$$\Rightarrow \tan^{-1}\left[\frac{2y+1}{\sqrt{3}}\right] + \tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] = \frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2y+1}{\sqrt{3}} + \frac{2x+1}{\sqrt{3}}}{1 - \frac{(2y+1)}{\sqrt{3}}\frac{(2x+1)}{\sqrt{3}}}\right] = \frac{\sqrt{3}C}{2}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x+2y+2}{\sqrt{3}}}{1 - \left(\frac{4xy+2x+2y+1}{3}\right)}\right] = \frac{\sqrt{3}C}{2}$$

Hence, the given result is proved.

8:

Find the equation of the curve passing through the point 0, $\left(0, \frac{\pi}{4}\right)$ whose differential equation is, sin x cos ydx + cos x sin ydy = 0

Solution:

The differential equation of the given curve is:

sin x cos ydx + cos x sin ydy = 0

$$\Rightarrow \frac{\sin x \cos y dx + \cos x \sin y dy}{\cos x \cos y} = 0$$

$\Rightarrow$ tan xdx + tan ydy = 0

Integrating both sides, we get:

log (sec x) + log (sec y) = log C
log (sec x. sec y) = log C
$\Rightarrow$ sec x. sec y = C …(1)

The curve passes through point $\left(0, \dfrac{\pi}{4}\right)$

$$\therefore 1 \times \sqrt{2} = C$$

$$\Rightarrow C = \sqrt{2}$$

On substituting in equation (1), we get:

$$\sec x . \sec y = \sqrt{2}$$

$$\Rightarrow \sec x . \frac{1}{\cos y} = \sqrt{2}$$

$$\Rightarrow \cos\ y = \frac{\sec x}{\sqrt{2}}$$

Hence, the required equation of the curve is $\text{os}\ y = \dfrac{\sec x}{\sqrt{2}}$

9:
Find the particular solution of the differential equation
$\left(1+e^x\right)dy + \left(1+y^2\right)e^x dx = 0$, given that y = 1 when x = 0

Solution:

$$\left(1+e^x\right)dy + \left(1+y^2\right)e^x dx = 0$$

$$\Rightarrow \frac{dy}{1+y^2} + \frac{e^x dx}{1+e^{2x}} = 0$$

Integrating both sides, we get:

$$\tan^{-1} y + \int \frac{e^x dx}{1+e^{2x}} = C \qquad \text{….(1)}$$

Let $e^x = t \Rightarrow e^{2x} = t^2$

$$\Rightarrow \frac{d}{dx}\left(e^x\right) = \frac{dt}{dx}$$

$$\Rightarrow e^x = \frac{dt}{dx}$$

Substituting these values in equation (1), we get:

$$\tan^{-1} y + \int \frac{dt}{1+t^2} = C$$

$$\Rightarrow \tan^{-1} y + \tan^{-1} t = C$$

$$\Rightarrow \tan^{-1} y + \tan^{-1}\left(e^x\right) = C \qquad \text{....(2)}$$

Now, y = 1 at x = 0.

Therefore, equation (2) becomes:

$$\tan^{-1} 1 + \tan^{-1} 1 = C$$

$$\Rightarrow \frac{\pi}{4} + \frac{\pi}{4} = C$$

$$\Rightarrow C = \frac{\pi}{4}$$

Substituting $C = \frac{\pi}{4}$ in equation (2), we get:

$$\tan^{-1} y + \tan^{-1}\left(e^x\right) = \frac{\pi}{4}$$

This is the required particular solution of the given differential equation.

10:

Solve the differential equation $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy\left(y \neq 0\right)$

Solution:

$$ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y^2\right)dy$$

$$\Rightarrow ye^{\frac{x}{y}}\frac{dx}{dy} = xe^{\frac{x}{y}} + y^2$$

$$\Rightarrow e^{\frac{x}{y}}\left[y.\frac{dx}{dy} - x\right] = y^2$$

$$\Rightarrow e^{\frac{x}{y}}\frac{\left[y.\frac{dx}{dy} - x\right]}{y^2} = 1 \qquad \text{.... (1)}$$

Let $e^{\frac{x}{y}} = z$

Differentiating it with respect to y, we get:

$$\frac{d}{dy}\left(e^{\frac{x}{y}}\right)=\frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}.\frac{d}{dy}\left(\frac{x}{y}\right)=\frac{dz}{dy}$$

$$\Rightarrow e^{\frac{x}{y}}\left[\frac{y.\frac{dx}{dy}-x}{y^2}\right]=\frac{dz}{dy} \quad \ldots.(2)$$

From equation (1) and equation (2), we get:

$$\frac{dz}{dy}=1$$

$$\Rightarrow dz=dy$$

Integrating both sides, we get:

$$z=y+C$$

$$\Rightarrow e^{\frac{x}{y}}=y+C$$

11:

Find a particular solution of the differential equation (x – y) (dx + dy) = (dx – dy, given that y = – 1, when x = 0 (Hint: put x – y = t)

Solution:

$$(x-y)(dx+dy)=dx-dy$$

$$\Rightarrow (x-y+1)dy=(1-x+y)dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-x+y}{x-y+1}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1-(x-y)}{1+(x-y)} \quad \ldots.(1)$$

Let $x-y=t$

$$\Rightarrow \frac{d}{dx}(x-y)=\frac{dt}{dx}$$

$$\Rightarrow 1-\frac{dy}{dx}=\frac{dt}{dx}$$

$$\Rightarrow 1-\frac{dt}{dx}=\frac{dy}{dx}$$

Substituting the values of x – y and $\frac{dy}{dx}$ in equation (1), we get:

$$1-\frac{dt}{dx}=\frac{1-t}{1+t}$$

$$\Rightarrow 1-\frac{dy}{dx}=\frac{dt}{dx}$$

$$\Rightarrow \frac{dt}{dx}=1-\left(\frac{1-t}{1+t}\right)$$

$$\Rightarrow \frac{dt}{dx}=\frac{(1+t)-(1-t)}{1+t}$$

$$\Rightarrow \frac{dt}{dx}=\frac{2t}{1+t}$$

$$\Rightarrow \left(\frac{1-t}{t}\right)dt=2dx$$

$$\Rightarrow \left(1+\frac{1}{t}\right)dt=2dx \quad .....(2)$$

Integrating both sides, we get:

$$t+\log|t|=2x+C$$

$$\Rightarrow (x-y)+\log|x-y|=2x+C$$

$$\Rightarrow \log|x-y|=x+y+C \quad .... (3)$$

Now, y = –1 at x = 0.

Therefore, equation (3) becomes:

log 1 = 0 – 1 + C

⇒ C = 1

Substituting C = 1 in equation (3) we get:

$$\log|x-y|=x+y+1$$

This is the required particular solution of the given differential equation.

12:

Solve the differential equation $\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\frac{dx}{dy}=1(x\neq 0)$

Solution:

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\frac{dx}{dy}=1$$

$$\Rightarrow \frac{dy}{dx}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

This equation is a linear differential equation of the form

$$\frac{dy}{dx}+py=Q,\text{ Where }P=\frac{1}{\sqrt{x}}\text{ and }Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Now, I.F. $e^{\int pdx}=e^{\int \frac{1}{\sqrt{x}}dx}=e^{2\sqrt{x}}$

The general solution of the given differential equation is given by,

$$y(\text{I.F})=\int(Q\times \text{I.F.})dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=\int\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\times e^{-2\sqrt{x}}\right)dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=\int\frac{1}{x}dx+C$$

$$\Rightarrow ye^{2\sqrt{x}}=2\sqrt{x}+C$$

13:

Find a particular solution of the differential equation $\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x\,(x\neq 0)$, given that $y=0$ when $x=\frac{\pi}{2}$

Solution:

The given differential equation is:

$$\frac{dy}{dx}+y\cot x=4x\operatorname{cosec}x$$

This equation is a linear differential equation of the form

$$\frac{dy}{dx}+Py=Q\text{ where }P=\cot x\text{ and }Qx\operatorname{cosec}x$$

Now, I.F = $e^{\int pdx}=e^{\int \cot.c.dx}=e^{\log|\sin x|}=\sin x$

The general solution of the given differential equation is given by,

$$y(\text{I.F}) = \int (Q \times \text{I.F.})dx + C$$
$$\Rightarrow y\sin x = \int (4x\,\text{cosec}x.\sin x)dx + C$$
$$\Rightarrow y\sin x = 4\int x dx + C$$
$$\Rightarrow y\sin x = 4.\frac{x^2}{2} + C$$
$$\Rightarrow y\sin x = 2x^2 + C \qquad \text{.... (1)}$$

Now, $y = 0$ at $x = \dfrac{\pi}{2}$

Therefore, equation (1) becomes:

$$0 = 2 \times \frac{\pi^2}{4} + C$$
$$\Rightarrow C = -\frac{\pi^2}{4}$$

Substituting $C = -\dfrac{\pi^2}{4}$ in equation (1), we get;

$$y\sin x = 2x^2 - \frac{\pi^2}{4}$$

This is the required particular solution of the given differential equation.

14:

Find a particular solution of the differential equation $(x+1)\dfrac{dy}{dx} = 2e^{-y} - 1$, given that y = 0 **when x = 0**

Solution:

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$$
$$\Rightarrow \frac{dy}{2e^{-y} - 1} = \frac{dx}{x+1}$$
$$\Rightarrow \frac{e^y dy}{2 - e^y} = \frac{dx}{x+1}$$

Integrating both sides, we get:

$$\int \frac{e^y dy}{2 - e^y} = \log|x+1| + \log C \qquad \text{... (1)}$$

Let $2-e^{y}=t$

$$\therefore \frac{d}{dy}\left(2-e^{y}\right)=\frac{dt}{dy}$$

$$\Rightarrow -e^{x}=\frac{dt}{dy}$$

$$\Rightarrow e^{y}dt=-dt$$

Substituting this value in equation (1), we get:

$$\int\frac{-dt}{t}=\log|x+1|+\log C$$

$$\Rightarrow -\log|t|=\log|C(x+1)|$$

$$\Rightarrow -\log\left|2-e^{y}\right|=\log|C(x+1)|$$

$$\Rightarrow \frac{1}{2-e^{y}}=C(x+1)$$

$$\Rightarrow 2-e^{y}=\frac{1}{C(x+1)} \qquad \text{.... (2)}$$

Now, at x = 0 and y = 0, equation (2) becomes:

$$\Rightarrow 2-1=\frac{1}{C}$$

$$\Rightarrow C=1$$

Substituting C = 1 in equation (2), we get:

$$2-e^{y}=\frac{1}{x+1}$$

$$\Rightarrow e^{y}=2-\frac{1}{x+1}$$

$$\Rightarrow e^{y}=\frac{2x+2-1}{x+1}$$

$$\Rightarrow e^{y}=\frac{2x+1}{x+1}$$

$$\Rightarrow y=\log\left|\frac{2x+1}{x+1}\right|,(x\neq-1)$$

This is the required particular solution of the given differential equation.

15:

The population of a village increases continuously at the rate proportional to the number of its inhabitants present at any time. If the population of the village was 20000 in 1999 and 25000 in the year 2004, what will be the population of the village in 2009?

Solution:

Let the population at any instant (t) be y.

It is given that the rate of increase of population is proportional to the number of inhabitants at any instant.

$$\therefore \frac{dy}{dx} \propto y$$

$$\Rightarrow \frac{dy}{dt} = ky \quad \text{(k is a constant)}$$

$$\Rightarrow \frac{dy}{y} = kdt$$

Integrating both sides, we get:

log y = kt + C … (1)

In the year 1999, t = 0 and y = 20000.

Therefore, we get:

log 20000 = C … (2)

In the year 2004, t = 5 and y = 25000.

Therefore, we get:

log25000 = k.5+ C

$\Rightarrow$ log 25000 = 5k + log 20000

$$\Rightarrow 5k = \log\left(\frac{25000}{20000}\right) = \log\left(\frac{5}{4}\right)$$

$$\Rightarrow k = \frac{1}{5}\log\left(\frac{5}{4}\right)$$

In the year 2009, t = 10 years.

Now, on substituting the values of t, k, and C in equation (1), we get:

$$\log y = 10 \times \frac{1}{5}\log\left(\frac{5}{4}\right) + \log(20000)$$

$$\Rightarrow \log y = \log\left[20000 \times \left(\frac{5}{4}\right)^2\right]$$

$$\Rightarrow y = 20000 \times \frac{5}{4} \times \frac{5}{4}$$

$$\Rightarrow y = 31250$$

Hence, the population of the village in 2009 will be 31250.

16:

The general solution of the differential equation $\frac{ydx - xdy}{y} = 0$ is

A. xy = C
B. $x = Cy^2$
C. y = Cx
D. $y = Cx^2$

Solution:

$$\frac{ydx - xdy}{y} = 0$$

$$\Rightarrow \frac{ydx - xdy}{xy} = 0$$

$$\Rightarrow \frac{1}{x}dx - \frac{1}{y}dy = 0$$

Integrating both sides, we get:

$$\log|x| - \log|y| = \log k$$

$$\Rightarrow \log\left|\frac{x}{y}\right| = \log k$$

$$\Rightarrow \frac{x}{y}$$

$$\Rightarrow y = \frac{1}{k}x$$

$$\Rightarrow y = Cx \text{ where } C = \frac{1}{k}$$

Hence, the correct answer is C.

17:

The general solution of a differential equation of the type $\frac{dx}{dy} + P_1x = Q_1$ is

A. $y.e^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$

B. $y.e^{\int P_1 dx} = \int\left(Q_1 e^{\int P_1 dx}\right)dx + C$

C. $x.e^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dy + C$

D. $x.e^{\int P_1 dx} = \int\left(Q_1 e^{\int P_1 dx}\right)dx + C$

Solution:

The integrating factor of the given differential equation $\frac{dx}{dy} + P_1 x = Q_1$ is $.e^{\int P_1 dy}$

The general solution of the differential equation is given by,

$$x(\text{I.F.}) = \left(\int Q \times \text{I.F}\right)dy + C$$

$$x.e^{\int P_1 dy} = \int\left(Q_1 e^{\int P_1 dy}\right)dx + C$$

Hence, the correct answer is C.

18:

The general solution of the differential equation $e^x dy + \left(ye^x + 2x\right)dx = 0$ is

A. $xe^y + x^2 = C$

B. $xe^y + y^2 = C$

C. $ye^x + x^2 = C$

D. $ye^y + x^2 = C$

Solution:

The given differential equation is:

$$e^x dy + \left(ye^x + 2x\right)dx = 0$$

$$\Rightarrow e^x \frac{dy}{dx} + ye^x + 2x = 0$$

$$\Rightarrow \frac{dy}{dx} + y = -2xe^{-x}$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, Where P = 1 and $Q = -\ 2xe^{-x}$

Now, I,F = $e^{\int p dx} = e^{\int dx} = e^x$

The general solution of the given differential equation is given by,

$$y(\text{I.F.}) = \left(\int Q \times \text{I.F}\right)dx + C$$

$$\Rightarrow ye^{x} = \int\left(-2xe^{-x}.x^{x}\right)dx + C$$

$$\Rightarrow ye^{x} = -\int 2xdx + c$$

$$\Rightarrow ye^{x} = -x^{2} + C$$

$$\Rightarrow ye^{x} + x^{2} = C$$

Hence, the correct answer is C.